Gulf Journal of Mathematics ISSN: 2309-4966
Vol 23, Issue 1 (2026) 1-33 https://doi.org/10.56947/fe3jq212

ON MULTIPLE SOLUTIONS GENERATED BY A PARAMETRIC SUBCRITICAL NONLINEARITY IN A QUASILINEAR ELLIPTIC DIRICHLET PROBLEM

TCHALLA AYÉKOTAN MESSAN JOSEPH[1]

Abstract. his paper deals with the existence of weak solutions for a quasilinear Dirichlet elliptic problem with a more general subcritical reaction involving a parameter. According to the values of the parameter, we clearly show the non-existence and multiplicity results. Specifically, we first exhibited a critical positive value such that the problem admits at least a nontrivial non-negative weak solution if and only if the parameter is greater than or equal to this critical value. Furthermore, for the values of parameter greater than a second critical positive value, we prove the existence of a second independent nontrivial non-negative weak solution of the problem.

Keywords. Quasilinear elliptic equations, Dirichlet problem, parametric nonlinearity, Variational methods, N-functions, Orlicz spaces, Ekeland's principle.
2020 Mathematics Subject Classification. 35A01; 35A15; 35B38; 35D30; 35J92; 46E30.

Introduction

For several decades, various papers have focused on the study of the existence, non-existence or even the multiplicity of nontrivial weak solutions of nonlinear elliptic problems, depending on a parameter, in a bounded domain of $\mathbb{R}^N$ or on the entire $\mathbb{R}^N$ space. Azorero and Alonso in [6] studied the existence of solutions for the following nonlinear degenerate elliptic problems in a bounded domain

$$\begin{cases} -div\,(|\nabla u|^{p-2}\nabla u) = |u|^{p^*-2}u + \lambda |u|^{q-2}u \text{ in } \Omega, \\ \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{0.1}$$

where $1 < p < N$, $1 < q < p^* = \frac{Np}{N-p}$, $\lambda > 0$, Ω a bounded domain of $\mathbb{R}^N$ with a boundary $\partial\Omega$. In [6], for $p < q < p^*$, one exihbited $\lambda_0 > 0$ such that problem (0.1) has a nontrivial solution for all $\lambda > \lambda_0$. On the other hand in [6], Azorero and Alonso proved that if $1 < q < p$ in problem (0.1), then (0.1) possesses an infinitely many solutions for $\lambda \in (0, \lambda_1)$, where $\lambda_1 > 0$ is a constant.

Date: Received: Feb 27, 2026; Revised: Mar 25, 2026; Accepted: Apr 8, 2026.
* Corresponding author

In [3], Ambrosetti, Brézis and Cerami consider the semilinear elliptic Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u^q + u^p & x \in \Omega \\ u > 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases} \tag{0.2}$$

where $0 < q < 1 < p$, λ is a real parameter, Ω a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. In [3], the authors proved that, for $N \geqslant 3$, if $0 < q < 1 < p \leqslant \frac{N+2}{N-2}$, there exists $\Lambda > 0$ such that problem (0.2) admits at least two solutions for all $\lambda \in (0, \Lambda)$, has one solution for $\lambda = \Lambda$, and no solution exists provided that $\lambda > \Lambda$.

Moreover, in the same paper, they proved, for example, that there exists $\Lambda^* > 0$ such that for all $\lambda \in (0, \Lambda^*)$, the problem

$$\begin{cases} -\Delta u = \lambda |u|^{q-1}u + |u|^{p-1}u & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

has an infinitely many solutions with negative energy when when $0 < q < 1 < p \leqslant \frac{N+2}{N-2}$. For their part, Alama and Tarantello later studied, in [2], according to the values of the real parameter λ the existence and multiplicity of non-negative weak solutions of the semilinear Dirichlet problem

$$\begin{cases} -\Delta u - \lambda u = \omega(x)u^q - h(x)u^r & x \in \Omega \\ u > 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}, \tag{0.3}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 3)$ is a bounded domain with smooth boundary, ω and h are functions in $L^1(\Omega)$ and $1 < q < r$. More specifically, for λ in a neighborhood of λ_1 (the first eigenvalue of operator $-\Delta$ in $H_0^1(\Omega)$), they obtained the solvability of (0.3) (and corresponding multiplicities) under different kinds of hypotheses about ω and h.

In the literature, one finds both several semilinear and quasilinear elliptic problems with one parametric nonlinearity characterized by nonlinear functions of the form $|u|^{p-2}u$ and dealing with existence and multiplicity results for nontrivial weak solutions depending on the values taken by parameter. Among semilinear elliptic problems we refer to [11],[18], [19], [4], [24], [10] and the references therein. Concerning the case of quasilinear elliptic equations in bounded domains or in entire space $\mathbb{R}^N$, we can mark the works [15], [16], [20], [13], [7], [21] and [5].
Like the works cited previously, this paper deals with the existence and multiplicity of solutions to the problem depending on the values of parameter. However, in terms of results, our work is much closer to the articles [21, 5]. The major novelty of our work compared to the previous ones lies in the fact that the nonlinear source term of our equation involves nonlinear terms that generalize those found,

for example, in [2, 21, 5]. More precisely, in this paper, we study the quasilinear elliptic equation with a single parameter

$$\begin{cases} -div\,(|\nabla u|^{m-2}\nabla u) + |u|^{m-2}u = \lambda f(|u|)u - g(|u|)u \text{ in } \Omega, \\ \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_\lambda$}$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, is a bounded domain with smooth boundary, f, g suitable positive functions and λ is a real parameter. This work that we do here is the quasilinear version of the semilinear problem developed in [23]. Indeed, by setting $m = 2$ in ($\mathcal{P}_\lambda$), we recover the problem studied in [23]. The solutions of the semilinear problem of [23] were considered in a space X that is continuously embedded in both the Sobolev space $H_0^1(\Omega)$ and the Orlicz space $L_G(\Omega)$ (the space $L_G(\Omega)$ will be defined later). For the problem ($\mathcal{P}_\lambda$), which is the focus of this paper, the solutions are found in the space X, the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_X = \left(\|u\|^m_{W_0^{1,m}(\Omega)} + \|u\|^m_G\right)^{1/m}$ where $\|u\|_{W_0^{1,m}(\Omega)}$ denotes the standard the norm in the Sobolev space $W_0^{1,m}(\Omega)$ and $\|u\|_G$ is the norm in Orlicz space $L_G(\Omega)$. Thus, the main results of this paper generalize those of [23] to the space X, which is continuously embedded in $W_0^{1,m}(\Omega)$.
By taking inspiration from the method developed in [21],[5] and [23], we use variational arguments to study the existence and multiple nontrivial weak solutions of problem ($\mathcal{P}_\lambda$) according to the values of the parameter λ. To obtain our results in this work, we require in ($\mathcal{P}_\lambda$) the following assumptions:

($\mathscr{F}_1$) f is a function continuous on $\mathbb{R}_+$ and of $C^1((0,+\infty),\mathbb{R}_+)$ such that

$$f(t) > 0,\ (f(t)t)' > 0 \text{ for all } t > 0, \tag{0.4}$$

$$f(t) = 0 \Leftrightarrow t = 0, \tag{0.5}$$

$$\lim_{t\to+\infty} f(t) = +\infty. \tag{0.6}$$

Let's set

$$F(t) = \int_0^t sf(s)ds \text{ for all } t \geqslant 0; \tag{0.7}$$

($\mathscr{F}_2$) There exist $p, q \in \mathbb{R}_+$ such that $\max(2,m) < p \leqslant q < m^* = \frac{Nm}{N-m}$ and

$$p - 2 \leqslant \frac{f'(t)t}{f(t)} \leqslant q - 2, \text{ for all } t > 0; \tag{0.8}$$

($\mathscr{G}_1$) g is a continuous function on $\mathbb{R}_+$ and of $C^1((0,+\infty),\mathbb{R}_+)$ such that

$$g(t) > 0,\ (g(t)t)' > 0 \text{ for all } t > 0, \tag{0.9}$$

$$g(t) = 0 \Leftrightarrow t = 0, \tag{0.10}$$

$$\lim_{t\to+\infty} g(t) = +\infty. \tag{0.11}$$

Let's set

$$G(t) = \int_0^t sg(s)ds \text{ for } t \geqslant 0; \tag{0.12}$$

($\mathscr{G}_2$) There exists r such that $r > q$ and

$$q - 2 < \frac{g'(t)t}{g(t)} \leqslant r - 2, \text{ for all } t > 0. \tag{0.13}$$

Thus, it is clear that the functions f and g of the Dirichlet problem ($\mathcal{P}_\lambda$) of our present work generalize the functions $|u|^{q-2}$ and $|u|^{r-2}$ $(2 < q < r)$ which appear in the main equation studied in [13], [21] or [5].
The main goal of this paper is the proof of the following two theorems.

Theorem A. *By the fulfillment of assumptions ($\mathscr{F}_1$), ($\mathscr{F}_2$) ($\mathscr{G}_1$) and ($\mathscr{G}_2$), there exists a critical value $\hat{\lambda} > 0$ such that the Dirichlet problem ($\mathcal{P}_\lambda$) admits at least a nontrivial non-negative weak solution if and only if $\lambda \geqslant \hat{\lambda}$.*

Theorem B. *Suppose that the assumptions ($\mathscr{F}_1$), ($\mathscr{F}_2$) ($\mathscr{G}_1$) and ($\mathscr{G}_2$) are fulfilled. Then, there exists a critical value $\bar{\lambda}$ satisfying $\bar{\lambda} \geqslant \hat{\lambda}$ such that for all $\lambda > \overline{\lambda}$, the Dirichlet problem ($\mathcal{P}_\lambda$) admits at least two nontrivial non-negative weak solutions.*

In Section 1 we talk about Orlicz spaces which we will use in our work. In the Section 2 we give different embeddings between the working spaces of this paper and prove the non-existence of a nontrivial weak solution when λ in ($\mathcal{P}_\lambda$) is less than a positive number. The conditions for existence of weak solutions of ($\mathcal{P}_\lambda$) are established in Sections 3. Section 4 is devoted to proving Theorem A and section 5 deal with the proof of Theorem B.

1. Some notions on Orlicz spaces

This section repeats exactly section 2 of [23].

Definition 1 (Definition of a N-function)**.** *Let ψ be a real-valued function defined on $[0, \infty)$ and having the following properties:*

(a) $\psi(0) = 0, \psi(t) > 0$ *if* $t > 0$, $\lim\limits_{t \to +\infty} = +\infty$;
(b) ψ *is nondecreasing, that is,* $s > t$ *implies* $\psi(s) \geqslant \psi(t)$;
(c) ψ *is right continuous, that is, if* $t \geq 0$, *then* $\lim\limits_{s \to t^+} \psi(s) = \psi(t)$.

Then function Ψ defined on $[0, +\infty)$ by

$$\Psi(t) = \int_0^t \psi(s)ds$$

is a N-function.

Any such N-function Ψ has the following properties:

(i) Ψ is continuous on $[0, \infty)$;
(ii) Ψ is strictly increasing;
(iii) Ψ is convex;
(iv) $\lim\limits_{t \to 0} \frac{\Psi(t)}{t} = 0$ and $\lim\limits_{t \to +\infty} \frac{\Psi(t)}{t} = +\infty$;
(iv) the function $t \mapsto \frac{\Psi(t)}{t}$ is strictly increasing on $(0, \infty)$.

For any N-function $\Psi = \Psi(t)$ and an open set $\Omega \subset \mathbb{R}^N$, the Orlicz space $L_\Psi(\Omega)$ is defined. When Ψ satisfies Δ_2-condition, i.e,

$$\Psi(2t) \leqslant k\Psi(t), \text{ for all } t \geqslant 0,$$

for some constant $k > 0$, then

$$L_\Psi(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega \Psi(|u(x)|)dx < \infty \right\}.$$

Endowed with the norm

$$\|u\|_\Psi = \inf \left\{ k > 0 : \int_\Omega \Psi\left(\frac{|u(x)|}{k}\right) dx \leqslant 1 \right\}$$

which is called the Luxembourg norm, the Orlicz space $L_\Psi(\Omega)$ is a Banach space. It's known that if $\displaystyle\int_\Omega \Psi\left(\frac{|u(x)|}{k_0}\right) dx = 1$, then $\|u\|_\Psi = k_0$ with $k_0 > 0$.
The complement of Ψ is given by the Legendre transformation

$$\widetilde{\Psi}(s) = \max_{t \geqslant 0}(st - \Psi(t)) \quad \text{for } s \geqslant 0.$$

We say that Ψ and $\widetilde{\Psi}$ are complementary N-functions each other.
For all $t > 0$, we have the inequality $\widetilde{\Psi}\left(\frac{\Psi(t)}{t}\right) \leqslant \Psi(t)$.
From Young's inequality

$$st \leqslant \Psi(t) + \widetilde{\Psi}(s)$$

a generalized version of Hölder's inequality is obtained:

$$\left| \int_\Omega u(x)v(x)dx \right| \leqslant 2\|u\|_\Psi \|v\|_{\widetilde{\Psi}} \quad \text{for } u \in L_\Psi(\Omega),\ v \in L_{\widetilde{\Psi}}(\Omega). \tag{1.1}$$

2. Preliminaries and non-exitence of nontrivial solution for λ small

By condition $(\mathscr{F}_1)$ together with the definition of F, and condition $(\mathscr{G}_1)$ together with the definition of G, the functions F and G are N-functions, with $\psi(s) = sf(s)$ for F and $\psi(s) = sg(s)$ for G, respectively.

From inequalities (0.8) and (0.13), it respectivelty follows that

$$\text{for all } t > 0,\ p \leqslant \frac{f(t)t^2}{F(t)} \leqslant q \tag{2.1}$$

and

$$\text{for all } t > 0,\ q < \frac{g(t)t^2}{G(t)} \leqslant r. \tag{2.2}$$

Furthermore, by Lemmas 1 and 2 of [23], the N-functions F and G satisfy the Δ_2-condition.

Let's consider the Sobolev space $W_0^{1,m}(\Omega)$ which is the completion of $C_0^\infty(\Omega)$ with the norm

$$\|u\|_{W_0^{1,m}(\Omega)} = \left(\int_\Omega |\nabla u|^m\right)^{1/m}.$$

Denote by X the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_X = \left(\|u\|^m_{W_0^{1,m}(\Omega)} + \|u\|^m_G\right)^{1/m}$$

where

$$\|u\|_G = \inf\left\{\lambda > 0 : \int_\Omega G\left(\frac{|u(x)|}{\lambda}\right) dx \leqslant 1\right\}$$

is the Luxembourg norm in the Orlicz space $L_G(\Omega)$. The space X is the space in which we will find our nontrivial weak solutions.

Lemma 2.1. *The embeddings $X \hookrightarrow W_0^{1,m}(\Omega) \hookrightarrow L^{m^*}(\Omega)$ are continuous with $\|u\|_{W_0^{1,m}(\Omega)} \leqslant \|u\|_X$, for all $u \in X$ and $\|u\|_{m^*} \leqslant C_{m^*}\|u\|_{W_0^{1,m}(\Omega)}$, for all $u \in W_0^{1,m}(\Omega)$.*
Furthermore $W_0^{1,m}(\Omega) \hookrightarrow\hookrightarrow L^\rho(\Omega)$ and $X \hookrightarrow\hookrightarrow L^\rho(\Omega)$ are compact for all ρ such that $1 \leqslant \rho < m^*$.

Proof. The definition of the norm in X immediately implies the continuous embedding $X \hookrightarrow W_0^{1,m}(\Omega)$. The embedding $W_0^{1,m}(\Omega) \hookrightarrow L^{m^*}(\Omega)$ is due to Talenti [22], with C_{m^*} representing the Talenti constant. By the Rellich–Kondrachov theorem, it follows that $W_0^{1,m}(\Omega)$ is compactly embedded in $L^\rho(\Omega)$ for $\rho \in [1, m^*)$. Hence X is compactly embedded in $L^\rho(\Omega)$ for the same $\rho :\ X \hookrightarrow\hookrightarrow L^\rho(\Omega)$. □

Lemma 2.2. *Let $\zeta_0(t) = min(t^p, t^q)$, $\zeta_1(t) = max(t^p, t^q)$, $\zeta_2(t) = min(t^q, t^r)$ and $\zeta_3(t) = max(t^q, t^r)$ for all $t \geqslant 0$. Then*

$$\zeta_0(\rho)F(t) \leqslant F(\rho t) \leqslant \zeta_1(\rho)F(t) \ \ \textit{for}\ \rho, t \geqslant 0; \tag{2.3}$$

$$\zeta_0(\|u\|_F) \leqslant \int_\Omega F(|u(x)|)dx \leqslant \zeta_1(\|u\|_F) \ \ \textit{for}\ u \in L_F(\Omega); \tag{2.4}$$

$$\zeta_2(\rho)G(t) \leqslant G(\rho t) \leqslant \zeta_3(\rho)G(t) \ \ \textit{for}\ \rho, t \geqslant 0; \tag{2.5}$$

$$\zeta_2(\|u\|_G) \leqslant \int_\Omega G(|u(x)|)dx \leqslant \zeta_3(\|u\|_G) \ \ \textit{for}\ u \in L_G(\Omega). \tag{2.6}$$

Proof. For proof, we simply follow the procedure of Lemma 2.1 of Article [14]. In fact By integrating the inequalities (2.1) and (2.2) respectively, we get inéqualities (2.3) and (2.5). From (2.3) and (2.5) and the definition of Luxembourg norm we get respectively (2.4) and (2.6). □

Lemma 2.3 (Lemma 5 of [23])**.** *The space $L_G(\Omega)$ is continuously embedded in $L_F(\Omega)$ with $\|u\|_F \leqslant \mathcal{H}\|u\|_G$, where*

$$\mathcal{H} = 2k\omega$$

with $k = \max\left\{1, \left(\frac{F(1)}{G(1)}\right)^{1/q}\right\}$ *and* $\omega = \max\left\{1, \frac{F(1)}{G\left(kF^{-1}\left(\frac{1}{2meas(\Omega)}\right)\right)}\right\}$.

Proof. For the proof of this Lemma, see the proof of Lemma 5 of the article [23]. □

Lemma 2.4. *The $N-function$ F satisfies the following*

$$\lim_{t\to+\infty}\frac{F(kt)}{t^{m^*}}=0 \text{ for all } k\geqslant 0.$$

Proof. the path of this demonstration is the same as that of the Lemma 3.6 of [23]. In fact, by (2.3), for all $k\geqslant 0,$ for all $t\geqslant 1$, we have

$$0\leqslant\frac{F(kt)}{t^{m^*}}\leqslant\frac{\zeta_1(t)F(k)}{t^{m^*}}=t^{q-m^*}F(k).$$

□

Lemma 2.5. *The embedding $L^{m^*}(\Omega)\hookrightarrow L_F(\Omega)$ is continuous with*

$$\|u\|_F\leqslant C_F^*\|u\|_{m^*}, \text{ for all } u\in L^{m^*}(\Omega)$$

where $C_F^*=2\max\left\{1,\left(F^{-1}\left(\frac{1}{2meas(\Omega)}\right)\right)^{-m^*}\right\}(F(1))^{\frac{1}{m^*}}.$

Proof. The N-function F increases more slowly than $t\mapsto t^{m^*}$ near infinity and by (2.3) it follows that for all $t\geqslant 1$, $F(t)\leqslant F(1)t^{m^*}$.

By the proof of Theorem 8.12 ([1]),

$$\text{for all } t\geqslant F^{-1}\left(\frac{1}{2meas(\Omega)}\right),\ F(t)\leqslant\omega F(1)t^{m^*},$$

where $\omega=\max\left\{1,\left(F^{-1}\left(\frac{1}{2meas(\Omega)}\right)\right)^{-m^*}\right\}$. Let $u\in L^{m^*}(\Omega)$ such that $u\neq 0$.

Let $\Omega'(u)=\left\{x\in\Omega:\frac{|u(x)|}{2\omega\left(F(1)\right)^{\frac{1}{m^*}}\|u\|_{m^*}}\leqslant F^{-1}\left(\frac{1}{2meas(\Omega)}\right)\right\}$ and $\Omega''(u)=\Omega-\Omega'(u)$.

$$\begin{aligned}\int_\Omega F\left(\frac{|u(x)|}{2\omega(F(1))^{\frac{1}{m^*}}\|u\|_{m^*}}\right)dx&=\int_{\Omega'(u)}F\left(\frac{|u(x)|}{2\omega\left(F(1)\right)^{\frac{1}{m^*}}\|u\|_{m^*}}\right)dx\\&\quad+\int_{\Omega''(u)}F\left(\frac{|u(x)|}{2\omega(F(1))^{\frac{1}{m^*}}\|u\|_{m^*}}\right)dx\\&\leqslant\frac{1}{2meas(\Omega)}\int_{\Omega'(u)}dx+\omega\int_{\Omega''(u)}\frac{|u(x)|^{m^*}}{(2\omega\|u\|_{m^*})^{m^*}}dx\\&\leqslant\frac12+\frac12\int_\Omega\frac{|u(x)|^{m^*}}{\|u\|_{m^*}^{m^*}}dx.\\&\leqslant 1.\end{aligned}$$

It follows that $\|u\|_F\leqslant 2\omega\left(F(1)\right)^{\frac{1}{m^*}}\|u\|_{m^*}$ and consequently $L^{m^*}(\Omega)$ is imbedded continously in $L_F(\Omega)$. □

Lemma 2.6. *The embedding $W_0^{1,m}(\Omega) \hookrightarrow L_F(\Omega)$ is continuous with*

$$\|u\|_F \leqslant C_F \|u\|_{W_0^{1,m}(\Omega)} \text{ for all } u \in W_0^{1,m}(\Omega), \text{ where } C_F = C_{m^*}C_F^*. \tag{2.7}$$

Moreover, the embeddings $W_0^{1,m}(\Omega) \hookrightarrow\hookrightarrow L_F(\Omega)$ and $X \hookrightarrow\hookrightarrow L_F(\Omega)$ are compacts.

Proof. The continuity of the embedding $W_0^{1,m}(\Omega) \hookrightarrow L_F(\Omega)$ is followed from Lemmas 2.1 and 2.5. By Lemma 2.4 and Theorem 8.36 of [1], the embedding $W_0^{1,m}(\Omega) \hookrightarrow\hookrightarrow L_F(\Omega)$ is compact. Henceforward, $X \hookrightarrow\hookrightarrow L_F(\Omega)$ is compact since $X \hookrightarrow W_0^{1,m}(\Omega)$ is continuous by Lemma 2.1. □

Lemma 2.7. *If u belongs to $X \setminus \{0\}$ and λ is a real such that*

$$\int_\Omega |\nabla u|^m dx + \int_\Omega |u|^m dx + \int_\Omega g(|u|)u^2 dx = \lambda \int_\Omega f(|u|)u^2 dx, \tag{2.8}$$

then $\lambda > 0$ and there exists two positive constants k_1 and k_2 independent of u such that

$$k_1 f_1(\lambda) \leqslant \|u\|_F \leqslant k_2 f_2(\lambda) \tag{2.9}$$

where f_1 and f_2 are two functions of λ.

Proof. Let's take $u \in X \setminus \{0\}$ and $\lambda \in \mathbb{R}$ such that (2.8) hold. Thus we have $0 < \|u\|^m_{W_0^{1,m}(\Omega)} \leqslant \lambda \int_\Omega f(|u|)u^2 dx$. As f is a nonnegative function in $(0, +\infty)$, it follows that $\lambda > 0$. Let show the second part of the Lemma. Firstly, by using respectively (2.7), (2.8), (2.1) and (2.4) , we get

$$\|u\|_F^m \leqslant q\lambda C_F^m \zeta_1(\|u\|_F). \tag{2.10}$$

Thus, this last inequality yields

$$\|u\|_F \geqslant \min\left\{ \left(qC_F^m\right)^{1/(m-p)} \lambda^{1/(m-p)}, \left(qC_F^m\right)^{1/(m-q)} \lambda^{1/(m-q)} \right\}. \tag{2.11}$$

Secondly, by (2.7), (2.8), (2.1) and (2.2) we have

$$\begin{aligned} \|u\|_F^m &\leqslant C_F^m \|u\|^m_{W_0^{1,m}(\Omega)} \\ &\leqslant C_F^m \left(\lambda \int_\Omega f(|u|)u^2 dx - \int_\Omega g(|u|)u^2 dx \right) \\ &\leqslant C_F^m q \int_\Omega \lambda F(|u(x)|) - G(|u(x)|) dx. \end{aligned} \tag{2.12}$$

Let's set $\Omega_u = \{x \in \Omega : |u(x)| \leqslant 1\}$ and $\Omega'_u = \Omega - \Omega_u$. Since $\lambda > 0$, (2.3) and (2.5) imply

$$\lambda F(|u(x)|) - G(|u(x)|) \leqslant \begin{cases} \lambda F(1)|u(x)|^p - G(1)|u(x)|^r & \text{for } x \in \Omega_u \\ \lambda F(1)|u(x)|^q - G(1)|u(x)|^r & \text{for } x \in \Omega'_u \end{cases}. \tag{2.13}$$

Let $s = p$ or $s = q$. By applying Young's inequality $ab \leqslant \frac{a^\alpha}{\alpha} + \frac{b^\beta}{\beta}$ for $a, b > 0$ and $\alpha > 1$ and $\beta > 1$, we have

$$\lambda F(1)|u(x)|^s \leqslant \frac{s}{r} G(1)|u(x)|^r + \frac{r-s}{r} \left(\frac{\lambda F(1)}{(G(1))^{s/r}} \right)^{r/(r-s)}. \tag{2.14}$$

where $a = (G(1))^{s/r}|u(x)|^s$, $b = \lambda F(1)(G(1))^{-s/r}$, $\alpha = \frac{r}{s} > 1$ and $\beta = \frac{r}{r-s} > 1$. Inequality (2.11) yields:

$$\lambda F(1)|u(x)|^s - G(1)|u(x)|^r \leqslant G(1)|u(x)|^r \left(\frac{s}{r} - 1\right) + \frac{r-s}{r}\left(\frac{\lambda F(1)}{(G(1))^{s/r}}\right)^{r/(r-s)} \tag{2.15}$$

$$\leqslant \lambda^{r/(r-s)} \frac{r-s}{r}\left(\frac{F(1)}{(G(1))^{s/r}}\right)^{r/(r-s)}, \tag{2.16}$$

being $s < r$. Hence (2.13) yields

$$\lambda F(|u(x)|) - G(|u(x)|) \leqslant \begin{cases} \lambda^{r/(r-p)} \frac{r-p}{r}\left(\frac{F(1)}{(G(1))^{p/r}}\right)^{r/(r-p)} & \text{for } x \in \Omega_u \\ \\ \lambda^{r/(r-q)} \frac{r-q}{r}\left(\frac{F(1)}{(G(1))^{q/r}}\right)^{r/(r-q)} & \text{for } x \in \Omega'_u \end{cases}. \tag{2.17}$$

Thus, by (2.17) and (2.12), we get

$$\|u\|_F \leqslant C_F \left(2q\, meas(\Omega)\right)^{\frac{1}{m}} \max\left\{\lambda^{r/m(r-p)}\varpi, \lambda^{r/m(r-q)}\varrho\right\}, \tag{2.18}$$

where

$$\varpi = \left(\frac{(r-p)}{r}\right)^{1/m}\left(\frac{F(1)}{(G(1))^{p/r}}\right)^{\frac{r}{m(r-p)}},\ \varrho = \left(\frac{(r-q)}{r}\right)^{1/m}\left(\frac{F(1)}{(G(1))^{q/r}}\right)^{\frac{r}{m(r-q)}}.$$

Finally, inequalities (2.11) and (2.18) yield the result (2.9).

□

Definition 2. *An element u of X is a weak solution of $(\mathcal{P}_\lambda)$ if*

$$\int_\Omega |\nabla u|^{m-2}\nabla u \nabla v dx + \int_\Omega |u|^{m-2}uvdx = \lambda\int_\Omega f(|u|)uvdx - \int_\Omega g(|u|)uvdx \text{ for all } v \in X. \tag{2.19}$$

Consequetly, the weak solutions of $(\mathcal{P}_\lambda)$ are exactly the critical points of the energy functional $\Phi_\lambda : X \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) = \frac{1}{m}\int_\Omega |\nabla u|^m dx + \frac{1}{m}\int_\Omega |u|^m dx - \lambda\int_\Omega F(|u|)dx + \int_\Omega G(|u|)dx.$$

Lemma 2.8. *If $(\mathcal{P}_\lambda)$ has a nontrivial weak solution $u \in X$, then $\lambda \geqslant \lambda_0$, where*

$$\lambda_0 = \min\left\{\left(\frac{\eta}{\kappa\varpi}\right)^{\frac{m(r-p)(p-m)}{p(r-m)}}, \left(\frac{\eta}{\kappa\varrho}\right)^{\frac{m(r-q)(p-m)}{rp-mq}}, \left(\frac{\eta}{\xi\varpi}\right)^{\frac{m(r-p)(q-m)}{rq-mp}}, \left(\frac{\eta}{\xi\varrho}\right)^{\frac{m(r-q)(q-m)}{q(r-m)}}\right\} > 0 \tag{2.20}$$

with

$$\eta = C_F^{-1}(2\, q\, meas(\Omega))^{-1/m},\ \kappa = (qC_F^m)^{1/(p-m)},\ \xi = (qC_F^m)^{1/(q-m)},$$

$$\varpi = \left(\frac{(r-p)}{r}\right)^{1/m}\left(\frac{F(1)}{(G(1))^{p/r}}\right)^{\frac{r}{m(r-p)}} \quad \textit{and} \quad \varrho = \left(\frac{(r-q)}{r}\right)^{1/m}\left(\frac{F(1)}{(G(1))^{q/r}}\right)^{\frac{r}{m(r-q)}}.$$

Proof. If $(\mathcal{P}_\lambda)$ admits a nontrivial weak solution $u \in X$, then equality (2.8) is satisfy and $\lambda > 0$ by Lemma 2.7. Let's now show that $\lambda \geqslant \lambda_0$. By inequalities (2.11) and (2.18) of the proof of Lemma 2.7 we get the result. □

Let define

$$\hat{\lambda} = \sup \left\{ \lambda > 0 : (\mathcal{P}_\mu) \quad \text{admits only the trivial solution for all } \mu < \lambda \right\}.$$

It's clear that $\hat{\lambda} \geqslant \lambda_0 > 0$. In Section 4, we will prove that $\hat{\lambda}$ is the required critical value of the Theorem A.

3. Basic results for existence of nontrivial solution

The results in previous section require us to work from now with $\lambda > 0$.

Lemma 3.1. *The energy functional Φ_λ in coercive on X.*

Proof. Let $u \in X$.

$$\begin{aligned}
\Phi_\lambda(u) &= \frac{1}{m}\int_\Omega |\nabla u|^m dx + \frac{1}{m}\int_\Omega |u|^m dx - \lambda \int_\Omega F(|u|)dx + \int_\Omega G(|u|)dx, \\
&\geqslant \frac{1}{m}\|u\|^m_{W_0^{1,m}(\Omega)} - \int_\Omega \left[\lambda F(|u|)dx - \frac{1}{2}G(|u|)\right] dx + \frac{1}{2}\int_\Omega G(|u|)dx.
\end{aligned}$$

By proceeding, as in the previous section to establish (2.17) (through Young's inequality), we obtain

$$\lambda F(|u(x)|) - \frac{1}{2}G(|u(x)|) \leqslant \begin{cases} 2^{\frac{p}{(r-p)}} \frac{r-p}{r}\left(\frac{\lambda F(1)}{(G(1))^{p/r}}\right)^{r/(r-p)} & \text{for } x \in \Omega_u \\ 2^{\frac{q}{(r-q)}} \frac{r-q}{r}\left(\frac{\lambda F(1)}{(G(1))^{q/r}}\right)^{r/(r-q)} & \text{for } x \in \Omega'_u \end{cases}. \tag{3.1}$$

In consequence, since $p \leqslant q < r$, we have

$$\int_\Omega \left[\lambda F(|u|)dx - \frac{1}{2}G(|u|)\right] dx \leqslant 2^{\frac{q}{(r-q)}} \frac{(r-p)}{r} meas\,(\Omega)\, C_\lambda. \tag{3.2}$$

where $C_\lambda = \left\{\lambda^{r/(r-p)}\left(\frac{F(1)}{(G(1))^{p/r}}\right)^{r/(r-p)} + \lambda^{r/(r-q)}\left(\frac{F(1)}{(G(1))^{q/r}}\right)^{r/(r-q)}\right\}$. Therefore, we have

$$\begin{aligned}
\Phi_\lambda(u) &\geqslant \frac{1}{m}\|u\|^m_{W_0^{1,m}(\Omega)} + \frac{1}{2}\int_\Omega G(|u|)dx - 2^{\frac{q}{r-q}}\frac{r-p}{r}C_\lambda\, meas\,(\Omega)\,, \\
&\geqslant \frac{1}{m}\|u\|^m_{W_0^{1,m}(\Omega)} + \frac{1}{2}\zeta_2(\|u\|_G) - 2^{\frac{q}{r-q}}\frac{r-p}{r}C_\lambda\, meas\,(\Omega)\,, \\
&\geqslant \frac{1}{m}\|u\|^m_{W_0^{1,m}(\Omega)} + \frac{1}{2}(\|u\|^m_G - 1) - 2^{\frac{q}{r-q}}\frac{r-p}{r}C_\lambda\, meas\,(\Omega)\,, \\
&\geqslant \Theta\|u\|^m_X - 2^{\frac{q}{r-q}}\frac{r-p}{r}C_\lambda\, meas\,(\Omega) - \frac{1}{2}, \text{ where } \Theta = \min\left\{m^{-1}, \frac{1}{2}\right\}.
\end{aligned}$$

In conclusion, Φ_λ is coercive in X. □

Lemma 3.2. *Let $\widetilde{F}$ and $\widetilde{G}$ be respectively the complements of N-functions F and G. Then we have*

$$\frac{p}{p-1} \leqslant \frac{\widetilde{F}'(s)s}{\widetilde{F}(s)} \leqslant \frac{q}{q-1} \quad \text{for } s>0, \tag{3.3}$$

$$\frac{q}{q-1} \leqslant \frac{\widetilde{G}'(s)s}{\widetilde{G}(s)} \leqslant \frac{r}{r-1} \quad \text{for } s>0. \tag{3.4}$$

where $\widetilde{F}'(s)$ and $\widetilde{G}'(s)$ are respectively the derivatives of $\widetilde{F}(s)$ and $\widetilde{G}(s)$.

Proof. The results are obtained by using (2.1) and (2.2) and the proof of the point (2.7) of Lemma 2.5 of [14]. □

Lemma 3.3. *Let $p'=\frac{p}{p-1}$, $q'=\frac{q}{q-1}$, $r'=\frac{r}{r-1}$ and consider the functions $\bar{\zeta}_0(t) = min(t^{p'},t^{q'})$, $\bar{\zeta}_1(t) = max(t^{p'},t^{q'})$, $\bar{\zeta}_2(t) = min(t^{q'},t^{r'})$ and $\bar{\zeta}_3(t) = max(t^{q'},t^{r'})$ define on $[0;\infty)$. Then*

$$\bar{\zeta}_0(\rho)\widetilde{F}(t) \leqslant \widetilde{F}(\rho t) \leqslant \bar{\zeta}_1(\rho)\widetilde{F}(t) \ \text{ for } \rho, t \geqslant 0; \tag{3.5}$$

$$\bar{\zeta}_0(\|u\|_{\widetilde{F}}) \leqslant \int_\Omega \widetilde{F}(|u(x)|)dx \leqslant \bar{\zeta}_1(\|u\|_{\widetilde{F}}) \ \text{ for } u \in L_{\widetilde{F}}(\Omega); \tag{3.6}$$

$$\bar{\zeta}_2(\rho)\widetilde{G}(t) \leqslant \widetilde{G}(\rho t) \leqslant \bar{\zeta}_3(\rho)\widetilde{G}(t) \ \text{ for } \rho, t \geqslant 0; \tag{3.7}$$

$$\bar{\zeta}_2(\|u\|_{\widetilde{G}}) \leqslant \int_\Omega \widetilde{G}(|u(x)|)dx \leqslant \bar{\zeta}_3(\|u\|_{\widetilde{G}}) \ \text{ for } u \in L_{\widetilde{G}}(\Omega). \tag{3.8}$$

Proof. The proof is similar to that of Lemma 2.5 of [14]. As it is said in the proof that of Lemma 2.2, It is enough to start by integrating inequalities (3.3) and (3.4) of Lemma 3.2 to deduce the results of the Lemma. □

Proposition 3.1. *The spaces $L_F(\Omega)$ and $L_G(\Omega)$ and complementary their $L_{\widetilde{F}}(\Omega)$ and $L_{\widetilde{G}}(\Omega)$ are reflexives Banach spaces.*

Proof. The N-functions F and G satisfy Δ_2-condition. Moreover, the inequalities (3.3) and (3.4) of Lemma 3.2 imply that the N-functions $\widetilde{F}$ and $\widetilde{G}$ satisfy the Δ_2-condition. Thus, we obtain the results by Theorem 8.20 and remark 8.22 of [1]. □

Proposition 3.2. $(X, \|.\|_X)$ *is a reflexive Banach space.*

Proof. Let $Y = W_0^{1,m}(\Omega) \times L_G(\Omega)$ which we endow with the norm $\|u\|_Y = \|u\|_{W_0^{1,m}(\Omega)} + \|u\|_G$. Since $W_0^{1,m}(\Omega)$ and $L_G(\Omega)$ are reflexives Banach spaces for $m>1$, then $(Y, \|.\|_Y)$ is a reflexive Banach space by Theorem 1.23 of [1]. Let's consider the operator $\Gamma : (X, \|.\|_X) \to (Y, \|.\|_Y)$ defined by $\Gamma(u) = (u,u)$. Γ is well defined and linear. Moreover, Γ is isometric if we endowed X to the equivalent norm $\|.\|_Y$. Thus, $\Gamma(X)$ is closed subspace of Y and so $\Gamma(X)$ is reflexive by Theorem 1.22 of [1]. Then it follows that $(X, \|.\|_Y)$ is a reflexive Banach space, being isomorphic to a reflexive space. Finally we conclude that $(X, \|.\|_X)$ is a reflexive Banach space due to relexivity is preserved under equivalent norms.

□

Lemma 3.4. *Let* (u_n) *be a sequence in* X *such that* $(\Phi_\lambda(u_n))_n$ *is bounded. Then* (u_n) *admits a weakly convergent subsequent in* X.

Proof. The proof comes from the coercivity of Φ_λ in X and the reflexivity of the space X. □

Lemma 3.5. *The functional* $\Phi_\triangledown(u) = \dfrac{1}{m}\displaystyle\int_\Omega |\nabla u|^m dx$ *is convex, of class* C^1 *and is particularly sequentially weakly lower semicontinuous in* X.

Proof. The convexity of functional $\Phi_\triangledown(u) = \frac{1}{m}\int_\Omega |\nabla u|^m dx$ is followed from the convexity of function $t \mapsto |t|^m$ in $\mathbb{R}$ for $m > 1$.
Let's show the continuity of $\Phi_\triangledown(u)$ on X. Let $(u_n)_n$ be a sequence of which elements are in X and let $u \in X$ such that $u_n \to u$ in X. Let $(u_{n_k})_k$ be an arbitrary subsequence of $(u_n)_n$. The subsequence $u_{n_k} \to u$ in X and hence, by Lemma 2.1, $\nabla u_{n_k} \to \nabla u$ in $(L^m(\Omega))^N$. By the Theorem 4.9 of [9], there exists a subsequence $\left(u_{n_{k_j}}\right)_j$ of u_{n_k} and a function $\psi \in L^m(\Omega)$ such that

$$\nabla u_{n_{k_j}} \to \nabla u \text{ a.e in } \Omega \text{ as } j \to \infty \text{ and } |\nabla u_{n_{k_j}}| \leqslant \psi \text{ a.e in } \Omega \text{ for all } j \in \mathbb{N}. \quad (3.9)$$

So $|\nabla u_{n_{k_j}}|^m \to |\nabla u|^m$ a.e in Ω as $j \to \infty$ and $|\nabla u_{n_{k_j}}|^m \leqslant \psi^m \in L^1(\Omega)$ a.e in Ω for all $j \in \mathbb{N}$.

The dominated convergence theorem implies that $|\nabla u_{n_{k_j}}|^m \to |\nabla u|^m$ in $L^1(\Omega)$ as $j \to \infty$. The subsequence $(u_{n_k})_k$ being arbitrary, we deduce that $|\nabla u_n|^m \to |\nabla u|^m$ in $L^1(\Omega)$ as $n \to \infty$. Then we get the continuity of $\Phi_\triangledown$ on X and also, $\Phi_\triangledown$ is sequentially weakly lower semicontinous in X by corollary 3.9 of [9]. Moreover $\Phi_\triangledown$ is Gateaux-differentiable in X and for all $u, \varphi \in X$, we have

$$\langle \Phi'_\triangledown(u), \varphi\rangle = \int_\Omega |\nabla u|^{m-2}\nabla u \nabla\varphi dx.$$

Let finish the proof by showing that $\Phi'_\triangledown$ is continuous. Let $(u_n)_n$ be a sequence of functions in X and $u \in X$ such that $u_n \to u$ in X when $n \to \infty$. Then we have

$$\begin{aligned}
\|\Phi'_\triangledown(u_n) - \Phi'_\triangledown(u)\|_{X'} &= \sup_{\substack{\varphi\in X\\ \|\varphi\|_X=1}} \left|\int_\Omega \left[|\nabla u_n|^{m-2}\nabla u_n - |\nabla u|^{m-2}\nabla u\right]\nabla\varphi dx\right| \\
&\leqslant \sup_{\substack{\varphi\in X\\ \|\varphi\|_X=1}} \||\nabla u_n|^{m-2}\nabla u_n - |\nabla u|^{m-2}\nabla u\|_{\frac{m}{m-1}}\|\varphi\|_{W_0^{1,m}(\Omega)} \\
&\leqslant \||\nabla u_n|^{m-2}\nabla u_n - |\nabla u|^{m-2}\nabla u\|_{\frac{m}{m-1}}. \quad (3.10)
\end{aligned}$$

We just need to show that $\||\nabla u_n|^{m-2}\nabla u_n - |\nabla u|^{m-2}\nabla u\|_{\frac{m}{m-1}} \to 0$ as $n \to \infty$. For that, let $(u_{n_k})_k$ be an arbitrary subsequence of $(u_n)_n$. By proceeding exactly as in the case of the proof of continuity functional $\Phi_\triangledown$, one show that there exist a subsequence $\left(u_{n_{k_j}}\right)_j$ of u_{n_k} and a function $\psi \in L^m(\Omega)$ such that (3.9) be verified.

Then, we have

$$\begin{aligned}\Big||\nabla u_{n_{k_j}}|^{m-2}\nabla u_{n_{k_j}} - |\nabla u|^{m-2}\nabla u\Big|^{\frac{m}{m-1}} &\leqslant 2^{\frac{1}{m-1}}\left(\left(|\nabla u_{n_{k_j}}|^{m-1}\right)^{\frac{m}{m-1}} + \left(|\nabla u|^{m-1}\right)^{\frac{m}{m-1}}\right), \forall j \in \mathbb{N}.\\ &\leqslant 2^{\frac{1}{m-1}}\left(|\nabla u_{n_{k_j}}|^{m} + |\nabla u|^{m}\right),\\ &\leqslant 2^{\frac{m}{m-1}}\psi^m, \quad \forall j \in \mathbb{N}, \quad \text{a.e } \mathit{in} \in \Omega, \text{with } \ 2^{\frac{m}{m-1}}\psi^m \in L^1(\Omega).\end{aligned} \tag{3.11}$$

On the other hand, since the function $\xi \mapsto |\xi|^{m-2}\xi$ is continuous function from $\mathbb{R}^N$ to $\mathbb{R}^N$, then

$$\Big||\nabla u_{n_{k_j}}|^{m-2}\nabla u_{n_{k_j}} - |\nabla u|^{m-2}\nabla u\Big| \to 0, \text{ a.e in } \Omega, \text{ as } j \to \infty.$$

Thus, by (3.11) and the Dominated Convergence theorem, it follows that $|\nabla u_{n_{k_j}}|^{m-2}\nabla u_{n_{k_j}} \to |\nabla u|^{m-2}\nabla u$ in $L^{\frac{m}{m-1}}(\Omega)$. Hence the main sequence $|\nabla u_n|^{m-2}\nabla u_n$ converge to $|\nabla u|^{m-2}\nabla u$ in $L^{\frac{m}{m-1}}(\Omega)$ and then we get the expected result by (3.10). In conclusion, $\Phi_\triangledown$ is of class C^1 in X. □

Lemma 3.6. *The functional* $\Phi_m(u) = \dfrac{1}{m}\displaystyle\int_\Omega |u|^m dx$ *is convex, of class* C^1 *and sequentially weakly lower semicontinuous in* X*. Furthermore, if* (u_n) *is a sequence of elements of* X *and* u *belongs to* X *such that* u_n *converge weakly to* u *in* X*, then* $\Phi'_m(u_n) \to \Phi'_m(u)$ *in* X'.

Proof. The convexity of the functional Φ_m is obvious. The continuity of Φ_m in $X \subset W_0^{1,m}(\Omega)$ follows from $\|u\|_{W_0^{1,m}(\Omega)} \leqslant \|u\|_X$, for all $u \in X$ (see Lemma 2.1) and Poincaré inequality. Thus, Φ_m is sequentially weakly lower semicontinuous in X by corollary 3.9 of [9]. Furthermore, Φ_m is Gateaux differentiable in X and for all $u, v \in X$, we have

$$< \Phi'_m(u), v >= \int_\Omega |u(x)|^{m-2}u(x)v(x)dx.$$

Let's take $(u_n) \subset X$ and $u \in X$ such that $u_n \rightharpoonup u$ weakly in X as $n \to \infty$. Since $X \hookrightarrow\hookrightarrow L^m(\Omega)$ is compact by lemme 2.1.Thus we have :

$$\|\Phi'_m(u_n) - \Phi'_m(u)\|_{X'} \leqslant \| \quad |u_n|^{m-2}u_n - |u|^{m-2}u \quad \|_{\frac{m}{m-1}}. \tag{3.12}$$

By reflexivity of X and Since $X \hookrightarrow\hookrightarrow L^{m/(m-1)}(\Omega)$ is particularly compact by Lemma 2.1, then $u_n \to u$ strongly in $L^{m/(m-1)}(\Omega)$. Let set $v_n = |u_n|^{m-2}u_n$ and let choise $(v_{n_k})_k$ a subsequence of $(v_n)_n$. One have $v_{n_k} = |u_{n_k}|^{m-2}u_{n_k}$ with $(u_{n_k})_k$ a subsequence of $(u_n)_n$. Then there exists a subsequence $\left(u_{n_{k_j}}\right)_j$ of $(u_{n_k})_k$ such that $\left|u_{n_{k_j}}(x) - u(x)\right| \to 0$ almost averywhere for x in Ω and $\left|u_{n_{k_j}}(x) - u(x)\right| \leqslant g(x)$ a.e for $x \in \Omega$ where $g \in L^{m'}(\Omega)$. By continuity of the function $t \mapsto |t|^{m-2}t$ over $\mathbb{R}$, there exists $C > 0$ such that $\left|\ v_{n_{k_j}}(x) - |u(x)|^{m-2}u(x)\ \right| \leqslant C\left|u_{n_{k_j}}(x) - u(x)\right|$ a.e in Ω. Thus $\left|\ v_{n_{k_j}}(x) - |u(x)|^{m-2}u(x)\ \right|$ tends to 0 a.e for $x \in \Omega$ and $\left|\ v_{n_{k_j}}(x) - |u(x)|^{m-2}u(x)\ \right|^{m'} \leqslant C^{m'}g^{m'}(x) \in L^1(\Omega)$ a.e for $x \in \Omega$. Henceforth, by Lebesgue's dominated convergence theorem, it follows that $v_{n_{k_j}}$ converges to

$v = |u|^{m-2}u$ in $L^{m/(m-1)}(\Omega)$. Then v_{n_k} converges to v in $L^{m/(m-1)}(\Omega)$. Being $(v_{n_k})_k$ is an arbitrary subsequence of $(v_n)_n$, we get that v_n converges to v in $L^{m/(m-1)}(\Omega)$. Therefore, $\Phi'_m(u_n) \to \Phi'_m(u)$ in X' by (3.12). In particular, this last proof shows that Φ'_m is continuous in X and then Φ'_m is of class $C^1(X)$. So the Lemma ends. □

Proposition 3.3 (Proposition 3 of [23]). i) *If u_n converge strongly to u in $L_F(\Omega)$, then $f(|u_n|)u_n$ converge to $f(|u|)u$ in $L_{\widetilde{F}}(\Omega)$.*

ii) *If u_n converge strongly to u in $L_G(\Omega)$, then $g(|u_n|)u_n$ converge to $g(|u|)u$ in $L_{\widetilde{G}}(\Omega)$.*

Lemma 3.7. *The functional $\Phi_F(u) = \int_\Omega F(|u|)dx$ is convex, of class C^1 and sequentially weakly lower semi-continuous in X. Furthermore, if $(u_n)_n$ is a sequence of elements in X and u belongs to X such that u_n converge weakly to $u \in X$, then $\Phi'_F(u_n) \to \Phi'_F(u)$ in X'.*

Proof. The convexity of Φ_F follows from the convexity of the positive function $t \mapsto F(|t|)$ defined on $\mathbb{R}$.

Let's show that Φ_F is continuous in X. Let $(u_n)_n$ be a sequence of X and u an element of X such that $u_n \to u$ in X. Then, by Lemma 2.6, u_n converges to u in $L_F(\Omega)$. For n fixed in $\mathbb{N}$ and x fixed in Ω and since the function $t \mapsto f(t)t$ is increasing in $[0, +\infty)$, we have respectively, through the mean value inequality and Hölder's inequality, the two following inequalities:

$$|F(|u_n(x)|) - F(|u(x)|)| \leqslant |u_n(x) - u(x)| \left[f\left(|u_n(x)| + |u(x)|\right)\right] \left(|u_n(x)| + |u(x)|\right),$$

$$\int_\Omega |F(|u_n(x)|) - F(|u(x)|)|dx \leqslant 2\left\|u_n - u\right\|_F \left\|\left[f\left(|u_n| + |u|\right)\right]\left(|u_n| + |u|\right)\right\|_{\widetilde{F}}. \tag{3.13}$$

Being u_n converges to u in $L_F(\Omega)$, it is clear that $|u_n|$ converges to $|u|$ in $L_F(\Omega)$ due to the nondecreasing of the function F on $[0, \infty)$. Thus $\left[f\left(|u_n| + |u|\right)\right]\left(|u_n| + |u|\right)$ tends to $2|u|f(2|u|)$ in $L_{\widetilde{F}}(\Omega)$ by proposition 3.3 i). By inequality (3.13), it follows that

$\int_\Omega |F(|u_n(x)|) - F(|u(x)|)|dx \to 0$ and we deduce the continuity of the functional Φ_F in X. Consequently Φ_F is sequentially weakly lower semi-continuous in X by Corollary 3.9 of [9]. Futhermore, Φ_F is Gateaux-differentiable in X and for all $u, \varphi \in X$,

$$\langle \Phi'_F(u), \varphi\rangle = \int_\Omega f(|u(x)|)u(x)\varphi(x)dx.$$

Let $(u_n) \subset X$ and $u \in X$ such that $u_n \rightharpoonup u$ in X as $n \to \infty$. Let show that $\Phi'_F(u_n) \to \Phi'_F(u)$ in X'. By Hölder's inequality, we have

$$\begin{aligned}\|\Phi_F'(u_n)-\Phi_F'(u)\|_{X'} &= \sup_{\substack{\varphi\in X\\ \|\varphi\|_X=1}}\left|\int_\Omega \left[f(|u_n|)u_n-f(|u|)u\right]\varphi dx\right|\\ &\leqslant 2\sup_{\substack{\varphi\in X\\ \|\varphi\|_X=1}}\|f(|u_n|)u_n-f(|u|)u\|_{\widetilde{F}}\|\varphi\|_F\\ &\leqslant 2\|f(|u_n|)u_n-f(|u|)u\|_{\widetilde{F}}.\end{aligned}$$

As $u_n \rightharpoonup u$ in X, by Lemma 2.6 and proposition 3.3 i), it follows that $\|f(|u_n|)u_n - f(|u|)u\|_{\widetilde{F}} \to 0$ and consequently, $\Phi_F'(u_n) \to \Phi_F'(u)$ in X'. In particular, this shows that Φ_F is class C^1 in X and the proof of the Lemma is finished. □

Proposition 3.4. *If (u_n) is bounded in $L_F(\Omega)$ (resp. $L_G(\Omega)$) , then $(f(|u_n|)u_n)$ (resp. $(g(|u_n|)u_n)$) is bounded in $L_{\widetilde{F}}(\Omega)$ (resp. $L_{\widetilde{G}}(\Omega)$).*

Proof. By inequalities (3.6), (2.1) and (2.4) we have

$$\begin{aligned}\bar{\zeta}_0(\|f(|u_n|).|u_n|\|_{\widetilde{F}}) &\leqslant \int_\Omega \widetilde{F}(f(|u_n(x)|).|u(x)|)dx\\ &\leqslant \int_\Omega f(|u_n(x)|).|u(x)|^2dx\\ &\leqslant q\int_\Omega F(|u(x)|)dx\\ &\leqslant q.\zeta_1(\|u_n\|_F)\\ &\leqslant q.\zeta_1\left(\sup_{n\in\mathbb{N}}\|u_n\|_F\right)\quad\text{due to monotony of }\ \zeta_1.\end{aligned}$$

Hence, being $\bar{\zeta}_0$ is a strictly increasing function, we have

$$\|f(|u_n|).|u_n|\|_{\widetilde{F}} \leqslant \bar{\zeta}_0^{-1}\left[q.\zeta_1\left(\sup_{n\in\mathbb{N}}\|u_n\|_F\right)\right]. \tag{3.14}$$

Similarly, by using inequalities (3.8), (2.2) and (2.6) and monotony of functions ζ_3 and $\bar{\zeta}_2$, we have

$$\|g(|u_n|).|u_n|\|_{\widetilde{G}} \leqslant \bar{\zeta}_2^{-1}\left[r.\zeta_3\left(\sup_{n\in\mathbb{N}}\|u_n\|_G\right)\right]. \tag{3.15}$$

□

Proposition 3.5. *Let (u_n) a sequence of $L_G(\Omega)$ and $u \in L_G(\Omega)$ such that $u_n \rightharpoonup u$ in $L_G(\Omega)$ and $u_n \to \widetilde{u}$ a.e in Ω as $n \to \infty$. Then $u = \widetilde{u}$ a.e in Ω.*

Proof. Denote by $U = \{x \in \Omega : u(x) \neq \widetilde{u}\}$ and assume that $meas(S) > 0$. Then by the Severini–Egoroff theorem one can find a measurable set $V \subset U$ such that $u_n \to \widetilde{u}$ uniformly in V. Then $u_n \to \widetilde{u}$ in $L^\infty(V)$. Being $meas(V) < \infty$ and by noting $\|.\|_{G,V}$ the Luxembourg norm in the Orlicz space $L_G(V)$ define by the N-function G on V, we have

$$\|u_n-\widetilde{u}\|_{G,V} \leqslant \frac{1}{G^{-1}(1/meas(V))}\|u_n-\widetilde{u}\|_{L^\infty(V)}.$$

Henceforth, $u_n \to \widetilde{u}$ in $L_G(V)$ and u_n converges weakly to $\widetilde{u}$ in $L_G(V)$. In consequence, $\widetilde{u} = u$ a.e in V, since the weak limit is unique.This last assertion is a contradiction since $V \subset U$ and $meas(V) > 0$. □

Proposition 3.6. *If (u_n) is bounded in $L_G(\Omega)$ and $u_n \to u$ a.e in Ω, then*

$$u_n \rightharpoonup u \text{ in } L_G(\Omega) \qquad \text{and} \qquad g(|u_n|)u_n \rightharpoonup g(|u|)u \text{ in } L_{\widetilde{G}}(\Omega).$$

Proof. Fix a subsequence $(u_{n_k})_k$ of (u_n). Then there exist a subsequence $\left(u_{n_{k_j}}\right)$ of (u_{n_k}) and a function $w \in L_G(\Omega)$ such that $u_{n_{k_j}} \rightharpoonup w \in L_G(\Omega)$, being $(u_n)_n$ is bounded in the reflexive Banach space $L_G(\Omega)$. Thus $w = u$ by Proposition 3.5. By the arbitrariness of (u_{n_k}), we deduce that the entire sequence $(u_n)_n$ converges weakly to u in $L_G(\Omega)$ as $n \to \infty$.

As $u_n \to u$ a.e in Ω, it follows that $g(|u_n|)u_n \to g(|u|)u$ a.e in Ω since the function $t \mapsto g(|t|)t$ is continuous in $(-\infty, +\infty)$. On the other hand, $g(|u_n|)u_n$ is bounded in $L_{\widetilde{G}}(\Omega)$ according to the Proposition 3.4, since (u_n) is bounded in $L_G(\Omega)$. Thus, using the same reasoning from the first part of this proof to the sequence $n \mapsto g(|u_n|)u_n$, we obtain that $g(|u_n|)u_n$ converges weakly to $g(|u|)u$ in $L_{\widetilde{G}}(\Omega)$. □

Lemma 3.8. *The functional $\Phi_G(u) = \int_\Omega G(|u|)dx$ is convex, of class C^1 and sequentially weakly lower semi-continuous in X. Moreover, if $(u_n)_n$ is a sequence of elements in X and $u \in X$ such that $u_n \rightharpoonup u \in X$ as $n \to \infty$, then $\Phi'_G(u_n) \overset{*}{\rightharpoonup} \Phi'_G(u)$ in X'.*

Proof. The convexity of Φ_G follows from the convexity of the positive function $t \mapsto G(|t|)$ defined on $\mathbb{R}$. The proof of continuity of Φ_G in X is analogous to the proof of continuity of Φ_F in X in the previous Lemma. In fact,consider a sequence $(u_n)_n$ in X and an element u of X such that $u_n \to u$ in X. Then, by the definition of the norm in X, u_n converges to u in $L_G(\Omega)$. By reasoning similar to that used to establish 3.13, we have

$$\int_\Omega |G(|u_n(x)|) - G(|u(x)|)|dx \leqslant 2 \left\|u_n - u\right\|_G \left\|\left[g\left(|u_n| + |u|\right)\right]\left(|u_n| + |u|\right)\right\|_{\widetilde{G}}. \tag{3.16}$$

Being G is nondecreasing on $[0, \infty)$, $|u_n|$ converges to $|u|$ in $L_G(\Omega)$ since u_n converges to u in $L_G(\Omega)$. Thus $\left[g\left(|u_n| + |u|\right)\right]\left(|u_n| + |u|\right)$ converges to $2|u|g(2|u|)$ in $L_{\widetilde{G}}(\Omega)$ by proposition 3.3 ii). Finally $\int_\Omega |G(|u_n(x)|) - G(|u(x)|)|dx$ converge to zero by (3.16). Thus the continuity of functional Φ_G in X follows. Consequently Φ_G is sequentially weakly lower semi-continuous in X by Corollary 3.9 of [9]. Futhermore, Φ_G is Gateaux-differentiable in X and for all $u, \varphi \in X$,

$$\langle \Phi'_G(u), \varphi \rangle = \int_\Omega g(|u(x)|)u(x)\varphi(x)dx.$$

Let $(u_n) \subset X$ and $u \in X$ such that $u_n \to u$ in X as $n \to \infty$. Let show that $\Phi'_G(u_n) \to \Phi'_G(u)$ in X'. By Hölder's inequality, we have

$$\begin{aligned}\|\Phi'_G(u_n) - \Phi'_G(u)\|_{X'} &= \sup_{\substack{\varphi \in X\\ \|\varphi\|_X=1}} \left| \int_\Omega [g(|u_n|)u_n - g(|u|)u]\, \varphi dx \right| \\ &\leqslant 2\|g(|u_n|)u_n - g(|u|)u\|_{\widetilde{G}}. \end{aligned} \tag{3.17}$$

According to proposition 3.3 ii), it follows that $\|g(|u_n|)u_n - g(|u|)u\|_{\widetilde{G}} \to 0$ and consequently, $\Phi'_G(u_n) \to \Phi'_G(u)$ in X' by (3.17). Thus Φ_G is class C^1 in X.
Let's show the last part of this Lemma. Let $(u_n) \subset X$ and $u \in X$ such that $u_n \rightharpoonup u$ in X. We shall proof that $\Phi'_G(u_n) \overset{*}{\rightharpoonup} \Phi'_G(u)$ in X', as $n \to \infty$. Let $\varphi \in X$. For any integer n, let's set $v_n = g(|u_n|)u_n$ and consider an arbitrary subsequence $(v_{n_k})_k$ of the sequence (v_n). We have $v_{n_k} = g(|u_{n_k}|)u_{n_k}$, where $(u_{n_k})_k$ is a subsequence of (u_n) which converges weakly to u in X when $k \to \infty$. By Lemma 2.1, it follows that $u_{n_k} \to u$ in $L^m(\Omega)$. Thus there exists a subsequence $\left(u_{n_{k_j}}\right)_j$ of $(u_{n_k})_k$ such that $u_{n_{k_j}} \to u$ a.e in Ω. Moreover, by the continuous embedding $X \hookrightarrow L_G(\Omega)$, it follows that $\left(u_{n_{k_j}}\right)_j$ is bounded in $L_G(\Omega)$. Thus, by Proposition 3.6, it follows that $v_{n_{k_j}} \rightharpoonup g(|u|)u$ in $L_{\widetilde{G}}(\Omega)$. Due to the arbitrariness of $(v_{n_k})_k$, we conclude that the entire sequence $(v_n)_n$ weakly converges to v in $L_{\widetilde{G}}(\Omega)$ as $n \to \infty$. Particularly,

$$\text{for all } \ \varphi \in X, \ \int_\Omega g(|u_n|)u_n\varphi dx \to \int_\Omega g(|u|)u\varphi dx, \ \text{ as } \ n \to \infty.$$

In conclusion, $\Phi'_G(u_n) \overset{*}{\rightharpoonup} \Phi'_G(u)$ in X', as $n \to \infty$. □

For any $(x, u) \in \Omega \times \mathbb{R}$, consider

$$\mathfrak{M}(x,u) = \lambda f(|u|)u - g(|u|)u \text{ and} \tag{3.18}$$

$$M(x,u) = \int_0^u \mathfrak{M}(x,t)dt = \lambda F(|u|) - G(|u|). \tag{3.19}$$

Lemma 3.9. *For any fixed $u \in X$, the functional $\mathscr{M}_u$ defined on X by $\mathscr{M}_u(v) = \int_\Omega \mathfrak{M}(x,u(x))v(x)dx$ belongs to X' (X' is the Banach dual space of X). In particular, if $v_n \rightharpoonup v$ in X, then $\mathscr{M}_u(v_n) \to \mathscr{M}_u(v)$.*

Proof. Let take $u \in X$. It easy to show that $\mathscr{M}_u$ is linear. Moreover, for all $v \in X$, by using Hölder inequality and imbbedings properties of X in $L_F(\Omega)$ and of X in $L_G(\Omega)$, we have

$$\begin{aligned}|\mathscr{M}_u(v)| &\leqslant \lambda \int_\Omega f(|u|)|u||v|dx + \int_\Omega g(|u|)|u||v|dx \\ &\leqslant 2\lambda\|f(|u|)|u|\|_{\widetilde{F}}\|v\|_F + 2\|g(|u|)|u|\|_{\widetilde{G}}\|v\|_G \\ &\leqslant 2\left[C_F\lambda\|f(|u|)|u|\|_{\widetilde{F}} + \|g(|u|)|u|\|_{\widetilde{G}}\right]\|v\|_X.\end{aligned}$$

Thus, we conclude that $\mathscr{M}_u$ is continuous on X. Then, it follows that, $v_n \rightharpoonup v$ in X yields $\mathscr{M}_u(v_n) \to \mathscr{M}_u(v)$. □

Lemma 3.10. *The functional Φ_λ is of class C^1 and sequentially weakly lower semicontnuous in X, that is if $u_n \rightharpoonup u$ in X, then*

$$\Phi_\lambda(u) \leqslant \lim_{n\to\infty} \inf \Phi_\lambda(u_n). \tag{3.20}$$

Proof. Lemmas 3.5-3.8 imply that $\Phi_\lambda \in C^1(X)$. Let $(u_n)_n \subset X$ and $u \in X$ such that $u_n \rightharpoonup u$ in X. The definition of Φ_λ and (3.19) allow us to write

$$\begin{aligned}\Phi_\lambda(u) - \Phi_\lambda(u_n) =& \frac{1}{m}\left(\|u\|^m_{W_0^{1,m}(\Omega)} - \|u_n\|^m_{W_0^{1,m}(\Omega)}\right) + \frac{1}{m}\left(\|u\|^m_m - \|u_n\|^m_m\right) \\ &+ \int_\Omega [M(x,u_n(x)) - M(x,u(x))]\, dx. \end{aligned}\tag{3.21}$$

Inasmuch as $u_n \rightharpoonup u$ in X, Lemmas 3.5 and 3.6 yield respectively that

$$\|u\|^m_{W_0^{1,m}(\Omega)} \leqslant \liminf_{n\to\infty} \|u_n\|^m_{W_0^{1,m}(\Omega)}, \tag{3.22}$$

$$\|u\|^m_m \leqslant \liminf_{n\to\infty} \|u_n\|^m_m. \tag{3.23}$$

Hence, by (3.21), we get

$$\limsup_{n\to\infty} [\Phi_\lambda(u) - \Phi_\lambda(u_n)] \leqslant \limsup_{n\to\infty} \int_\Omega [M(x,u_n(x)) - M(x,u(x))]\, dx. \tag{3.24}$$

By (3.18) and (3.19), for all $s \in [0,1]$,

$$\begin{aligned} M_u(x, u + s(u_n - u)) &= \mathfrak{M}(x, u + s(u_n - u)) \\ &= \mathfrak{M}(x,u) + (u_n - u)\int_0^s \mathfrak{M}_u(x, u + t(u_n - u))dt \end{aligned}\tag{3.25}$$

where $\mathfrak{M}_u(x,u) = \frac{\partial \mathfrak{M}}{\partial u}(x,u) = \lambda\,[\,|u|f'(|u|) + f(|u|)\,] - [\,|u|g'(|u|) + g(|u|)\,]$, for all $u \in X$ and $M_u(x,u) = \frac{\partial M}{\partial u}(x,u)$.
Multiplying (3.25) by $(u_n - u)$ and integrating the result over $[0,1]$ with respect to s, we obtain

$$M(x,u_n) - M(x,u) = \mathfrak{M}(x,u)(u_n - u) + (u_n - u)^2 \int_0^1 \left(\int_0^s \mathfrak{M}_u(x, u + t(u_n - u))dt\right) ds. \tag{3.26}$$

According to the assumptions $(\mathscr{F}_2)$ and $(\mathscr{G}_2)$, one has:

$$\text{for all } x \in \Omega, \text{ for all } u \in X,\ \mathfrak{M}_u(x,u) \leqslant (q-1)\,[\lambda f(|u(x)|) - g(|u(x)|)]\,. \tag{3.27}$$

Afterwards, (2.1) and (2.2) imply

$$\text{for all } u \in X \setminus \{0\}\,, \text{for all } x \in \Omega,\ \mathfrak{M}_u(x,u) \leqslant q(q-1)\,[\lambda F(|u(x)|) - G(|u(x)|)]\,|u(x)|^{-2}. \tag{3.28}$$

By (2.13), for all $u \in X \setminus \{0\}$, we have

$$[\lambda F(|u(x)|) - G(|u(x)|)]\,|u(x)|^{-2} \leqslant \begin{cases} \lambda F(1)|u(x)|^{p-2} - G(1)|u(x)|^{r-2} & \text{for } x \in \Omega_u \\ \lambda F(1)|u(x)|^{q-2} - G(1)|u(x)|^{r-2} & \text{for } x \in \Omega'_u \end{cases}. \tag{3.29}$$

Put $s=p-2$ or $s=q-2$. Applying Young's inequality $ab \leqslant \frac{a^\alpha}{\alpha}+\frac{b^\beta}{\beta}$ for $a,b>0$ and $\alpha>1$ and $\beta>1$, we have

$$\lambda F(1)|u(x)|^s \leqslant \frac{s}{r-2}G(1)|u(x)|^{r-2}+\frac{r-2-s}{r-2}\left(\frac{\lambda F(1)}{(G(1))^{s/(r-2)}}\right)^{(r-2)/(r-2-s)} \tag{3.30}$$

where we take $a=(G(1))^{s/(r-2)}|u(x)|^s$, $b=\lambda F(1)(G(1))^{-s/(r-2)}$, $\alpha=\frac{r-2}{s}>1$ and
$\beta=\frac{r-2}{r-2-s}>1$. Inequality (3.30) yields:

$$\begin{aligned}\lambda F(1)|u(x)|^s - G(1)|u(x)|^{r-2} &\leqslant G(1)|u(x)|^{r-2}\left(\frac{s}{r-2}-1\right)\\ &\quad+\frac{r-2-s}{r-2}\left(\frac{\lambda F(1)}{(G(1))^{s/(r-2)}}\right)^{(r-2)/(r-2-s)} \qquad (3.31)\\ &\leqslant \lambda^{(r-2)/(r-2-s)}\,\frac{(r-2-s)}{r-2}\left(\frac{F(1)}{(G(1))^{s/(r-2)}}\right)^{(r-2)/(r-2-s)}, \qquad (3.32)\end{aligned}$$

being $s<r-2$. Hence (3.29) yields

$$[\lambda F(|u(x)|)-G(|u(x)|)]\,|u(x)|^{-2} \leqslant \begin{cases} \lambda^{(r-2)/(r-p)}\,\frac{r-p}{r-2}\left(\frac{F(1)}{(G(1))^{(p-2)/(r-2)}}\right)^{(r-2)/(r-p)} & \text{for } x\in\Omega_u \\ \lambda^{(r-2)/(r-q)}\,\frac{r-q}{r-2}\left(\frac{F(1)}{(G(1))^{(q-2)/(r-2)}}\right)^{(r-2)/(r-q)} & \text{for } x\in\Omega_u' \end{cases}. \tag{3.33}$$

As $\frac{r-p}{r-2}>\frac{r-q}{r-2}$, it follows, from (3.33), that

$$[\lambda F(|u(x)|)-G(|u(x)|)]\,|u(x)|^{-2} \leqslant \frac{r-p}{r-2}\mathfrak{T}_\lambda, \text{ for all } x\in\Omega, \tag{3.34}$$

where

$$\mathfrak{T}_\lambda=\max\left\{\lambda^{(r-2)/(r-p)}\left(\frac{F(1)}{(G(1))^{(p-2)/(r-2)}}\right)^{(r-2)/(r-p)},\lambda^{(r-2)/(r-q)}\left(\frac{F(1)}{(G(1))^{(q-2)/(r-2)}}\right)^{(r-2)/(r-q)}\right\}.$$

From (3.28) and (3.34), we obtain

$$\text{for all } u\in X,\ \mathfrak{M}_u(x,u) \leqslant \frac{q(q-1)(r-p)}{r-2}\mathfrak{T}_\lambda. \tag{3.35}$$

Consequently, (3.26) yields

$$\begin{aligned}\int_\Omega [M(x,u_n)-M(x,u)]\,dx &\leqslant \int_\Omega \mathfrak{M}(x,u)(u_n-u)dx+\frac{q(q-1)(r-p)\,\mathfrak{T}_\lambda}{2(r-2)}\int_\Omega (u_n-u)^2dx\\ &\leqslant \mathscr{M}_u(u_n-u)+\frac{q(q-1)(r-p)\,\mathfrak{T}_\lambda}{2(r-2)}\|u_n-u\|_2^2.\end{aligned}$$

Since $u_n \rightharpoonup u$ in X, Lemmas 3.9 and 2.1 yield respectively that $\mathscr{M}_u(u_n - u) \to 0$ and $\|u_n - u\|_2 \to 0$. Thus $\limsup_{n\to\infty} \int_\Omega [M(x, u_n(x)) - M(x, u(x))]\, dx \leqslant 0$ and then, by (3.24), we get the claim (3.20). □

4. Existence of weak solutions of $(\mathcal{P}_\lambda)$ for large values of parameter

Let

$$\bar{\lambda} = \inf_{\substack{u\in X\\ \Phi_F(u)=1}} \left\{ \frac{1}{m}\int_\Omega (|\nabla u|^m + |u|^m)\, dx + \int_\Omega G(|u(x)|)dx \right\} \text{ with } \Phi_F(u) = \int_\Omega F(|u(x)|)dx.$$

Note that $\bar{\lambda} > 0$. Indeed, by the fact that $\Phi_F(u) = 1 \Rightarrow \|u\|_F = 1$, we get, from Lemma 2.6, the inequality

$$\int_\Omega (|\nabla u|^m + |u|^m)\, dx \geqslant \|u\|^m_{W_0^{1,m}(\Omega)} \geqslant \frac{1}{C_F^m}, \text{ for all } u \in X \text{ with } \Phi_F(u) = 1. \quad (4.1)$$

By the property (2.6), Lemma 2.3 and the monotony of function ζ_2, we have

$$\int_\Omega G(|u(x)|)dx \geqslant \zeta_2(\|u\|_G) \geqslant \zeta_2\left(\frac{1}{\mathcal{H}}\right), \text{ for all } u \in X \text{ with } \Phi_F(u) = 1. \quad (4.2)$$

Thus, by (4.1) and (4.2), we get

$$\bar{\lambda} \geqslant \frac{1}{mC_F^m} + \zeta_2\left(\frac{1}{\mathcal{H}}\right) > 0. \quad (4.3)$$

Lemma 4.1. *For all $\lambda > \bar{\lambda}$, there exists a global nontrivial non-negative minimizer $\boldsymbol{\omega} \in X$ of Φ_λ with negative energy, that is $\Phi_\lambda(\boldsymbol{\omega}) < 0$.*

Proof. By Lemmas 3.1, 3.10 and corollary 3.23 of [9], for each $\lambda > 0$, there exists a global minimizer $\boldsymbol{\omega} \in X$ of Φ_λ, that is

$$\Phi_\lambda(\boldsymbol{\omega}) = \inf_{v\in X} \Phi_\lambda(v).$$

So $\boldsymbol{\omega}$ is a solution of $(\mathcal{P}_\lambda)$. We shall prove that $\boldsymbol{\omega}$ is nontrivial provided that $\lambda > \bar{\lambda}$. For that we shall show that $\inf_{v\in X} \Phi_\lambda(v) < 0$. Let $\lambda > \bar{\lambda}$. By the definiton of $\bar{\lambda}$, there exixts a function $\varphi \in X$, with $\int_\Omega F(|\varphi(x)|)dx = 1$, such that

$$\lambda \int_\Omega F(|\varphi(x)|)dx = \lambda > \frac{1}{m}\int_\Omega (|\nabla \varphi|^m + |\varphi(x)|^m)\, dx + \int_\Omega G(|\varphi(x)|)dx \geqslant \bar{\lambda}.$$

Hence,

$$\frac{1}{m}\int_\Omega (|\nabla \varphi|^m + |\varphi(x)|^m)\, dx - \lambda \int_\Omega F(|\varphi(x)|)dx + \int_\Omega G(|\varphi(x)|)dx < 0$$

and then

$$\Phi_\lambda(\varphi) < 0.$$

Therefore,

$$\Phi_\lambda(\boldsymbol{\omega}) = \inf_{v\in X} \Phi_\lambda(v) \leqslant \Phi_\lambda(\varphi) < 0.$$

In conclusion, for $\lambda > \bar{\lambda}$, problem $(\mathcal{P}_\lambda)$ has a nontrivial solution $\boldsymbol{\omega} \in X$ such that $\Phi_\lambda(\boldsymbol{\omega}) < 0$. Finally, we may assume $\boldsymbol{\omega} \geqslant 0$ a.e in Ω, since $|\boldsymbol{\omega}| \in X$ and $\Phi_\lambda(\boldsymbol{\omega}) = \Phi_\lambda(|\boldsymbol{\omega}|)$. $\square$

Lemma 4.2. *Let $(u_n)_n$ be bounded in X and $(\lambda_n)_n$ bounded in $\mathbb{R}$. Let's consider the sequence of functions $(R_n)_n$ defined in Ω by*

$$R_n(x) = -|u_n(x)|^{m-2}u_n(x) + \lambda_n f\left(|u_n(x)|\right) u_n(x) - g\left(|u_n(x)|\right) u_n(x), \quad \forall n \in \mathbb{N}. \tag{4.4}$$

Then for all compact set $K \subset \Omega$, there exist $C_K > 0$ such that

$$\sup_{n \in \mathbb{N}} \int_K |R_n(x)| dx \leqslant C_K.$$

Proof. Let $(u_n)_n$ be a bounded sequence in X and $(\lambda_n)_n$ a bounded sequence in $\mathbb{R}$. Consider K, a compact set in Ω and φ_K the characteristic function of K defined on Ω. Then the sequence $(u_n)_n$ is bounded in $L^m(\Omega)$ by Lemma (2.1) and in $L_G(\Omega)$ by the definition of norm on X. It also bounded in $L_F(\Omega)$ by Lemma (2.6).
Thus, by Hölder inequality, one has

$$\int_K |u_n(x)|^{m-1} dx \leqslant (meas(K))^{1/m}.\|u_n\|_m^{m-1} \leqslant (meas(K))^{1/m}.\sup_{n \in \mathbb{N}} \|u_n\|_m^{m-1} = C_1(K). \tag{4.5}$$

Also, by Hölder inequality, equality (9.23) of [17] (page 79) and inequality (3.14) in the proof of Proposition 3.4, we have

$$\begin{aligned}\int_K f\left(|u_n(x)|\right).|u_n(x)| dx &\leqslant 2\|\varphi_K\|_F.\|f\left(|u_n|\right).|u_n|\|_{\widetilde{F}} \\ &\leqslant 2\left[F^{-1}\left(\frac{1}{meas(K)}\right)\right]^{-1} \bar{\zeta}_0^{-1}\left(q.\zeta_1\left(\sup_{n \in \mathbb{N}} \|u_n\|_F\right)\right) = C_2(K).\end{aligned} \tag{4.6}$$

In the same direction, using Hölder's inequality, equality (9.23) of [17] and inequality (3.15) in the proof of Proposition 3.4, we have

$$\int_K g\left(|u_n(x)|\right).|u_n(x)| dx \leqslant 2\left[G^{-1}\left(\frac{1}{meas(K)}\right)\right]^{-1} \bar{\zeta}_2^{-1}\left(r.\zeta_3\left(\sup_{n \in \mathbb{N}} \|u_n\|_G\right)\right) = C_3(K). \tag{4.7}$$

Combining (4.5), (4.6),(4.7) and taking into account that the sequence $(\lambda_n)_n$ is bounded, we get the result of the Lemma. $\square$

Let define

$$\check{\lambda} = \inf\left\{\lambda > 0 : (\mathcal{P}_\lambda) \text{ has a nontrivial weak solution}\right\}.$$

Lemma 4.1 assures that this definition is meaningful. Moreover, it is obvious that $\bar{\lambda} > \check{\lambda}$.

Theorem 4.1. *For any $\lambda > \check{\lambda}$, the problem $(\mathcal{P}_\lambda)$ admits a nontrivial non-negative weak solution $u_\lambda \in X$.*

Proof. Fix $\lambda > \check{\lambda}$. By definition of $\check{\lambda}$, there exists $\mu \in (\check{\lambda}, \lambda)$ such that Φ_μ has a nontrivial critical point $u_\mu \in X$. We assume, without loss of generality that $u_\mu \geqslant 0$ a.e in Ω, since $|u_\mu|$ is also a solution of (ε_μ). We can easily see that u_μ is a sub-solution for $(\mathcal{P}_\lambda)$. Let consider the following minimization problem:

$$\inf_{v\in\mathcal{M}} \Phi_\lambda(v) \text{ where } \mathcal{M} = \{v \in X : \text{ for all } x \in \Omega, v(x) \geqslant u_\mu(x)\}. \tag{4.8}$$

We remark easily that $\mathcal{M}$ is closed and convex. So $\mathcal{M}$ is weakly closed. Moreover, Φ_λ is coercive in $\mathcal{M}$, being it's coercive in X by Lemma 3.1. Finally Φ_λ is sequentially weakly lower semicontinuous in X and so in $\mathcal{M}$. Hence, corollary 3.23 of [9] implies that there exists $u_\lambda \in \mathcal{M}$ such that

$$\Phi_\lambda(u_\lambda) = \inf_{v\in\mathcal{M}} \Phi_\lambda(v). \tag{4.9}$$

Now we shall prove that u_λ is a solution of $(\mathcal{P}_\lambda)$. Let $\varphi \in C_0^\infty(\Omega)$ and $\epsilon > 0$. Let set $\varphi_\epsilon = \max\{0, u_\mu - u_\lambda - \epsilon\varphi\}$ and $v_\epsilon = u_\lambda + \epsilon\varphi + \varphi_\epsilon$. It is clear that $v_\epsilon \in \mathcal{M}$ and

$$0 \leqslant \langle \Phi'_\lambda(u_\lambda), v_\epsilon - u_\lambda\rangle = \epsilon\langle \Phi'_\lambda(u_\lambda), \varphi\rangle + \langle \Phi'_\lambda(u_\lambda), \varphi_\epsilon\rangle.$$

Hence

$$\langle \Phi'_\lambda(u_\lambda), \varphi\rangle \geqslant -\frac{1}{\epsilon}\langle \Phi'_\lambda(u_\lambda), \varphi_\epsilon\rangle. \tag{4.10}$$

Let set

$$\Omega_\epsilon = \{x \in \Omega : u_\lambda(x) + \epsilon\varphi(x) \leqslant u_\mu(x) < u_\lambda(x)\}.$$

It's obvious that $\Omega_\epsilon \subset \operatorname{supp}\varphi$. Since u_μ is a subsolution of $(\mathcal{P}_\lambda)$ and $\varphi_\epsilon \geqslant 0$, it turns out that $\langle \Phi'_\lambda(u_\mu), \varphi_\epsilon\rangle \leqslant 0$. Consequently, as

$$\langle \Phi'_\lambda(u_\lambda), \varphi_\epsilon\rangle = \langle \Phi'_\lambda(u_\mu), \varphi_\epsilon\rangle + \langle \Phi'_\lambda(u_\lambda) - \Phi'_\lambda(u_\mu), \varphi_\epsilon\rangle$$

and

$$\mathfrak{M}(u(x)) = \lambda f(|u(x)|)u(x) - g(|u(x)|)u(x),$$

we have

$$\begin{aligned}
\langle \Phi'_\lambda(u_\lambda), \varphi_\epsilon\rangle &\leqslant \langle \Phi'_\lambda(u_\lambda) - \Phi'_\lambda(u_\mu), \varphi_\epsilon\rangle \\
&\leqslant \int_{\Omega_\epsilon} \left(|\nabla u_\lambda|^{m-2}\nabla u_\lambda - |\nabla u_\mu|^{m-2}\nabla u_\mu\right).\nabla(u_\mu - u_\lambda - \epsilon\varphi)dx \quad (4.11)\\
&+ \int_{\Omega_\epsilon} \left(|u_\lambda|^{m-2}u_\lambda - |u_\mu|^{m-2}u_\mu\right)(u_\mu - u_\lambda - \epsilon\varphi)dx \\
&- \int_{\Omega_\epsilon} [\mathfrak{M}(u_\lambda) - \mathfrak{M}(u_\mu)].(u_\mu - u_\lambda - \epsilon\varphi)dx.
\end{aligned}$$

The convexity of the function $\xi \mapsto \dfrac{|\xi|^m}{m}$ on $\mathbb{R}^N$ yields $\left(|\nabla u_\lambda|^{m-2}\nabla u_\lambda - |\nabla u_\mu|^{m-2}\nabla u_\mu\right).\nabla(u_\mu - u_\lambda) \leqslant 0$ and in consequence, one has

$$\begin{aligned}\langle \Phi'_\lambda(u_\lambda), \varphi_\epsilon\rangle \leqslant & \int_{\Omega_\epsilon} \left(|\nabla u_\lambda|^{m-2}\nabla u_\lambda - |\nabla u_\mu|^{m-2}\nabla u_\mu\right).\nabla(-\epsilon\varphi)dx \\ & + \int_{\Omega_\epsilon} \left(|u_\lambda|^{m-2}u_\lambda - |u_\mu|^{m-2}u_\mu\right)(u_\mu - u_\lambda - \epsilon\varphi)dx \\ & - \int_{\Omega_\epsilon} \left[\mathfrak{M}(u_\lambda) - \mathfrak{M}(u_\mu)\right].(u_\mu - u_\lambda - \epsilon\varphi)dx.\end{aligned} \tag{4.12}$$

- For $\Omega_\epsilon = \emptyset$, it is clear that $\langle \Phi'_\lambda(u_\lambda), \varphi_\epsilon\rangle \leqslant 0$. Hence by (4.10), we have $\langle \Phi'_\lambda(u_\lambda), \varphi\rangle \geqslant 0$.
- Also when $\Omega_\epsilon \neq \emptyset$, we have $0 \leqslant u_\mu - u_\lambda - \epsilon\varphi = u_\mu - u_\lambda + \epsilon|\varphi| < \epsilon|\varphi|$ in Ω_ϵ because in this case $\varphi \leqslant 0$ in Ω_ϵ. Then (4.11) yields

$$\begin{aligned}\langle \Phi'_\lambda(u_\lambda), \varphi_\epsilon\rangle \leqslant & \, \epsilon\Bigg(\int_{\Omega_\epsilon} \left(|\nabla u_\mu|^{m-2}\nabla u_\mu - |\nabla u_\lambda|^{m-2}\nabla u_\lambda\right).\nabla(\varphi)dx \\ & + \int_{\Omega_\epsilon} \left(u_\lambda^{m-1} - u_\mu^{m-1}\right)|\varphi|dx + \int_{\Omega_\epsilon} |\mathfrak{M}(u_\lambda) - \mathfrak{M}(u_\mu)|\,.\,|\varphi|\,dx\Bigg) \\ \leqslant & \, \epsilon\int_{\Omega_\epsilon}\psi(x)dx,\end{aligned} \tag{4.13}$$

where

$$\begin{aligned}\psi(x) = & \left(|\nabla u_\mu|^{m-2}\nabla u_\mu - |\nabla u_\lambda|^{m-2}\nabla u_\lambda\right).\nabla(\varphi) \\ & + \left(\left(u_\lambda^{m-1} - u_\mu^{m-1}\right) + |\mathfrak{M}(u_\lambda) - \mathfrak{M}(u_\mu)|\right)|\varphi|\,.\end{aligned}$$

We claim that $\psi \in L^1(\mathrm{supp}(\varphi))$. Indeed the terms $\left(|\nabla u_\mu|^{m-2}\nabla u_\mu - |\nabla u_\lambda|^{m-2}\nabla u_\lambda\right)$ and $\left(u_\lambda^{m-1} - u_\mu^{m-1}\right)$ belong to $L^1_{loc}(\Omega)$ since $|\nabla u_\mu|^{m-2}\nabla u_\mu$ and $|\nabla u_\lambda|^{m-2}\nabla u_\lambda$ belong to $\left[L^{m/(m-1)}(\Omega)\right]^N$ and u_λ^{m-1} and u_μ^{m-1} are in $L^{m/(m-1)}(\Omega)$ being $u_\lambda, u_\mu \in X \subset W_0^{1,m}(\Omega) \subset L^m(\Omega)$. Moreover, the functions $f(|u_\lambda|)|u_\lambda|$, $f(|u_\mu|)|u_\mu|$ belong to $L_{\widetilde{F}}(\Omega)$ and $g(|u_\lambda|)|u_\lambda|$ and $g(|u_\mu|)|u_\mu|$ belong to $L_{\widetilde{G}}(\Omega)$. Thus, referring to the proof of Lemma 4.2, one can easily see that the functions $f(|u_\lambda|)|u_\lambda|$, $f(|u_\mu|)|u_\mu|$, $g(|u_\lambda|)|u_\lambda|$ and $g(|u_\mu|)|u_\mu|$ belong to $L^1_{loc}(\Omega)$. Finally, $|\mathfrak{M}(u_\lambda) - \mathfrak{M}(u_\mu)| \in L^1_{loc}(\Omega)$ inasmuch as

$$|\mathfrak{M}(u_\lambda) - \mathfrak{M}(u_\mu)| \leqslant \lambda\left(f(|u_\lambda|)|u_\lambda| + f(|u_\mu|)|u_\mu|\right) + \left(g(|u_\lambda|)|u_\lambda| + g(|u_\mu|)|u_\mu|\right).$$

Hence, the claim is obtained since φ and $\nabla\varphi$ are bounded in $\mathrm{supp}(\varphi)$. Thus,

$$\lim_{\epsilon\to 0^+}\int_{\Omega_\epsilon}\psi(x)dx = 0,$$

because $meas(\Omega_\epsilon) \to 0$ as $\epsilon \to 0^+$. Then (4.13) implies that $\langle \Phi'_\lambda(u_\lambda), \varphi_\epsilon\rangle \leqslant \circ(\epsilon)$ as $\epsilon \to 0^+$. So, by (4.10), it follows that $\langle \Phi'_\lambda(u_\lambda), \varphi\rangle \geqslant \circ(1)$ as $\epsilon \to 0^+$. We deduce that, $\langle \Phi'_\lambda(u_\lambda), \varphi\rangle \geqslant 0$ for all $\varphi \in C_0^\infty(\Omega)$, that is $\langle \Phi'_\lambda(u_\lambda), \varphi\rangle = 0$ for all $\varphi \in C_0^\infty(\Omega)$. Since X is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$\|.\|_X$, one obtains that u_λ is a solution of $(\mathcal{P}_\lambda)$. Finally, u_λ is nontrivial and non-negative, since $u_\lambda > u_\mu$. $\square$

Lemma 4.3. $\check{\lambda} = \hat{\lambda}$.

Proof. This proof is the same to that one did in the step 5 of the proof of Theorem 1.1 in [13]. In fact, by Theorem 4.1, we have $\check{\lambda} \geqslant \hat{\lambda}$. Indeed, for all λ such that $(\mathcal{P}_\lambda)$ has a weak nontrivial solution, we have $\lambda > \hat{\lambda}$ and thus, $\check{\lambda} \geqslant \hat{\lambda}$.
Let's show that $\check{\lambda} \leqslant \hat{\lambda}$. suppose that $\check{\lambda} > \hat{\lambda}$. Problem $(\mathcal{P}_\lambda)$ cannot admit a nontrivial solution $u \in X$ if $\lambda < \check{\lambda}$, since this would contradict the minimality of $\check{\lambda}$. Hence, for all $\lambda \in [\hat{\lambda}, \check{\lambda})$ the unique solution of $(\mathcal{P}_\lambda)$ is $u \equiv 0$. But this is again impossible since it would contradict the maximality of $\hat{\lambda}$. Hence $\check{\lambda} = \hat{\lambda}$. $\square$

Theorem 4.2. *The problem $(\mathcal{P}_{\hat{\lambda}})$ admits a nontrivial non-negative weak solution in X.*

Proof. Let $(\lambda_n)_{n\in\mathbb{N}^*}$ be a strictly decreasing sequence converging to $\hat{\lambda}$ and $u_n \in X$ be a nontrivial non-negative weak solution of $(\mathcal{P}_{\lambda_n})$ for all $n \in \mathbb{N}^*$. Then we have, for all integer n,

$$\int_\Omega |\nabla u_n|^{m-2}\nabla u_n \nabla v dx = -\int_\Omega |u_n|^{m-2}u_n v dx + \lambda_n \int_\Omega f(|u_n|)u_n v dx - \int_\Omega g(|u_n|)u_n v dx, \tag{4.14}$$

for all $v \in X$. Then, for any fixed n, relation (2.8) holds for u_n:

$$\int_\Omega |\nabla u_n|^m dx + \int_\Omega |u_n|^m dx + \int_\Omega g(|u_n|)u_n^2 dx = \lambda_n \int_\Omega f(|u_n|)u_n^2 dx. \tag{4.15}$$

Using a combination of relations (2.1), (2.2), inequalities (2.4) and (2.6) of Lemma 2.2 and (4.15), we have

$$\|u_n\|^m_{W_0^{1,m}(\Omega)} + q\,\zeta_2(\|u_n\|_G) \leqslant q\lambda_n\zeta_1(\|u_n\|_F).$$

Furthermore, by inequality (2.18) and the monotony of ζ_1 this last inequality implies

$$\|u_n\|^m_{W_0^{1,m}(\Omega)} + q\,\zeta_2(\|u_n\|_G) \leqslant q\,\zeta_1\left(C_F\left\{\frac{2q(r-p)}{r}meas(\Omega)\right\}^{1/m}\right)K_{\lambda_n}, \text{ for all } n \in \mathbb{N}^*, \tag{4.16}$$

where

$$K_{\lambda_n} = \max\left\{\lambda_n^{1+pr/m(r-p)}\left(\frac{F(1)}{(G(1))^{p/r}}\right)^{\frac{pr}{m(r-p)}}, \lambda_n^{1+qr/m(r-p)}\left(\frac{F(1)}{(G(1))^{p/r}}\right)^{\frac{qr}{m(r-p)}},\right.$$
$$\left.\lambda_n^{1+pr/m(r-q)}\left(\frac{F(1)}{(G(1))^{q/r}}\right)^{\frac{pr}{m(r-q)}}, \lambda_n^{1+qr/m(r-q)}\left(\frac{F(1)}{(G(1))^{q/r}}\right)^{\frac{qr}{m(r-q)}}\right\}, \text{ for } n \in \mathbb{N}^*.$$

Being $(\lambda_n)_{n\in\mathbb{N}^*}$ is strictly decreasing, we get the following majoration from (4.16):

$$\|u_n\|^m_{W_0^{1,m}(\Omega)}+q\,\zeta_2(\|u_n\|_G) \leqslant q\,\zeta_1\left(C_F\left\{\frac{2q(r-p)}{r}meas(\Omega)\right\}^{1/m}\right)K_{\lambda_1},\ \text{for all } n\in\mathbb{N}^*. \tag{4.17}$$

Thus the sequences $\left(\|u_n\|_{W_0^{1,m}(\Omega)}\right)_n$ and $\left(\zeta_2\left(\|u_n\|_G\right)\right)_n$ are bounded. In particular, $(|u_n\|_G)_n$ is bounded due to the monotony of the inverse of ζ_2. Consequently, the sequence $(\|u_n\|_X)_n$ is bounded. According to the Theorem 3.18 of [9], Propositions 3.2 and 3.1 and Lemma 2.6, one can extract a subsequence from $(u_n)_n$ which is still relabeled $(u_n)_n$ and which satisfies these differents convergences:

$$\begin{aligned}
&u_n \rightharpoonup u \text{ in } X, && u_n \to u \text{ in } L_F(\Omega),\\
&u_n \rightharpoonup u \text{ in } L_G(\Omega), && u_n \to u \text{ a.e in } \ \Omega,\\
&\nabla u_n \rightharpoonup \nabla u \text{ in } \left[L^m(\Omega)\right]^N, && |\nabla u_n|^{m-2}\nabla u_n \rightharpoonup \Lambda \text{ in } \left[L^{m/(m-1)}(\Omega)\right]^N
\end{aligned} \tag{4.18}$$

for some $u\in X$ and $\Lambda\in\left[L^{m/(m-1)}(\Omega)\right]^N$.
We assert that $\Lambda=|\nabla u|^{m-2}\nabla u$ and u, which is clearly non-negative by (4.18), is the solution we are looking for.

Let's show this assertion by the following steps. The proof of $\Lambda=|\nabla u|^{m-2}\nabla u$ is done in the image of that of theorem 2.1 of [8].

Step 1. Let $K\subset\Omega$, be a compact set and J_K a function of $C_0^\infty(\Omega)$ such that $0\leqslant J_K\leqslant 1$ and $J_K=1$ on K.

For a given constant $\delta>0$, we define the function $S_\delta:\mathbb{R}\to\mathbb{R}$ by

$$S_\delta(s)=\begin{cases} s, & \text{if } \ |s|\leqslant\delta,\\ \delta\frac{s}{|s|} & \text{if } \ |s|>\delta.\end{cases}$$

For all $n\in\mathbb{N}$, let $\omega_n=J_K S_\delta\circ(u_n-u)$. $\omega_n\in X$, since $(u_n-u)\in X$, $S_\delta\circ(u_n-u)$ is bounded and Ω is a finite measure set. By substituting the function test ω_n for v in (4.14), we have

$$\begin{aligned}\int_\Omega J_K\left[|\nabla u_n|^{m-2}\nabla u_n-|\nabla u|^{m-2}\nabla u\right]\nabla(S_\delta\circ(u_n-u))dx = &-\int_\Omega S_\delta\circ(u_n-u)|\nabla u_n|^{m-2}\nabla u_n\nabla J_K dx\\ &-\int_\Omega J_K|\nabla u|^{m-2}\nabla u\nabla(S_\delta\circ(u_n-u))dx\\ &+\int_\Omega R_n(x)\omega_n(x)dx,\end{aligned} \tag{4.19}$$

with

$$R_n(x)=-|u_n(x)|^{m-2}u_n(x)+\lambda_n f\left(|u_n(x)|\right)u_n(x)-g\left(|u_n(x)|\right)u_n(x), \text{for } n\in\mathbb{N} \text{ and } x\in\Omega. \tag{4.20}$$

Let's remark that $\int_\Omega S_\delta\circ(u_n-u)|\nabla u_n|^{m-2}\nabla u_n\nabla J_K dx\to 0$ as $n\to+\infty$. In fact, since $u_n\to u$ a.e in Ω by (4.18), there exists $n_\delta\in\mathbb{N}$, for all $n\in\mathbb{N}$, $n>n_\delta$,

implies $meas\left(\{x\in\Omega:|u_n(x)-u(x)|>\delta\}\right)=0$.
Thus $S_\delta\circ(u_n-u)\nabla J_K\to 0$ in $\left(L^m(supp(J_K))\right)^N$, as $n\to\infty$ because $u_n-u\to 0$ in $L^m(\Omega)$ since $u_n\rightharpoonup u$ in X (see (4.18)) and $X\hookrightarrow\hookrightarrow L^m(\Omega)$ by Lemma 2.1. As $|\nabla u_n|^{m-2}\nabla u_n\rightharpoonup\Lambda$ in $\left[L^{m/(m-1)}(\Omega)\right]^N$, then $\displaystyle\int_\Omega S_\delta\circ(u_n-u)|\nabla u_n|^{m-2}\nabla u_n\nabla J_K dx\to 0$ as $n\to+\infty$. Furthermore, by the definition of S_δ and since $\nabla(u_n-u)\rightharpoonup 0$ in $[L^m(\Omega)]^N$ (see (4.18)), the sequence of functions $\left(\nabla S_\delta\circ(u_n-u)\right)_n$ weakly converge to zero in $[L^m(\Omega)]^N$. Consequently,

$$\int_\Omega J_K|\nabla u|^{m-2}\nabla u\nabla(S_\delta\circ(u_n-u))dx\to 0,$$

inasmuch as $|\nabla u|^{m-2}\nabla u\in\left[L^{m/(m-1)}(\Omega)\right]^N$. Thus, the first two terms in the right-side of (4.19) tend to zero when n tends to infinity. Let's now focus on the third term of the right-side of (4.19). According to the definitions of the functions J_K, S_δ and the fact that the sequence $(R_n)_n$ is bounded in $L^1_{\text{loc}}(\Omega)$ by the Lemma 4.2, we have

$$\int_\Omega R_n(x)\omega_n(x)dx\leqslant\int_{\text{supp}J_K}|R_n(x)|.|S_\delta\circ(u_n-u)(x)|dx\leqslant\delta\int_{\text{supp}J_K}|R_n(x)|dx\leqslant\delta.C_K. \tag{4.21}$$

Thus, from (4.19), we have just proved that

$$\begin{aligned}\ell(u):=&\limsup_{n\to+\infty}\int_K\left[|\nabla u_n|^{m-2}\nabla u_n-|\nabla u|^{m-2}\nabla u\right].\nabla(S_\delta\circ(u_n-u))dx\\ &\leqslant\limsup_{n\to+\infty}\int_\Omega J_K\left[|\nabla u_n|^{m-2}\nabla u_n-|\nabla u|^{m-2}\nabla u\right].\nabla(S_\delta\circ(u_n-u))dx\\ &\leqslant\delta.C_K\end{aligned} \tag{4.22}$$

since $\left[|\nabla u_n|^{m-2}\nabla u_n(x)-|\nabla u|^{m-2}\nabla u(x)\right].\nabla(S_\delta\circ(u_n-u)(x))\geqslant 0$ for all $x\in\Omega$ and $J_K\equiv 1$ on K.

Step2. Define now the non-negative function ξ_n by

$$\xi_n(x)=\left[|\nabla u_n|^{m-2}\nabla u_n(x)-|\nabla u|^{m-2}\nabla u(x)\right].\left[\nabla u_n(x)-\nabla u(x)\right] \tag{4.23}$$

and fix $\alpha\in(0,1)$. Considering the two complementaries subsets
$\Lambda^\delta_\alpha=\{x\in K:|u_n(x)-u(x)|\leqslant\delta\}$ and $\Sigma^\delta_\alpha=K\setminus\Lambda^\delta_\alpha$ of the set K and using Hölder inequality one has

$$\int_K\xi_n^\alpha(x)dx\leqslant\left(\int_{\Lambda^\delta_\alpha}\xi_n(x)dx\right)^\alpha\left(meas\left(\Lambda^\delta_\alpha\right)\right)^{1-\alpha}+\left(\int_{\Sigma^\delta_\alpha}\xi_n(x)dx\right)^\alpha\left(meas\left(\Sigma^\delta_\alpha\right)\right)^{1-\alpha}. \tag{4.24}$$

For δ fixed, $meas\left(\Sigma^\delta_\alpha\right)$ tends to 0 when n tends to infinity. Thus, since ξ_n is bounded in $L^1(\Omega)$, it follows from (4.22) and (4.24) that

$$\limsup_{n\to+\infty}\int_K\xi_n^\alpha(x)dx\leqslant(\delta.C_K)^\alpha\left(meas\left(\Lambda^\delta_\alpha\right)\right)^{1-\alpha}.$$

Letting δ tend to 0 implies that sequence ξ_n^α converge to 0 in $L^1(K)$ and thus, using a sequence of compact sets K, there exists a subsequence of $(\xi_{n_k})_k$ of (ξ_n) such that

$$\xi_{n_k}(x) \to 0 \text{ a.e. } x \; in \;\; \Omega.$$

Henceforth, $\nabla u_{n_k}(x) \to \nabla u(x)$ a.e $for \, x \; in \;\; \Omega$ by the following inequality which holds for a certain constant $\kappa > 0$ and for all $m \geqslant 2$:

$$|\nabla u_{n_k}(x) - \nabla u(x)|^m \leqslant \kappa \xi_{n_k}(x) = \left[|\nabla u_{n_k}(x)|^{m-2} \nabla u_{n_k}(x) - |\nabla u(x)|^{m-2} \nabla u(x) \right] . \left[\nabla u_{n_k}(x) - \nabla u(x) \right].$$

Consequently, $|\nabla u_{n_k}(x)|^{m-2} \nabla u_{n_k}(x) \to |\nabla u(x)|^{m-2} \nabla u(x)$ a.e. $for \, x \; in \;\; \Omega$. Moreover, the sequence (∇u_n) is bounded in $(L^m(\Omega))^N$ since $\nabla u_n \rightharpoonup \nabla u$ in $[L^m(\Omega)]^N$ (see (4.18)). Thus, by Vitali's Theorem, $|\nabla u_{n_k}|^{m-2} \nabla u_{n_k} \to |\nabla u|^{m-2} \nabla u$ in $\left[L^{m/(m-1)}(\Omega) \right]^N$. Being the limit $|\nabla u|^{m-2} \nabla u$ is independent of the subsequent $|\nabla u_{n_k}|^{m-2} \nabla u_{n_k}$, it follows that $|\nabla u_n|^{m-2} \nabla u_n \to |\nabla u|^{m-2} \nabla u$ in $\left[L^{m/(m-1)}(\Omega) \right]^N$. Finally, it follows that

$$|\nabla u_n|^{m-2} \nabla u_n \to |\nabla u|^{m-2} \nabla u = \Lambda \text{ in } \left[L^{m/(m-1)}(\Omega) \right]^N.$$

This yields that

$$\int_\Omega |\nabla u_n|^{m-2} \nabla u_n \nabla v dx \to \int_\Omega |\nabla u|^{m-2} \nabla u \nabla v dx, \text{ for all } v \in X \text{ as } n \to \infty. \quad (4.25)$$

Since $u_n \rightharpoonup u$ in X by (4.18), then Lemma 3.6 yields in particular that, for all $v \in X$,

$$\int_\Omega |u_n|^{m-2} u_n v dx \to \int_\Omega |u|^{m-2} u v dx, \quad \text{as } n \to \infty. \quad (4.26)$$

Moreover, the Lemmas 3.7 and 3.8 imply that, for all $v \in X$,

$$\int_\Omega f(|u_n|) u_n v dx \to \int_\Omega f(|u|) u v dx, \quad (4.27)$$

$$\int_\Omega g(|u_n|) u_n v dx \to \int_\Omega g(|u|) u v dx, \quad (4.28)$$

as $n \to \infty$.
Consequently, if we make n tend to ∞ in (4.14), then, by the convergences (4.25)-(4.28), we have

$$\int_\Omega |\nabla u|^{m-2} \nabla u \nabla v dx = - \int_\Omega |u|^{m-2} u v dx + \hat{\lambda} \int_\Omega f(|u|) u v dx - \int_\Omega g(|u|) u v dx, \quad (4.29)$$

for all $v \in X$.

Hence, u is a weak non-negative solution of $(\mathcal{P}_{\hat{\lambda}})$.
Step3. It remains to show that the solution u is nontrivial. Since $u_n \rightharpoonup u$ in X by (4.18), Lemma 2.6, in particular, show that $\|u\|_F = \lim \|u_n\|_F$. Moreover, by applying (2.11) to each $u_n \neq 0$, one has

$$\|u_n\|_F \geqslant \min \left\{ \left(q C_F^m \right)^{1/(m-p)} \lambda_n^{1/(m-p)}, \left(q C_F^m \right)^{1/(m-q)} \lambda_n^{1/(m-q)} \right\}. \quad (4.30)$$

Thus, by passing to the limit in this last inequality as $n \to \infty$, we get

$$\|u\|_F \geqslant \min\left\{ (qC_F^m)^{1/(m-p)} \left(\hat{\lambda}\right)^{1/(m-p)}, (qC_F^m)^{1/(m-q)} \left(\hat{\lambda}\right)^{1/(m-q)} \right\}, \tag{4.31}$$

since $(\lambda_n)_n$ is a strictly decreasing sequence converging to $\hat{\lambda}$ and $\hat{\lambda} > 0$. Consequently, u is nontrivial and non-negative by (4.18). $\square$

Proof of Theorem A. The combination of Section 2, Lemma 4.3 and Theorems 4.1 and 4.2 show the existence of $\hat{\lambda} > 0$ such that for all $\lambda \geqslant \hat{\lambda}$, $(\mathcal{P}_\lambda)$ admits at least a nontrivial non-negative weak solution in X.

5. Existence of a second non-negative weak solution for large values of the parameter

Using variational methods, we prove, in this section, that the Dirichlet problem $(\mathcal{P}_\lambda)$ admits at least a second nontrivial weak solution if λ is sufficiently large.

Lemma 5.1. *For any $\boldsymbol{\omega} \in X \setminus \{0\}$ and $\lambda > 0$ there exist $\varrho \in (0, \|\boldsymbol{\omega}\|_{W_0^{1,m}(\Omega)})$ and $\alpha = \alpha(\varrho) > 0$ such that $\Phi_\lambda(u) \geqslant \alpha$ for all $u \in X$, with $\|u\|_{W_0^{1,m}(\Omega)} = \varrho$.*

Proof. Let $\boldsymbol{\omega} \in X \setminus \{0\}$ and $\lambda > 0$. Let u be in X. By (2.4), (2.7) and monotony of the function ζ_1,

$$\begin{aligned}
\Phi_\lambda(u) &\geqslant \frac{1}{m}\|u\|^m_{W_0^{1,m}(\Omega)} - \lambda \int_\Omega F(|u|)dx \\
&\geqslant \frac{1}{m}\|u\|^m_{W_0^{1,m}(\Omega)} - \lambda\zeta_1\left(\|u\|_F\right) \\
&\geqslant \frac{1}{m}\|u\|^m_{W_0^{1,m}(\Omega)} - \lambda\zeta_1\left(C_F\|u\|^m_{W_0^{1,m}(\Omega)}\right) \\
&\geqslant \left[\frac{1}{m} - \lambda \max\left\{C_F^p\|u\|^{p-m}_{W_0^{1,m}(\Omega)}, C_F^q\|u\|^{q-m}_{W_0^{1,m}(\Omega)}\right\}\right]\|u\|^m_{W_0^{1,m}(\Omega)}.
\end{aligned}$$

So, to find the result of the Lemma, it is enough to take ϱ such that

$$0 < \varrho < \min\left\{ (m\lambda C_F^p)^{1/(m-p)}, (m\lambda C_F^q)^{1/(m-q)}, \|\boldsymbol{\omega}\|_{W_0^{1,m}(\Omega)} \right\}$$

and

$$\alpha = \alpha(\varrho) = \left[\frac{1}{m} - \lambda \max\left\{C_F^p\varrho^{p-m}, C_F^q\varrho^{q-m}\right\}\right]\varrho^m > 0.$$

$\square$

In Lemma 4.1 we have shown that, for all $\lambda > \overline{\lambda}$, there exists a nontrivial non-negative weak solution $\boldsymbol{\omega} \in X$ of $(\mathcal{P}_\lambda)$, which is a global minimizer for Φ_λ in X, with $\Phi_\lambda(\boldsymbol{\omega}) < 0$. In this section we are looking for a second nontrivial weak solution of $(\mathcal{P}_\lambda)$, when $\lambda > \overline{\lambda}$.

Theorem 5.1. *Let $(X, \|.\|)$ and $(X, \|.\|_E)$ be two Banach spaces such that $X \hookrightarrow E$. Let $\Phi : X \to \mathbb{R}$ be a C^1 functional with $\Phi(0) = 0$. Suppose that there exist*

$\varrho, \alpha > 0$ and $\boldsymbol{\omega} \in X$ such that $\|\boldsymbol{\omega}\|_E > \varrho$, $\Phi(\boldsymbol{\omega}) < \alpha$ and $\Phi(u) \geqslant \alpha$ for all $u \in X$ with $\|u\|_E = \varrho$. Then there exists a sequence $(u_n)_n \in X$ such that for all n

$$\rho \leqslant \Phi(u_n) \leqslant \rho + \frac{1}{n^2} \text{ and } \|\Phi'(u_n)\|_{X'} \leqslant \frac{2}{n},$$

where

$$\rho = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t)) \text{ and } \Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = 0,\ \gamma(1) = \boldsymbol{\omega}\}. \tag{5.1}$$

Proof. The proof of this theorem is given by the proof of Theorem A.3. of [5]. □

Proof of Theorem B. Let λ be a strictly positive fixed number such that $\lambda > \overline{\lambda}$. Let's designate by $\boldsymbol{\omega} \in X$ the global minimizer of Φ_λ obtained by Lemma 4.1. Thanks to Lemma 5.1 and the fact that $\Phi_\lambda(\boldsymbol{\omega}) < 0$, the assumptions of Theorem 5.1 which is a variant of Ekeland's variational principle, are fufilled for the functional Φ_λ. Consequently, there exists a sequence $(u_n)_n \subset X$ such that

$$\Phi_\lambda(u_n) \to \rho \text{ and } \|\Phi'(u_n)\|_{X'} \to 0$$

where

$$\rho = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t)) \text{ and } \Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = 0,\ \gamma(1) = \boldsymbol{\omega}\}. \tag{5.2}$$

We shall prove that $(u_n)_n$ strongly converges to some u in X and that u is a second nontrivial non-negative solution of $(\mathcal{P}_\lambda)$. To do this, follow the steps below:

Step1. By Lemma 3.1 (coercivity of Φ_λ on X), the sequence $(u_n)_n$ is bounded in X and consequently, the sequence $(|\nabla u_n|^{m-2}\nabla u_n)_n$ is bounded in $\left(L^{m/(m-1)}(\Omega)\right)^N$. From now we can use the argument of the proof of Theorem 4.2. In order not to repeat everything that is done in the proof of the Theorem 4.2, we report here the essentials needed to the demonstration of this theorem. By Propositions 3.2 and 3.1 and Lemma 2.6 we can extract, from the sequence $(u_n)_n$, a subsequence still relabeled $(u_n)_n$ and satisfying (4.18) (see in the proof of theorem 4.2), for some $u \in X$ and $\Lambda \in \left(L^{m/(m-1)}(\Omega)\right)^N$. The fundamental point of the proof is to show that $\Lambda = |\nabla u|^{m-2}\nabla u$ and u is a nontrivial solution of $(\mathcal{P}_\lambda)$, with $u \neq \boldsymbol{\omega}$. We have

$$\langle \Phi'_\lambda(u_n), v\rangle = \int_\Omega |\nabla u_n|^{m-2}\nabla u_n \nabla v dx - \int_\Omega R_n(x)v(x)dx, \text{ for all } n, \tag{5.3}$$

for any $v \in X$, where the sequence $(R_n)_n$ is the one defined in (4.4) with λ_n which is identically equal to λ for all n. $(R_n)_n$ belongs to $L^1_{\text{loc}}(\Omega)$ by Lemma 4.2, being $(\|u_n\|_X)_n$ is bounded.

Fix a compact subset K in Ω and a constant $\delta > 0$. Consider J_K the function test of $C_0^\infty(\Omega)$ such that $0 \leqslant J_K \leqslant 1$ and $J_K = 1$ on K and S_δ the function definned in the proof of Theorem 4.2.

We continuous also to set $\omega_n = J_K S_\delta \circ (u_n - u)$ which belongs to X for all $n \in \mathbb{N}$, as in the proof Theorem 4.2. By replacing v by the function test ω_n in

(5.3), we have

$$\begin{aligned}\int_\Omega J_K\left[|\nabla u_n|^{m-2}\nabla u_n-|\nabla u|^{m-2}\nabla u\right]\nabla(S_\delta\circ(u_n-u))dx=&-\int_\Omega S_\delta\circ(u_n-u)|\nabla u_n|^{m-2}\nabla u_n\nabla J_K dx\\&-\int_\Omega J_K|\nabla u|^{m-2}\nabla u\nabla(S_\delta\circ(u_n-u))dx\\&+\langle\Phi'_\lambda(u_n),\omega_n\rangle+\int_\Omega R_n(x)\omega_n(x)dx.\end{aligned} \tag{5.4}$$

Proceeding as for Theorem 4.2, we show that the first two terms in the right-side of (5.4) tend to zero when n tends to infinity. Moreover,

$$\langle\Phi'_\lambda(u_n),\omega_n\rangle\to 0, \text{when } n\to\infty,$$

since $\Phi'_\lambda(u_n)\to 0$ in X' and $\omega_n\rightharpoonup 0$ as $n\to\infty$ due to the fact that $u_n\rightharpoonup u$ as $n\to\infty$. Finally, as we show it in the proof of Theorem 4.2, being $(R_n)_n$ is bounded in $L^1_{loc}(\Omega)$ by Lemma 4.2, we have

$$\int_\Omega R_n(x)\omega_n(x)dx\leqslant\delta.C_K, \tag{5.5}$$

where C_K is a suitable constant depend only on the compact set K. From now on, operating as in the proof of Theorem 4.2, one show that

$$|\nabla u_n(x)|^{m-2}\nabla u_n(x)\to|\nabla u(x)|^{m-2}\nabla u(x), \text{ as } n\to\infty, \text{ a.e for } x\in\Omega.$$

Thus, through Vitali's Theorem, we show that

$$|\nabla u_n|^{m-2}\nabla u_n\to|\nabla u|^{m-2}\nabla u=\Lambda \text{ in } \left[L^{m/(m-1)}(\Omega)\right]^N,$$

as it does in Theorem 4.2.

This yields in particular that

$$\int_\Omega|\nabla u_n|^{m-2}\nabla u_n\nabla v dx\to\int_\Omega|\nabla u|^{m-2}\nabla u\nabla v dx, \text{ for all } v\in X \text{ as } n\to\infty. \tag{5.6}$$

Also, as in the proof of Theorem 4.2, since $u_n\rightharpoonup u$ in X by (4.18), it follows by Lemma 3.6 that, for all $v\in X$,

$$\int_\Omega|u_n|^{m-2}u_n v dx\to\int_\Omega|u|^{m-2}uv dx, \quad \text{as } n\to\infty. \tag{5.7}$$

Moreover, the Lemmas 3.7 and 3.8 imply that, for all $v\in X$,

$$\int_\Omega f(|u_n|)u_n v dx\to\int_\Omega f(|u|)uv dx, \tag{5.8}$$

$$\int_\Omega g(|u_n|)u_n v dx\to\int_\Omega g(|u|)uv dx, \tag{5.9}$$

as $n\to\infty$.

Henceforth, passing to the limit in (5.3), as $n\to\infty$, using (5.6), (5.7) (5.8), (5.9) and the fact that $\langle \Phi'_\lambda(u_n), v\rangle \to 0$ as $n \to \infty$ for all $v \in X$, we get

$$\int_\Omega |\nabla u_n|^{m-2}\nabla u_n \nabla v dx + \int_\Omega |u_n|^{m-2} u_n v dx = \lambda \int_\Omega f(|u_n|)u_n v dx - \int_\Omega g(|u_n|)u_n v dx, \tag{5.10}$$

for all $v \in X$. So, u is a weak solution of ($\mathcal{P}_\lambda$).

Step2. We claim that

$$\mathcal{I}_f(n) = \int_\Omega \left[f(|u_n|)u_n - f(|u|)u\right](u_n - u)dx \to 0 \tag{5.11}$$

as $n \to \infty$. By hypothesis **($\mathscr{F}_1$)**, the function $t \mapsto f(|t|)t$ is strictly increase in $[0; +\infty)$ and consequently it is strictly increase in $(-\infty; +\infty)$ because it is an odd function. Thus, we have

$$\mathcal{I}_f(n) \geqslant 0.$$

By Lemma (2.6), $u_n \to u$ in $L_F(\Omega)$, since $u_n \rightharpoonup u$ in X. Thus, $f(|u_n|)u_n \to f(|u|)u$ in $L_{\widetilde{F}}(\Omega)$ by proposition 3.3. Therefore, by Hölder inequality, we get

$$0 \leqslant \int_\Omega \left[f(|u_n|)u_n - f(|u|)u\right](u_n - u)dx \leqslant 2\|f(|u_n|)u_n - f(|u|)u\|_{\widetilde{F}}\|u_n - u\|_F \to 0,$$

as $n \to \infty$. This give the proof of the claim.

Step3. In this step, we show that $\|u_n - u\|_X \to 0$ as $n \to \infty$. By convexity, we have:

$$\mathcal{I}_\nabla(n) = \int_\Omega (|\nabla u_n|^{m-2}\nabla u_n - |\nabla u|^{m-2}u)(\nabla u_n - \nabla u)dx \geqslant 0$$

$$\mathcal{I}_m(n) = \int_\Omega (|u_n|^{m-2}u_n - |u|^{m-2}u)(u_n - u)dx \geqslant 0$$

$$\mathcal{I}_g(n) = \int_\Omega \left[g(|u_n|)u_n - g(|u|)u\right](u_n - u)dx \geqslant 0.$$

Then we have

$$\mathcal{W}_n := \mathcal{I}_\nabla(n) + \mathcal{I}_m(n) + \mathcal{I}_g(n) = \langle \Phi'_\lambda(u_n) - \Phi'_\lambda(u), u_n - u\rangle + \lambda \mathcal{I}_f(n).$$

Since $u_n \rightharpoonup u$ in X and $\Phi'_\lambda(u_n) \to 0$ in X' when $n \to \infty$, then $\langle \Phi'_\lambda(u_n) - \Phi'_\lambda(u), u_n - u\rangle \to 0$, as $n \to \infty$. Hence, by (5.11)

$$\mathcal{W}_n = \langle \Phi'_\lambda(u_n) - \Phi'_\lambda(u), u_n - u\rangle + \lambda \mathcal{I}_f(n) \to 0 \tag{5.12}$$

as $n \to \infty$. Furthermore, there exists a constant $\mathfrak{H} > 0$ such that we have

$$\|u_n - u\|^m_{W_0^{1,m}(\Omega)} \leqslant \mathfrak{H}\,\mathcal{I}_\nabla(n) \to 0, \tag{5.13}$$

as $n \to \infty$. By Lemma 23 of [23], there exists $\hat{\kappa} > 0$ such that

$$\text{for all } (x, y) \in \mathbb{R}^2, (g(|x|)x - g(|y|)y)(x - y) \geqslant \hat{\kappa} G(|x - y|) \geqslant 0.$$

Thus, by (5.12) it follows that

$$\int_\Omega G(|u_n - u|)dx \leqslant \frac{1}{\hat{\kappa}}\mathcal{I}_g(n) \to 0,$$

as $n \to \infty$. Therefore,

$$\|u_n - u\|_G \to 0, \text{ as } n \to \infty, \tag{5.14}$$

since the N-function G satisfied the Δ_2-condition. Finally, by (5.13) and (5.14), we have

$$\|u_n - u\|_X \to 0, \text{ when } n \to \infty.$$

Step4. Since $u_n \to u$ in X and $\Phi_\lambda \in C^1(X)$, we get $\Phi_\lambda(u) = \rho = \lim_{n\to\infty} \Phi_\lambda(u_n)$. So, u is a second independent nontrivial weak solution of $(\mathcal{P}_\lambda)$, with $\Phi_\lambda(u) = \rho > 0 > \Phi_\lambda(\boldsymbol{\omega})$. We can assume $u \geqslant 0$ a.e. in Ω, since $|u|$ is also a solution of $(\mathcal{P}_\lambda)$ due to $\Phi_\lambda(|u|) = \Phi_\lambda(u)$. This concludes the proof.

Conclusion

In this paper we studied, according to the values of parameter λ, the existence of multiple nontrivial non-negative weak solutions of the quasilinear Dirichlet elliptic equation $(\mathcal{P}_\lambda)$ involving two more general nonlinearities functions $f(|t|)t$ and $g(|t|)t$ with appropriate conditions. Initially, we showed that if $\lambda \geqslant \hat{\lambda}$, with $\hat{\lambda}$ a critical value, then the problem $(\mathcal{P}_\lambda)$ admits at least a nontrivial non-negative weak solutions. Secondly, we exhibited a second critical value $\overline{\lambda} > \hat{\lambda}$ and showed, by a variant of Ekeland's variational principle, that for all $\lambda \geqslant \overline{\lambda}$, $(\mathcal{P}_\lambda)$ admits at least a second nontrivial non-negative weak solution different from the first one.

References

1. Adams, R.A., & Fournier, J.J.F. (2003). *Sobolev Spaces*. Academic Press. New York, London.
2. Alama, S., & Tarantello, G. (1996). *Elliptic problems with nonlinearities indefinite in sign.* J. Funct. Anal. 141, 159-215. https://doi.org/10.1006/jfan.1996.0125
3. Ambrosetti, A., Brezis, H., & Cerami, G. (1994). *Combined effects of concave and convex nonlinearities in some elliptic problems.* J. Funct. Anal., 122, 519-543. https://doi.org/10.1006/jfan.1994.1078
4. Ambrosetti, A., Azorero, J. G., & Peral, I. (1996). *Multiplicity results for some nonlinear elliptic equations*, J. Funct. Anal., 137, 219–242. 10.1006/jfan.1996.0045
5. Autuori, G., & Pucci, P. (2013). *Existence of entire solutions for a class of quasilinear elliptic equations.* NoDEA Nonlinear Differential Equations Appl., 20, 977-1009. https://doi.org/10.1007/s00030-012-0193-y
6. Azorero, J. G., & Alonso I. P. (1991). *Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term.* Trans. Amer. Math. Soc., 323, 877-895. https://doi.org/10.1090/S0002-9947-1991-1083144-2
7. Baldelli, L., & Filippucci, R. (2022). Existence of solutions for critical (p, q)-Laplacian equations in $\mathbb{R}^N$. Communications in Contemporary Mathematics, 2150109. https://doi.org/10.1142/S0219199721501091
8. Boccardo, L., & Murat, F. (1992). *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations* Nonlinear Anal., Theory, Methods & Applications, Vol. 19, issue 6,pp. 581-597. https://doi.org/10.1016/0362-546X(92)90023-8
9. Brezis, H. (2011). *Functional analysis, Sobolev spaces and partial differential equations.* Universitext, p. xiv+599. Springer, New York.
10. Chabrowski, J., & Bezzera Do Ó, J. M. (2002). *On semilinear elliptic equations involving concave and convex nonlinearities*, Math. Nachr. 233–234 , 55–76. https://doi.org/10.1002/1522-2616(200201)233:1¡55::AID-MANA55¿3.0.CO;2-R
11. De Figueiredo, D.G., Gossez, J.-P., & Ubilla, P. (2003). *Local superlinearity and sublinearity for indefinite semilinear elliptic problems.* J. Funct. Anal., 199, 452-467. https://doi.org/10.1016/S0022-1236(02)00060-5

12. De Figueiredo, D.G., Gossez, J.-P., & Ubilla, P. (2006). *Multiplicity results for a family of semilinear elliptic problems under local superlinearity and sublinearity.* J. Eur. Math. Soc, 8 (2), 269–286.
13. Filippucci, R., Pucci, P., & Rădulescu, V. (2008). *Existence and non-existence results for quasilinear elliptic exterior problems with nonlinear boundary conditions.* Comm. Partial Differential Equations 33, no. 4-6, 706-717. https://doi.org/10.1080/03605300701518208
14. Fukagai, N., Ito, M., & Narukawa K. (2006). *Positive solutions of quasilinear elliptic equations with critical Orlicz-Sobolev non-linearity on* $\mathbb{R}^N$, Funkcialaj Ekvacioj, vol. 49, no. 2, pp. 235–267. https://doi.org/10.1619/fesi.49.235
15. García, J., Peral, I., & Manfredi, J. (2000) *Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations.* Commun. Contemp. Math., 2 (3), 385–404. https://doi.org/10.1142/S0219199700000190
16. Ghoussoub, N., & Yuan, C. (2000). *Multiple Solutions for Quasi-Linear PDEs Involving the Critical Sobolev and Hardy Exponents.* Transactions of the American Mathematical Society, volume 352, number 12, pp. 5703-5743. https://doi.org/10.1090/S0002-9947-00-02560-5
17. Krasnosielskii, M. A., & Rutickii, Ya B. (1961). *Convex Functions and Orlicz Spaces.* Noordhoff Ltd, Groningen, Holland.
18. Kristály, A., & Morosanu, G. (2010). *New competition phenomena in Dirichlet problems.* J. Math. Pures Appl., 94, 555-570. https://doi.org/10.1016/j.matpur.2010.03.005
19. Li, S., & Wang, Z. Q. (2000). *Mountain pass theorem in order intervals and multiple solutions for semilinear elliptic Dirichlet problems.* J. Anal. Math., 81, 373–396. https://doi.org/10.1007/BF02788997
20. Perera, K. (2024). *A general perturbation theorem with applications to nonhomogeneous critical growth elliptic problems.* Journal of Differential Equations, Volume 389, Pages 150-189. https://doi.org/10.1016/j.jde.2024.01.014
21. Pucci, P., & Rădulescu, V. (2011). *Combined effects in quasilinear elliptic problems with lack of compactness.* Rend. Lincei Mat. Appl., 22, 189-205 (special volume dedicated to the memory of Prof. G. Prodi). 10.4171/RLM/595
22. Talenti, G. (1976) *Best constant in Sobolev inequality.* Ann. Mat. Pura Appl., 110 , 353–372. https://doi.org/10.1007/BF02418013
23. Tchalla, A. M. J., & Tcharie, K. (2022) *Multiplicity results for weak solutions of a semilinear Dirichlet elliptic problem with a parametric nonlinearity.* International Journal of Mathematics and Mathematical Sciences, 6011860, 19 pages. https://doi.org/10.1155/2022/6011860.
24. Wu, T.-f. (2010). *Multiple positive solutions for a class of concave–convex elliptic problems in* $\mathbb{R}^N$ *involving sign-changing weight.* Journal of Functional Analysis, Volume 258, Issue 1,Pages 99-131. https://doi.org/10.1016/j.jfa.2009.08.005

[1] Département de Mathématiques, Université de Lomé 01 B.P 1515 Lomé 1 (Togo)

Email address: atchalla@univ-lome.tg