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KNESER'S PROPERTY OF A CLASS OF AN INTEGRODIFFERENTIAL EQUATIONS IN α-NORM

GAPILY DOUKA[1], JUDÉON KABRE[1], PAUL OUEDRAOGO[1], and ISSA ZABSONRE[1*]

Abstract. In this work, we prove some properties of the set of solutions for some integrodifferential equations called Kneser's property.

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1. Introduction

This work gives some properties of the set of solutions for the following system

$$\begin{cases} \zeta'(x) = -B\zeta(x) + \displaystyle\int_0^x \Pi(x-t)\zeta(t)dt + f(x,\zeta_x) \text{ for } x \geq 0 \\ \\ \zeta_0 = \phi \in \Upsilon, \end{cases} \tag{1.1}$$

where $-B$ generates a compact analytic semigroup, Υ_α, is a subset of Υ, where Υ is a Banach space of functions defined from $]-\infty,0]$ to X and for $0<\alpha<1$, the operator B^α is a α-power of B and it will be described later. For $x \geq 0$, $\Pi(x)$ is closed linear operator with domain $D(\Pi) \supset D(B)$, $f : \mathbb{R}^+ \times \Upsilon_\alpha \to X$ is a continuous function, where Υ_α is defined by

$$\Upsilon_\alpha = \{\phi \in \Upsilon;\ \phi(\theta) \in D(B^\alpha) \text{ for } \theta < 0 \text{ and } B^\alpha\phi \in \Upsilon\}$$

with the norm

$$\|\phi\|_{\Upsilon_\alpha} = \|B^\alpha \phi\|_\Upsilon.$$

For every $x \geq 0$, $\zeta_x \in \Upsilon_\alpha$ is a function defined by

$$\zeta_x(\theta) = \zeta(x+\theta) \text{ for } \theta \in]-\infty,0],$$

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* Corresponding author
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Using semigroup's theory, the authors in [9], proved that the set of solutions of the following

$$\begin{cases} \zeta'(x) = -B\zeta(x) + f(x, \zeta_x) \text{ for } x \geq 0 \\ \\ \zeta_0 = \phi \in \Upsilon, \end{cases}$$

is connected in the space of continuous functions. The aim of this work is to generalize the work done in [9] by using resolvent operator's theory which is more general than semigroup theory.

A brief reminder of this theory is provided in Section 2. In section 3, we establish the Kneser's property of equation (1.1). Last section is devoted to an application.

2. **Preliminary results**

In what follows $(X, \|.\|)$ is a Banach space, α is a constant such that $0 < \alpha < 1$ and $-A$ generates a bounded analytic semigroup of linear operator $(S(x))_{x\geq 0}$ on X. The fractional power B^α, with domain $D(B^\alpha)$, is a closed linear invertible operator dense in X. $D(B^\alpha)$ and endowed with the norm $|.|_\alpha$, defined by $|x|_\alpha = \|B^\alpha x\|$ for each $x \in D(B^\alpha)$ is a Banach space, which is denoted by X_α.

Proposition 2.1. [6] *Let $0 < \alpha < 1$. Assume that the operator $-B$ is the infinitesimal generator of an analytic semigroup $(S(x))_{x\geq 0}$ on the Banach space X satisfying $0 \in \rho(A)$. Then we have*

- *i)* $S(x) : X \to D(B^\alpha)$ *for every* $x > 0$.
- *ii)* $S(x)B^\alpha y = B^\alpha S(x)y$ *for every* $y \in D(B^\alpha)$ *and* $x \geq 0$.
- *iii)* *for every* $x > 0$, $B^\alpha S(x)$ *is bounded on* X *and there exist* $M_\alpha > 0$ *and* $\omega > 0$ *such that*

$$\|B^\alpha S(x)\| \leq M_\alpha e^{-\omega x} x^{-\alpha} \text{ for } x > 0.$$

- *iv)* *If* $0 < \alpha \leq \beta < 1$, $D(B^\beta) \hookrightarrow D(B^\alpha)$.
- *v)* *There exists* $N_\alpha > 0$ *such that*

$$\|(S(x) - I)B^{-\alpha}\| \leq N_\alpha x^\alpha \text{ for } x > 0.$$

- *vi)* *If* $S(x)$ *is compact for each* $x > 0$, *then* $B^{-\alpha}$ *is compact.*

Now, we give some basic results about resolvent operators.
Consider the following equation

$$\begin{cases} \vartheta'(x) = -B\vartheta(x) + \displaystyle\int_0^x \Pi(x - t)\vartheta(t)dt + g(x) \text{ for } x \geq 0 \\ \\ \vartheta(0) = \vartheta_0 \in X, \end{cases} \tag{2.1}$$

where $-B$ and $\Pi(x)$ are closed linear operators on X.

Definition 2.2. [3] A resolvent operator for equation (2.1) is a bounded operator valued-function $\Phi(x) \in \Upsilon(X)$ such that for $x \geq 0$ we have
(a) $\Phi(0) = I$ and $|\Phi(x)| \leq Ne^{-\beta x}$ for some positive constants N and β.
(b) For all $y \in X$, $\Phi(x)y$ is strongly continuous for $y \geq 0$.
(c) $\Phi(x) \in \Upsilon(D(A))$ for $x \geq 0$. For $y \in Y = D(A)$, $\Phi(.)y \in C^1([0, +\infty[; X) \cap C([0, +\infty[; D(A))$ and

$$\begin{aligned}\Phi'(x)y &= -A\Phi(x)y + \int_0^x \Pi(x-t)\Phi(t)ydt \\ &= -\Phi(x)Ay + \int_0^x \Phi(x-t)\Pi(t)xdt \text{ for } x \geq 0.\end{aligned}$$

Under some assumptions given in [3], we obtain the existence of an analytic resolvent operator, where f^* is the Laplace transform of f.

$(\mathbf{V}_1)$ $-B$ generates an analytic semigroup $(S(x))_{x\geq 0}$ on X. $(\Pi(x))_{x\geq 0}$ is a closed operator on X with domain at least $D(B)$ a.e $t \geq 0$ with $\Pi(x)y$ strongly measurable for each $y \in D(A)$ and $\|\Pi(x)y\| \leq \Pi(x)\|y\|_1$ for $b \in L^1_{loc}(0, \infty)$ with $b^*(\lambda)$ absolutely convergent for $Re\lambda > 0$.

$(\mathbf{V}_2)$ $\rho(\lambda) = (\lambda I + A - \Pi^*(\lambda))^{-1}$ exists as a bounded operator on X which is analytic for $\lambda \in \Lambda = \{\lambda \in \mathbf{C} : \ |arg(\lambda)| < \frac{\pi}{2} + \delta\}$, where $0 < \delta < \frac{\pi}{2}$. In Λ if $|\lambda| \geq \varepsilon > 0$ there exists $M = M(\varepsilon) > 0$ so that $\|\rho(\lambda)\| \leq \frac{M}{|\lambda|}$.

$(\mathbf{V}_3)$ $B\rho(\lambda) \in \mathcal{L}(X)$ for $\lambda \in \Lambda$ and is analytic from Λ to $\mathcal{L}(X)$. $\Pi^*(\lambda) \in \mathcal{L}(Y, X)$ and $\Pi^*(\lambda)\rho(\lambda) \in \mathcal{L}(Y, X)$ for $\lambda \in \Lambda$. Given $\varepsilon > 0$, there exists a positive constant $M = M(\varepsilon)$ so that for $y \in Y$ and $\lambda \in \Lambda$ with $|\lambda| \geq \varepsilon$ $\|A\rho(\lambda)y\| + \|\Pi^*(\lambda)\rho(\lambda)y\| < \frac{M}{|\lambda|}\|y\|$ and $\|\Pi^*(\lambda)\| \to 0$ as $|\lambda| \to +\infty$ in Λ. In addition, $\|A\rho(\lambda)y\| \leq M|\lambda|^n$ for some $n > 0$, $\lambda \in \Lambda$ with , $|\lambda| \geq \varepsilon$. Further, there exists $D \subset D(B^2)$ which is dense in Y such that $B(D)$ and $\Pi^*(\lambda)(D)$ are contained in Y and $\|\Pi^*(\lambda)y\|$ is bounded for each $y \in D$ and $\lambda \in \Lambda$ with $|\lambda| \geq \varepsilon$.

From [3], there exists a resolvent operator $(\Phi(x))_{x\geq 0}$ for linear equation (2.1) which is analytic. Moreover for $a > 0$, there exist $K_\alpha > 0$ such that

$$\|B^\alpha \Phi(x)\| \leq K_\alpha x^{-\alpha} \text{ for } x \in (0, a] \text{ and } 0 < \alpha < 1. \tag{2.2}$$

Theorem 2.3. [2] *Assume that $(\boldsymbol{V}_1)$, $(\boldsymbol{V}_2)$ and $(\boldsymbol{V}_3)$ hold. Let $(S(x))_{x\geq 0}$ be a compact semigroup for $x > 0$. Then the corresponding resolvent operator $(\Phi(x))_{x\geq 0}$ of equation (2.1) is also compact for $x > 0$.*

Now consider the following system:

$$\begin{cases} z'(x) = Az(x) + \displaystyle\int_0^x [\Pi_1(x-t) - \Pi_2(x-t)]z(t)dt \text{ for } x \geq 0 \\ \\ z(0) = z_0 \in X. \end{cases} \tag{2.3}$$

where $\Pi_1(x)$ and $\Pi_2(x)$ are closed linear operators in X and satisfy $(\mathbf{V}_3)$. Then we have the following Lemma and corollary coming from [2].

Lemma 2.4. [2] *(Perturbation result) Assume that* $(\boldsymbol{V}_1)$, $(\boldsymbol{V}_2)$ *and* $(\boldsymbol{V}_3)$ *hold and Let* $(\Phi_{\Pi_1}(x))_{x\geq 0}$ *be a resolvent operator of equation* (2.1) *and* $(\Phi_{\Pi_1+\Pi_2}(x))_{x\geq 0}$ *be a resolvent operator of equation* (2.3). *Then*

$$\Phi_{\Pi_1+\Pi_2}(x)z - \Phi_{\Pi_1}(x)z = \int_0^x \Phi_{\Pi_1}(x-t)Q(t)zdt,$$

where the operator Q is defined by

$$Q(x)z = \int_0^x \Pi_2'(x-t)\Big(\int_0^s \Phi_{\Pi_1+\Pi_2}(\tau)zd\tau\Big)dt + \Pi_2(0)\int_0^x \Phi_{\Pi_1+\Pi_2}(t)zdt,$$

Q is uniformly bounded on bounded intervals, and for each $z \in X$, $Q(.)z$ belongs to $C([0,\infty);X)$.

Corollary 2.5. [2] *Let B be a closed, densely defined linear operator in X, $\Pi(x) = 0$ for all $x \geq 0$, and $(\Phi(x))_{x\geq 0}$ be a resolvent operator for equation* (2.1). *Then $(\Phi(x))_{x\geq 0}$ is a C_0-semigroup with infinitesimal generator B.*

Let

$$Q(x)z = \int_0^x \Pi'(x-t)\Big(\int_0^s \Phi(\tau)xd\tau\Big)dt + \Pi(0)\int_0^x \Phi(t)zdt,$$

By Lemma 2.4 we deduce that the operators $(Q(x))_{x\geq 0}$ are uniformly bounded for t in a bounded interval and that

$$\Phi(x)z = S(x)z + \int_0^x S(x-t)Q(t)zdt \tag{2.4}$$

where $(S(x))_{x\geq 0}$ in a C_0-semigroup generates by A on X.

The following the theorem is an important tool to prove our main result.

Theorem 2.6. [5] *Let $\mathcal{F} : C([0,\tau];X) \to C([0,\tau];X)$, $\tau > 0$ be continuous and $\mathbb{S}$ be the set of fixed points of $\mathcal{F}$. Assume that there is a compact set $K \subset C([0,\tau];X)$ such that for each $\varepsilon > 0$, there is a set $K_\varepsilon \subset K$ with the following properties:*

(i) *the sets K_ε are connected;*

(ii) $d(z, K_\varepsilon) < \varepsilon$ *for all* $z \in \mathbb{S}$;

(iii) $|y - \mathcal{F}y|_\infty < \delta(\varepsilon)$ *for all* $y \in K_\varepsilon$, *where* $\delta(\varepsilon) \to 0$, *as* $\varepsilon \to 0$.

Then $\mathbb{S}$ is connected.

3. Kneser's Property

In what follows, we give some important axioms from [4].

$(\mathbf{A}_1)$ There exist a positive constant H and functions $K(.)$, $P(.) : \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and $\mathcal{M}$ locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $u :]-\infty, a] \to X$, $\vartheta_\sigma \in \Upsilon$, and $\vartheta(.)$ is continuous on $[\sigma, \sigma + a]$, then for every $x \in [\sigma, \sigma + a]$ the following conditions hold

(**i**) $\vartheta_x \in \Upsilon$.

(**ii**) $|\vartheta(x)| \leq H|\vartheta_x|_\Upsilon \Leftrightarrow |\phi(0)| \leq H|\phi|_\Upsilon$ for all $\phi \in \Upsilon$.

(**iii**) $|\vartheta_x|_\Upsilon \leq K(x-\sigma) \sup_{\sigma \leq t \leq x} |\vartheta(t)| + \mathcal{M}(x-\sigma)|\vartheta_\sigma|_\Upsilon$.

($\mathbf{A}_2$) If $\vartheta(.)$ in ($\mathbf{A}_1$), $x \mapsto \vartheta_x$ is a Υ-valued continuous function for $x \in [\sigma, \sigma + a]$.

(**B**) Υ is a complete space.

In the sequel, we give the conditions of the existence of mild solutions of equation (1.1). For this, we suppose that:

($\mathbf{H}_1$) The operator $-B$ generates an analytic semigroup $(S(x))_{x\geq 0}$ on X and $0 \in \rho(A)$.

($\mathbf{H}_2$) $((S(x))_{x\geq 0}$ is compact semigroup on X for $x > 0$.

($\mathbf{H}_3$) $B^{-\alpha}\phi \in \Upsilon$ for all $\phi \in \Upsilon$ and $B^{-\alpha}\phi$ is given by

$$(B^{-\alpha}\phi)(\theta) = B^{-\alpha}(\phi(\theta)) \text{ for } \theta \in]-\infty, 0].$$

Lemma 3.1. [1] *Assume that* ($\boldsymbol{H}_1$) *and* ($\boldsymbol{H}_3$) *hold. Then* Υ_α *is a Banach space.*

We give the definition of mild solution of this work.

Definition 3.2. A function $z :]-\infty, +\infty[\to X$ is a mild solution of equation (1.1) if z is given by

$$\begin{cases} z(x) = \Phi(x)\phi(0) + \displaystyle\int_0^x \Phi(x-t) f(t, z_t)dt \text{ for } x \geq 0 \\ \\ z_0 = \phi \in \Upsilon. \end{cases}$$

Theorem 3.3. [8] *Assume that* ($\boldsymbol{V}_1$), ($\boldsymbol{V}_2$) *and* ($\boldsymbol{V}_3$), ($\boldsymbol{H}_1$), ($\boldsymbol{H}_2$) *and* ($\boldsymbol{H}_3$) *hold. For all* $\phi \in \Upsilon_\alpha$ *such that* $\phi(0) \in X_\alpha$, *equation* (1.1) *has a solution* $z :]-\infty, \ell] \to X$ *for some given* $\ell > 0$.

Since $(S(x))_{x\geq 0}$ is a C_0-semigroup, then there exists $M > 0$ such that $|S(x)| \leq M$ for all $t \in [0, \ell]$. We define $\mathcal{E}_\ell$ and $\mathcal{D}_\ell$ by $\mathcal{E}_\ell = \sup_{0\leq t\leq \ell} K(x)$ and $\mathcal{D}_\ell = \sup_{0\leq t\leq \ell} \mathcal{M}(x)$.

We assume that:

($\mathbf{H}_4$) For all $\xi > 0$, there exists an increasing continuous function $\chi_\xi \in L^1([0, +\infty[, X)$

such that for almost $t \in I = [0, \ell]$, $\sup\{\|f(x, \psi)\| : \|\psi\|_{\Upsilon_\alpha} \leq \xi\} \leq x^{-\alpha}\chi_\xi(x)$.

$(\mathbf{H}_5)$ $\displaystyle\lim_{\alpha\to+\infty} \frac{K_\delta}{\xi}\int_0^\ell \frac{\chi_\xi(t)}{(\ell - t)^\alpha}dt < 1.$

Theorem 3.4. *Assume that* $(\boldsymbol{V}_1)$, $(\boldsymbol{V}_2)$ *and* $(\boldsymbol{V}_3)$, $(\boldsymbol{H}_1)$, $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_3)$, $(\boldsymbol{H}_4)$ *and* $(\boldsymbol{H}_5)$ *hold. Then the set* $\mathbb{S}$ *of mild solutions of equation* (1.1) *is a compact set in* $C([0, \ell]; X_\alpha)$.

Proof. Let $\xi > \|\phi\|_{\Upsilon_\alpha}$ and Λ_{ξ_α} be the set given by

$$\Lambda_{\xi_\alpha} = \{z \in C([0, \ell]; X_\alpha) \text{ such that } z(0) = \phi(0) \text{ and } \|z\|_\infty \leq \xi\},$$

where $\|z\|_\infty = \sup\limits_{t\in[0,\ell]} |z(x)|_\alpha$. Let $\mathcal{H}$ be a function given on Λ_ϕ by

$$(\mathcal{H}z)(x) = \Phi(x)\phi(0) + \int_0^x \Phi(x - t)f(t, z_t)dt \quad \text{for} \quad t \in [0, \ell].$$

Suppose that there exists a positive number $\xi > \|\phi\|_{\Upsilon_\alpha}$ satisfying $\mathcal{H}(\Lambda_{\xi_\alpha}) \subset \Lambda_{\xi_\alpha}$. In fact, if $\mathcal{H}(\Lambda_{\xi_\alpha}) \not\subseteq \Lambda_{\xi_\alpha}$, there exist $x^\xi \in \Lambda_{\xi_\alpha}$ and $x_\xi \in [0, \ell]$ such that $\|(\mathcal{H}z^\xi)(x_\xi)\|_\alpha > \xi$. Using inequality 2.2 and $(\mathbf{H}_4)$, it follows that

$$\begin{aligned}
\xi &< |(\mathcal{H}z_\xi)(x_\xi)|_\alpha \leq |\Phi(x_\xi)\phi(0)|_\alpha + \int_0^{x_\xi} |\Phi(x_\xi - t)f(t, z_t)|_\alpha dt \\
&< N|\phi(0)|_\alpha + K_\alpha \int_0^{x_\xi} \frac{\chi_\xi(t)}{(x_\xi - t)^\alpha}dt.
\end{aligned}$$

On the other hand, we claim that the function $g : x \mapsto \displaystyle\int_0^x \frac{\chi_\xi(t)}{(x - t)^\alpha}dt$ is an increasing function on $[0, \ell]$. In fact, let $x, x' \in [0, \ell]$ be such that $x < x'$. Then we have

$$\int_0^x \frac{\chi_\xi(t)}{(x - t)^\alpha}dt = \int_0^x \frac{\chi_\xi(x - t)}{t^\alpha}dt \leq \int_0^x \frac{\chi_\xi(x' - t)}{t^\alpha}dt \leq \int_0^{x'} \frac{\chi_\xi(x' - t)}{t^\alpha}dt = g(x').$$

Consequently

$$1 < \frac{N|\phi(0)|_\alpha}{\xi} + \frac{K_\alpha}{\xi}\int_0^\ell \frac{\chi_\xi(t)}{(\ell - t)^\alpha}dt,$$

thent

$$1 \leq \liminf_{\xi\to+\infty} \frac{K_\alpha}{\xi}\int_0^\ell \frac{\chi_\xi(t)}{(\ell - t)^\alpha}dt,$$

which gives a contradiction by $(\mathbf{H}_5)$.

It remains to prove that the set

$$\{(\mathcal{H}z)(x) - \Phi(x)\phi(0) : \|z\|_\infty \leq \xi\}$$

is relatively compact in X_α for all $x \in]0,\ell]$ and Range($\mathcal{H}$) is equicontinuous on $[0,\ell]$. Let $x \in]0,\ell]$ be a given number and $\beta > 0$ such that $\alpha < \beta < 1$. Inequality 2.2, implies

$$\left\| B^\beta \int_0^x \Phi(x-t) f(t,z_t)dt \right\| \leq K_\beta \int_0^\ell \frac{\chi_\xi(t)}{(\ell-t)^\alpha} dt,$$

which implies that

$$\left\{ B^\beta \int_0^x \Phi(x-t) f(t,z_t)dt, \quad \|z\|_\infty \leq \xi \right\}$$

is bounded in X. $(\mathbf{H}_2)$ and Proposition 2.1 imply that $B^{-\beta} : X \to X_\alpha$ is a compact oprator. Consequently

$$\left\{ \int_0^x \Phi(x-t) f(t,z_t)dt, \quad \|z\|_\infty \leq \xi \right\}$$

is relatively compact set in X_α.

let $x, x' \in [0,\ell]$ be such that $x < x'$, then for all $\varepsilon > 0$, we have

$$\int_{x-\varepsilon}^x \frac{\chi_\xi(t)}{(x-t)^\alpha} dt = \int_0^\varepsilon \frac{\chi_\xi(x-t)}{t^\alpha} dt \leq \int_0^\varepsilon \frac{\chi_\xi(x'-t)}{t^\alpha} dt = \int_{x'-\varepsilon}^{x'} \frac{\chi_\xi(t)}{(x'-t)^\alpha} dt = g(x'),$$

which implies that the function $g_\varepsilon : x \mapsto \displaystyle\int_{x-\varepsilon}^x \frac{\chi_\xi(t)}{(x-t)^\alpha} dt$ is an increasing function on $[0,\ell]$.

Thus for every $0 \leq x_0 < x \leq \ell$, if $\varepsilon = x - x_0$, then $x_0 = x - \varepsilon$ and equation (2.4) implies

$$\begin{aligned} (\mathcal{H}z)(x) - (\mathcal{H}z)(x_0) &= (\Phi(x) - \Phi(x_0))\phi(0) + \int_0^x \Phi(x-t) f(t,z_t)dt - \int_0^{x_0} \Phi(x_0-t) f(t,z_t)dt \\ &= (\Phi(x) - \Phi(x_0))\phi(0) + \int_{x_0}^x \Phi(x-t) f(t,z_t)dt \\ &\quad + \int_0^{x_0} \left[S(x-t) f(t,z_t)dt + \int_0^{x-t} S(x-t-\tau) Q(\tau) f(\tau,z_\tau)d\tau \right] dt \\ &\quad - \int_0^{x_0} \left[S(x_0-t) f(t,z_t)dt + \int_0^{x_0-t} S(x_0-t-\tau) Q(\tau) f(\tau,z_\tau)d\tau \right] dt \end{aligned}$$

$$
\begin{aligned}
&= (\Phi(x)-\Phi(x_0))\phi(0)+\int_{x_0}^{x}\Phi(x-t)f(t,z_t)dt\\
&\quad+\left[\int_0^{x_0}[S(x-t)-S(x_0-t)]f(t,z_t)dt\right]\\
&\quad+\int_{x_0}^{x}\left[\int_{x_0-t}^{x-t}S(x-t-\tau)Q(\tau)f(\tau,z_\tau))d\tau\right]dt\\
&\quad+\int_0^{x_0}\left[\int_0^{x_0-t}[S(x-t-\tau)-S(x_0-t-\tau)]Q(\tau)f(\tau,z_\tau))d\tau\right]dt.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
|(\mathcal{H}z)(x)-(\mathcal{H}z)(x_0)|_\alpha &\leq |(\Phi(x)-\Phi(x_0))\phi(0)|_\alpha+\left\|(S(x-x_0)-I)\int_0^{x_0}B^\alpha S(x_0-t)f(t,z_t)dt\right\|\\
&\quad+K_\alpha\int_{x-\varepsilon}^{x}\frac{\chi_\xi(t)}{(x-t)^\alpha}dt+K_\alpha\mathcal{L}_b\int_{x-\varepsilon}^{x}\left[\int_{x_0-t}^{x-t}\frac{\chi_\xi(\tau)}{(t-\tau)^\alpha}d\tau\right]dt\\
&\quad+\left\|(S(x-x_0)-I)\int_0^{x_0}\left[\int_0^{x_0-t}B^\alpha S(x_0-t-\tau)Q(\tau)f(\tau,z_\tau))d\tau\right]dt\right\|\\
&\leq |(\Phi(x)-\Phi(x_0))\phi(0)|_\alpha+\left\|(S(x-x_0)-I)\int_0^{x_0}B^\alpha S(x_0-t)f(t,z_t)dt\right\|\\
&\quad+K_\alpha\int_{\ell-\varepsilon}^{b}\frac{\chi_\xi(t)}{(x-t)^\alpha}dt+K_\alpha\mathcal{L}_b\int_{\ell-\varepsilon}^{b}\left[\int_{x_0-t}^{x-t}\frac{\chi_\xi(\tau)}{(t-\tau)^\alpha}d\tau\right]dt\\
&\quad+\left\|(S(x-x_0)-I)\int_0^{x_0}\left[\int_0^{x_0-t}B^\alpha S(x_0-t-\tau)Q(\tau)f(\tau,z_\tau))d\tau\right]dt\right\|,
\end{aligned}
$$

where $\mathcal{L}_b=\sup\{\|Q(x)\|:\quad x\in[0,\ell]\}$. The continuity of $(\Phi(.)x)$, implies that the first part tend to zero as $|x-x_0|\to 0$. Like previously, we can prove that for $x_0>0$ the sets

$$\left\{\int_0^{x_0}B^\alpha S(x_0-t)f(t,z_t)dt,\ \ \|z\|_\infty\leq\xi\right\}$$

and

$$\left\{\int_0^{x_0}\left[\int_0^{x_0-t}B^\alpha S(x_0-t-\tau)Q(\tau)f(\tau,z_\tau))d\tau\right]dt,\ \ \|z\|_\infty\leq\xi\right\}$$

are relatively compact in X. Thus

$$\left\{\int_0^{x_0}B^\alpha S(x_0-t)f(t,z_t)dt,\ \ \|z\|_\infty\leq\xi\right\}\subset\Omega_1$$

and

$$\left\{\int_0^{x_0}\left[\int_0^{x_0-t}B^\alpha S(x_0-t-\tau)Q(\tau)f(\tau,z_\tau))d\tau\right]dt,\ \ \|z\|_\infty\leq\xi\right\}\subset\Omega_2,$$

where Ω_1 and Ω_2 are two compact subsets of X. By Banach-Steinhaus's Theorem, we have

$$\left\| (S(x-x_0)-I)\int_0^{x-\varepsilon} B^\alpha S(x_0-t)f(t,z_s)dt \right\| \to 0 \ \text{ as } \ x_0 \to x$$

and

$$\left\| (S(x-x_0)-I)\int_0^{x_0}\left[\int_0^{x_0-t} B^\alpha S(x_0-t-\tau)Q(\tau)f(\tau,z_\tau))d\tau\right]dt \right\| \to 0 \ \text{ as } \ x_0 \to t$$

uniformly in z. Using similar argument for $0 \le x < x_0 \le b$, we can conclude that Range$\mathcal{H}z$ is equicontinuous. Arzela-Ascoli's Theorem implies that Range$\{\mathcal{H}z : \|z\|_\infty \le \xi\}$ is relatively compact in $C([0,\ell];X_\alpha)$. □

Theorem 3.5. *Assume that* $(\boldsymbol{V}_1)$, $(\boldsymbol{V}_2)$ *and* $(\boldsymbol{V}_3)$, $(\boldsymbol{H}_1)$, $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_3)$, $(\boldsymbol{H}_4)$ *and* $(\boldsymbol{H}_5)$ *hold. Then the* $\mathbb{S}$ *of mild solutions of equation* (1.1) *is connected in* $C([0,\ell];X)$.

Proof. Define the function g byb $g(x) = \Phi(x)\phi(0)$. Thus $(\mathbf{H}_4)$, implies we might select a number $\eta > 0$ large enough such that $\|z\|_\infty \le \eta$ for $x \in \mathbb{S}$ and

$$\|g\|_\infty + 3K_\alpha \int_0^\ell \frac{\chi_\mu(t)}{(\ell-t)^\alpha}dt \le \eta. \tag{3.1}$$

Consequently, we have

$$\mathcal{E}_\ell |g|_{\Upsilon_\alpha} + \mathcal{D}_\ell |\phi|_{\Upsilon_\alpha} + 3K_\alpha \int_0^\ell \frac{\chi_\mu(t)}{(\ell-t)^\alpha}dt \le \mu, \tag{3.2}$$

where $\mathcal{E}_\ell = \sup\limits_{0\le x\le \ell} K(x)$, $\mathcal{D}_\ell = \sup\limits_{0\le x\le \ell} \mathcal{M}(x)$ and $\mu = \mathcal{E}_\ell\eta + \mathcal{D}_\ell|\phi|_{\Upsilon_\alpha}$. We define the Δ by

$$\Delta = \left\{ \int_0^x \Phi(x-t)f(t,z_t)dt : \ 0 \le x \le \ell, \ y \in C([0,\ell];X_\alpha), \|z\|_\infty \le \mu \right\}$$

Using the same reasoning like above (see proof of Theorem 3.3), we deduce that Δ is relatively compact in X_α. Without loss of generality, suppose that Δ is absolutely convex. Let us pose $\Gamma = 2\Delta$ and $\Gamma_1 = 3\Delta$ and $N_1 = 2K_\alpha \int_0^\ell \chi_\mu(t)dt$. The proof is made in several steps.

Step 1. We define the function $y \in \Lambda_{\xi_\alpha}$ by

$$\begin{cases} y(x) = g(x) + \displaystyle\sum_{k=1}^{i-1}\Big(\int_{x_{k-1}}^{x_k} \Phi(x-t)f(t,y_{x_{k-1}})dt + (x_k - x_{k-1})\vartheta_k\Big) \\ \qquad\qquad + \displaystyle\int_{x_{i-1}}^{x} \Phi(x-t)f(t,y_{x_{i-1}})dt + (x-x_{i-1})\vartheta_i \ \text{ for } x_{i-1} < x < x_i \\ y(0) = \phi(0). \end{cases} \tag{3.3}$$

We choose ϑ_k such that $\displaystyle\sum_{k=1}^{i}(x_k - x_{k-1})\vartheta_k \in \Gamma$ and $\displaystyle\left|\sum_{k=1}^{i}(x_k - x_{k-1})\vartheta_k\right|_\alpha \leq N_1$ for all $j = 1, ..., n$ and $0 = x_0 < x_1 < .... < x_{n-1} < x_n = b$ is a division d of $[0, \ell]$. Show that $|y(x)|_\alpha \leq \eta$ for $0 \leq x \leq \ell$ independent of the division d of $[0, \ell]$ and the points ϑ_i.

y defined by equation (3.3) is a continuous function and we also define $\psi(.)$ and $\vartheta(.)$ the steps functions by $\psi(0) = \phi,\ \vartheta(0) = \vartheta_1,\ \psi(x) = y_{x_{k-1}}$ and $u(x) = \vartheta_k$ for $x_{k-1} < x < x_k$ and $k = 1, ..., n$. Thus $y(.)$ takes the following form

$$y(x) = g(x) + \int_0^x \Phi(x-t) f(t, \psi(t))dt + \int_0^x \vartheta(t)dt. \tag{3.4}$$

Equation (3.1) implies that $|y(x)|_\alpha \leq \eta$ for $0 \leq x \leq x_1$. We assume that this inequality holds on $[0, x_{j-1}]$ and we prove that it is also satisfied for $x_{i-1} < x < x_i$. Phase's axioms and equation (3.2), imply that $|y|_{\Upsilon_\alpha} \leq \mu$ for $0 < t < x_{i-1}$. Since $\displaystyle\sum_{k=1}^{i-1}(x_k - x_{k-1})\vartheta_k + (x - x_{i-1})\vartheta_i$ is a convex combination of $\displaystyle\sum_{k=1}^{i-1}(x_k - x_{k-1})\vartheta_k$ and $\displaystyle\sum_{k=1}^{i}(x_k - x_{k-1})\vartheta_k$, then by equation (3.4), $(\mathbf{H}_4)$ and the fact that $\displaystyle\left|\sum_{k=1}^{i}(x_k - x_{k-1})\vartheta_k\right|_\alpha \leq N_1$, we have

$$\begin{aligned} |y(x)|_\alpha &\leq |g(x)|_\alpha + \left|\int_0^x \Phi(x-t) f(t, \psi(t))dt\right|_\alpha + \left|\sum_{k=1}^{i-1}(x_k - x_{k-1})\vartheta_k + (x - x_{i-1})\vartheta_i\right|_\alpha \\ &\leq \|g\|_\infty + K_\alpha \int_0^\ell \frac{\chi_\mu(t)}{(\ell - t)^\alpha}dt + 2K_\alpha \int_0^\ell \frac{\chi_\mu(t)}{(\ell - t)^\alpha}dt, \end{aligned}$$

then we get our result.

Step 2. We denote by W the set of functions y given by equation (3.3) and we claim that W is relatively compact in $C([0, \ell]; X)$.

We suppose that the points x_k equally spaced with $\delta = x_k - x_{k-1}$. Moreover, we suppose that

$$\Big|\delta \sum_{k=i+1}^{j} \vartheta_k - \Big[\Phi((j-i)\delta) - I\Big]\delta \sum_{k=1}^{i} \vartheta_k\Big|_\alpha \leq 2K_\alpha \int_{\ell - \varepsilon_{ij}}^{b} \frac{\chi_Q(t)}{(\ell - t)^\alpha}dt \tag{3.5}$$

for $1 \leq i + 1 \leq j \leq n$, with $\varepsilon_{ij} = x_j - x_i$.

We define the function $\widetilde{y}$ by $\widetilde{y} = y - g$ and we show that $W_0 = \{\widetilde{y} :\ y \in W\}$ is relatively compact.

Equation (3.4) implies that

$$\widetilde{y}(x) = \int_0^x \Phi(x-t)f(t,\psi(t))dt + \delta\sum_{k=1}^{i-1}\vartheta_k + (x-x_{i-1})\vartheta_i \in \Delta + \Gamma \subseteq \Gamma_1$$

for all $y \in W$ and $x \in [0,\ell]$.
We now show that W_0 is equicontinuous. Let $0 \leq x' \leq x \leq \ell$. Equation (3.4) implies that

$$\widetilde{y}(x) - \widetilde{y}(x') = \left(\int_0^x \Phi(x-t)f(t,\psi(t))dt - \int_0^{x'} \Phi(x'-t)f(t,\psi(t))dt\right) + \int_{x'}^x \vartheta(t)dt$$

$$\begin{aligned}
&= \left(\int_0^x S(x-t)f(t,\psi(t))dt - \int_0^{x'} S(x'-t)f(t,\psi(t))dt\right) + \int_{x'}^x \vartheta(t)dt \\
&\quad +\Big(\int_0^x \int_0^{x-t} S(x-t-\tau)Q(\tau)f(\tau,\psi(\tau))d\tau]dt \\
&\quad -\int_0^{x'} \int_0^{x'-t} S(x'-t-\tau)Q(\tau)f(\tau,\psi(\tau))d\tau]dt\Big) \\
&= [S(x-x')-I]\int_0^{x'}\left(S(x'-t)f(t,\psi(t)) + \int_0^{x'-t} S(x'-t-\tau)Q(\tau)f(\tau,\psi(\tau))d\tau\right)dt \\
&\quad + \int_{x'}^x \left(S(x-t)f(t,\psi(t)) + \int_0^{x-t} S(x-t-\tau)Q(\tau)f(\tau,\psi(\tau))d\tau\right)dt + \int_{x'}^x \vartheta(t)dt \\
&\quad + \int_0^{x'}\left(\int_{x'-t}^{x-t} S(x-t-\tau)Q(\tau)f(\tau,\psi(\tau))d\tau\right)dt \\
&= [S(x-x')-I]\Big(\int_0^{x'} \Phi(x'-t)f(t,\psi(t))dt\Big) + \int_{x'}^x \Phi(x-t)f(t,\psi(t))dt + \int_{x'}^x \vartheta(t)dt \\
&\quad + \int_0^{x'}\left(\int_{x'-t}^{x-t} S(x-t-\tau)Q(\tau)f(\tau,\psi(\tau))d\tau\right)dt \\
&= [S(x-x')-I]\Big(\widetilde{y}(x') - \int_0^{x'} \vartheta(t)dt\Big) + \int_{x'}^x \Phi(x-t)f(t,\psi(t))dt + \int_{x'}^x \vartheta(t)dt \\
&\quad + \int_0^{x'}\left(\int_{x'-t}^{x-t} S(x-t-\tau)Q(\tau)f(\tau,\psi(\tau))d\tau\right)dt
\end{aligned}$$

Since Δ is a compact set and $\left(\widetilde{y}(x') - \int_0^{x'}\vartheta(t)dt\right) \in \Delta$, we might prove that $\int_{x'}^x \vartheta(t)dt \to 0$ as $x - x' \to 0$ no matter the function y.
The compactness of Γ implies for $\varepsilon > 0$ there is $\eta_0 > 0$ such that $|(\Phi(t)-I)\vartheta| \leq \dfrac{\varepsilon}{2}$

for all $x \in [0, \eta_0]$, $\vartheta \in \Gamma$ and $2K_\alpha \displaystyle\int_{\ell-(x-x')}^{b} \frac{\chi_\mu(t)}{(\ell-t)^\alpha} dt \leq \frac{\varepsilon}{2}$ for all $x, x' \in [0, \ell]$ such that $|x - x'| \leq \eta_0$. We define a constant c_1 by

$$c_1 = \sup \left\{ \frac{1}{s} |(\Phi(t) - I)\vartheta| : \ \eta_0 \leq s \leq \ell, \ \vartheta \in \Gamma \right\}$$

and we choose $0 < \rho \leq \min \left\{ \eta_0, \ \dfrac{\varepsilon \eta_0}{2N_1}, \ \dfrac{\varepsilon}{2c_1} \right\}$.
First, we suppose that x and x' coincide with some points of the division. Thus, we take $x' = x_i$ and $x = x_j$ and we have

$$\begin{aligned} \int_{x'}^{x} \vartheta(t) dt &= \delta \sum_{k=i+1}^{j} \vartheta_k \\ &= \delta \sum_{k=i+1}^{j} \vartheta_k - \Big[\Phi((j-i)\delta) - I \Big] \delta \sum_{k=1}^{i} \vartheta_k + \Big[\Phi((j-i)\delta) - I \Big] \delta \sum_{k=1}^{i} \vartheta_k \end{aligned}$$

and equation (3.5) implies that

$$\left| \int_{x'}^{x} \vartheta(t) dt \right|_\alpha \leq 2K_\alpha \int_{\ell-\delta}^{\ell} \frac{\chi_\mu(t)}{(\ell-t)^\alpha} dt + \left| \Big[\Phi((j-i)\delta) - I \Big] \delta \sum_{k=1}^{i} \vartheta_k \right|. \tag{3.6}$$

Since $\delta \displaystyle\sum_{k=1}^{i} \vartheta_k \in \Gamma$, if $x_j - x_i \leq \eta_0$, it follows that $\left| \displaystyle\int_{x'}^{x} \vartheta(t) dt \right|_\alpha \leq \varepsilon$.
Let $x - x' \leq \eta$. The position of points x_k gives three possible situations. First, we suppose that there is any point x_k between x' and x. Thus, there exists an index i such that $x_i < x' < x \leq x_{i+1}$ and $\displaystyle\int_{x'}^{x} \vartheta(t) dt = (x - x')\vartheta_{i+1}$. Equation (3.5) and equation (3.6) give

$$|\vartheta_{i+1}|_\alpha \leq \frac{2K_\alpha}{\delta} \int_{b-\delta}^{b} \frac{\chi_Q(t)}{(\ell-t)^\alpha} dt + \left| \Big[\Phi(\delta) - I \Big] \sum_{k=1}^{i} \vartheta_k \right|_\alpha.$$

Since $x - x' \leq \rho \leq \delta$, we have

$$\begin{aligned} (x - x')|\vartheta_{i+1}|_\alpha &\leq \frac{\varepsilon}{2} + \frac{x - x'}{\delta} \left| \Big[\Phi(\delta) - I \Big] \delta \sum_{k=1}^{i} \vartheta_k \right|_\alpha \\ &\leq \varepsilon, \end{aligned}$$

which gives the result in this case.
Finally, we suppose that there is an index i such that $x_{i-1} < x' < x_i < x < x_{i+1}$. Then, we have

$$\int_{x'}^{x} \vartheta(t) dt = \int_{x'}^{x_i} \vartheta(t) dt + \int_{x_i}^{x} \vartheta(t) dt = \int_{x'}^{x_i} \vartheta_i dt + \int_{x_i}^{x} \vartheta_{i+1} dt = (x_i - x')\vartheta_i + (x - x_i)\vartheta_{i+1}.$$

Since $x_i - x' < x - x'$ and $x - x_i < x - x'$, we proceed like in the preceding case. From the Ascoli-Arzela's Theorem, we conclude that W_0 is relatively compact and so is $W = h + W_0$.

Step 3. We now give $\varepsilon > 0$ and we suppose, without loss of generality, that $\varepsilon \le \min\left\{\frac{\eta_0}{2},\ 2N_1\right\}$ and we define ε_1 by $\varepsilon_1 = \frac{(1-\alpha)\varepsilon}{2M_\alpha b^{1-\alpha}}$. The compactness of sets $\mathbb{S}$ and W and the continuity of f imply that there is $0 \le \delta_1 \le 2\mathcal{E}_\ell \varepsilon$ such that

$$|f(x,\psi_1) - f(x,\psi_2)|_\alpha \le \varepsilon_1 \tag{3.7}$$

for all $x \in [0,\ell]$ and for every $\psi_1,\ \psi_2 \in H(W \cup \mathbb{S})$ such that $|\psi_1 -\ \psi_2|_{\Upsilon_\alpha} \le \delta_1$. In the same way, there exists $\delta_2 > 0$ such that

$$|z(s) - z(t)|_\alpha \le \frac{\delta_1}{4\mathcal{E}_\ell}, \tag{3.8}$$

for all $z \in W \cup \mathbb{S}$ and $s, t \in [0,\ell]$ such that $|t-s| \le \delta_2$. We now suppose that $\delta = \frac{\ell}{n} \le \min\left\{\frac{(\delta_1 \ell)^{1-\alpha}}{2\mathcal{E}_\ell \varepsilon},\ \delta_2\right\}$. In what follows, we choose the subdivision d such that $x_i = i\delta,\ i = 0, ..., n$. For the set W_ε gives by the functions defined by equation (3.3) with $(\vartheta_1, ..., \vartheta_n) \in Z_\varepsilon$ where Z_ε is the set of points $(\vartheta_1, ..., \vartheta_n) \in \left(\frac{2}{\delta}\Gamma\right)^n$ such that:

(i) $\delta \sum_{k=1}^{i} \vartheta_k \in \Gamma$;

(ii) $\left|\delta \sum_{k=1}^{i} \vartheta_k\right|_\alpha \le \frac{M_\alpha \varepsilon_1 x_i^{1-\alpha}}{1-\alpha}$;

(iii) $\left|\delta \sum_{k=i+1}^{j} \vartheta_k - \left[S((j-i)\delta) - I\right]\delta \sum_{k=1}^{i} \vartheta_k\right|_\alpha \le 2K_\alpha \int_{\ell-\varepsilon_{ij}}^{b} \frac{\chi_Q(t)}{(\ell-t)^\alpha} dt$, for all $i = 1, ..., n$ and $j \ge i+1$, with $\varepsilon_{ij} = (j-i)\delta$

By **(ii)** we have $\left|\delta \sum_{k=1}^{i} \vartheta_k\right|_\alpha \le N_1$. We next give some properties of W_ε.

Step 4. We claim that W_ε is connected.

This comes from of the fact that the functions $y \in W_\varepsilon$ depend continuously on the choice of $(\vartheta_1, ..., \vartheta_n) \in Z_\varepsilon$ and Z_ε is convex by our construction.

Step 5. Here, we prove that the elements of W_ε can be approximations of solutions of equation (1.1).

For $z \in \mathbb{S}$ fixed, we choose $y \in W_\varepsilon$, defined on $[x_{i-1}, x_i]$ such that $|z - y|_\infty \le \varepsilon$. For $i = 1$, we have $x_1 = \delta$ and we pose

$$\vartheta_1 = \frac{1}{\delta}\int_0^{x_1} \Phi(x_1 - t)[f(t, z_t) - f(t, \phi)]dt.$$

We can see that $\vartheta_1 \in \frac{1}{\delta}\Gamma$. One the other hand, equation (3.8) and axiom $(\mathbf{A}_1)$ imply that $|z_s - \phi|_{\Upsilon_\alpha} \leq \dfrac{\delta_1}{4}$ and equation (3.7) implies that $|f(x, z_x) - f(x, \phi)|_\alpha \leq \varepsilon_1$ for all $x \in [0, \delta]$. By $(\mathbf{H}_4)$, we have

$$
\begin{aligned}
|\vartheta_1|_\alpha &\leq \frac{1}{x_1}\int_0^{x_1} |\Phi(x_1 - t)[f(t, z_t) - f(t, \phi)]|_\alpha dt \\
&\leq \frac{K_\alpha \varepsilon_1}{x_1}\int_0^{x_1} (x_1 - t)^{-\alpha} dt \\
&\leq \frac{K_\alpha \varepsilon_1}{x_1} \times \frac{x_1^{1-\alpha}}{1-\alpha} \\
&\leq \frac{K_\alpha \varepsilon_1 x_1^{-\alpha}}{1-\alpha}.
\end{aligned}
$$

We define

$$
y(x) = g(x) + \int_0^x \Phi(x-t) f(t, \phi) dt + t\vartheta_1,
$$

for $0 \leq x \leq x_1$. The we have

$$
\begin{aligned}
y(x_1) &= g(x_1) + \int_0^{x_1} \Phi(x-t) f(t, \phi) dt + x_1\vartheta_1 \\
&= g(x_1) + \int_0^{x_1} \Phi(x-t) f(t, z_t) dt + x_1\vartheta_1 \\
&= z(x_1).
\end{aligned}
$$

For $0 \leq x \leq x_1 = \delta$, the definition of ε_1, δ and since $\delta \leq \min\left\{\dfrac{(\delta_1 \ell)^{1-\alpha}}{2\mathcal{E}_\ell \varepsilon},\ \delta_2\right\}$, we have

$$
\begin{aligned}
|z(x) - y(x)|_\alpha &\leq \int_0^x |\Phi(x-t)[f(t, z_t) - f(t, \phi)]|_\alpha dt + |t\vartheta_1|_\alpha \\
&\leq \frac{K_\alpha \varepsilon_1 t^{1-\alpha}}{1-\alpha} + t_1|\vartheta_1|_\alpha \\
&\leq \frac{2K_\alpha \varepsilon_1 x_1^{1-\alpha}}{1-\alpha} \\
&\leq \frac{2K_\alpha}{1-\alpha} \times \frac{(1-\alpha)\varepsilon}{2M_\alpha b^{1-\alpha}} \times \frac{(\delta_1 b)^{1-\alpha}}{2\mathcal{E}_\ell \varepsilon} \\
&\leq \frac{\delta_1}{2\mathcal{E}_\ell}.
\end{aligned}
$$

Using the sane reasoning, we have choose the elements ϑ_k, $k = 1, ..., i-1$ such that $(\vartheta_1, ..., \vartheta_{i-1}, 0, ..., 0) \in Z_\varepsilon$ and the function $y(x)$ for $x \in [0, x_{i-1}]$ such that

$y(x_k) = z(x_k)$ and we get

$$|z(x) - y(x)|_\alpha \quad \leq \quad \frac{\delta_1}{2\mathcal{E}_\ell} \text{ for } 0 \leq x \leq x_{i-1}.$$

Let be y defined on $[x_{i-1}, x_i]$ and select

$$\begin{aligned}\vartheta_i \quad &= \quad \frac{1}{\delta}\sum_{k=1}^{i-1}\Big(\int_{x_{k-1}}^{x_k}[\Phi(x_i - t) - \Phi(x_{i-1} - t)][f(t, z_t) - f(t, y_{t_{k-1}})]dt\Big) \\ &\quad +\frac{1}{\delta}\int_{x_{i-1}}^{x_i}\Phi(x_i - s)[f(t, z_t) - f(t, y_{x_{i-1}})]dt.\end{aligned}$$

The function $\psi(.)$ defined previously implies

$$\vartheta_i \quad = \quad \frac{1}{\delta}\int_0^{x_i}\Phi(x_i - t)[f(t, z_t) - f(t, \psi)]dt - \frac{1}{\delta}\int_0^{x_{i-1}}\Phi(x_{i-1} - t)[f(t, z_t) - f(t, \psi)]dt.$$

We prove that $(\vartheta_1, ..., \vartheta_i, 0, ..., 0) \in Z_\varepsilon$. We can see $\delta\vartheta_i \in 2\Gamma$ and

$$\delta\sum_{k=1}^{i}\vartheta_k = \int_0^{x_i}\Phi(x_i - t)[f(t, z_t) - f(t, \psi)]dt,$$

which implies that $\delta\sum_{k=1}^{i}\vartheta_k \in 2\Gamma$ and $\left|\delta\sum_{k=1}^{i}\vartheta_k\right|_\alpha \leq N_1$. Thus for $m + 1 \leq j \leq i$, we have

$$\begin{aligned}\delta\sum_{k=m+1}^{j}\vartheta_k \quad &= \quad \delta\sum_{k=1}^{j}u_k - \delta\sum_{k=1}^{m}\vartheta_k \\ &= \quad \int_0^{x_j}\Phi(x_j - t)[f(t, z_t) - f(t, \psi)]dt - \int_0^{x_m}\Phi(x_m - t)[f(t, z_t) - f(t, \psi)]dt.\end{aligned}$$

Consequently as in **Step 2**, it follows

$$\begin{aligned}\delta\sum_{k=m+1}^{j}\vartheta_k \quad &= \quad \Big(\int_0^{x_j}\Phi(x_j - t)[f(t, z_t) - f(t, \psi)]dt - \int_0^{x_m}\Phi(x_m - t)[f(t, z_t) - f(s, \psi)]dt\Big) \\ &= \quad \Big[S(x_j - x_m) - I\Big]\int_0^{x_m}\Phi(t_m - t)[f(t, z_t) - f(t, \psi)]dt \\ &\quad + \int_{x_m}^{t_j}\Phi(x_j - t)[f(t, z_t) - f(t, \psi)]dt \\ &\quad + \int_0^{x_m}\left(\int_{x_m - t}^{x_j - t}S(x_j - t - \tau)Q(\tau)[f(\tau, z_\tau) - f(\tau, \psi)]d\tau\right)dt,\end{aligned}$$

which yields

$$\begin{aligned}
&\Big(\delta \sum_{k=m+1}^{j} \vartheta_k - \Big[S(x_j - x_m) - I\Big] \int_0^{x_m} \Phi(x_m - t)[f(t, z_t) - f(t, \psi)]dt\Big) = \\
&\int_{x_m}^{x_j} \Phi(x_j - t)[f(t, z_t) - f(t, \psi)]dt \\
&+ \int_0^{x_m} \Big(\int_{x_m - t}^{t_j - t} S(x_j - t - \tau)Q(\tau)[f(\tau, z_\tau) - f(\tau, \psi)]d\tau\Big)\, dt.
\end{aligned}$$

This implies that

$$\begin{aligned}
\Big|\delta \sum_{k=m+1}^{j} \vartheta_k - [S((j-m)\delta) - I]\, \delta \sum_{k=1}^{m} \vartheta_k\Big|_\alpha \quad &\leq \quad 2K_\alpha \int_{\ell - \varepsilon_{mj}}^{\ell} \frac{\chi_\mu(t)}{(\ell - t)^\alpha} dt \\
&+2K_\alpha \mathcal{L}_b \int_0^{x_m} \Big(\int_{x_m - t}^{x_j - t} \frac{\chi_\mu(t)}{(t - \tau)^\alpha} d\tau\Big)\, dt.
\end{aligned}$$

Moreover for $x_{i-1} < t \leq x_i$, we have $z_s - \psi(t) = z_s - z_{x_{i-1}} + z_{x_{i-1}} - y_{x_{i-1}}$. Equation (3.8) implies that $|z_s - z_{x_{i-1}}|_{\Upsilon_\alpha} \leq \dfrac{\delta_1}{4}$ and by induction, we have $|z_{x_{i-1}} - y_{x_{i-1}}|_{\Upsilon_\alpha} \leq \dfrac{\delta_1}{2}$. Thus equation (3.7), implies that $\delta\Big|\sum_{k=1}^{i} \vartheta_k\Big|_\alpha \leq \dfrac{K_\alpha \varepsilon_1 x_i^{1-\alpha}}{1-\alpha}$ and we can see that $(\vartheta_1, ..., \vartheta_i, 0, ..., 0) \in Z_\varepsilon$.
Define now $y(x)$ for $x_{i-1} < x \leq x_i$. Using equation (3.3) and by the choice of ϑ_k, $k = 1, ..., i$, we have

$$\begin{aligned}
z(x_i) - y(x_i) \quad &= \quad \int_0^{x_i} \Phi(x_i - t)[f(t, z_t) - f(t, \psi)]dt - \delta \sum_{k=1}^{i} \vartheta_k \\
&= \quad \int_0^{x_i} \Phi(x_i - t)[f(t, z_t) - f(t, \psi)]dt - \int_0^{x_i} \Phi(x_i - t)[f(t, z_t) - f(t, \psi)]dt \\
&= \quad 0.
\end{aligned}$$

In addition, by equation (3.8) and the choice of δ, we have

$$\begin{aligned}
|z(x) - y(x)|_\alpha \quad &\leq \quad |z(x) - z(x_{i-1})|_\alpha + |z(x_{i-1}) - y(x)|_\alpha \\
&\leq \quad |z(x) - z(x_{i-1})|_\alpha + |y(x) - y(x_{i-1})|_\alpha \\
&\leq \quad \frac{\delta_1}{2\mathcal{E}_\ell},
\end{aligned}$$

which proves this result.

Step 6. In this step, we prove that the elements of W_ε are approximate solutions of equation (1.1), that is to say

$$\left| y(x) - g(x) - \int_0^x \Phi(x-t) f(t, z_t) dt \right|_\alpha \leq \varepsilon,$$

for $x \in [0, \ell]$ and $z \in W_\varepsilon$.

For $x_{i-1} < x \leq x_i$, equation (3.3) implies that

$$y(x) - g(x) - \int_0^x \Phi(x-t) f(t, z_t) dt \;=\; \int_0^t \Phi(x-t)[f(t,\psi) - f(t, z_t)] dt + \sum_{k=1}^{i} \vartheta_k + (x - x_{i-1})\vartheta_i$$

and by equation (3.7), equation (3.8) and the choice of δ, it follows that

$$\begin{aligned}\left| y(x) - g(x) - \int_0^x \Phi(x-t) f(t, z_t) dt \right|_\alpha &= \left| \int_0^x \Phi(x-t)[f(t,\psi) - f(t, z_t)] dt \right|_\alpha \\ &\quad + \left| \sum_{k=1}^{i} \vartheta_k + (x - x_{i-1}) u_i \right|_\alpha \\ &\leq \frac{2K_\alpha \varepsilon_1 \ell^{1-\alpha}}{1-\alpha},\end{aligned}$$

which gives our assertion.
Since $\mathbb{S}$ is the set of fixed points of the function $\mathcal{H}$, then by **Step 1** and **Step 6** and Theorem 2.6, we conclude that $\mathbb{S}$ is connected in the space $C([0,\ell]; X_\alpha)$. $\square$

4. **Application**

For an application, we study the following model

$$\begin{cases} \dfrac{\partial}{\partial s}\psi(x,\rho) = \dfrac{\partial^2}{\partial \rho^2}\psi(x,\rho) + \displaystyle\int_0^x \eta(x-u) \dfrac{\partial^2}{\partial z^2}\psi(u,z) du + x e^{-|x|} \int_{-\infty}^0 h(\lambda, z(x+\lambda, \xi)) d\lambda \text{ for } x \geq 0 \\ \qquad\qquad \text{and } \rho \in [0,\pi] \\ \psi(x,0) = \psi(x,\pi) = 0 \text{ for } x \geq 0 \\ \psi(\lambda,\rho) = \phi(\lambda)(\rho) \text{ for } \lambda \in]-\infty, 0] \text{ and } \rho \in [0,\pi], \end{cases} \tag{4.1}$$

where $h :]-\infty, 0] \to X$ is a positive integrable function and $\eta : \mathbb{R}^+ \to \mathbb{R}$ is uniformly bounded continuous. To rewrite equation (4.1) in the abstract form, we introduce the space X of $L^2([0,\pi];\mathbb{R})$ functions such that for all $z \in X$ $z(0) = z(\pi) = 0$ and

$$\|z\|_{L^2} = \left(\int_0^\pi |z(t)|^2 dt \right)^{\frac{1}{2}}.$$

Defined $B : X \to X$ by

$$\left\{\begin{array}{l} D(B) = H^2(0,\pi) \cap H_0^1(0,\pi) \\ B\psi = -\psi''. \end{array}\right.$$

$-B$ generates a compact analytic semigroup $(S(x))_{x\geq 0}$ on X. Thus $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ are satisfied. In what follows, we take $\alpha = \dfrac{1}{2}$. Let $\mu > 0$, the phase space is given by

$$\Upsilon = C_\mu = \{\phi \in C(]-\infty,0]; X) : \lim_{\lambda\to-\infty} e^{\mu\lambda}\phi(\lambda) \text{ exist in } X\},$$

with the norm

$$\|\phi\|_\mu = \sup_{\lambda\leq 0} e^{\mu\lambda}\|\phi(\lambda)\|, \text{ for } \phi \in C_\mu$$

This space satisfies $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ and $(\mathbf{B})$. $\Upsilon_{\frac{1}{2}}$ is equipped with the norm given by

$$\|\phi\|_{\Upsilon_{\frac{1}{2}}} = \sup_{\lambda\leq 0} e^{\mu\lambda}\|A^{\frac{1}{2}}\phi(\lambda)\| = \sup_{\lambda\leq 0} e^{\mu\theta}\left(\int_0^\pi \left(\frac{\partial}{\partial x}(\phi)(\lambda)(\rho)\right)^2 d\rho\right)^{\frac{1}{2}}.$$

Let $\Pi : D(A) \to X$ given by

$$\Pi(x)(\psi) = \eta(x)B\psi \; for \; t \geq 0$$

and function f is defined by

$$f(x,\phi)(\rho) = xe^{-|x|}\int_{-\infty}^0 g(\lambda)\phi(\lambda)(\rho)d\lambda \text{ for } \rho \in [0,\pi] \text{ and } x \geq 0.$$

If $\vartheta(x) = z(x,\rho)$, equation (4.1) becomes

$$\left\{\begin{array}{l} \vartheta(x) = -B\vartheta(x) + \displaystyle\int_0^x \Pi(x-t)\vartheta(t)dt + f(x,\vartheta_x), \; x \geq 0 \\ \\ \vartheta_0 = \phi \in \Upsilon \end{array}\right. \tag{4.2}$$

We suppose that

$(\mathbf{H}_6)$: $e^{-2\mu}h \in L^2(\mathbb{R}^-)$.

By Hölder's inegality, one has

$$\begin{aligned}
\|f(x,\phi)\|^2 &= \int_0^\pi \left| xe^{-|x|}\int_{-\infty}^0 h(\lambda)\phi(\lambda)(\xi)d\lambda\right|^2 d\rho \\
&= x^2e^{-2|x|}\int_0^\pi \left|\int_{-\infty}^0 h(\lambda)e^{-2\mu\lambda}e^{2\mu\lambda}\phi(\lambda)(\rho)d\lambda\right|^2 d\rho \\
&\le x^2e^{-2|x|}\int_0^\pi \left(\int_{-\infty}^0 h^2(\lambda)e^{-4\mu\lambda}d\lambda\right)\left(\int_{-\infty}^0 e^{4\mu\lambda}\phi^2(\lambda)(\rho)d\theta\right)d\rho \\
&\le \left(\int_{-\infty}^0 h^2(\lambda)e^{-4\mu\lambda}d\lambda\right)x^2e^{-2|x|}\int_0^\pi \left(\int_{-\infty}^0 e^{2\mu\lambda}e^{2\mu\lambda}\phi^2(\lambda)(\rho)d\lambda\right)d\rho \\
&\le \left(\int_{-\infty}^0 h^2(\lambda)e^{-4\mu\lambda}d\lambda\right)x^2e^{-2|x|}\int_0^\pi \sup_{\lambda\le 0} e^{2\mu\lambda}[\phi^2(\lambda)(\rho)]\left(\int_{-\infty}^0 e^{2\mu\lambda}d\lambda\right)d\rho \\
&\le \left(\int_{-\infty}^0 h^2(\lambda)e^{-4\mu\lambda}d\lambda\right)x^2e^{-2|x|}\sup_{\lambda\le 0} e^{2\mu\lambda}\int_0^\pi \phi^2(\lambda)(\rho)\left(\int_{-\infty}^0 e^{2\mu\lambda}d\lambda\right)d\rho \\
&\le \frac{1}{2\mu}\left(\int_{-\infty}^0 h^2(\lambda)e^{-4\mu\lambda}d\lambda\right)x^2e^{-2|x|}\sup_{\lambda\le 0} e^{2\mu\lambda}\int_0^\pi \phi^2(\lambda)(\rho)d\rho \\
&\le \frac{1}{2\mu}\left(\int_{-\infty}^0 h^2(\lambda)e^{-2\mu\lambda}d\lambda\right)x^2e^{-2|x|}\|\phi\|^2_{\Upsilon_{\frac{1}{2}}}
\end{aligned}$$

Let $\rho > 0$, for all $\phi \in \Upsilon_{\frac{1}{2}}$ such that $\|\phi\|_{\Upsilon_{\frac{1}{2}}} \le \rho$, we define

$$\chi_\xi(x) = \left[\frac{1}{2\mu}\left(\int_{-\infty}^0 h^2(\lambda)e^{-2\mu\lambda}d\lambda\right)\right]^{\frac{1}{2}} xe^{-x}\rho^{\frac{1}{2}} \quad \text{for} \quad x \in [0,\ell].$$

Then $(\mathbf{H}_4)$ and $(\mathbf{H}_5)$ are satisfied. Since $(\mathbf{H}_2)$ is satisfied, then the linear part of equation (4.2) has a resolvent operator which is analytic. Using [1] (Section 6), for all $\psi \in \Upsilon_{\frac{1}{2}}$, $B^{\frac{-1}{2}}\psi \in \Upsilon_{\frac{1}{2}}$, i.e $(\mathbf{H}_3)$ is satisfied. Then we have the following result.

Theorem 4.1. *For $\phi \in \Upsilon_{\frac{1}{2}}$, there exists $\ell > 0$ such that the set $\mathbb{S}$ of mild solution of equation (4.1) is connected in $C([0,\ell]; X_\alpha)$.*

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[1] Université Joseph KI-ZERBO, Unité de Recherche et de Formation en Sciences Exactes et Appliquées, Département de Mathématiques, B.P.7021 Ouagadougou 03, Burkina Faso

Email address: zabsonreissa@yahoo.fr