

STUDY OF A NONLOCAL ELLIPTIC TRANSMISSION SYSTEM WITH SINGULAR WEIGHTS

AYOUB ZAKI¹, ABDELAH MOALI² and MUSTAPHA HADDAOUI^{3*}

ABSTRACT. This paper investigates the multiplicity of weak solutions for a transmission problem governed by a coupled system of two nonlinear equations of $p(x)$ -Kirchhoff type. Under nonstandard growth conditions, we establish the existence of at least two distinct non-trivial weak solutions. The proof relies on a variational approach, combining the classical Mountain Pass Theorem with Ekeland's Variational Principle to overcome the lack of regularity inherent to the problem.

Keywords. Transmission Problems; $p(x)$ -laplacian; Singular Weights
2020 Mathematics Subject Classification. Primary 35J35 ; Secondary 35J60

1. INTRODUCTION

The objective of this work is to investigate the existence and multiplicity of weak solutions for a nonlocal elliptic system of gradient type, characterized by non-standard growth conditions. More specifically, we analyze the following system:

$$\begin{cases} -\mathcal{M}_1 \left(\int_{\Omega_1} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla u|^{p(x)} dx \right) \operatorname{div} \left(\frac{1}{\delta^{\gamma(x)}} |\nabla u|^{p(x)-2} \nabla u \right) = f(x, u) & \text{in } \Omega_1 \\ -\mathcal{M}_2 \left(\int_{\Omega_2} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla v|^{p(x)} dx \right) \operatorname{div} \left(\frac{1}{\delta^{\gamma(x)}} |\nabla v|^{p(x)-2} \nabla v \right) = g(x, v) & \text{in } \Omega_2 \\ v = 0, & \text{on } \Gamma, \end{cases} \quad (1.1)$$

subject to the transmission conditions on the interface Σ :

$$u = v \quad \text{and} \quad \mathcal{M}_1 \left(\frac{1}{p(x)} \int_{\Omega_1} \frac{1}{\delta^{\gamma(x)}} |\nabla u|^{p(x)} dx \right) \frac{\partial u}{\partial \eta} = \mathcal{M}_2 \left(\frac{1}{p(x)} \int_{\Omega_2} \frac{1}{\delta^{\gamma(x)}} |\nabla v|^{p(x)} dx \right) \frac{\partial v}{\partial \eta}$$

The domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is bounded and smooth. It is partitioned into two subdomains Ω_1 and Ω_2 such that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, with $\Sigma = \partial\Omega_1$ a smooth interface and $\Gamma = \partial\Omega_2 \setminus \Sigma$ the external boundary. The unit normal η points inward to Ω_1 (and thus outward to Ω_2). The function $p(\cdot) \in C(\bar{\Omega})$ satisfies $p(x) > 1$. The

Date: Received: Mar 2, 2026; Revised: Mar 29, 2026; Accepted: Apr 13, 2026.

* Corresponding author

© The Author(s) 2026. This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License. To view a copy of the licence, visit <https://creativecommons.org/licenses/by-nc-nd/4.0/>.

function $\delta(x)$ represents the distance from x to the interface Σ . The Kirchhoff functions $\mathcal{M}_i : \mathbb{R}^+ \rightarrow \mathbb{R}$ are continuous and satisfy the following hypotheses:

Assumed to be continuous, the Kirchhoff function $\mathcal{M}_i : \mathbb{R}^+ \rightarrow \mathbb{R}$ meets the following requirements:

(H_1) There exist real numbers $m_1, m_2 > 0$ and $\theta \geq 1$ such that

$$m_1 t^{\theta-1} \leq \overline{M}_1(t) \leq m_2 t^{\theta-1} \quad \forall t \geq 0.$$

(H_2) There exist real numbers $m'_1, m'_2 > 0$ and $\theta \geq 1$ such that

$$m'_1 t^{\theta-1} \leq \overline{M}_2(t) \leq m'_2 t^{\theta-1} \quad \forall t \geq 0.$$

where For all $t \geq 0$,

$$\overline{M}_i(t) = \int_0^t \mathcal{M}_i(s) ds.$$

(H_3) $a : \overline{\Omega}_1 \rightarrow \mathbb{R}$, satisfies $a \in L^{\alpha_0(x)}(\Omega_1)$ and $\alpha_0 \in C_+(\overline{\Omega}_1)$ such that

$$\frac{Np(x)}{Np(x) - \alpha(x)(N - p(x))} < \alpha_0(x) < \frac{p(x)}{p(x) - \alpha(x)} \quad x \in \overline{\Omega}_1.$$

(H_4) $b : \overline{\Omega}_2 \rightarrow \mathbb{R}$, satisfies $b \in L^{\beta_0(x)}(\Omega_1)$ and $\alpha_0 \in C_+(\overline{\Omega}_2)$ such that

$$\frac{p(x)}{p(x) - \beta(x)} < \beta_0(x) < \frac{Np(x)}{Np(x) - \beta(x)(N - p(x))} \quad \text{for all } x \in \overline{\Omega}_2,$$

where

$$C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}) \text{ and } h(x) > 1 \text{ for all } x \in \overline{\Omega}\}$$

and $h^+ := \max_{\overline{\Omega}} h(x)$, $h^- := \min_{\overline{\Omega}} h(x)$ for any $h \in C_+(\overline{\Omega})$.

We see that the stationary problem with two wave equations of Kirchhoff type is generalized in problem 1.1, with the transmission condition.

$$\begin{cases} u_{tt} - M \left(\int_{\Omega_1} |\nabla u|^2 dx \right) \Delta_2 u = f(x, u) & \text{in } \Omega_1 \\ v_{tt} - N \left(\int_{\Omega_2} |\nabla v|^2 dx \right) \Delta_2 v = g(x, v) & \text{in } \Omega_2 \end{cases} \quad (1.2)$$

This simulates a membrane made of two distinct materials' transverse vibrations in Ω_1 and Ω_2 . For wave equations, controllability and stability of transmission problems are found in [19, 16]. We direct the reader to [18] for Kirchhoff-type stationary issues, [6] for p -Kirchhoff-type elliptic equations, and [4, 7, 8, 22, 21, 20] for $p(x)$ -Kirchhoff type equation.

In order to model and describe a wide range of phenomena, including fluid dynamics, quantum physics, sound, heat, electrostatics, diffusion, gravitation, chemistry, biology, airplane simulation, calculator charts, and time prediction, elliptic systems of differential equations are essential.

Numerous applications in biology and physics include transmission issues. The electromagnetic process seen in ferromagnetic media with varying dielectric constants, for example, is a significant issue in the electrodynamics of solid media (see, for example, [2, 4, 5, 13, 17]).

Inspired by the aforementioned study in [4, 10, 22], we demonstrate in this paper that there are solutions for the system (1.1) when the nonlinear terms f and g are specified by

$$f(x, u) = \frac{\lambda_1 a(x)}{\delta^{\gamma_2(x)}} |u|^{\alpha(x)-2} u \quad \text{and} \quad g(x, v) = \frac{\lambda_2 b(x)}{\delta^{\gamma_4(x)}} |v|^{\beta(x)-2} v$$

with $\alpha, \beta \in C_+(\overline{\Omega})$ such that

$$\alpha^- \leq \alpha^+ < \theta p^- < \theta p^+ < \beta^- \leq \beta^+ < \min\{N, \frac{Np^-}{N-p^-}\} \quad (1.3)$$

More specifically, we demonstrate the existence and non-triviality of the solutions using the compact embedding $W_0^{1,p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right) \hookrightarrow L^{p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right)$ (see the lemma 3.6 below).

Let

$$E := \left\{ (u, v) \in W^{1,p(x)}\left(\Omega_1, \frac{1}{\delta^{\gamma(x)}}\right) \times W_\Gamma^{1,p(x)}\left(\Omega_2, \frac{1}{\delta^{\gamma(x)}}\right) : u = v \quad \text{on} \quad \Gamma \right\},$$

where

$$W_\Gamma^{1,p(x)}\left(\Omega_2, \frac{1}{\delta^{\gamma(x)}}\right) = \left\{ v \in W^{1,p(x)}\left(\Omega_2, \frac{1}{\delta^{\gamma(x)}}\right) : v = 0 \quad \text{on} \quad \Gamma \right\}.$$

Definition 1.1. We recall that $(u, v) \in E$ is a weak solution of problem (1.1), if it verifies

$$\begin{aligned} & \mathcal{M}_1 \left(\int_{\Omega_1} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla u|^{p(x)} dx \right) \int_{\Omega_1} \frac{1}{\delta^{\gamma(x)}} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx \\ & + M_2 \left(\int_{\Omega_2} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla v|^{p(x)} dx \right) \int_{\Omega_2} \frac{1}{\delta^{\gamma(x)}} |\nabla v|^{p(x)-2} \nabla v \nabla \psi dx \\ & - \lambda_1 \int_{\Omega_1} \frac{1}{\delta^{\gamma_2(x)}} a(x) |u|^{\alpha(x)-2} u \varphi dx - \lambda_2 \int_{\Omega_2} \frac{1}{\delta^{\gamma_4(x)}} b(x) |v|^{\beta(x)-2} v \psi dx = 0 \quad \forall (\varphi, \psi) \in E \end{aligned}$$

The main result is stated below

Theorem 1.2. Assume that the following criteria are met: $0 \leq \gamma_i^+ < \alpha^- + p^-$ ($i = 2, 4$), (1.3) and $(H_0) - (H_4)$. There are then $\lambda^* > 0$ such that the problem (1.1) has at least two different nontrivial weak solutions for any $\lambda_1 + \lambda_2 \in (0, \lambda^*)$.

There are three sections to this study. We provide some essential information on variable-exponent Sobolev spaces with weights in Section 2. Prior to examining the possibility of a weak solution to the problem 1.1, we introduce and examine a variable-exponent Sobolev space with singular weights in Section 3.

2. PRELIMINARIES

The weighted Sobolev space $W^{1,p(\cdot)}\left(\Omega, \frac{1}{\delta^\alpha}\right)$ is introduced in this section, along with some initial findings. The standard weighted variable exponents are represented by $\left(L^{p(\cdot)}(\Omega, \vartheta(x)), \|\cdot\|_{p(x)}\right)$ and $\left(W^{1,p(\cdot)}(\Omega, \vartheta(x)), \|\cdot\|_{1,p(x)}\right)$, respectively. For $p(\cdot) \in L_+^\infty(\Omega)$ and $\vartheta \in L_{Loc}^1(\Omega)$ such that $\vartheta > 0$ is present nearly everywhere

in Ω . The weighted variable exponent Lebesgue space is taken into consideration by:

$$L^{p(x)}(\Omega, \vartheta) = \left\{ u : \Omega \longrightarrow \mathbb{R}, \quad \text{measurable and} \quad \int_{\Omega} \vartheta(x) |u(x)|^{p(x)} dx < \infty \right\},$$

with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega, \vartheta)} = \|u\|_{p(x)} = \inf \left\{ \mu > 0 / \rho_{p(\cdot), \vartheta} \left(\frac{u}{\mu} \right) = \int_{\Omega} \vartheta(x) \left| \frac{u(x)}{\mu} \right|^{p(x)} dx = 1 \right\},$$

and the space $(L^{p(x)}(\Omega, \vartheta), \|\cdot\|_{p(x)})$ is a Banach.

Moreover, $u \in L^{p(x)}(\Omega, \vartheta)$ if and only if $\|u\|_{L^{p(x)}(\Omega, \vartheta)} = \|u \vartheta^{\frac{1}{p(\cdot)}}\|_{L^{p(x)}(\Omega)} < \infty$. It is known that we have the relationships between the modular $\rho_{p(\cdot), \vartheta(\cdot)}$ and the norm $\|\cdot\|_{L^{p(x)}(\Omega, \vartheta)}$ are as follows

$$\min\{\rho_{p(\cdot), \vartheta(\cdot)}(u)^{\frac{1}{p^-}}, \rho_{p(\cdot), \vartheta(\cdot)}(u)^{\frac{1}{p^+}}\} \leq \|u\|_{L^{p(x)}(\Omega, \vartheta)} \leq \max\{\rho_{p(\cdot), \vartheta(\cdot)}(u)^{\frac{1}{p^-}}, \rho_{p(\cdot), \vartheta(\cdot)}(u)^{\frac{1}{p^+}}\}$$

and

$$\min\{\|u\|_{L^{p(x)}(\Omega, \vartheta)}^{p^-}, \|u\|_{L^{p(x)}(\Omega, \vartheta)}^{p^+}\} \leq \rho_{p(\cdot), \vartheta(\cdot)}(u) \leq \max\{\|u\|_{L^{p(x)}(\Omega, \vartheta)}^{p^-}, \|u\|_{L^{p(x)}(\Omega, \vartheta)}^{p^+}\}.$$

Also, if $0 < c \leq \vartheta$, then we have $L^{p(x)}(\Omega, \vartheta) \hookrightarrow L^{p(x)}(\Omega)$, since one easily sees that

$$c \int_{\Omega} |u(x)|^{p(x)} dx \leq \int_{\Omega} \vartheta(x) |u(x)|^{p(x)} dx$$

and

$$c \|u\|_{L^{p(x)}(\Omega)} \leq \|u\|_{L^{p(x)}(\Omega, \vartheta)}.$$

Additionally, the dual space of $L^{p(x)}(\Omega, \vartheta)$ is $L^{p'(\cdot)}(\Omega, \vartheta^*)$, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$ and $\vartheta^* = \vartheta^{1-p'(\cdot)} = \vartheta^{\frac{-1}{1-p(\cdot)}}$. For further information, see [3].

Lemma 2.1. (cf. [14]) *Let ϑ be a weight function and Ω be a bounded subset of $\mathbb{R}^N$. If and only if the inequality integral*

$$\int_{\Omega} |u(x)| \vartheta(x) dx \leq c_1 (\text{diam}(\Omega)) \int_{\Omega} |\nabla u(x)| \vartheta(x) dx$$

holds, then a function $u \in C_0^\infty(\Omega)$ satisfies Poincaré inequality in $L_\vartheta^1(\Omega)$.

We define $W_{p(\cdot)}(\Omega) = \{\vartheta \in L_{Loc}^1(\Omega) / \vartheta^{\frac{-1}{p(\cdot)-1}} \in L_{Loc}^1(\Omega)\}$.

Proposition 2.2. (cf. [3])

$$L^{p(x)}(\Omega, \vartheta) \hookrightarrow L_{Loc}^1(\Omega) \hookrightarrow D'(\Omega),$$

where $D'(\Omega)$ is the space of distributions, if $\vartheta \in W_{p(\cdot)}(\Omega)$.

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega, \vartheta) : \Delta^\alpha u \in L^{p(x)}(\Omega, \vartheta), |\alpha| \leq k\},$$

is how we set the weighted variable Sobolev spaces.

$$\Delta^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}},$$

is the derivative in the sense of distributions, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$.

Equipped with the norm

$$\|u\|_{W^{k,p(x)}(\Omega,\vartheta)} = \sum_{|\alpha| \leq k} \|\Delta^\alpha u\|_{L^{p(x)}(\Omega,\vartheta)},$$

also turns into a reflexive, separable, Banach space. Specifically, the space $W^{1,p(x)}(\Omega, \vartheta)$ is defined as the weighted Sobolev space with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega,\vartheta)} = \|u\|_{L^{p(x)}(\Omega,\vartheta)} + \|\nabla u\|_{L^{p(x)}(\Omega,\vartheta)}.$$

A closure of the set $C_0^\infty(\Omega)$ (smooth functions with compact support in Ω) with regard to the norm $\|\cdot\|_{W^{1,p(\cdot)}(\Omega,\vartheta)}$ is what we also define as the $W_0^{1,p(x)}(\Omega, \vartheta)$ subset of $W^{1,p(x)}(\Omega, \vartheta)$.

Proposition 2.3. *Assume that $p \in C_+(\overline{\Omega})$ and the cone property are present at the boundary of Ω . Assume that for a.e. $x \in \Omega$, $c(x) > 0$ and $c \in L^{\beta(x)}(\Omega)$. If $q(x) \in C_+(\overline{\Omega})$ and*

$$q(x) < \frac{\beta(x) - 1}{\beta(x)} p^*(x), \quad \forall x \in \overline{\Omega}.$$

A compact embedding follows.

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega, c(x)).$$

The embedding becomes continuous when $\leq$ is substituted for $<$.

Proposition 2.4. *(Fan and Zhao [12]) Let $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$. Then there is a continuous embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

If we replace $\leq$ with $<$, the embedding is compact.

Furthermore, we define $\rho : L^{p(x)}(\Omega) \rightarrow \mathbb{R}$ by

$$\rho_{p(x)}(u) = \int_{\Omega} |u|^{p(x)} dx,$$

Consequently, the following relationships are valid:

Proposition 2.5 ([9, 12, 15]). *If $u, u_n \in L^{p(x)}(\Omega)$, ($n = 1, 2, \dots$) then the following statements are equivalent:*

- (i) $\lim_{n \rightarrow \infty} |u_n - u|_{p(x)} = 0$;
- (ii) $\lim_{n \rightarrow \infty} \rho_{p(x)}(u_n - u) = 0$;
- (iii) $u_n \rightarrow u$ in measure in Ω and $\lim_{n \rightarrow \infty} \rho_{p(x)}(u_n) = \rho_{p(x)}(u)$.

Proposition 2.6. ([9, 12, 15]) *If $u, u_n \in L^{p(x)}(\Omega)$, ($n = 1, 2, \dots$), we obtain*

- (1) $|u|_{p(x)} < 1$ (respectively $=1; > 1$) $\iff \rho_{p(x)}(u) < 1$ (respectively $=1; > 1$);
- (2) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}$;
- (3) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}$;

$$(4) \quad |u_n|_{p(x)} \rightarrow 0 \text{ (respectively } \rightarrow \infty) \iff \rho_{p(x)}(u_n) \rightarrow 0 \text{ (respectively } \rightarrow \infty).$$

Remark 2.7. The claims of Propositions 2.5 and Proposition 2.6 also apply to $u, u_n \in W^{1,p(x)}(\Omega, \vartheta(x))$. if we take into account $\rho_{p(\cdot), \vartheta(\cdot)}(u) = \int_{\Omega} \vartheta(x) |u|^{p(x)} dx$, rather than $\rho_{p(x)}(u)$.

3. MAIN RESULT

3.1. The weighted variable exponent Sobolev space.

We introduce and examine the weighted variable exponent Sobolev spaces before examining the possibility of a weak solution to problem 1.1.

For $\gamma \in C_+(\overline{\mathbb{R}})$, we define the following the weighted variable exponent space

$$L^{p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right) := \left\{ u = u(x), x \in \Omega, \int \frac{1}{\delta^{\gamma(x)}} |u|^{p(x)} < \infty \right\}$$

equipped with the norm

$$|u|_{\gamma(x), p(x)} := \inf \left\{ \mu > 0 / \int_{\Omega} \frac{1}{\delta^{\gamma(x)}} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}, \quad (3.1)$$

and

$$W^{1,p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right) := \left\{ u, \int_{\Omega} \frac{1}{\delta^{\gamma(x)}} |\nabla u|^{p(x)} + |u|_{\gamma(x), p(x)}^{p(x)} < \infty \right\} \quad (3.2)$$

equipped with the norm

$$\|u\|_X = |u|_{1, \gamma(x), p(x)} := |\nabla u|_{\gamma(x), p(x)} + |u|_{\gamma(x), p(x)}. \quad (3.3)$$

Lemma 3.1. *Let $\gamma \in C_+(\mathbb{R})$. The space $W^{1,p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right)$, equipped with the norm $|\cdot|_{1, \gamma(x), p(x)}$, is Banach space.*

Proof. Let $(u_n)_n$ be a sequence in $X_{\gamma(x)}$. Assume that $(u_n)_n$ is Cauchy sequence in $X_{\gamma(x)}$. One can easily observe that $(u_n)_n$ (resp. $((\partial u_n / \partial x_1, \dots, \partial u_n / \partial x_N))_n$) is also Cauchy sequence in $L^{p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right)$ (resp. $\left(L^{p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right)\right)^N$). Using the fact that the space $L^{p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right)$ is Banach space, we obtain that there exist u and w_i , $i = 1, 2, \dots, N$, such that the sequence $(u_n)_n$ converges to u in $L^{p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right)$ and the sequence $(\partial u_n / \partial x_i)$ converges to w_i in $L^{p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right)$ for $i = 1, \dots, N$. For every $\psi \in D(\Omega)$, we have

$$\left| \int_{\Omega} u_n \psi dx - \int_{\Omega} u \psi dx \right| \leq \int_{\Omega} |u_n - u| |\psi| dx \quad (3.4)$$

and

$$\begin{aligned} \int_{\Omega} |u_n - u| |\psi| dx &= \int_{\Omega} \frac{1}{\delta^{\gamma(x)p(x)}} |u_n - u| \frac{1}{\delta^{-\gamma(x)p(x)}} |\psi| dx \\ &\leq |u_n - u|_{\gamma(x), p(x)} \left\| \frac{1}{\delta^{-\gamma(x)p(x)}} \psi \right\|_{p'(x)} \\ &\leq |u_n - u|_{\gamma(x), p(x)} \|\psi\|_{\infty} \left\| \frac{1}{\delta^{-\gamma(x)p(x)}} \right\|_{L^{p'(x)}(\text{supp } \psi)} \end{aligned}$$

where $\text{supp}\psi \subset \Omega$ denotes the support of ψ and $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. We deduce

$$\left| \int_{\Omega} u_n \psi dx - \int_{\Omega} u \psi dx \right| \leq |u_n - u|_{\gamma(x), p(x)} \|\psi\|_{\infty} \left\| \frac{1}{\delta - \gamma(x)p(x)} \right\|_{L^{p'(x)}(\text{supp}\psi)} \quad (3.5)$$

Since $\left\| \frac{1}{\delta - \gamma(x)p(x)} \right\|_{L^{p'(x)}(\text{supp}\psi)} < \infty$ and $u_n \rightarrow u$ in $L^{p(x)}(\Omega, \frac{1}{\delta\gamma(x)})$ we get

$$\int_{\Omega} u_n \psi dx \rightarrow \int_{\Omega} u \psi dx \quad \text{as } n \rightarrow +\infty. \quad (3.6)$$

Similarly, by $\partial u_n / \partial x_i \rightarrow w_i$ in $L^{p(x)}(\Omega, \frac{1}{\delta\gamma(x)})$, we obtain

$$\int_{\Omega} \frac{\partial u_n}{\partial x_i} \psi dx \rightarrow \int_{\Omega} w_i \psi dx, \quad i = 1, \dots, N \quad (3.7)$$

as $n \rightarrow +\infty$. Then,

$$\int_{\Omega} w_i \psi dx = \lim_{n \rightarrow \infty} \int_{\Omega} \frac{\partial u_n}{\partial x_i} \psi dx \quad (3.8)$$

$$= - \lim_{n \rightarrow \infty} \int_{\Omega} u_n \frac{\partial \psi}{\partial x_i} dx \quad (3.9)$$

$$= - \int_{\Omega} u \frac{\partial \psi}{\partial x_i} dx \quad (3.10)$$

$$= \int_{\Omega} \frac{\partial u}{\partial x_i} \psi dx, \quad i = 1, \dots, N. \quad (3.11)$$

We deduce that $\nabla u = w$ and $u_n \rightarrow u$ in $W^{1,p(x)}(\Omega, \frac{1}{\delta\gamma(x)})$. The proof is finished with this. $\square$

Lemma 3.2. *Let $\gamma \in C_+(\mathbb{R})$. The coordinate system $X_{\gamma(x)} := W^{1,p(x)}(\Omega, \frac{1}{\delta\gamma(x)})$, which is reflexive, is furnished with the norm $|\cdot|_{1,p(x),\gamma(x)}$.*

Proof. It is easy to verify that $L^{p(x)}(\Omega, \frac{1}{\delta\gamma(x)})$ is uniformly convex. Consequently, $L^{p(x)}(\Omega, \frac{1}{\delta\gamma(x)})$ is reflexive. Think about the functional

$$T : X_{\gamma(x)} \rightarrow F := L^{p(x)}\left(\Omega, \frac{1}{\delta\gamma(x)}\right) \times L^{p(x)}\left(\Omega, \frac{1}{\delta\gamma(x)}\right)^N$$

$$u \rightarrow T(u) = (u, \nabla u)$$

Since T is an isometry, we obtain that $T(X_{\gamma(x)})$ is a closed subspace of F . Because $T(X_{\gamma(x)})$ is reflexive, therefore $X_{\gamma(x)}$ is reflexive. $\square$

Lemma 3.3. *Assume that $p, q \in L_+^\infty(\Omega)$. If $q(x) \leq p(x)$. Next, the embedding*

$$L^{p(x)}\left(\Omega, \frac{1}{\delta\gamma(x)}\right) \hookrightarrow L^{q(x)}\left(\Omega, \frac{1}{\delta\gamma(x)}\right)$$

is continuously.

Lemma 3.4. Assume that $p, q \in L_+^\infty(\Omega)$. If $q(x) \leq p(x)$. Next, the embedding

$$W^{m,p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right) \hookrightarrow W^{m,q(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right)$$

is continuously.

Proof. Lemma 3.5 proof is symmetric with the proofs of Lemma 3.3 and 3.4. $\square$

The lemma that follows relates to a compact embedding result.

Lemma 3.5. Let $p, q \in L_+^\infty(\Omega)$, we assume $mp(x) \leq n$ and $q(x) < \frac{np(x)}{n-mp(x)}$ $\forall x \in \bar{\Omega}$. Next,

$$X_{\gamma(x)} =: W^{m,p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right) \hookrightarrow L^{q(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right)$$

is a compact and continuous embedding. We write $X_{\gamma(x)} \hookrightarrow L^{q(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right)$.

Proof. To represent a positive constant r with $mr < n$, denote $r^* = \frac{nr}{n-mr}$.

It is clear from the assumptions that, for each given $x \in \bar{\Omega}$, we can locate a neighborhood U_x in $\bar{\Omega}$ such that $q^+(U_x) < (p^-(U_x))^*$, where

$$p^-(U_x) = \inf\{p(y) \mid y \in U_x\}; \quad q^+(U_x) = \sup\{q(y) \mid y \in U_x\}.$$

Currently, $\{U_x\}_{x \in \bar{\Omega}}$ is an open covering that indicates $\{U_i \mid i = 1, 2, \dots, s\}$ and denoting

$$p_i^- = p^-(U_i) \quad q_i^+ = q^+(U_i),$$

It is well known that if $u \in W^{m,p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right)$ then $u \in W^{m,p(x)}\left(U_i, \frac{1}{\delta^{\gamma(x)}}\right)$, and thus from the lemma 3.4, $u \in W^{m,p_i^-}\left(U_i, \frac{1}{\delta^{\gamma(x)}}\right)$. Therefore by the well-known Sobolev imbedding theorem we have continuous and compact imbedding,

$$W^{m,p_i^-}\left(U_i, \frac{1}{\delta^{\gamma(x)}}\right) \hookrightarrow L^{q_i^+}\left(U_i, \frac{1}{\delta^{\gamma(x)}}\right),$$

so for every $U_i, i = 1, 2, \dots, s$ (see [10, 22]), we have $u \in L^{q(x)}\left(U_i, \frac{1}{\delta^{\gamma(x)}}\right)$ and therefore $u \in L^{q(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right)$. We can now assert that $W^{m,p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right) \subset L^{q(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right)$ and the imbedding is continuous and compact. $\square$

Lemma 3.6. Let $0 \leq \gamma^+ < \alpha^- + p^-$. Then

$$W_0^{1,p(x)}\left(\Omega, \frac{1}{\delta^{\alpha(x)}}\right) \hookrightarrow L^{p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right) \hookrightarrow L^{p^*(x)}(\Omega)$$

Proof. Let $\sigma > 0$ small enough we define $\Omega_\sigma := \{x \in \Omega \mid d(x) > \sigma\}$. Consider the maps:

$$I_\sigma : W_0^{1,p(x)}\left(\Omega, \frac{1}{\delta^{\alpha(x)}}\right) \longrightarrow L^{p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right),$$

$$I_\sigma^1 : W_0^{1,p(x)}\left(\Omega, \frac{1}{\delta^{\alpha(x)}}\right) \longrightarrow W_0^{1,p(x)}(\Omega_\sigma),$$

$$I_\sigma^2 : W_0^{1,p(x)}(\Omega_\sigma) \longrightarrow L^{p(x)}(\Omega_\sigma)$$

and

$$I_\sigma^3 : L^{p(x)}(\Omega_\sigma) \longrightarrow L^{p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right).$$

Here $I_\sigma^1 u(x) = u(x)$, $x \in \Omega_\sigma$ is the "restriction" (bounded and linear map) from $W_0^{1,p(x)}\left(\Omega, \frac{1}{\delta^{\alpha(x)}}\right)$ into $W_0^{1,p(x)}(\Omega_\sigma)$; $I_\sigma^2 u(x) = u(x)$, $x \in \Omega_\sigma$ is "embedding" (compact linear map) from $W_0^{1,p(x)}(\Omega_\sigma)$ into $L^{p(x)}(\Omega_\sigma)$; $I_\sigma^3 u(x) = u(x)$, $x \in \Omega_\sigma$ and $I_\sigma^3 u(x) = 0$, $x \in \Omega \setminus \Omega_\sigma$ is "zero extension" (bounded and linear map) from $L^{p(x)}(\Omega_\sigma)$ into $L^{p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right)$.

Then for any $\sigma > 0$, $I_\sigma : W_0^{1,p(x)}\left(\Omega, \frac{1}{\delta^{\alpha(x)}}\right) \longrightarrow L^{p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right)$ is a compact linear map. Denote by

$$I : W_0^{1,p(x)}\left(\Omega, \frac{1}{\delta^{\alpha(x)}}\right) \hookrightarrow L^{p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right),$$

the embedding of $W_0^{1,p(x)}\left(\Omega, \frac{1}{\delta^{\alpha(x)}}\right)$ into $L^{p(x)}\left(\Omega, \frac{1}{\delta^{\gamma(x)}}\right)$. Then for $u \in W_0^{1,p(x)}\left(\Omega, \frac{1}{\delta^{\alpha(x)}}\right)$ there exists $K > 0$ such that

$$\begin{aligned} |(I_\sigma - I)(u)|_{\gamma(x), p(x)} &= \int_{\Omega \setminus \Omega_\sigma} \frac{1}{\delta^{\gamma(x)}(x)} |u(x)|^{p(x)} dx = \int_{\Omega \setminus \Omega_\sigma} \frac{1}{\delta^{\gamma(x)}} \delta \alpha(x) + p(x) \frac{|u(x)|^{p(x)}}{\delta \alpha(x) + p(x)} dx \\ &\leq \max_{x \in \Omega \setminus \Omega_\sigma} \delta \alpha^- + p^- - \gamma^+ \int_{\Omega \setminus \Omega_\sigma} \frac{1}{\delta \alpha^+ + p^+} |u(x)|^{p(x)} dx \\ &\leq \max_{x \in \Omega \setminus \Omega_\sigma} \delta \alpha^- + p^- - \gamma^+ |u|_{p(x), \alpha^+ + p^+}^{p^-} \\ &\leq \max_{x \in \Omega \setminus \Omega_\sigma} \delta \alpha^- + p^- - \gamma^+ |u|_{p(x), \alpha^+}^{p^-} \\ &\leq K \max_{x \in \Omega \setminus \Omega_\sigma} \delta \alpha^- + p^- - \gamma^+ \|u\|_X^{p^-}. \end{aligned}$$

The claim is now supported by the observation that $\max_{x \in \Omega \setminus \Omega_\sigma} \delta \alpha^- + p^- - \gamma^+ \longrightarrow 0$ as $\sigma \longrightarrow 0$. $\square$

3.2. The existence of weak solution of problem 1.1:

$J : E \longrightarrow \mathbb{R}$ is the definition of the energy functional related to problem 1.1

$$J(u, v) = \phi(u, v) - \psi(u, v)$$

where

$$\phi(u, v) = \overline{M}_1 \left(\int_{\Omega_1} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla u|^{p(x)} dx \right) + \overline{M}_2 \left(\int_{\Omega_2} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla u|^{p(x)} dx \right)$$

and

$$\psi(u, v) = \lambda_1 \int_{\Omega_1} \frac{1}{\alpha(x)} \frac{a(x)}{\delta^{\gamma_2(x)}} |u|^{\alpha(x)} dx + \lambda_2 \int_{\Omega_2} \frac{1}{\beta(x)} \frac{b(x)}{\delta^{\gamma_4(x)}} |v|^{\beta(x)} dx$$

In a standard way, it can be proved that $J \in C^1(E, \mathbb{R})$. Moreover, we have, for all $(\rho, \xi) \in E$,

$$\langle J'(u, v), (\rho, \xi) \rangle = \langle \phi'(u, v), (\rho, \xi) \rangle - \langle \psi'(u, v), (\rho, \xi) \rangle$$

where

$$\begin{aligned} \langle \phi'(u, v), (\rho, \xi) \rangle &= \mathcal{M}_1 \left(\int_{\Omega_1} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla u|^{p(x)} dx \right) \int_{\Omega_1} \frac{1}{\delta^{\gamma(x)}} |\nabla u|^{p(x)-2} \nabla u \nabla \rho dx \\ &+ \mathcal{M}_2 \left(\int_{\Omega_2} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla v|^{p(x)} dx \right) \int_{\Omega_1} \frac{1}{\delta^{\gamma(x)}} |\nabla v|^{p(x)-2} \nabla v \nabla \xi dx \end{aligned}$$

and

$$\langle \psi'(u, v), (\rho, \xi) \rangle = \lambda_1 \int_{\Omega_1} \frac{1}{\delta^{\gamma_2(x)}} a(x) |u|^{\alpha(x)-2} u \rho dx - \lambda_2 \int_{\Omega_2} \frac{1}{\delta^{\gamma_4(x)}} b(x) |v|^{\beta(x)-2} v \xi dx$$

Therefore, the critical points of the functional J are precisely the weak solutions of problem 1.1.

Lemma 3.7. *In E , the functional J is semicontinuous but weakly lower.*

Proof. Assume that $\{(u_n, v_n)\} \subset E$ is a sequence such that

$$(u_n, v_n) \rightharpoonup (u, v)$$

$\|u\|_{X_1} = \int_{\Omega_1} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla u_n|^{p(x)} dx$ is the norm in X_1 , then $\|u\|_{X_1}$ is weakly lower semicontinuous

$$\|u\|_{X_1} \leq \liminf \|u_n\|_{X_1}$$

Since $\overline{M}_i$ is non-decreasing, then

$$\overline{M}_i \left(\int_{\Omega_1} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla u|^{p(x)} dx \right) \leq \overline{M}_i \left(\liminf \int_{\Omega_1} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla u_n|^{p(x)} dx \right)$$

Moreover, the functional $\overline{M}_i$ is continuous, thus

$$\overline{M}_i \left(\int_{\Omega_1} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla u|^{p(x)} dx \right) \leq \liminf \overline{M}_i \left(\int_{\Omega_1} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla u_n|^{p(x)} dx \right).$$

Additionally, the functional ψ is weakly lower semicontinuous in E according to the dominated convergence theorem. $\square$

Lemma 3.8. *There exist λ^* and $\rho, r > 0$ such that for any $\lambda_1 + \lambda_2 \in (0, \lambda^*)$ we have*

$$J(u, v) \geq \rho \quad \forall (u, v) \in E, \quad \text{with} \quad \|(u, v)\| = r$$

Proof. By using the condition $(H_3) - (H_4)$, Proposition 2.5 and the Lemma 3.6, the embedding from E to $L^{\alpha(x)}(\Omega_1, \frac{a(x)}{\delta^{\gamma_2(x)}})$ and $L^{\beta(x)}(\Omega_2, \frac{b(x)}{\delta^{\gamma_4(x)}})$ are compact, (see [15], Theorems 2.7, 2.8). Then there exists $C > 0$ such that for all $(u, v) \in E$

$$|u|_{1, \alpha(x), \gamma_2(x), \Omega_1} + |v|_{1, \beta(x), \gamma_4(x), \Omega_2} \leq C_0 \|(u, v)\|_E \quad \text{for all } (u, v) \in E$$

Consider $r \in (0, 1)$ such that $r < \frac{1}{C_0}$. Then we get

$$|u|_{1, \alpha(x), \gamma_2(x), \Omega_1} + |v|_{1, \beta(x), \gamma_4(x), \Omega_2} < 1 \quad \text{with} \quad \|(u, v)\|_E = r. \quad (3.12)$$

Furthermore, Proposition 2.3 yields

$$\int_{\Omega_1} \frac{a(x)}{\delta^{\gamma_2(x)}} |u|^{\alpha(x)} dx \leq C_1 (|u|^{\alpha^-}_{1, \alpha(x), \gamma(x), \Omega_1} + |u|^{\alpha^+}_{1, \alpha(x), \gamma(x), \Omega_1}), \quad \forall u \in W^{1, p(x)} \left(\Omega_1, \frac{1}{\delta^{\gamma(x)}} \right),$$

and

$$\int_{\Omega_2} \frac{b(x)}{\delta, \gamma_4(x)} |v|^{\beta(x)} dx \leq C_2(|v|_{1, \beta(x), \gamma(x), \Omega_2}^{\beta^+} + |v|_{1, \beta(x), \gamma(x), \Omega_2}^{\beta^-}), \quad \forall v \in W^{1, p(x)} \left(\Omega_2, \frac{1}{\delta^{\gamma(x)}} \right).$$

Then

$$\int_{\Omega_1} \frac{a(x)}{\delta, \gamma_2(x)} |u|^{\alpha(x)} dx + \int_{\Omega_2} \frac{b(x)}{\delta, \gamma_4(x)} |v|^{\beta(x)} dx \leq C_3(|u|_{1, \alpha(x), \gamma(x), \Omega_1} + |v|_{1, \beta(x), \gamma(x), \Omega_2}).$$

Hence, from 3.12, we deduce

$$\int_{\Omega_1} \frac{a(x)}{\delta^{\gamma_2(x)}} |u|^{\alpha(x)} dx + \int_{\Omega_2} \frac{b(x)}{\delta^{\gamma_4(x)}} |v|^{\beta(x)} dx \leq C_4 \|(u, v)\|_E. \quad (3.13)$$

By $(H_0) - (H_2)$ and by help of the following elementary inequality

$$|a + b|^s \leq 2^{s-1}(|a|^s + |b|^s) \quad \text{for } a, b \in \mathbb{R}^n$$

the following inequalities hold true:

$$\begin{aligned} J(u, v) &= \overline{M}_1 \left(\int_{\Omega_1} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla u|^{p(x)} dx \right) + \overline{M}_2 \left(\int_{\Omega_2} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla v|^{p(x)} dx \right) \\ &\quad - \lambda_1 \int_{\Omega_1} \frac{1}{\alpha(x)} \frac{a(x)}{\delta^{\gamma_2(x)}} |u|^{\alpha(x)} dx - \lambda_2 \int_{\Omega_2} \frac{1}{\beta(x)} \frac{b(x)}{\delta^{\gamma_4(x)}} |v|^{\beta(x)} dx \\ &\geq \frac{m_1}{(p^+)^{\theta-1}} \left(\int_{\Omega_1} \frac{1}{\delta^{\gamma(x)}} |\nabla u|^{p(x)} dx \right)^{\theta-1} + \frac{m'_1}{(p^+)^{\theta-1}} \left(\int_{\Omega_2} \frac{1}{\delta^{\gamma(x)}} |\nabla v|^{p(x)} dx \right)^{\theta-1} \\ &\quad - \frac{\lambda_1 + \lambda_2}{\alpha^-} \left(\int_{\Omega_1} \frac{a(x)}{\delta^{\gamma_2(x)}} |u|^{\alpha(x)} dx + \int_{\Omega_2} \frac{b(x)}{\delta^{\gamma_4(x)}} |v|^{\beta(x)} dx \right) \end{aligned}$$

thus

$$\begin{aligned} J(u, v) &\geq \frac{m_1}{\theta(p^+)^{\theta-1}} \|u\|_{X_1}^{(\theta-1)p^+} + \frac{m'_1}{(p^+)^{\theta-1}} \|v\|_{X_2}^{(\theta-1)p^+} - \frac{C_5(\lambda_1 + \lambda_2)}{\alpha^-} \|(u, v)\|_X \\ &\geq \frac{\min(m_1, m'_1)}{(p^+)^{(\theta-1)}} \left(\|u\|_{X_1}^{(\theta-1)p^+} + \|v\|_{X_2}^{(\theta-1)p^+} \right) - \frac{C_5(\lambda_1 + \lambda_2)}{\alpha^-} \|(u, v)\| \end{aligned}$$

thus

$$J(u, v) \geq \frac{\min(m_1, m'_1)}{2^{(\theta-1)p^+-1} (p^+)^{(\theta-1)}} \|(u, v)\|^{(\theta-1)p^+} - \frac{C_5(\lambda_1 + \lambda_2)}{\alpha^-} \|(u, v)\|.$$

Define

$$\lambda^* = \frac{\min(m_1, m'_1) \alpha^- r^{(\theta-1)p^+-1}}{2^{(\theta-1)p^+-1} C_5 (p^+)^{(\theta-1)}}$$

According to the aforementioned inequality, there exists $\rho > 0$ such that $J(u, v) \geq \rho$ for each $\lambda_1 + \lambda_2 \in (0, \lambda^*)$ and $(u, v) \in E$ with $\|(u, v)\| = r$. $\square$

Lemma 3.9. *There exist $(\widehat{\varphi}, \widehat{\psi}) \in E$, $\widehat{\varphi}, \widehat{\psi} \geq 0$ such that*

$$\lim_{t \rightarrow \infty} J(t\widehat{\varphi}, t\widehat{\psi}) = -\infty.$$

Proof. Let us choose $\widehat{\varphi}, \widehat{\psi} \in C_0^\infty(\Omega)$ such that $\widehat{\varphi}, \widehat{\psi} \neq 0$. Then, for any $t > 1$, by (H_1) and (H_2) it follows

$$\begin{aligned} J(t\widehat{\varphi}, t\widehat{\psi}) &= \overline{M}_1 \left(\int_{\Omega_1} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |t\nabla\widehat{\varphi}|^{p(x)} dx \right) + \overline{M}_2 \left(\int_{\Omega_2} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |t\nabla\widehat{\psi}|^{p(x)} dx \right) \\ &\quad - \lambda_1 \int_{\Omega_1} \frac{1}{\alpha(x)} \frac{a(x)}{\delta^{\gamma_2(x)}} |t\widehat{\varphi}|^{\alpha(x)} dx - \lambda_2 \int_{\Omega_2} \frac{1}{\beta(x)} \frac{b(x)}{\delta^{\gamma_4(x)}} |t\widehat{\psi}|^{\beta(x)} dx \\ &\leq \frac{\max\{m_2, m'_2\} t^{(\theta-1)p^+}}{\theta(p^+)^{(\theta-1)}} \left\{ \rho_{p(x), \gamma(x), \Omega_1}(\nabla\widehat{\varphi})^{(\theta-1)} + \rho_{p(x), \gamma(x), \Omega_2}(\nabla\widehat{\psi})^{(\theta-1)} \right\} \\ &\quad - \frac{\lambda_1 t^{\alpha^-}}{\alpha^+} \rho_{p(x), \gamma_2(x), \Omega_1}(\widehat{\varphi}) - \frac{\lambda_2 t^{\beta^-}}{\beta^+} \rho_{p(x), \gamma_4(x), \Omega_2}(\widehat{\psi}) \end{aligned}$$

Since $\beta^- > (\theta-1)p^+$, we get $J(t\widehat{\varphi}, t\widehat{\psi}) \rightarrow -\infty$ as $t \rightarrow \infty$. □

Lemma 3.10. *For every sufficiently small $t > 0$, there exist $(\widehat{\varphi}, \widehat{\psi}) \in E$ such that $\widehat{\varphi}, \widehat{\psi} > 0$ and $J(t\widehat{\varphi}, t\widehat{\psi}) < 0$.*

Proof. Let choose $\widehat{\varphi}, \widehat{\psi} \in C_0^\infty(\Omega)$ and $0 \leq \widehat{\varphi} \leq 1$ in Ω_1 , $0 \leq \widehat{\psi} \leq 1$ in Ω_2 . Then, for any $t \in (0, 1)$, by (H_1) and (H_2) it follows

$$\begin{aligned} J(t\widehat{\varphi}, t\widehat{\psi}) &= \overline{M}_1 \left(\int_{\Omega_1} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |t\nabla\widehat{\varphi}|^{p(x)} dx \right) + \overline{M}_2 \left(\int_{\Omega_2} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |t\nabla\widehat{\psi}|^{p(x)} dx \right) \\ &\quad - \lambda_1 \int_{\Omega_1} \frac{1}{\alpha(x)} \frac{a(x)}{\delta^{\gamma_2(x)}} |t\widehat{\varphi}|^{\alpha(x)} dx - \lambda_2 \int_{\Omega_2} \frac{1}{\beta(x)} \frac{b(x)}{\delta^{\gamma_4(x)}} |t\widehat{\psi}|^{\beta(x)} dx \\ &\leq \frac{\max\{m_2, m'_2\} t^{(\theta-1)p^+}}{(p^-)^{(\theta-1)}} \left\{ (\rho_{p(x), \gamma(x), \Omega_1}(\nabla\widehat{\varphi}))^{(\theta-1)} + (\rho_{p(x), \gamma(x), \Omega_2}(\nabla\widehat{\psi}))^{(\theta-1)} \right\} \\ &\quad - \frac{\lambda_1 t^{\alpha^-}}{\alpha^+} \rho_{p(x), \gamma_2(x), \Omega_1}(\widehat{\varphi}) - \frac{\lambda_2 t^{\beta^-}}{\beta^+} \rho_{p(x), \gamma_4(x), \Omega_2}(\widehat{\psi}) \\ &\leq \frac{\max\{m_2, m'_2\} t^{(\theta-1)p^+}}{(p^-)^{(\theta-1)}} \left\{ (\rho_{p(x), \gamma(x), \Omega_1}(\nabla\widehat{\varphi}))^{(\theta-1)} + (\rho_{p(x), \gamma(x), \Omega_2}(\nabla\widehat{\psi}))^{(\theta-1)} \right\} \\ &\quad - \frac{\lambda_1 t^{\beta^+}}{\beta^+} \rho_{p(x), \gamma_2(x), \Omega_1}(\widehat{\varphi}) - \frac{\lambda_2 t^{\beta^+}}{\beta^+} \rho_{p(x), \gamma_4(x), \Omega_2}(\widehat{\psi}) \end{aligned}$$

Let

$$R_\theta = \max\{m_2, m'_2\} \left\{ (\rho_{p(x), \gamma(x), \Omega_1}(\nabla\widehat{\varphi}))^{(\theta-1)} + (\rho_{p(x), \gamma(x), \Omega_2}(\nabla\widehat{\psi}))^{(\theta-1)} \right\}$$

and

$$S_{1,2} = \lambda_1 \rho_{p(x), \gamma_2(x), \Omega_1}(\widehat{\varphi}) + \lambda_2 \rho_{p(x), \gamma_4(x), \Omega_2}(\widehat{\psi}).$$

Then, for $0 < t < \sigma^{\frac{1}{p^- - \beta^+}}$ providing that

$$\sigma = \text{Min} \left\{ 1, \frac{(p^-)^{(\theta-1)} S_{1,2}}{\beta^+ R_\theta} \right\}$$

we conclude that

$$J(t\widehat{\varphi}, t\widehat{\psi}) < 0.$$

The proof of Lemma 3.10 is complete. $\square$

Lemma 3.11. *We suppose that $\overline{M}_i(t) \geq \frac{1}{\theta} t \mathcal{M}_i(t)$ then The functional J satisfies the Palais-Smale condition in E .*

Proof. Let $\{(u_n, v_n)\} \subset E$ be a sequence such that

$$J(u_n, v_n) \longrightarrow c > 0, \quad J'(u_n, v_n) \longrightarrow 0 \quad \text{in } E^*. \quad (3.14)$$

where E^* is the dual space of E .

We first demonstrate that $\{(u_n, v_n)\}$ is bounded in E . By contradiction, assume the opposite. When we finally reach a subsequence, which is still represented by (u_n, v_n) , we can presume that $\|(u_n, v_n)\| \longrightarrow \infty$. Consequently, we can think that

$$\|u_n\| > 1 \quad \|v_n\| > 1 \quad \text{for any integer } n,$$

so that

$$\begin{aligned} c + 1 + \|(u_n, v_n)\| &\geq J(u_n, v_n) - \frac{1}{\beta^-} < J'(u_n, v_n), (u_n, v_n) > \\ &\geq \overline{M}_1 \left(\int_{\Omega_1} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla u_n|^{p(x)} dx \right) + \overline{M}_2 \left(\int_{\Omega_2} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla v_n|^{p(x)} dx \right) \\ &\quad - \lambda_1 \int_{\Omega_1} \frac{1}{\alpha(x)} \frac{a(x)}{\delta^{\gamma_2(x)}} |u_n|^{\alpha(x)} dx - \lambda_2 \int_{\Omega_2} \frac{1}{\beta(x)} \frac{b(x)}{\delta^{\gamma_4(x)}} |v_n|^{\beta(x)} dx \\ &\quad - \frac{1}{\beta^-} \mathcal{M}_1 \left(\int_{\Omega_1} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla u_n|^{p(x)} dx \right) \int_{\Omega_1} \frac{1}{\delta^{\gamma(x)}} |\nabla u_n|^{p(x)} dx \\ &\quad - \frac{1}{\beta^-} \mathcal{M}_2 \left(\int_{\Omega_2} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla v_n|^{p(x)} dx \right) \int_{\Omega_2} \frac{1}{\delta^{\gamma(x)}} |\nabla v_n|^{p(x)} dx \\ &\quad - \frac{1}{\beta^-} \left[-\lambda_1 \int_{\Omega_1} \frac{a(x)}{\delta^{\gamma_2(x)}} |u_n|^{\alpha(x)} dx - \lambda_2 \int_{\Omega_2} \frac{b(x)}{\delta^{\gamma_4(x)}} |v_n|^{\beta(x)} dx \right] \\ &\geq \left(\frac{\beta^- - p^+}{(p^+)^{\theta} \beta^-} \right) \left[\left(\int_{\Omega_1} \frac{1}{\delta^{\gamma(x)}} |\nabla u|^{p(x)} dx \right)^{\theta} + \left(\int_{\Omega_2} \frac{1}{\delta^{\gamma(x)}} |\nabla v|^{p(x)} dx \right)^{\theta} \right] \\ &\quad + \left(\frac{1}{\beta^-} - \frac{1}{\alpha^-} \right) \left[\lambda_1 \int_{\Omega_1} \frac{a(x)}{\delta^{\gamma_2(x)}} |u_n|^{\alpha(x)} dx + \lambda_1 \int_{\Omega_1} \frac{a(x)}{\delta^{\gamma_2(x)}} |u_n|^{\alpha(x)} dx \right] \\ &\geq \left(\frac{\beta^- - p^+}{(p^+)^{(\theta-1)} \beta^-} \right) \left[\left(\int_{\Omega_1} \frac{1}{\delta^{\gamma(x)}} |\nabla u|^{p(x)} dx \right)^{(\theta-1)} + \left(\int_{\Omega_2} \frac{1}{\delta^{\gamma(x)}} |\nabla v|^{p(x)} dx \right)^{(\theta-1)} \right] \\ &\quad - 2\lambda_1 \left(\frac{1}{\alpha^-} - \frac{1}{\beta^-} \right) \int_{\Omega_1} \frac{a(x)}{\delta^{\gamma_2(x)}} |u_n|^{\alpha(x)} dx \\ &\geq 2^{1-(\theta-1)p^+} \left(\frac{\beta^- - p^+}{(p^+)^{(\theta-1)} \beta^-} \right) \|(u_n, v_n)\|^{(\theta-1)p^-} - 2C\lambda_1 \left(\frac{1}{\alpha^-} - \frac{1}{\beta^-} \right) \|(u_n, v_n)\|_E^{\alpha^+} \end{aligned}$$

Since $\alpha^+ < (\theta - 1)p^-$ and $p^+ < (\theta - 1)p^+ < \beta^-$, dividing the above inequality by $\|(u, v)\|$ and passing to the limit as $n \longrightarrow \infty$ we obtain a contradiction. It follows that the sequence $\{(u_n, v_n)\}$ is bounded in E .

Thus, there exists $(u, v) \in E$ such that passing to a subsequence, still denoted by (u_n, v_n) , it converges weakly to (u, v) in E . The embedding from E to the weighted spaces $L^{\alpha(x)}(\Omega_1, \frac{1}{\delta^{\gamma_2(x)}})$ and $L^{\beta(x)}(\Omega_2, \frac{1}{\delta^{\gamma_4(x)}})$ are compact. Then, using the Hölder inequalities, Propositions

$$\begin{aligned}
\left| \int_{\Omega_1} \frac{a(x)}{\delta^{\gamma_2(x)}} |u_n|^{\alpha(x)-2} u_n (u_n - u) dx \right| &\leq \int_{\Omega_1} \frac{1}{\delta^{\gamma_2(x)}} a(x) |u_n|^{\alpha(x)-1} |u_n - u| dx \\
&\leq K_1 \left| \left(\int_{\Omega_1} \frac{a(x)}{\delta^{\gamma_2(x)}} |u_n|^{\alpha(x)} dx \right)^{\frac{\alpha(x)-1}{\alpha(x)}} \left(\int_{\Omega_1} \frac{1}{\delta^{\gamma_2(x)}} |u_n - u|^{\alpha(x)} dx \right)^{\frac{1}{\alpha(x)}} \right| \\
&\leq K_2 |u_n|_{a(x), \alpha(x), \gamma_2(x), \Omega_1}^{\alpha^+-1} |u_n - u|_{a(x), \alpha(x), \gamma_2(x), \Omega_1} \\
&\leq K_4 |u_n|_{1, \alpha(x), \gamma(x), \Omega_1}^{\alpha^+-1} |u_n - u|_{1, \alpha(x), \gamma(x), \Omega_1}
\end{aligned}$$

where $\frac{1}{\alpha(x)} + \frac{1}{\alpha'(x)} = 1$ for a.e $x \in \Omega_1$. As $n \rightarrow \infty$, we deduce

$$\int_{\Omega_1} \frac{a(x)}{\delta^{\gamma_2(x)}} |u_n|^{\alpha(x)-2} u_n (u_n - u) dx \rightarrow 0 \quad (3.15)$$

Similarly, we get

$$\int_{\Omega_2} \frac{b(x)}{\delta^{\gamma_4(x)}} |v_n|^{\beta(x)-2} v_n (v_n - v) dx \rightarrow 0 \quad (3.16)$$

By (3.14), we have

$$\langle J'(u_n, v_n), (u_n - u, v_n - v) \rangle \rightarrow 0 \quad (3.17)$$

From (3.14), (3.16) and (3.17), we get

$$\langle \Phi'(u_n, v_n), (u_n - u, v_n - v) \rangle \rightarrow 0 \quad (3.18)$$

Hence,

$$\mathcal{M}_1 \left(\int_{\Omega_1} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla u_n|^p dx \right) \int_{\Omega_1} \frac{1}{\delta^{\gamma(x)}} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) dx \rightarrow 0$$

and

$$\mathcal{M}_2 \left(\int_{\Omega_2} \frac{1}{p(x)} \frac{1}{\delta^{\gamma(x)}} |\nabla v_n|^p dx \right) \int_{\Omega_2} \frac{1}{\delta^{\gamma(x)}} |\nabla v_n|^{p(x)-2} \nabla v_n (\nabla v_n - \nabla v) dx \rightarrow 0$$

It can be inferred from (H_1) and (H_2) that

$$\int_{\Omega_1} \frac{1}{\delta^{\gamma(x)}} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) dx \rightarrow 0$$

$$\int_{\Omega_2} \frac{1}{\delta^{\gamma(x)}} |\nabla v_n|^{p(x)-2} \nabla v_n (\nabla v_n - \nabla v) dx \rightarrow 0$$

Since p -Laplacian is of type (S_+) , we can conclude that functional J meets the Palais-Smale criterion since (u_n, v_n) converges strongly to (u, v) in X . $\square$

3.2.1. *Proof of theorem 1.2.* All of the assumptions of the mountain pass theorem in [1] are met by Lemmas 3.8, 3.9 and 3.11.

Then, using $J(u_1, v_1) = c$, we derive (u_1, v_1) as a nontrivial critical point of the functional J and, consequently, a nontrivial weak solution of problem (1.1).

We demonstrate the existence of a nontrivial weak solution $(u_2, v_2) \neq (u_1, v_1)$. In fact, Lemmas 3.8 and 3.10 give us

$$\inf_{\partial B_\rho(0)} J_\lambda \geq r > 0 \quad \text{and} \quad \inf_{\overline{B_\rho(0)}} J_\lambda < 0$$

Let's select $\epsilon > 0$ so that

$$0 < \epsilon < \inf_{\partial B_\rho(0)} J - \inf_{\overline{B_\rho(0)}} J. \quad (3.19)$$

The functional $J : \overline{B_\rho(0)} \rightarrow \mathbb{R}$ can therefore be subjected to Ekeland's variational principle, which yields $(u_\epsilon, v_\epsilon) \in \overline{B_\rho(0)} \subset X$ such that

$$J(u_\epsilon, v_\epsilon) < \inf_{\overline{B_\rho(0)}} J + \epsilon$$

and

$$J(u_\epsilon, v_\epsilon) < J(u, v) + \epsilon \|(u - u_\epsilon, v - v_\epsilon)\|, \quad u \neq u_\epsilon, \quad v \neq v_\epsilon$$

In light of (3.19), it follows that $J(u_\epsilon, v_\epsilon) < \inf_{\partial B_\rho(0)} J$ and thus, $(u_\epsilon, v_\epsilon) \in B_\rho(0)$. Let's now define $I : \overline{B_\rho(0)} \rightarrow \mathbb{R}$ using the formula $I(u, v) = J(u, v) + \epsilon \|(u - u_\epsilon, v - v_\epsilon)\|$. It is evident that (u_ϵ, v_ϵ) is a minimum point of I , and as a result

$$\frac{I(u_\epsilon + t\zeta, v_\epsilon + t\xi) - I(u_\epsilon, v_\epsilon)}{t} \geq 0,$$

for any $(\zeta, \xi) \in B_\rho(0)$ and $t > 0$ small enough. The statement above indicates that

$$\frac{J(u_\epsilon + t\zeta, v_\epsilon + t\xi) - J(u_\epsilon, v_\epsilon)}{t} + \epsilon \|(\zeta, \xi)\| \geq 0,$$

Assuming $t \rightarrow 0^+$, we conclude that

$$\langle J'(u_\epsilon, v_\epsilon), (u, v) \rangle \geq -\epsilon \|(u, v)\|.$$

Note that $-(u, v)$ also belongs to $B_\rho(0)$. Therefore, substituting $-(u, v)$ for (u, v) , we obtain

$$\langle J'(u_\epsilon, v_\epsilon), (u, v) \rangle \leq \epsilon \|(u, v)\|$$

which implies that $\|J'(u_\epsilon, v_\epsilon)\| \leq \epsilon$.

Consequently, there is a sequence $\{(u_n, v_n)\} \subset B_\rho(0)$ such that

$$J(u_n, v_n) \rightarrow \underline{c} = \inf_{\overline{B_\rho(0)}} J < 0 \quad \text{and} \quad J'(u_n, v_n) \rightarrow 0 \quad \text{in } X^* \quad \text{as } n \rightarrow \infty. \quad (3.20)$$

As $n \rightarrow +\infty$, the sequence $\{(u_n, v_n)\}$ converges strongly to (u_2, v_2) . Furthermore, by (3.20), it is implied that $J(u_2, v_2) = \underline{c}$ and $J'(u_2, v_2) = 0$, since $J \in C^1(X, \mathbb{R})$. Thus, a nontrivial weak solution to issue (1.1) is (u_2, v_2) .

Last but not least, we highlight the fact that $(u_2, v_2) \neq (u_1, v_1)$ since $J(u_1, v_1) = c > 0 > J(u_2, v_2)$.

REFERENCES

1. Ambrosetti, A., & Rabinowitz, P. H. (1973). Dual variational methods in critical points theory and applications. *Journal of Functional Analysis*, 14(4), 349-381. [https://doi.org/10.1016/0022-1236\(73\)90051-7](https://doi.org/10.1016/0022-1236(73)90051-7)
2. Athanasiadis, C., & Stratis, I. G. (1996). On some elliptic transmission problems. *Annales Polonici Mathematici*, 63(2), 137-154. <http://eudml.org/doc/262580>
3. Aydin, I. (2012). Weighted variable Sobolev spaces and capacity. *Journal of Function Spaces and Applications*, 2012, Article 132690. <https://doi.org/10.1155/2012/132690>
4. Bilal, C., & Rabil, A. M. (2006). Nontrivial solution for a nonlocal elliptic transmission problem in variable exponent Sobolev spaces. *Bulletin of the Australian Mathematical Society*, 74(2), 263-277. <https://doi.org/10.1155/2010/385048>
5. Carvalho Pereira, D., & Perla Menzala, G. (1989). Exponential decay of solutions to a coupled system of equations of linear thermoelasticity. *Matemática Aplicada e Computacional*, 8(3), 193-204.
6. Corrêa, F. J. S. A., & Figueiredo, G. M. (2006). On an elliptic equation of p -Kirchhoff type via variational methods. *Bulletin of the Australian Mathematical Society*, 74(2), 263-277. <https://doi.org/10.1017/S000497270003570X>
7. Dai, G., & Hao, R. (2009). Existence of solutions for a $p(x)$ -Kirchhoff-type equation. *Journal of Mathematical Analysis and Applications*, 359(1), 275-284. <https://doi.org/10.1016/j.jmaa.2009.05.031>
8. Dai, G., & Liu, D. (2009). Infinitely many positive solutions for a $p(x)$ -Kirchhoff-type equation. *Journal of Mathematical Analysis and Applications*, 359(2), 704-710. <https://doi.org/10.1016/j.jmaa.2009.06.012>
9. Diening, L., Harjulehto, P., Hästö, P., & Ruicka, M. (2011). *Lebesgue and Sobolev spaces with variable exponents*. Springer-Verlag. <https://doi.org/10.1007/978-3-642-18363-8>
10. Drábek, P., & Hernández, J. (2019). Quasilinear eigenvalue problems with singular weights for the p -Laplacian. *Annali di Matematica Pura ed Applicata*, 198, 1069-1086. <https://doi.org/10.1007/s10231-018-0811-3>
11. El Minsari, A., El Alami, O., & Ourraoui, A. (2025). Some result of the existence and behavior of solutions for p -Laplacian parabolic type equation on networks. *Gulf Journal of Mathematics*, 22(1), 1-15. <https://doi.org/10.56947/gjom.v22i1.3926>
12. Fan, X. L., & Zhao, D. (2001). On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. *Journal of Mathematical Analysis and Applications*, 263(2), 424-446. <https://doi.org/10.1006/jmaa.2000.7617>
13. Fatori, L. H., Lueders, E., & Muñoz Rivera, J. E. (2003). Transmission problem for hyperbolic thermoelastic systems. *Journal of Thermal Stresses*, 26(7), 739-763. <https://doi.org/10.1080/713855994>
14. Heinonen, J., Kilpelainen, T., & Martio, O. (1993). *Nonlinear potential theory of degenerate elliptic equations*. Oxford University Press.
15. Kováčik, O., & Rákosník, J. (1991). On spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. *Czechoslovak Mathematical Journal*, 41(4), 592-618. <https://doi.org/10.21136/CMJ.1991.102493>
16. Marzocchi, A., Muñoz Rivera, J. E., & Naso, M. G. (2003). Transmission problem in thermoelasticity with symmetry. *IMA Journal of Applied Mathematics*, 68(1), 23-46. <https://doi.org/10.1093/imamat/68.1.23>
17. Muñoz Rivera, J. E., & Naso, M. G. (2007). About asymptotic behavior for a transmission problem in hyperbolic thermoelasticity. *Acta Applicandae Mathematicae*, 99(1), 1-27. <https://doi.org/10.1007/s10440-007-9152-8>
18. Muñoz Rivera, J. E., & Oquendo, H. P. (2000). The transmission problem of viscoelastic waves. *Acta Applicandae Mathematicae*, 62(1), 1-21. <https://doi.org/10.1023/A:1006449032100>
19. Muñoz Rivera, J. E., & Oquendo, H. P. (2003). Positive solutions for a nonlinear nonlocal elliptic transmission problem. *Applied Mathematics Letters*, 16(2), 243-248.

20. Zaki, A., Tsouli, N., Baladi, H., & Agalzim, A. (2025). On a bi-nonlocal $p(x)$ -Kirchhoff equation with dependence on the gradient via mountain pass techniques. *Journal of Elliptic and Parabolic Equations*. <https://doi.org/10.1007/s41808-025-00344-5>
21. Zaki, A., Hamydy, A., & Tsouli, N. (2022). Existence of solution for a $p(x)$ -Kirchhoff type with singular weights. *Advanced Studies: Euro-Tbilisi Mathematical Journal, Special Issue*(10), 191-202.
22. Zaki, A., Haddaoui, M., & Bochiha, W. (2025). On an elliptic equation of $p(x)$ -Kirchhoff type with convection term and singular weights. *Discontinuity, Nonlinearity, and Complexity*, 14(2), 269-278. <https://doi.org/10.5890/DNC.2025.06.002>

¹ SMART LABORATORY, THE HIGHER SCHOOL OF MANAGEMENT, TELECOMMUNICATIONS AND COMPUTER SCIENCE, OUJDA, MOROCCO.

Email address: a.zaki@ump.ac.ma

² M2A TEAM, LMPA LABORATORY, FST ERRACHIDIA, MOULAY ISMAIL UNIVERSITY, MEKNES, MOROCCO.

Email address: a.moali@edu.umi.ac.ma

³ MADiNE TEAM, APPLIED MATHEMATICS LABORATORY OF THE ORIENTAL (LMAO), MOHAMMED PREMIER UNIVERSITY, OUJDA, MOROCCO.

Email address: m.haddaoui@ump.ac.ma