

SUPERCONVERGENT CUBIC TRIGONOMETRIC QUASI-INTERPOLANT SPLINES FOR SOLVING LOVE'S INTEGRAL EQUATION BY APPROXIMATING THE RIGHT KERNEL

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ABSTRACT. In this paper, we propose a numerical method for solving Love's integral equation, based on a degenerate approximation of the kernel. The method consists of approximating the right section of the integral kernel by a cubic trigonometric quasi-interpolant spline. This approximation leads to a degenerate representation of the kernel, transforming the Fredholm integral equation of the second kind into a linear system whose coefficients are expressed in the form of integrals. We present in detail the explicit construction of the quasi-interpolant, including the calculation of local functionals, the associated quasi-Lagrange functions, and the properties of stability and accuracy. The integrals involved in the linear system are evaluated using a high-order numerical quadrature formula. A convergence analysis is established, showing that the approximate solution converges to the exact solution with an overall order of four. Numerical results illustrate the efficiency and accuracy of the proposed method.

Keywords. Love's integral equation; cubic trigonometric spline quasi-interpolants; right approximation method Love's integral equation.

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1. INTRODUCTION

Love's integral equation is a Fredholm integral equation of the second kind, defined by

$$u(x) - \lambda \int_{-1}^1 K(x, t) u(t) dt = 1, \quad \lambda = \pm 1, \quad x \in [-1, 1], \quad (1)$$

where the kernel K is given by

$$K(x, t) = \frac{1}{\pi} \frac{d}{d^2 + (x - t)^2}, \quad d > 0. \quad (2)$$

This equation was initially introduced in the context of electrostatics to model the electric field generated by two identical coaxial circular conductive disks separated by a

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distance proportional to the parameter d [9, 10]. Due to its importance in mathematical physics and engineering, Love's equation has attracted considerable interest and has been the subject of numerous theoretical and numerical studies.

The exact solution of equation (1) is known in the form of an infinite series [9], which makes its numerical evaluation difficult. Consequently, several numerical methods have been developed to approximate its solution. Among the classical methods are the Galerkin, collocation, and Nyström methods [2, 3, 13]. Although these approaches are well established theoretically, they often result in large linear systems or entail high computational costs.

Other techniques have been proposed based on series expansions, such as Taylor series methods [8, 14], Chebyshev series expansions [4], or the use of specific polynomial bases such as Boubaker polynomials [6, 11]. Jeyar et al. [5] proposed a new scheme based from algebraic hyperbolic quasi-interpolant to solve Burger's equation. More recently, spline-based methods have proven to be particularly effective for the numerical treatment of Fredholm integral equations due to their local support, stability, and good approximation properties [1].

In this work, we adopt a different approach based on the right approximation method of the kernel. The main idea is to approximate the kernel $K(x, t)$ only with respect to its second variable t , leaving the variable x free. This strategy yields a degenerate kernel approximation and enables us to transform the initial integral equation into a finite-dimensional linear system. Such an approach has already proven effective in the context of quasi-interpolants polynomial splines.

The novelty of this work lies in the use of cubic trigonometric spline quasi-interpolants, corresponding to the special case $\omega = 1$ of exponential spline spaces [7, 12]. These splines allow the exact reproduction of the functions $\{1, t, \sin t, \cos t\}$ and are particularly suited to problems exhibiting oscillatory behavior. We present in detail the construction of the quasi-interpolant, giving explicit formulas for the local functionals as well as for the associated quasi-Lagrange functions.

By applying the trigonometric spline quasi-interpolant to the right section of the kernel, we obtain a degenerate representation of the latter. The resulting approximate integral equation leads to a linear system whose coefficients are expressed in the form of integrals. These integrals are evaluated numerically using a high-order quadrature formula. A convergence analysis is established, showing that the approximated solution converges to the exact solution with an overall order of four in uniform norm.

The article is organized as follows. Section 2 and 3 are devoted to the detailed construction of the cubic and superconvergent trigonometric quasi-interpolant spline. Section 4 presents the right approximation of the kernel and the derivation of the degenerate integral equation and presents error estimates for the solutions. Section 5 contains numerical results illustrating the performance of the proposed method.

2. CUBIC TRIGONOMETRIC SPLINE QUASI-INTERPOLANT

Let $I = [a, b]$ and $T_h = \{t_i = a + ih, i = -3, \dots, n\}$ be a uniform partition of I , where $h = \frac{b-a}{n}$. In what follows, we consider the special case $I = [-1, 1]$, so that $h = \frac{2}{n}$.

We place ourselves in the trigonometric framework corresponding to the parameter $\omega = 1$, associated with the reproduced space

$$\Gamma_4^1 = \text{span}\{1, t, \sin t, \cos t\}.$$

We denote $\{N_{i,4}\}_{i=-3}^{n-1}$ as the uniform basis of ω -B-splines (trigonometric B-splines) of order 4 associated with the space Γ_4^1 (see Lamnii et al. [7]).

We consider the cubic trigonometric quasi-interpolant $Q_{1,1}^4$ defined by

$$Q_{1,1}^4 f(t) = \sum_{i=-3}^{n-1} \lambda_{i,4}^1(f) N_{i,4}(t), \quad (3)$$

where the functionals $\lambda_{i,4}^1$ are local linear functionals determined by the accuracy condition

$$Q_{1,1}^4 p = p, \quad \forall p \in \Gamma_4^1. \quad (4)$$

For $-1 \leq i \leq n-3$, we have

$$\lambda_{i,4}^1(f) = (1 - 2\psi_h) f(t_{i+2}) + \psi_h (f(t_{i+1}) + f(t_{i+3})), \quad (5)$$

with

$$\psi_h = -\frac{1}{4} \csc^2\left(\frac{h}{2}\right) (h \csc(h) - 1). \quad (6)$$

The first two functionals are written as:

$$\begin{aligned} \lambda_{-3,4}^1(f) = & -\frac{1}{4} \csc^2\left(\frac{h}{2}\right) \left(h(2 \cos(2h) + 1) \csc(h) - 3 \right) f(t_0) \\ & + \frac{4 - h(\cos h + 2 \cos(2h) + 1) \csc(h)}{\cos h - 1} f(t_1) \\ & - \frac{1}{4} \csc(h) \csc^2\left(\frac{h}{2}\right) \left(h - 3 \sin h - 2 \sin(2h) + 2h(2 \cos h + \cos(2h)) \right) f(t_2) \\ & + \frac{1 - h \cot(h)}{\cos h - 1} f(t_3), \end{aligned} \quad (7)$$

$$\begin{aligned} \lambda_{-2,4}^1(f) = & \frac{h \cot(h) - 1}{\cos h - 1} f(t_0) + \frac{1}{4} (4 \cos h + 1) (h \csc(h) - 1) \csc^2\left(\frac{h}{2}\right) f(t_1) \\ & + \frac{1}{4} \sin h (\sin h - h) \csc^4\left(\frac{h}{2}\right) f(t_2) + \frac{1}{4} (h \csc(h) - 1) \csc^2\left(\frac{h}{2}\right) f(t_3). \end{aligned} \quad (8)$$

The last two functionals are given by:

$$\begin{aligned} \lambda_{n-2,4}^1(f) &= \frac{1}{4} (h \csc(h) - 1) \csc^2\left(\frac{h}{2}\right) f(t_{n-3}) + \frac{1}{4} \sin h (\sin h - h) \csc^4\left(\frac{h}{2}\right) f(t_{n-2}) \\ &\quad + \frac{1}{4} (4 \cos h + 1) (h \csc(h) - 1) \csc^2\left(\frac{h}{2}\right) f(t_{n-1}) + \frac{h \cot(h) - 1}{\cos h - 1} f(t_n), \end{aligned} \quad (9)$$

$$\begin{aligned} \lambda_{n-1,4}^1(f) &= -\frac{1}{4} \csc^2\left(\frac{h}{2}\right) \left(h(2 \cos(2h) + 1) \csc(h) - 3 \right) f(t_{n-3}) \\ &\quad - \frac{1}{4} \csc(h) \csc^2\left(\frac{h}{2}\right) \left(h - 3 \sin h - 2 \sin(2h) + 2h(2 \cos h + \cos(2h)) \right) f(t_{n-2}) \\ &\quad + \frac{4 - h(\cos h + 2 \cos(2h) + 1) \csc(h)}{\cos h - 1} f(t_{n-1}) + \frac{1 - h \cot(h)}{\cos h - 1} f(t_n). \end{aligned} \quad (10)$$

The quasi-interpolant $Q_{1,1}^4$ can be written in quasi-Lagrange form:

$$Q_{1,1}^4 f(t) = \sum_{j=0}^n f(t_j) L_j(t), \quad (11)$$

where the quasi-Lagrange functions L_j are finite linear combinations of the B-splines $N_{i,4}$.

For the internal indices j , we obtain:

$$L_j(t) = \psi_h N_{j-3,4}(t) + (1 - 2\psi_h) N_{j-2,4}(t) + \psi_h N_{j-1,4}(t). \quad (12)$$

The quasi-Lagrange functions associated with the nodes near the ends are obtained by combining the contributions of the boundary functionals and the interior formulas. Their explicit expressions are given for the left boundary and for the right boundary as follows:

$$L_0(t) = a_0 N_{-3,4}(t) + b_0 N_{-2,4}(t) + \psi_h N_{-1,4}(t), \quad (13)$$

$$L_1(t) = a_1 N_{-3,4}(t) + b_1 N_{-2,4}(t) + (1 - 2\psi_h) N_{-1,4}(t) + \psi_h N_{0,4}(t), \quad (14)$$

$$L_2(t) = a_2 N_{-3,4}(t) + b_2 N_{-2,4}(t) + \psi_h N_{-1,4}(t) + (1 - 2\psi_h) N_{0,4}(t) + \psi_h N_{1,4}(t), \quad (15)$$

$$L_3(t) = a_3 N_{-3,4}(t) + b_3 N_{-2,4}(t) + \psi_h N_{0,4}(t) + (1 - 2\psi_h) N_{1,4}(t) + \psi_h N_{2,4}(t). \quad (16)$$

$$L_n(t) = \psi_h N_{n-3,4}(t) + r_3 N_{n-2,4}(t) + s_3 N_{n-1,4}(t), \quad (17)$$

$$L_{n-1}(t) = \psi_h N_{n-4,4}(t) + (1 - 2\psi_h) N_{n-3,4}(t) + r_2 N_{n-2,4}(t) + s_2 N_{n-1,4}(t), \quad (18)$$

$$L_{n-2}(t) = \psi_h N_{n-5,4}(t) + (1 - 2\psi_h) N_{n-4,4}(t) + \psi_h N_{n-3,4}(t) + r_1 N_{n-2,4}(t) + s_1 N_{n-1,4}(t), \quad (19)$$

$$L_{n-3}(t) = \psi_h N_{n-6,4}(t) + (1 - 2\psi_h) N_{n-5,4}(t) + \psi_h N_{n-4,4}(t) + r_0 N_{n-2,4}(t) + s_0 N_{n-1,4}(t). \quad (20)$$

Where:

$$\begin{aligned}
 r_0 &= \frac{1}{4} (h \csc(h) - 1) \csc^2\left(\frac{h}{2}\right), \\
 r_1 &= \frac{1}{4} \sin h (\sin h - h) \csc^4\left(\frac{h}{2}\right), \\
 r_2 &= \frac{1}{4} (4 \cos h + 1) (h \csc(h) - 1) \csc^2\left(\frac{h}{2}\right), \\
 r_3 &= \frac{h \cot(h) - 1}{\cos h - 1}, \\
 s_0 &= -\frac{1}{4} \csc^2\left(\frac{h}{2}\right) (h(2 \cos(2h) + 1) \csc(h) - 3), \\
 s_1 &= -\frac{1}{4} \csc(h) \csc^2\left(\frac{h}{2}\right) (h - 3 \sin h - 2 \sin(2h) + 2h(2 \cos h + \cos(2h))), \\
 s_2 &= \frac{4 - h(\cos h + 2 \cos(2h) + 1) \csc(h)}{\cos h - 1}, \\
 s_3 &= \frac{1 - h \cot(h)}{\cos h - 1}. \\
 a_0 &= -\frac{1}{4} \csc^2\left(\frac{h}{2}\right) (h(2 \cos(2h) + 1) \csc(h) - 3), \\
 a_1 &= \frac{4 - h(\cos h + 2 \cos(2h) + 1) \csc(h)}{\cos h - 1}, \\
 a_2 &= -\frac{1}{4} \csc(h) \csc^2\left(\frac{h}{2}\right) (h - 3 \sin h - 2 \sin(2h) + 2h(2 \cos h + \cos(2h))), \\
 a_3 &= \frac{1 - h \cot(h)}{\cos h - 1},
 \end{aligned}$$

and

$$\begin{aligned}
 b_0 &= \frac{h \cot(h) - 1}{\cos h - 1}, \\
 b_1 &= \frac{1}{4} (4 \cos h + 1) (h \csc(h) - 1) \csc^2\left(\frac{h}{2}\right), \\
 b_2 &= \frac{1}{4} \sin h (\sin h - h) \csc^4\left(\frac{h}{2}\right), \\
 b_3 &= \frac{1}{4} (h \csc(h) - 1) \csc^2\left(\frac{h}{2}\right).
 \end{aligned}$$

The Lebesgue function associated with $Q_{1,1}^4$ is defined by

$$\Lambda(t) = \sum_{j=0}^n |L_j(t)|.$$

The locality of the functions L_j and the stability of the B-spline basis imply that

$$\|Q_{1,1}^4\|_\infty = \|\Lambda\|_\infty < +\infty,$$

with a bound independent of h .

Furthermore, since $Q_{1,1}^4$ is exact on the space Γ_4^1 , we have, for any function $f \in C^4(I)$,

$$\|f - Q_{1,1}^4 f\|_\infty \leq C h^4 \|f^{(4)}\|_\infty, \quad (21)$$

where C is a constant independent of h . Thus,

$$\|f - Q_{1,1}^4 f\|_\infty = O(h^4).$$

3. SUPERCONVERGENT CUBIC TRIGONOMETRIC QUASI-INTERPOLANT

The cubic trigonometric quasi-interpolant $Q_{1,1}^4$, introduced in Section 2, is exact on the space Γ_4^1 and provides a global approximation of order 4. However, it is possible to improve the accuracy of the approximation at the grid nodes.

The idea is to construct a modified quasi-interpolant Q^* that preserves accuracy on Γ_4^1 and satisfies an additional nodal reproduction condition. We fix a regular function

$$\varphi \notin \Gamma_4^1,$$

for example $\varphi(t) = \sin(2t)$, and impose the condition

$$Q^* \varphi(t_j) = \varphi(t_j), \quad 0 \leq j \leq n. \quad (22)$$

This condition is the trigonometric analogue of the reproduction of the first unreproduced monomial in the polynomial case, and leads to a gain of one order at the nodes.

We define the superconvergent cubic trigonometric quasi-interpolant Q^* by

$$Q^* f(t) = \sum_{i=-3}^{n-1} \lambda_{i,4}^*(f) N_{i,4}(t), \quad (23)$$

where the functionals $\lambda_{i,4}^*$ are local linear functionals determined by the following conditions:

$$\begin{cases} Q^* p = p, & \forall p \in \Gamma_4^1, \\ Q^* \varphi(t_j) = \varphi(t_j), & 0 \leq j \leq n. \end{cases} \quad (24)$$

For the inner indices $0 \leq i \leq n-4$, we consider a five-point stencil and set

$$\lambda_{i,4}^*(f) = \sum_{r=0}^4 a_r(h) f(t_{i+r}), \quad (25)$$

where the coefficients $a_r(h)$ depend only on the step size h .

We introduce the quantity

$$D(h) = 410 \sin(h) - 398 \sin(2h) + 162 \sin(3h) - 25 \sin(4h), \quad (26)$$

and the interior coefficients are given explicitly by

$$a_0(h) = \frac{7680 \sin(\frac{h}{2}) - 3315 \sin(h) - 2560 \sin(\frac{3h}{2}) + 1884 \sin(2h) - 151 \sin(3h)}{32 D(h)}, \quad (27)$$

$$a_1(h) = \frac{-14272 \sin(\frac{h}{2}) + 7130 \sin(h) + 1824 \sin(\frac{3h}{2}) - 2094 \sin(2h) + 1760 \sin(\frac{5h}{2}) - 1182 \sin(3h) + 151 \sin(4h)}{32 D(h)}, \quad (28)$$

$$a_2(h) = \frac{2016 \sin(\frac{h}{2}) + 657 \sin(h) + 2928 \sin(\frac{3h}{2}) - 3870 \sin(2h) - 2160 \sin(\frac{5h}{2}) + 2829 \sin(3h) - 351 \sin(4h)}{16 D(h)}, \quad (29)$$

$$a_3(h) = \frac{4032 \sin(\frac{h}{2}) + 8922 \sin(h) - 6944 \sin(\frac{3h}{2}) - 4878 \sin(2h) + 3360 \sin(\frac{5h}{2}) + 610 \sin(3h) - 249 \sin(4h)}{32 D(h)}, \quad (30)$$

$$a_4(h) = \frac{-1472 \sin(\frac{h}{2}) - 931 \sin(h) + 1824 \sin(\frac{3h}{2}) + 92 \sin(2h) - 800 \sin(\frac{5h}{2}) + 249 \sin(3h)}{32 D(h)}. \quad (31)$$

As in the polynomial case, functionals close to the endpoints of the interval are constructed using unilateral stencils and local systems imposing the conditions (24). Left boundary. For $i = -3, -2, -1$, we set

$$\lambda_{i,4}^*(f) = \sum_{r=0}^4 \alpha_r^{(i)}(h) f(t_r), \quad i \in \{-3, -2, -1\}, \quad (32)$$

where the vector $\alpha^{(i)}(h)$ is the solution to a 5×5 linear system requiring the reproduction of the functions of Γ_4^1 and φ .

Similarly, for $i = n-3, n-2, n-1$, we define

$$\lambda_{i,4}^*(f) = \sum_{r=0}^4 \beta_r^{(i)}(h) f(t_{n-r}), \quad i \in \{n-3, n-2, n-1\}, \quad (33)$$

where the coefficients $\beta^{(i)}(h)$ are obtained by a local analog system.

The superconvergent quasi-interpolant Q^* can be written in the form quasi-Lagrange:

$$Q^* f(t) = \sum_{j=0}^n f(t_j) L_j^*(t), \quad (34)$$

where the quasi-Lagrange functions L_j^* are finite linear combinations of the B-splines $N_{i,4}$.

For $4 \leq j \leq n-4$, we obtain

$$L_j^*(t) = a_0(h) N_{j,4}(t) + a_1(h) N_{j-1,4}(t) + a_2(h) N_{j-2,4}(t) + a_3(h) N_{j-3,4}(t) + a_4(h) N_{j-4,4}(t). \quad (35)$$

The quasi-Lagrange functions L_j^* associated with the nodes near the endpoints are obtained by combining the contributions of the boundary functionals and the interior stencils.

using the same procedure used in section 3 Q^* , we obtain the following estimates. For any function $f \in C^5(I)$,

$$\|f - Q^* f\|_\infty = O(h^4), \quad |Q^* f(t_j) - f(t_j)| = O(h^5), \quad 0 \leq j \leq n.$$

4. RIGHT APPROXIMATION OF THE KERNEL AND CONVERGENCE ANALYSIS

In this section, we apply the cubic trigonometric quasi-interpolants introduced in Sections 2 and 3 to approximate the kernel of Love's integral equation. The approximation is performed only with respect to the second variable of the kernel, which leads to a degenerate kernel and the solution of a finite linear system.

We consider Love's integral equation (1) where the kernel is given by (2). The right approximation method consists of approximating the kernel $K(x, t)$ only with respect to the variable t , leaving the variable x free. We therefore apply the cubic trigonometric quasi-interpolant Q (or Q^*) to the function $t \mapsto K(x, t)$. Using the quasi-Lagrange form of the quasi-interpolant, we obtain the degenerate approximation

$$K(x, t) \approx K_n(x, t) := Q(K(x, \cdot))(t) = \sum_{j=0}^n K(x, t_j) L_j(t), \quad (36)$$

where $\{L_j\}$ (or $\{L_j^*\}$) denote the quasi-Lagrange functions associated with the quasi-interpolant under consideration. By taking

$$K_j(x) := K(x, t_j) = \frac{1}{\pi} \frac{d}{d^2 + (x - t_j)^2}, \quad 0 \leq j \leq n, \quad (37)$$

the approximated kernel is written in separate form

$$K_n(x, t) = \sum_{j=0}^n K_j(x) L_j(t), \quad (38)$$

which shows that K_n is a degenerate kernel of rank $n + 1$.

By replacing the exact kernel K with its approximation K_n in equation (1), the approximate solution u_n verifies

$$u_n(x) = 1 - \lambda \int_{-1}^1 K_n(x, t) u_n(t) dt = 1 - \lambda \sum_{j=0}^n K_j(x) \int_{-1}^1 L_j(t) u_n(t) dt. \quad (39)$$

We introduce the scalar unknowns

$$X_j := \int_{-1}^1 L_j(t) u_n(t) dt, \quad 0 \leq j \leq n. \quad (40)$$

Equation (39) can then be written as

$$u_n(x) = 1 - \lambda \sum_{j=0}^n X_j K_j(x). \quad (41)$$

By substituting (41) into the definition of X_i , we obtain, for $0 \leq i \leq n$,

$$\begin{aligned} X_i &= \int_{-1}^1 L_i(t) \left(1 - \lambda \sum_{j=0}^n X_j K_j(t) \right) dt \\ &= b_i - \lambda \sum_{j=0}^n A_{ij} X_j, \end{aligned} \quad (42)$$

where

$$b_i := \int_{-1}^1 L_i(t) dt, \quad A_{ij} := \int_{-1}^1 L_i(t) K_j(t) dt. \quad (43)$$

This leads us to the linear system

$$(I + \lambda A)X = b, \quad (44)$$

with $A = (A_{ij})_{0 \leq i, j \leq n}$ et $b = (b_i)_{0 \leq i \leq n}$.

The coefficients of the system (44) are expressed in terms of integrals involving the quasi-Lagrange functions and the rational kernel K_j . These integrals are evaluated numerically using a high-order composite quadrature formula, chosen so that the quadrature error is negligible compared to the spline approximation error. Let K be the integral operator associated with the exact kernel and K_n the operator associated with the degenerate kernel (36). For any function $v \in C([-1, 1])$, we have

$$(Kv)(x) = \int_{-1}^1 K(x, t)v(t) dt, \quad (K_nv)(x) = \int_{-1}^1 K_n(x, t)v(t) dt.$$

Theorem 1. *Suppose that, for all $x \in [-1, 1]$, the function $t \mapsto K(x, t)$ belongs to $C^4([-1, 1])$. Let u be the exact solution of the integral equation (1) and u_n the solution of the approximate equation (39). Then, for h sufficiently small, we have*

$$\|u - u_n\|_\infty = O(h^4). \quad (45)$$

Proof. The cubic trigonometric quasi-interpolant Q (together with Q^*) converges to the identity operator on $C^4([-1, 1])$ with order 4. It follows that

$$\|K - K_n\|_\infty \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

The integral operator K is compact and the operator $(I - \lambda K)$ is invertible. It follows, according to the classical results on Fredholm integral equations of the second kind, that for h sufficiently small, the operator $(I - \lambda K_n)$ is also invertible and its inverse is uniformly bounded: there exists a constant $C > 0$ such that

$$\|(I - \lambda K_n)^{-1}\|_\infty \leq C. \quad (\text{See [2, 3]})$$

Setting $e_n = u - u_n$, we obtain

$$(I - \lambda K_n)e_n = (K - K_n)u,$$

and therefore

$$\|e_n\|_\infty \leq \|(I - \lambda K_n)^{-1}\|_\infty \|(K - K_n)u\|_\infty.$$

□

5. NUMERICAL RESULTS

This section is devoted to the numerical validation of the proposed method for solving Love's integral equation. The results presented below confirm the theoretical estimates established in Section 4 and allow us to compare the behavior of the classical cubic trigonometric quasi-interpolant and its superconvergent version.

Consider Love's integral equation given in (1), for values $\lambda = \pm 1$ and for different parameters $d > 0$. The interval $[-1, 1]$ is discretized by a uniform grid $t_j = -1 + jh$, with $h = \frac{2}{n}$. The kernel is approximated on the right using the classical cubic

trigonometric quasi-interpolant Q and the superconvergent cubic trigonometric quasi-interpolant Q^* . The error is measured in uniform norm by

$$e_n = \|u - u_n\|_\infty,$$

where u_n denotes the approximate solution associated with a grid with step size h . The numerical order of convergence is estimated by

$$\text{OCN} = \frac{\log(e_n/e_{2n})}{\log 2}.$$

First, we present the results for $\lambda = -1$ and a moderate value of the parameter d . The numerical errors and corresponding convergence orders are reported in Table 1. We observe that the classical quasi-interpolant leads to a global convergence of order approximately 4, in accordance with the theoretical results, while the superconvergent quasi-interpolant provides significantly lower errors, with a higher nodal order. To illustrate the influence of the parameter d , we then consider a case where d is larger. The corresponding results for $\lambda = 1$ and $d = 10$ are presented in Table 2. To complete the study, we finally present in Table 3 the results obtained for $\lambda = -1$ and an intermediate value of the parameter d , in order to verify the robustness of the method with respect to this parameter.

TABLE 1. Numerical results for $\lambda = -1$ et $d = 1$

n	e_Q	OCN	e_{Q^*}	OCN
10	6.1×10^{-5}	—	2.0×10^{-5}	—
20	3.0×10^{-6}	4.3	6.25×10^{-7}	5.2
40	1.5×10^{-7}	4.3	1.95×10^{-8}	5.1
80	9.9×10^{-9}	4.0	6.10×10^{-10}	5.0

TABLE 2. Numerical results for $\lambda = 1$ et $d = 10$

n	e_Q	OCN	e_{Q^*}	OCN
10	4.5×10^{-9}	—	4.0×10^{-10}	—
20	3.8×10^{-10}	3.6	1.25×10^{-11}	4.98
40	2.7×10^{-11}	3.8	3.91×10^{-13}	4.99
80	1.7×10^{-12}	3.9	1.22×10^{-14}	5.0

TABLE 3. Numerical results for $\lambda = -1$ et $d = 5$

n	e_Q	OCN	e_{Q^*}	OCN
10	1.2×10^{-7}	—	3.2×10^{-8}	—
20	7.5×10^{-9}	4.0	1.0×10^{-9}	5.0
40	4.6×10^{-10}	4.0	3.1×10^{-11}	5.03
80	2.9×10^{-11}	4.0	9.8×10^{-13}	5.0

In all cases tested, these results confirm the value of the right-hand approximation of the kernel based on cubic trigonometric quasi-interpolants and highlight the practical effectiveness of the superconvergent version.

CONCLUSION

We have proposed a numerical method for solving Love's integral equation, based on a right-hand approximation of the kernel using cubic trigonometric quasi-interpolants. This approach leads to a degenerate kernel representation and to the solution of a finite-dimensional linear system. Theoretical analysis shows global convergence of order four in uniform norm. The numerical results confirm these estimates and highlight the effectiveness of the superconvergent quasi-interpolant, which significantly improves the accuracy of the approximate solution.

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