

RELATIVISTIC HEAT EQUATION IN BOUNDED DOMAIN WITH DIRICHLET BOUNDARY CONDITION.

ROGER BASSOLE¹, MOHAMED GAZIBO KARIMOU^{2*} and AROUNA OUEDRAOGO¹

ABSTRACT. We study a one-dimensional relativistic heat equation with Dirichlet boundary conditions. We prove the equivalence between the entropy formulations introduced by Bénilan–Touré [8] and the one introduced by Mazón–Caselles–Moll [7]. The analysis relies on tools from nonlinear semigroup theory and functions of bounded variation. The existence of entropy solutions is obtained via an implicit time discretization scheme (Rothe’s method), combined with compactness arguments. Uniqueness follows from a Kruzhkov-type doubling of variables technique, yielding an L^1 -contraction principle.

Keywords. Relativistic heat equation, Dirichlet boundary conditions, entropy solutions, Rothe’s method, doubling of variables.

2020 Mathematics Subject Classification. 35K65, 35D30, 35K55, 35K59.

1. INTRODUCTION

The relativistic heat equation (RHE) in N spatial dimensions is given by

$$u_t = \vartheta \operatorname{div} \left(\frac{u Du}{\sqrt{u^2 + \left(\frac{\vartheta}{c}\right)^2 |Du|^2}} \right), \quad \text{in } Q = (0, T_{\max}) \times \Omega, \quad (1.1)$$

where ϑ denotes the kinematic viscosity, c is the speed of light, and $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain. This model arises as a modification of the classical heat equation

$$u_t = \vartheta \Delta u, \quad \text{in } Q = (0, T_{\max}) \times \Omega, \quad (1.2)$$

which predicts an infinite propagation speed. Such behavior contradicts the principles of special relativity, where the speed of light is a fundamental upper bound. The classical formulation is based on Fourier’s law, with heat flux

$$F = -c Du.$$

To overcome this inconsistency, several nonlinear models with flux limitation have been proposed. A first approach was introduced by Rosenau [24], who considered diffusion equations with bounded flux. Brenier [10] later proposed

Date: Received: Mar 5, 2026; Revised: Mar 30, 2026; Accepted: Apr 13, 2026.

* Corresponding author

© The Author(s) 2025. This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License. To view a copy of the licence, visit <https://creativecommons.org/licenses/by-nc-nd/4.0/>.

a formulation based on optimal transport theory. Further analytical properties of the relativistic heat equation were studied by Caselles and collaborators [12, 13]. Flux-limited diffusion equations are closely related to nonlinear degenerate parabolic equations. In particular, the relativistic heat equation converges to the classical heat equation as $c \rightarrow \infty$ under suitable assumptions [13]. More generally, such models have been widely studied in [3, 5, 6, 14]. The existence and uniqueness of entropy solutions for equations of the form

$$\frac{\partial u}{\partial t} = \operatorname{div} \mathbf{b}(u, Du) \quad \text{in } Q \quad (1.3)$$

have been developed in [4, 5, 9, 15, 16]. These approaches rely on entropy inequalities and contraction principles and include contributions by Andreianov, B enilan, Ouaro, Gazibo and Tour   [1, 18, 21, 22]. Moreover, entropy solutions and nonlinear evolution problems have been extensively investigated in the literature. In particular, Konat   and Ouaro [20] studied nonlinear multivalued problems with measure data in anisotropic variable exponent spaces. El Haji, El Moumni and Talha [17] established the existence of entropy solutions for nonlinear parabolic equations without the Δ_2 -condition. Related stability results for nonlinear degenerate equations can also be found in Ouaro and Sawadogo [23], where structural stability for $p(u)$ -Laplacian problems with Fourier boundary conditions is analyzed. In this work, we consider the one-dimensional relativistic heat equation with Dirichlet boundary conditions. More precisely, we study

$$\begin{cases} u_t = \mathbf{a}(u, u_x)_x, & \text{in } Q = (0, T_{\max}) \times I, \\ u = g, & \text{on } \Sigma = (0, T_{\max}) \times \partial I, \\ u(0, x) = u_0(x), & \text{in } I, \end{cases} \quad (1.4)$$

where $I \subset \mathbb{R}$ is a bounded interval. The data satisfy $u_0 \in BV(I) \cap L^\infty(I)$ and $g \in BV(\Sigma) \cap L^\infty(\Sigma)$. The flux is defined by

$$\mathbf{a}(u, u_x) = \vartheta \frac{uu_x}{\sqrt{u^2 + \left(\frac{\vartheta}{c}\right)^2 |u_x|^2}},$$

which ensures finite speed of propagation. The existence of entropy solutions is obtained using Rothe's method, based on implicit time discretization and compactness arguments. Uniqueness follows from Kruzhkov's doubling of variables technique, yielding an L^1 -contraction principle. The main contribution of this work is to establish the equivalence between the entropy formulations introduced by B enilan–Tour   [8] and that introduced by Maz  n–Caselles–Moll [7]. The paper is organized as follows. Section 2 contains preliminary results. Section 3 establishes the equivalence of entropy formulations. Section 4 is devoted to existence, and Section 5 addresses uniqueness.

2. PRELIMINARIES

We introduce the main notations used throughout the paper. For a function u , we denote by u_x and u_t its derivatives with respect to the space and time

variables, respectively. The truncation function is defined by

$$T_{a,b}(r) = \max(\min(b, r), a), \quad 0 < a < b,$$

and we set $T_k := T_{-k,k}$. We denote by $BV(I)$ the space of functions of bounded variation on I , and by $TBV(I)$ the class of functions whose truncations belong to $BV(I)$. The space of bounded Radon measures on I is denoted by $\mathcal{M}(I)$. We recall some basic notions that will be used throughout the paper. Since solutions may exhibit discontinuities, we work within the framework of functions of bounded variation.

A point $x_0 \in I$ is said to be a Lebesgue point of u if the local averages of u converge to $u(x_0)$ as the radius tends to zero. Points where one-sided limits exist but differ are called jump points. We denote by $BV(I)$ the space of functions of bounded variation, that is, functions $u \in L^1(I)$ whose distributional derivative Du is a finite Radon measure. This measure admits the decomposition

$$Du = \mathcal{R}_u^{ac} + \mathcal{R}_u^s,$$

where the singular part further splits into jump and Cantor components. We refer to [11, 12, 8] for the detailed properties of traces, truncations, and BV calculus used in this work. Following [11], given suitable functions v and w , one can define a Radon measure (v, Dw) which satisfies a Green-type formula and allows the treatment of nonlinear terms involving measure derivatives. We consider functionals with linear growth, which generate Radon measures admitting a decomposition into absolutely continuous and singular parts, the latter characterized by the recession function. These tools are suitable for the analysis of nonlinear fluxes and will be applied to the relativistic heat equation. We now examine the main properties of the flux function $\mathbf{a}$.

We consider the flux function defined by

$$\mathbf{a}(z, \xi) = \vartheta \frac{|z|\xi}{\sqrt{z^2 + \left(\frac{\vartheta}{c}\right)^2 |\xi|^2}},$$

which can be written as $\mathbf{a}(z, \xi) = \partial_\xi f(z, \xi)$, where

$$f(z, \xi) = \frac{c^2}{\vartheta} z \sqrt{z^2 + \left(\frac{\vartheta}{c}\right)^2 |\xi|^2}.$$

Since f is convex with respect to ξ , the flux $\mathbf{a}$ satisfies a monotonicity property and the inequality

$$(\eta - \xi)\mathbf{a}(z, \xi) \leq f(z, \eta) - f(z, \xi),$$

for all $(z, \xi, \eta) \in \mathbb{R}^3$. We also introduce the function

$$h(z, \xi) = \mathbf{a}(z, \xi) \cdot \xi,$$

which satisfies suitable growth conditions of linear type with respect to ξ . For functions u in $TBV^+(I) \cap L^\infty(I)$, one can define associated Radon measures involving truncations $T(u)$ and test functions $\phi \in \mathcal{C}_c(I)^+$. These constructions

rely on the framework developed in [11, 12]. Finally, for $S \in Lip_0^+$, we define the weighted quantities

$$h_S(u, \mathcal{R}_{T(u)}) = S(u) h(u, \mathcal{R}_{T(u)}), \quad f_S(u, \mathcal{R}_{T(u)}) = S(u) f(u, \mathcal{R}_{T(u)}),$$

which will be used in the entropy formulation.

3. ENTROPY FORMULATION

In this section, we present two notions of entropy solutions associated with problem (1.4). The first one, introduced by Bénilan–Touré [8], is formulated in a functional framework and is well adapted to possibly irregular data. The second notion, proposed by Mazón–Caselles–Moll [7], relies on the structure of the flux and on convexity arguments. Our main goal is to show that these two approaches are equivalent, in the sense that each entropy solution in one framework also satisfies the other formulation.

Definition 3.1. A nonnegative function $u \in L^\infty(Q) \cap BV(Q)$ is said to be an entropy solution in the sense of Bénilan–Touré [8] if the following conditions are satisfied:

- (1) $u(t, \alpha) = g(\alpha)$ for a.e. $t \in (0, T_{\max})$, $\alpha \in \partial I$.
- (2) $u(0, x) = u_0(x)$ in I .
- (3) For every $k \in \mathbb{R}$ and every nonnegative test function $\psi \in \mathcal{D}(Q)$,

$$\begin{aligned} \int_Q \text{sign}_0(u - k) \left[(u - k) \psi_t - \mathbf{a}(u, u_x) \psi_x \right] dx dt \\ \geq \sum_{\alpha \in \partial I} \delta(\alpha) \text{sign}_0(g(\alpha) - k) \langle \mathbf{b}(\cdot, \alpha), \psi(\cdot, \alpha) \rangle. \end{aligned}$$

Here $\delta(\alpha_1) = 1$, $\delta(\alpha_2) = -1$, and $\mathbf{b}$ denotes the boundary trace associated with the flux $\mathbf{a}$.

The stationary problem associated with (1.1) in one-dimensional space is given by

$$\begin{cases} u - \vartheta \left(\frac{uu_x}{\sqrt{u^2 + (\frac{\vartheta}{c})^2 |u_x|^2}} \right)_x = w, & x \in I, \\ u(x) = g(x), & x \in \partial I. \end{cases} \quad (3.1)$$

where $g \in BV(\partial I) \cap L^\infty(\partial I)$

Definition 3.2. A function u is called a weak solution of the stationary problem (3.1) in the sense of Bénilan–Touré [8], if $u \in L^1(Q)$ and satisfies the following inequality

$$\int_0^{T_{\max}} \int_{\mathbb{R}} (u(x_+) - u(x_-)) \left(\vartheta \frac{uu_x}{\sqrt{u^2 + (\frac{\vartheta}{c})^2 |u_x|^2}} - \frac{kD}{\sqrt{k^2 + |D|^2}} \right) dx dt \geq 0,$$

where

$$u(x_+) = \lim_{h \rightarrow 0} \text{ess sup } u(x + h), \quad u(x_-) = \lim_{h \rightarrow 0} \text{ess inf } u(x - h) \text{ and } k, D \in \mathbb{R}. \quad (3.2)$$

An entropy solution of (1.4) in the sense of Mazón-Caselles-Moll [7] is given as following :

Definition 3.3. Assume that $u_0 \in L^\infty(I)^+$. A measurable function $u \in C([0, T_{\max}], L^\infty(I))$ is an entropy solution of (1.4), if $T(u(\cdot)) \in L^1(0, T_{\max}; BV(I))$ and satisfies:

- (a) $u = l$ on $(0, T_{\max}) \times \partial I$,
- (b) $u(0) = u_0$,
- (c) the following inequality holds:

$$\begin{aligned} \int_0^{T_{\max}} \int_I \phi h_S(u, \mathcal{R}_{T(u)}) dt + \int_0^{T_{\max}} \int_I \phi h_T(u, \mathcal{R}_{S(u)}) dt \\ \leq \int_0^{T_{\max}} \int_I \{ J_{TS}(u(t)) \phi'(t) - \mathbf{a}(u, u_x) \phi_x T(u) S(u) \} dx dt, \end{aligned} \quad (3.3)$$

where $(S, T) \in \mathcal{T}$ with $h_T(u, \mathcal{R}_{S(u)}) = T(u)h(u, \mathcal{R}_{S(u)}) = T(u) \cdot \mathcal{R}_{S(u)} \cdot \mathbf{a}(u, \mathcal{R}_{S(u)})$, $TS, J_{TS} \in \text{Lip}_0$, and $\phi \mathcal{D}(Q)$.

To continue our work, we need the results of the following two lemmas.

Lemma 3.4. *if $u \in TVB(I)$ then $f(u) \in BV(I)$ for every $f \in \text{Lip}_0$ such that:*

$$\exists a > 0 \text{ with } f(s) = 0 \text{ for all } 0 \leq s \leq a.$$

Proof. The function $u \in L^1(I)$ is measurable and f is continuous, hence $f(u)$ is measurable. Moreover, since f vanishes near the origin, there exists $a > 0$ such that

$$0 \leq u(x) \leq a \Rightarrow f(u(x)) = 0,$$

and for $u(x) > a$, the Lipschitz continuity of f ensures that $f(u(x))$ remains bounded. Hence $f(u) \in L^1(I)$ and

$$f(u) \in L^1(I). \quad (3.4)$$

The distributional derivative of $f(u)$ admits the decomposition

$$\mathcal{R}_{f(u)} = \mathcal{R}_{f(u)}^{ac} + \mathcal{R}_{f(u)}^j + \mathcal{R}_{f(u)}^c. \quad (3.5)$$

Using the chain rule for BV functions, we obtain

$$|\mathcal{R}_{f(u)}^{ac}|(I) = \int_I |f'(u)u_x| dx \leq \|f'\|_\infty \int_I |u_x| dx \leq C_1 |\mathcal{R}_u^{ac}|(I) < \infty. \quad (3.6)$$

The jump part satisfies

$$\mathcal{R}_{f(u)}^j = (f(u_+) - f(u_-))\nu_u \mathcal{H}^0 \llcorner J_u.$$

Since f is Lipschitz, there exists $C_2 > 0$ such that

$$|\mathcal{R}_{f(u)}^j|(J_u) \leq C_2 |\mathcal{R}_u^j|(J_u) < \infty. \quad (3.7)$$

For the Cantor part, we have

$$|\mathcal{R}_{f(u)}^c|(I) \leq \|f'\|_\infty |\mathcal{R}_u^c|(I) \implies |\mathcal{R}_{f(u)}^c|(I) < \infty. \quad (3.8)$$

Combining (3.6), (3.7), and (3.8), we deduce that

$$|\mathcal{R}_{f(u)}|(I) < \infty. \quad (3.9)$$

Using (3.4) and (3.9), we deduce that $f(u) \in BV(I)$. $\square$

Lemma 3.5. *Let $S \in Lip_0(I)$, $(a, b) \in \mathbb{R}_+^2$, and $T = T_{a,b}^a$. For any $u \in TBV(I)$, one has*

$$S(u)T(u), \quad J_{T'S}(u), \quad J_{TS'}(u) \in BV(I).$$

Moreover,

$$D(S(u)T(u)) = DJ_{T'S}(u) + DJ_{TS'}(u).$$

Proof. Since $u \in TBV(I)$, it follows that $T(u) \in BV(I)$ for any truncation function T . Moreover, since $S \in Lip_0(I)$, we have $S(u) \in BV(I)$ (see [11, 12]). Using the Leibniz rule for BV functions, we obtain

$$D(S(u)T(u)) = S(u)\mathcal{R}_{T(u)} + T(u)\mathcal{R}_{S(u)}. \quad (3.10)$$

Hence,

$$\begin{aligned} |D(S(u)T(u))|(I) &\leq |S(u)\mathcal{R}_{T(u)}|(I) + |T(u)\mathcal{R}_{S(u)}|(I) \\ &\leq \|S(u)\|_\infty |\mathcal{R}_{T(u)}|(I) + \|T(u)\|_\infty |\mathcal{R}_{S(u)}|(I). \end{aligned}$$

Since $S(u)$ and $T(u)$ are bounded and $\mathcal{R}_{T(u)}$, $\mathcal{R}_{S(u)}$ are Radon measures with finite total variation, we deduce that

$$|D(S(u)T(u))|(I) < \infty. \quad (3.11)$$

Together with $S(u)T(u) \in L^1(I)$, this implies

$$S(u)T(u) \in BV(I).$$

Since $T'S, TS' \in Lip_0(I)$, it follows that

$$J_{T'S}(u), J_{TS'}(u) \in BV(I) \quad (3.12)$$

From (3.10), we rewrite

$$D(S(u)T(u)) = S(u)T'(u)D(u) + T(u)S'(u)D(u).$$

By definition of the functionals $J_{T'S}$ and $J_{TS'}$, we have

$$DJ_{T'S}(u) = S(u)T'(u)D(u), \quad DJ_{TS'}(u) = T(u)S'(u)D(u).$$

Therefore,

$$D(S(u)T(u)) = DJ_{T'S}(u) + DJ_{TS'}(u). \quad (3.13)$$

$\square$

We are now in a position to state the main result of this paper, which identifies the equivalence between the two entropy formulations under consideration.

Theorem 3.6. *Let $u_0 \in BV(I) \cap L^\infty(I)$ and $g \in BV(\partial I) \cap L^\infty(\partial I)$. Then a function u is an entropy solution to problem (1.4) in the sense of B enilan–Tour e if and only if it is an entropy solution in the sense of Maz on–Caselles–Moll.*

Proof. The proof is divided into two steps.

Step 1. We prove that the Bénilan–Touré formulation implies the Mazón–Caselles–Moll one.

(1) $\Rightarrow$ (a). From the Bénilan–Touré formulation, we have

$$u(t, \alpha) = g(\alpha) \quad \text{for a.e. } t \in (0, T_{\max}), \alpha \in \partial I.$$

Since $u \in BV(Q)$, its variation with respect to time is controlled in the sense of measures. This implies that the trace $u(t, \alpha)$ is essentially independent of t , hence there exists a function l such that

$$u(t, \alpha) = l(\alpha) \quad \text{for a.e. } t \in (0, T_{\max}).$$

Therefore, the boundary condition (a) holds.

(2) $\Rightarrow$ (b). Since $u \in L^\infty(Q) \cap BV(Q)$, the time variation of u yields

$$\|u(t, \cdot) - u_0\|_{L^\infty(I)} \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

Hence,

$$u(t, \cdot) \rightarrow u_0 \quad \text{in } L^\infty(I), \tag{3.14}$$

which proves the initial condition (b).

(3) $\Rightarrow$ (c). We rely on the entropy inequality satisfied in the sense of Bénilan–Touré. According to [8], one has $u_x, \mathbf{a}(u, u_x) \in L^\infty_{\text{loc}}(Q) \cap BV_{\text{loc}}(Q)$ and

$$\partial_t |u - k| \leq \partial_x (\text{sign}_0(u - k) \mathbf{a}(u, u_x)), \quad \forall k \in \mathbb{R}. \tag{3.15}$$

Multiplying (3.15) by a nonnegative test function $\psi \in \mathcal{D}^+(Q)$ and integrating by parts, we obtain

$$\int_Q (\psi_t |u - k| - \psi_x \text{sign}_0(u - k) \mathbf{a}(u, u_x)) \, dx dt \geq 0. \tag{3.16}$$

Let $S, T \in \tau^+$ be truncation functions. Using standard properties of truncations (see [12]), one can associate to TS a primitive J_{TS} satisfying

$$|u - k| \leq J_{TS}(u) \leq T(u)S(u). \tag{3.17}$$

Choosing ψ suitably and combining (3.16) with (3.17), we obtain

$$0 \leq \int_Q (\psi_t J_{TS}(u) - \psi_x \mathbf{a}(u, u_x) T(u) S(u)) \, dx dt. \tag{3.18}$$

To handle the boundary contributions, we use the trace formulation of Bénilan–Touré together with suitable complementary truncations. Arguing as in [8, 12], one shows that the corresponding boundary terms are nonpositive, leading to

$$\int_Q h_T(u, \mathcal{R}_{S(u)}) \phi \, dx dt + \int_Q h_S(u, \mathcal{R}_{T(u)}) \phi \, dx dt \leq 0, \tag{3.19}$$

for appropriate test functions ϕ . Combining (3.18) and (3.19), we obtain

$$\begin{aligned} \int_Q h_T(u, \mathcal{R}_{S(u)})\phi \, dxdt + \int_Q h_S(u, \mathcal{R}_{T(u)})\phi \, dxdt \\ \leq \int_Q (\psi_t J_{TS}(u) - \psi_x \mathbf{a}(u, u_x) T(u) S(u)) \, dxdt, \end{aligned} \quad (3.20)$$

which is exactly the entropy inequality (c) in the sense of Mazón–Caselles–Moll.

Step 2. We prove the converse implication, namely that the Mazón–Caselles–Moll formulation implies the Bénilan–Touré one.

(a) $\Rightarrow$ (1). Assume that

$$u(t, \alpha) = l(\alpha) \in C([0, T_{\max}]; L^\infty(\partial I)).$$

Then u is uniformly bounded on $(0, T_{\max}) \times \partial I$, hence

$$u \in L^\infty((0, T_{\max}) \times \partial I). \quad (3.21)$$

Moreover, since u is independent of t on the boundary, its variation with respect to time vanishes, and its spatial variation is finite. Therefore, the total variation of u on $\Sigma = (0, T_{\max}) \times \partial I$ is finite, which yields

$$u \in BV(\Sigma). \quad (3.22)$$

Combining (3.21) and (3.22), we obtain that

$$u(t, \alpha) = g(\alpha) \in BV(\Sigma) \cap L^\infty(\Sigma),$$

which proves condition (1).

(b) $\Rightarrow$ (2). Assume that $u(0) = u_0$ with $u_0 \in L^\infty(I)$. Using standard properties of BV functions, the finiteness of the total variation of u_0 implies

$$u_0 \in BV(I). \quad (3.23)$$

Therefore,

$$u_0 \in BV(I) \cap L^\infty(I), \quad (3.24)$$

which proves condition (2).

(c) $\Rightarrow$ (3). We start from the entropy inequality in the sense of Mazón–Caselles–Moll:

$$\begin{aligned} \int_Q h_T(u, \mathcal{R}_{S(u)})\phi \, dxdt + \int_Q h_S(u, \mathcal{R}_{T(u)})\phi \, dxdt \\ \leq \int_Q (\psi_t J_{TS}(u) - \psi_x \mathbf{a}(u, u_x) T(u) S(u)) \, dxdt. \end{aligned} \quad (3.25)$$

We denote by K the left-hand side and by L the right-hand side of (3.25). Choosing $S \equiv 1$ and a suitable truncation function T (as in [12]), we have $\mathcal{R}_{S(u)} = 0$ and $\mathcal{R}_{T(u)} = u_x$. Consequently,

$$K = \int_Q h_S(u, \mathcal{R}_{T(u)})\phi \, dxdt.$$

Using integration by parts and the properties of the flux, one obtains that all interior contributions are nonpositive, while the boundary term can be expressed through the trace of the flux. This yields

$$K \leq \sum_{\alpha \in \partial I} \delta(\alpha) \operatorname{sign}_0(g(\alpha) - k) \langle \mathbf{b}(\cdot, \alpha), \phi(\cdot, \alpha) \rangle \leq 0. \quad (3.26)$$

The right-hand side is estimated by using the intrinsic bounds of J_{TS} along with standard properties of truncation functions, we obtain

$$L \leq \int_Q (|u - k| \psi_t - \mathbf{a}(u, u_x) \psi_x \operatorname{sign}_0(u - k)) \, dx dt. \quad (3.27)$$

Combining (3.25), (3.26), and (3.27), and taking $\phi = \psi$, we deduce

$$\begin{aligned} \int_Q \operatorname{sign}_0(u - k) ((u - k) \psi_t - \mathbf{a}(u, u_x) \psi_x) \, dx dt \\ \geq \sum_{\alpha \in \partial I} \delta(\alpha) \operatorname{sign}_0(g(\alpha) - k) \langle \mathbf{b}(\cdot, \alpha), \psi(\cdot, \alpha) \rangle, \end{aligned}$$

which is precisely the entropy inequality (3) in the sense of B enilan–Tour e. $\square$

4. EXISTENCE OF ENTROPY SOLUTIONS

In this section, mild solutions are considered within the framework developed in [8] and are shown to satisfy the entropy formulation under appropriate structural assumptions, due to the preservation of monotonicity, energy dissipation, and L^1 -contractivity. This approach is used to establish the existence of entropy solutions of (1.4). We impose the following assumptions:

$$(H_1) : \quad \lim_{|\xi| \rightarrow \infty} \inf_{|k| \leq R} |\mathbf{a}(k, \xi)| = +\infty, \quad \forall R > 0,$$

$$(H_2) : \quad \lim_{(k, \xi) \rightarrow (0, 0)} \mathbf{a}(k, \xi) = \mathbf{b}_{\pm} \in \mathbb{R}.$$

We define the operator A on $L^\infty(I) \cap BV(I)$ by

$$Au = -(\mathbf{a}(u, u_x))_x, \text{ with } D(A) = \{u \in L^\infty(I) \cap BV(I) : Au \text{ is well defined}\}.$$

We recall the following result concerning the nonlinear operator A .

Lemma 4.1 (*m-accretivity, [3, 1]*). *Under assumptions (H_1) – (H_2) , the operator A is m-accretive in $L^\infty(I) \cap BV(I)$ and satisfies:*

(i) *A is accretive, i.e.*

$$\int_I (u_1 - u_2)^+ \leq \int_I (u_1 - u_2 + \lambda(Au_1 - Au_2))^+, \quad \forall u_1, u_2 \in D(A), \lambda > 0,$$

(ii) *the range $\mathcal{R}(I + \lambda A)$ is dense in $L^\infty(I) \cap BV(I)$,*

(iii) *$D(A)$ is dense in $L^\infty(I) \cap BV(I)$.*

Within the framework of nonlinear semigroup theory (see [3, 4]), problem (1.4) can be reformulated as

$$\frac{du}{dt} + Au \ni 0 \quad \text{in } [0, T_{\max}], \quad u(0) = u_0. \quad (4.1)$$

Theorem 4.2 (Existence of mild solutions, [3, 4]). *Under assumptions (H_1) – (H_2) , for every $u_0 \in L^\infty(I)$, there exists a unique mild solution*

$$u \in C([0, T_{\max}], L^\infty(I))$$

to problem (1.4). Moreover, this solution satisfies the L^1 -contraction property:

$$\max_{t \in [0, T_{\max}]} \int_I (u_1(t) - u_2(t))^+ \leq \int_I (u_{0,1} - u_{0,2})^+.$$

In particular,

$$u_{0,1} \leq u_{0,2} \Rightarrow u_1(t) \leq u_2(t), \quad \forall t \in [0, T_{\max}].$$

Let $((\mathbf{a}_n), (u_n), (g_n), (u_{0,n}))$ be sequences of functions satisfying (1.4) under the assumptions (H_1) – (H_2) .

Theorem 4.3 (Stability of mild solutions, [3, 4]). *Under the above assumptions, the sequence of mild solutions u_n associated with $(\mathbf{a}_n, g_n, u_{0,n})$ converges in*

$$C([0, T_{\max}], L^\infty(I))$$

to the mild solution u corresponding to $(\mathbf{a}, g, u_0)$.

If $u_0 \in L^1(I) \cap L^\infty(I)$, then the corresponding mild solution u belongs to $L^\infty(Q)$. More precisely, we have:

Proposition 4.4 (Uniform L^∞ estimate, [3, 1]). *Let $u_0 \in L^1(I) \cap L^\infty(I)$ and let u be the mild solution of (1.4). Then $u \in L^\infty(Q)$ and*

$$\|u\|_{L^\infty(Q)} \leq e^{wT_{\max}} \max(M, \|u_0\|_{L^\infty(I)}) + \int_0^{T_{\max}} c_0 e^{w(T_{\max}-t)} dt, \quad (4.2)$$

where $M = \max(|g(\alpha_1)|, |g(\alpha_2)|)$.

We introduce the function

$$V(k, \eta) = \sup_{\xi \in \mathbb{R}} \left(\eta \xi - \int_0^\xi \mathbf{a}(k, r) dr \right), \quad (4.3)$$

which is the convex conjugate of

$$V^*(k, \xi) = \int_0^\xi \mathbf{a}(k, r) dr. \quad (4.4)$$

We assume:

$$(H_3) \quad \begin{cases} \forall R > 0, \exists V_R : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ convex, increasing,} \\ \lambda_R > 0, M_R \in \mathbb{R}, \\ \lambda_R V_R(|\eta|) - M_R \leq V(k, \eta) \leq V_R(|\eta|), \quad \forall (k, \eta) \in [-R, R] \times \mathbb{R}. \end{cases}$$

Condition (H_3) implies (H_1) (see [8]). In particular, it ensures the coercivity of

V_R , namely

$$\lim_{\eta \rightarrow \infty} \frac{V_R(\eta)}{\eta} = +\infty.$$

Moreover, (H_3) yields local boundedness of the flux $\mathbf{a}$ on bounded sets, i.e.

$$|\mathbf{a}(k, \xi)| \leq C_R(1 + V_R^*(|\xi|)), \quad \forall (k, \xi) \in [-R, R] \times [-S, S],$$

for suitable constants $C_R > 0$.

Remark 4.5. Assumptions of Leray–Lions type imply (H_3) (see [8]), namely

$$c_0(k)|\xi|^p - C_0(k) \leq \xi \mathbf{a}(k, \xi) \leq C(k)(1 + |\xi|^p), \quad 1 < p < \infty.$$

Furthermore, (H_3) is satisfied for fluxes of the form

$$\mathbf{a}(k, \xi) = \mathbf{a}_0(\xi + l(k)) - f(k),$$

where f, l are bounded continuous functions and $\mathbf{a}_0$ is monotone, e.g.

$$\mathbf{a}_0(\xi) = |\xi|^{p-2} \xi.$$

We recall the following auxiliary results.

Lemma 4.6 (Lower semicontinuity, [8]). *Let $(I, \mathcal{B}, \mu)$ be a finite measure space and let $V_n : I \times \mathbb{R} \rightarrow \mathbb{R}^+$ be measurable in x and convex in η . Assume that*

$$V_n(x, \eta) \leq V_0(\eta),$$

for some coercive convex function V_0 . Then, for every $\phi \in L^\infty(I)^+$,

$$\int V(x, \mathbf{b}) \phi \, d\mu \leq \liminf_{n \rightarrow \infty} \int V_n(x, \mathbf{b}_n) \phi \, d\mu.$$

Lemma 4.7 (Compactness, [8]). *Let $\phi \in L^\infty(I)^+$ and assume that*

$$\int V_n(x, \mathbf{b}_n) \phi \, d\mu \rightarrow \int V(x, \mathbf{b}) \phi \, d\mu.$$

If $V(x, \cdot)$ is strictly convex and coercive for μ -a.e. x , then

$$\phi \mathbf{b}_n \rightarrow \phi \mathbf{b} \quad \text{in } BV(I) \cap L^\infty(I).$$

We now state the main existence result.

Theorem 4.8. *Under assumptions (H_2) – (H_3) , for any initial data $u_0 \in L^\infty(I)$, there exists a unique entropy solution*

$$u \in C([0, T_{\max}]; L^\infty(I))$$

to problem (1.4). Moreover, u satisfies:

- (1) $u(t, \alpha) = g(\alpha)$ for a.e. $t \in (0, T_{\max})$ and $\alpha \in \partial I$,
- (2) for all $k \in \mathbb{R}$ and all nonnegative test functions $\phi \in \mathcal{D}(Q)$,

$$\begin{aligned} \int_Q \text{sign}_0(u - k) \left((u - k) \phi_t - \mathbf{a}(u, u_x) \phi_x \right) dx dt \\ \geq \sum_{\alpha \in \partial I} \delta(\alpha) \text{sign}_0(g(\alpha) - k) \langle \mathbf{b}(\cdot, \alpha), \phi(\cdot, \alpha) \rangle. \end{aligned}$$

Here, $\delta(\alpha_1) = +1$, $\delta(\alpha_2) = -1$, and $\mathbf{b}$ denotes the boundary trace of the flux $\mathbf{a}$, defined by

$$\langle \mathbf{b}(\cdot, \alpha), \phi(\cdot, \alpha) \rangle = \delta(\alpha) \int_Q (u(t, x) \zeta'(t) \rho(x) - \mathbf{a}(u, u_x) \zeta(t) \rho(x)) dx dt,$$

for test functions of the form $\phi(t, x) = \zeta(t) \rho(x)$.

Proof. Let $R_0 < R$ be given by (4.2). For $\delta > 0$, consider a step function u_δ associated with a partition

$$t_0 = 0 < t_1 < \dots < t_n \leq T,$$

satisfying

$$\frac{u_\delta(t_i) - u_\delta(t_{i-1})}{t_i - t_{i-1}} + \bar{A}u_\delta(t_i) \ni 0. \quad (4.5)$$

Moreover,

$$\|u_\delta\|_{L^\infty(Q)} \leq R_0, \quad \lim_{\delta \rightarrow 0} \|u - u_\delta\|_{L^\infty(0, T; L^1(I))} = 0.$$

Define

$$\underline{u}_\delta(t) = \frac{(t - t_{i-1})u_\delta(t_i) + (t_i - t)u_\delta(t_{i-1})}{t_i - t_{i-1}}, \quad \mathbf{b}_\delta = \mathbf{a}(u_\delta, u_{\delta, x}),$$

so that

$$\underline{u}_\delta \in W^{1, \infty}(0, T; L^\infty(I)), \quad \underline{u}_{\delta, t} + A u_\delta \ni 0.$$

For all $\phi \in \mathcal{D}(Q)^+$, $k \in \mathbb{R}$,

$$\int_Q \text{sign}_0(u_\delta - k) ((u_\delta - k) \phi_t - \mathbf{b}_\delta \phi_x) \geq 0.$$

Using

$$\partial_t |\underline{u}_\delta - k| = \text{sign}_0(\underline{u}_\delta - k) \underline{u}_{\delta, t},$$

one obtains

$$\int_Q \text{sign}_0(u_\delta - k) \mathbf{b}_\delta \phi_x \leq \int_Q |\underline{u}_\delta - k| \phi_t.$$

For the boundary terms,

$$\int_{(0, T_{\max}) \times I} \text{sign}_0(u_\delta - k) (\underline{u}'_\delta(t) \phi + \mathbf{b}_\delta \phi_x) \geq \sum_{\alpha \in \partial I} \delta(\alpha) \text{sign}_0(g(\alpha) - k) \langle \mathbf{b}_\delta(\cdot, \alpha), \phi(\cdot, \alpha) \rangle. \quad (4.6)$$

This yields

$$\begin{aligned} \int_0^{T_{\max}} \int_I \text{sign}_0(u_\delta - k) \mathbf{b}_\delta \phi_x + \int_0^{T_{\max}} \int_I (\underline{u}_\delta - k) \phi_t &\geq \sum_{\alpha \in \partial I} \delta(\alpha) \text{sign}_0(g(\alpha) - k) \times \\ &\times \int_0^{T_{\max}} \mathbf{b}_\delta(t, \alpha) \phi(t, \alpha) dt. \end{aligned}$$

Moreover,

$$u_\delta(t, \alpha) = g(\alpha), \quad \underline{u}_\delta \rightarrow u \text{ in } C(0, T_{\max}; L^\infty(I)), \quad u_\delta \rightarrow u \text{ in } L^\infty(0, T_{\max}; L^1_{\text{loc}}(I)).$$

Using $(\mathbf{b}_\delta u_\delta)_x = \underline{u_{\delta,t}} u_\delta + \mathbf{b}_\delta u_{\delta,x}$, one has $\mathbf{b}_\delta u_{\delta,x} = (\mathbf{b}_\delta u_\delta)_x - \underline{u_{\delta,t}}$. Let

$$J_q(r) = \int_0^r q(s) ds, \quad \underline{J_{q\delta}} \in W^{1,\infty}(0, T; L^\infty(I)),$$

with $\underline{J_{q\delta,t}} \leq \underline{u_{\delta,t}} u_\delta$. Hence $\mathbf{b}_\delta u_{\delta,x} \leq (\mathbf{b}_\delta u_\delta)_x - \underline{J_{q\delta,t}}$. By (H_3) we have

$$\lambda_R V_R(|\mathbf{b}_\delta|) - M_R \leq V_R(0) + \mathbf{b}_\delta u_{\delta,x}, \quad V_R^*(|u_{\delta,x}|) \leq \mathbf{b}_\delta u_{\delta,x}.$$

For $\phi \in \mathcal{D}(I)^+$,

$$\begin{aligned} \int_0^{T_{\max}} \int_I \mathbf{b}_\delta u_{\delta,x} \phi + \int_0^{T_{\max}} \int_I \mathbf{b}_\delta u_\delta \phi_x &\leq \int_I (J_q(u_0) - J_q(u_\delta(T))) \phi - \sum_{\alpha \in \partial I} \delta(\alpha) \times \\ &\quad \times \int_0^{T_{\max}} \mathbf{b}_\delta(t, \alpha) \phi(t, \alpha) dt. \end{aligned} \quad (4.7)$$

Taking $\phi = 1$,

$$\int_0^{T_{\max}} \int_I \mathbf{b}_\delta u_{\delta,x} \leq \int_I (J_q(u_0) - J_q(u_\delta(T))) - \sum_{\alpha \in \partial I} g(\alpha) \int_0^{T_{\max}} \mathbf{b}_\delta(t, \alpha) dt.$$

This implies

$$\int_0^{T_{\max}} \int_I \mathbf{b}_\delta u_{\delta,x} \leq C_0 \left(1 + \int_0^{T_{\max}} \int_I |\mathbf{b}_\delta| \right).$$

Hence $\mathbf{b}_\delta u_{\delta,x}$ is bounded. Consequently,

$$\lambda_R \int_Q V_R(|\mathbf{b}_\delta|) \leq (M_R + V_R(0))|Q| + \int_Q \mathbf{b}_\delta u_{\delta,x}.$$

By Young's inequality, $\int_Q V_R(|\mathbf{b}_\delta|) \leq C$. Moreover,

$$\int_Q V_R^*(|u_{\delta,x}|) \leq \int_Q \mathbf{b}_\delta u_{\delta,x} \leq C_0 \left(1 + \int_Q |\mathbf{b}_\delta| \right).$$

Hence both $V_R(|\mathbf{b}_\delta|)$ and $V_R^*(|u_{\delta,x}|)$ are bounded. Since V_R and V_R^* are coercive, then $(\mathbf{b}_\delta)$, $(u_{\delta,x})$ are relatively weakly compact. Let $\delta_n \rightarrow 0$. For $w \in L_{\text{loc}}^\infty$,

$$\int_Q V(\mathbf{b})w \leq \liminf_{n \rightarrow \infty} \int_Q V(u_{\delta_n}, \mathbf{b}_{\delta_n})w.$$

For $\phi \in \mathcal{D}(Q)^+$, $\xi \in L^\infty(Q)$,

$$V(u_\delta, \mathbf{b}_\delta) \leq V(u_\delta, \xi) + u_{\delta,x}(\mathbf{b}_\delta - \xi).$$

Using

$$u_{\delta,x}(\mathbf{b}_\delta - \xi) \leq ((\mathbf{b}_\delta - \xi)u_\delta)_x - \underline{J_{q\delta,t}},$$

one obtains

$$\limsup_{\delta \rightarrow 0} \int_Q V(u_\delta, \mathbf{b}_\delta) \phi \leq \int_Q \left(V(u, \xi) + J_q(u) \phi_t - u \mathbf{b} \phi_x - u_x \xi \phi \right). \quad (4.8)$$

Combining with the lower semicontinuity inequality,

$$V(u, \mathbf{b}) + J_{q,t}(u) \leq V(u, \xi) + (u \mathbf{b})_x - u_x \xi \quad \text{in } \mathcal{D}'(Q).$$

Moreover, $\underline{u_{\delta,t}} = \mathbf{b}_{\delta,x} \implies u_t = \mathbf{b}_x$ in $\mathcal{D}'(Q)$. For $\theta > 0$, define

$$J_{q_\theta}(t, x) = \frac{1}{\theta} \int_t^{t+\theta} q(u(\tau, x)) d\tau.$$

Then

$$\frac{d}{dt} J_{q_\theta} \geq (u\mathbf{b})_x - u_x \frac{1}{\theta} \int_t^{t+\theta} \mathbf{b}(\tau) d\tau.$$

Passing to the limit $\theta \rightarrow 0$, $\frac{d}{dt} J_q(u) \geq (u\mathbf{b})_x - u_x \mathbf{b}$. Thus,

$$V(u, \mathbf{b}) \leq V(u, \xi) + u_x(\mathbf{b} - \xi), \quad \forall \xi \in L^\infty(Q),$$

Furthermore,

$$\liminf_{\delta \rightarrow 0} \int_Q V(u_\delta, \mathbf{b}_\delta) \phi \leq \limsup_{\delta \rightarrow 0} \int_Q V(u_\delta, \mathbf{b}_\delta) \phi \leq \int_Q (V(u, \xi) + u_x(\mathbf{b} - \xi)) \phi.$$

Applying Lemma 4.7 with $\xi = (\mathbf{b} \wedge n) \vee (-n)$ and passing to the limit as $n \rightarrow \infty$,

$$\int_Q V(u_\delta, \mathbf{b}_\delta) \phi \rightarrow \int_Q V(u, \mathbf{b}) \phi.$$

By strict convexity,

$$\phi \mathbf{b}_\delta \rightarrow \phi \mathbf{b} \quad \text{in } L^\infty(Q),$$

Finally, $\underline{u_{\delta,t}} = \mathbf{b}_{\delta,x}$ yields, for $\phi \in \mathcal{D}(Q)^+$, $\int_Q \underline{u_{\delta,t}} \phi = \int_Q \mathbf{b}_{\delta,x} \phi$, and passing to the limit,

$$\mathbf{b}_\delta(\cdot, \alpha) \rightarrow \mathbf{b}(\cdot, \alpha) \quad \text{in } \mathcal{D}'.$$

□

5. ON THE UNIQUENESS OF ENTROPY SOLUTIONS

The purpose of this section is to analyze the uniqueness of entropy solutions for (1.4). The argument relies on a doubling of variables technique inspired by Kruzhkov.

Theorem 5.1. *Assume that $g \in BV(\partial I) \cap L^\infty(\partial I)$ and let $u_0 \in BV(I) \cap L^\infty(I)$. Then, there exists a unique entropy solution u to (1.4) in Q satisfying*

$$\begin{aligned} & \int_0^{T_{\max}} \int_I \phi h_S(u, \mathcal{R}_T(u)) dx dt + \int_0^{T_{\max}} \int_I \phi h_T(u, \mathcal{R}_S(u)) dx dt \\ & \leq \int_0^{T_{\max}} \int_I \left(J_{TS}(u(t)) \phi'(t) - \mathbf{a}(u, u_x) \phi_x T(u) S(u) \right) dx dt. \end{aligned} \quad (5.1)$$

Furthermore, let u_1 and u_2 be two entropy solutions associated with initial data $u_{1,0}, u_{2,0} \in BV(I) \cap L^\infty(I)$. Then the following L^1 -contraction property holds:

$$\int_I (u_1(t, x) - u_2(t, x))^+ dx \leq \int_I (u_{1,0}(x) - u_{2,0}(x))^+ dx, \quad \forall t \geq 0. \quad (5.2)$$

Proof. Let

$$S_{\delta,k}(r) = T_{\delta}(r-k)^+ = T_{k,k+\delta}(r) - k, \quad S_{\delta}^k(r) = T_{\delta}(r-k)^- = T_{k-\delta,k}(r) + \delta - k.$$

Set

$$T(r) = T_{a,b}(r) - a, \quad b > a > 2\delta > 0, \quad J_{T,\delta,k}^+(r) = \int_k^r T(s)T_{\delta}(s-k)^+ ds,$$

$$J_{T,\delta,k}^-(r) = \int_k^r T(s)T_{\delta}(s-k)^- ds, \quad \delta J_T(r) = \delta \int_k^r T(s) ds.$$

Then

$$J_{TS_{\delta,k}} = J_{T,\delta,k}^+, \quad J_{TS_{\delta}^k} = J_{T,\delta,k}^- + \delta J_T.$$

Let u_1, u_2 be two entropy solutions of (1.4). There exist

$$\mathbf{b}_1 = \mathbf{a}(u_1, v_1), \quad \mathbf{b}_2 = \mathbf{a}(u_2, v_2), \quad \text{with } v_1 = \partial_x u_1, \quad v_2 = \partial_x u_2.$$

For $k_1, k_2 > \delta$, the entropy inequalities yield

$$\begin{aligned} \int_Q \phi \left(h_T(u_1, S'_{\delta,k_1}(u_1)) + h_{S_{\delta,k_1}}(u_1, T'(u_1)) \right) + \int_Q \mathbf{b}_1 \phi_x T(u_1) S_{\delta,k_1}(u_1) \\ - \int_Q J_{T,\delta,k_1}^+(u_1) \phi_t \leq 0, \end{aligned} \quad (5.3)$$

$$\begin{aligned} \int_Q \phi \left(h_T(u_2, S_{\delta}^k(u_2)) + h_{S_{\delta}^k}(u_2, T'(u_2)) \right) + \int_Q \mathbf{b}_2 \phi_x T(u_2) S_{\delta}^k(u_2) \\ - \int_Q (J_{T,\delta,k_2}^-(u_2) + J_T(u_2)) \phi_t \leq 0. \end{aligned} \quad (5.4)$$

Fix $(s, y) \in Q$ and take $k_1 = u_2(s, y)$ in (5.3):

$$\begin{aligned} \int_Q \phi(t, x) \left(h_T(u_1, S'_{\delta,u_2(s,y)}(u_1)) + h_{S_{\delta,u_2(s,y)}}(u_1, T'(u_1)) \right) \\ + \int_Q \mathbf{b}_1 \phi_x T(u_1) S_{\delta,u_2(s,y)}(u_1) - \int_Q J_{T,\delta,u_2(s,y)}^+(u_1) \phi_t \leq 0. \end{aligned} \quad (5.5)$$

Fix $(t, x) \in Q$ and take $k_2 = u_1(t, x)$ in (5.4):

$$\begin{aligned} \int_Q \phi(s, y) \left(h_T(u_2, S_{\delta}^{u_1(t,x)}(u_2)) + h_{S_{\delta}^{u_1(t,x)}}(u_2, T'(u_2)) \right) + \int_Q \mathbf{b}_2 \phi_y T(u_2) S_{\delta}^{u_1(t,x)}(u_2) \\ - \int_Q J_{T,\delta,u_1(t,x)}^-(u_2) \phi_s - \delta \int_Q J_T(u_2) \phi_s \leq 0. \end{aligned} \quad (5.6)$$

Let $\phi^{(m,n,i)}(t, x, s, y) = \rho_m(x-y) \tilde{\rho}_n(t-s) \xi\left(\frac{t+s}{2}\right) \psi^i\left(\frac{x+y}{2}\right)$. Substituting into (5.5):

$$\begin{aligned} \int_Q \phi^{(m,n,i)} \left(h_T(u_1, S'_{\delta,u_2}(u_1)) + h_{S_{\delta,u_2}}(u_1, T'(u_1)) \right) + \int_Q \mathbf{b}_1 \partial_x \phi^{(m,n,i)} T(u_1) S_{\delta,u_2}(u_1) \\ - \int_Q J_{T,\delta,u_2}^+(u_1) \partial_t \phi^{(m,n,i)} \leq 0, \end{aligned} \quad (5.7)$$

and into (5.6):

$$\begin{aligned} \int_Q \phi^{(m,n,i)} \left(h_T(u_2, S_\delta^{u_1}(u_2)) + h_{S_\delta^{u_1}}(u_2, T'(u_2)) \right) + \int_Q \mathbf{b}_2 \partial_y \phi^{(m,n,i)} T(u_2) S_\delta^{u_1}(u_2) \\ - \int_Q J_{T,\delta,u_1}^-(u_2) \partial_s \phi^{(m,n,i)} - \delta \int_Q J_T(u_2) \partial_s \phi^{(m,n,i)} \leq 0. \end{aligned} \quad (5.8)$$

Integrating (5.7) in (s, y) and (5.8) in (t, x) :

$$\begin{aligned} \int_{Q \times Q} \phi^{(m,n,i)} \left(h_T(u_1, D_y S_{\delta,u_2}(u_1)) + h_{S_{\delta,u_2}}(u_1, D_x T(u_1)) \right) \\ + \int_{Q \times Q} \mathbf{b}_1 \partial_x \phi^{(m,n,i)} T(u_1) S_{\delta,u_2}(u_1) - \int_{Q \times Q} J_{T,\delta,u_2}^+(u_1) \partial_t \phi^{(m,n,i)} \leq 0. \end{aligned} \quad (5.9)$$

and

$$\begin{aligned} \int_{Q \times Q} \phi^{(m,n,i)} \left(h_T(u_2, D_y S_\delta^{u_1}(u_2)) + h_{S_\delta^{u_1}}(u_2, D_y T(u_2)) \right) + \int_{Q \times Q} \mathbf{b}_2 \partial_y \phi^{(m,n,i)} T(u_2) S_\delta^{u_1}(u_2) \\ - \int_{Q \times Q} J_{T,\delta,u_1}^-(u_2) \partial_s \phi^{(m,n,i)} - \delta \int_{Q \times Q} J_T(u_2) \partial_s \phi^{(m,n,i)} \leq 0. \end{aligned} \quad (5.10)$$

Summing (5.9) and (5.10), we obtain

$$\begin{aligned} \int_{Q \times Q} \phi^{(m,n,i)} h_T(u_2, D_y S_\delta^{u_1}(u_2)) + \int_{Q \times Q} \left(h_T(u_1, D_x S_{\delta,u_2}(u_1)) + h_{S_\delta^{u_1}}(u_2, D_y T(u_2)) \right) \phi^{(m,n,i)} \\ + \int_{Q \times Q} \phi^{(m,n,i)} h_{S_{\delta,u_2}}(u_1, D_x T(u_1)) - \int_{Q \times Q} \left(J_{T,\delta,u_2}^+(u_1) \partial_t \phi^{(m,n,i)} + J_{T,\delta,u_1}^-(u_2) \partial_s \phi^{(m,n,i)} \right) \\ - \int_{Q \times Q} \delta J_T(u_2) \partial_s \phi^{(m,n,i)} - \int_{Q \times Q} \mathbf{b}_2 \partial_x \phi^{(m,n,i)} T(u_2) S_\delta^{u_1}(u_2) - \int_{Q \times Q} \mathbf{b}_1 \partial_y \phi^{(m,n,i)} T(u_1) S_{\delta,u_2}(u_1) \\ + \int_{Q \times Q} T_\delta^+(u_1 - u_2) (T(u_1) \mathbf{b}_1 - T(u_2) \mathbf{b}_2 + \delta T(u_2) \mathbf{b}_2) (\partial_x \phi^{(m,n,i)} \leq 0. \end{aligned} \quad (5.11)$$

Since $\phi^{(m,n,i)} \geq 0$, we reduce (5.11) to

$$\begin{aligned} - \int_{Q \times Q} \left(J_{T,\delta,u_2}^+(u_1) \partial_t \phi^{(m,n,i)} + J_{T,\delta,u_1}^-(u_2) \partial_s \phi^{(m,n,i)} \right) - \delta \int_{Q \times Q} J_T(u_2) \partial_s \phi^{(m,n,i)} \\ - \int_{Q \times Q} \mathbf{b}_2 \partial_x \phi^{(m,n,i)} T(u_2) S_\delta^{u_1}(u_2) - \int_{Q \times Q} \mathbf{b}_1 \partial_y \phi^{(m,n,i)} T(u_1) S_{\delta,u_2}(u_1) \\ + \int_{Q \times Q} T_\delta^+(u_1 - u_2) (T(u_1) \mathbf{b}_1 - T(u_2) \mathbf{b}_2) (\partial_x \phi^{(m,n,i)} + \partial_y \phi^{(m,n,i)}) \\ + \delta \int_{Q \times Q} T(u_2) \mathbf{b}_2 (\partial_x \phi^{(m,n,i)} + \partial_y \phi^{(m,n,i)}) \leq 0. \end{aligned} \quad (5.12)$$

Denote by A_1 the first two terms and by A_2 the remaining ones in (5.12). From [12], $\frac{1}{\delta}A_2 \geq C + o(\delta)$. Thus, (5.12) yields

$$\begin{aligned} & - \int_{Q \times Q} \left(J_{T,\delta,u_2}^+(u_1) \partial_t \phi^{(m,n,i)} + J_{T,\delta,u_1}^-(u_2) \partial_s \phi^{(m,n,i)} \right) \\ & \quad + \int_{Q \times Q} T_\delta(u_1 - u_2)^+ (T(u_1)\mathbf{b}_1 - T(u_2)\mathbf{b}_2) (\partial_x \phi^{(m,n,i)} + \partial_y \phi^{(m,n,i)}) \\ & \quad + \delta \int_{Q \times Q} T(u_2)\mathbf{b}_2 (\partial_x \phi^{(m,n,i)} + \partial_y \phi^{(m,n,i)}) \leq \delta o(\delta) + \delta \int_{Q \times Q} J_T(u_2) \partial_s \phi^{(m,n,i)}. \end{aligned} \quad (5.13)$$

Dividing by δ and letting $\delta \rightarrow 0$:

$$\begin{aligned} & - \int_{Q \times Q} \left(J_{T,\text{sign},u_2}^+(u_1) \partial_t \phi^{(m,n,i)} + J_{T,\text{sign},u_1}^-(u_2) \partial_s \phi^{(m,n,i)} \right) \\ & \quad + \int_{Q \times Q} \text{sign}_0^+(u_1 - u_2) (T(u_1)\mathbf{b}_1 - T(u_2)\mathbf{b}_2) (\partial_x \phi^{(m,n,i)} + \partial_y \phi^{(m,n,i)}) \\ & \quad + \int_{Q \times Q} T(u_2)\mathbf{b}_2 (\partial_x \phi^{(m,n,i)} + \partial_y \phi^{(m,n,i)}) \leq \int_{Q \times Q} J_T(u_2) \partial_s \phi^{(m,n,i)}. \end{aligned} \quad (5.14)$$

Passing to the limit $m \rightarrow \infty$:

$$\begin{aligned} & - \int_{(0,T_{\max})^2 \times I} \left(J_{T,\text{sign},u_2(s,x)}^+(u_1(t,x)) \partial_t \phi^{(\infty,n,i)} + J_{T,\text{sign},u_1(t,x)}^-(u_2(s,x)) \partial_s \phi^{(\infty,n,i)} \right) \\ & \quad + \int_{(0,T_{\max})^2 \times I} \text{sign}_0^+(u_1 - u_2) (T(u_1)\mathbf{b}_1 - T(u_2)\mathbf{b}_2) \partial_x \phi^{(\infty,n,i)} \\ & \quad + \int_{(0,T_{\max})^2 \times I} T(u_2)\mathbf{b}_2 \partial_x \phi^{(\infty,n,i)} \leq \int_{(0,T_{\max})^2 \times I} J_T(u_2) \partial_s \phi^{(\infty,n,i)}. \end{aligned} \quad (5.15)$$

Here $\phi^{(\infty,n,i)}(t, x, s) = \tilde{\rho}_n(t-s) \xi\left(\frac{t+s}{2}\right) \psi^i(x)$. Substituting $\phi^{(\infty,n,i)}$ into (5.13) and passing to the limit as $i \rightarrow \infty$:

$$\begin{aligned} & - \lim_{i \rightarrow \infty} \int_{(0,T_{\max})^2 \times I} \left(J_{T,\text{sign},u_2}^+(u_1) \partial_t \phi^{(\infty,n,i)} + J_{T,\text{sign},u_1}^-(u_2) \partial_s \phi^{(\infty,n,i)} \right) \\ & \quad + \liminf_{i \rightarrow \infty} \int_{(0,T_{\max})^2 \times I} \left(\text{sign}_0^+(u_1 - u_2) (T(u_1)\mathbf{b}_1 - T(u_2)\mathbf{b}_2) + T(u_2)\mathbf{b}_2 \right) \phi^{(\infty,n)} \partial_x \psi^i \\ & \quad \leq \lim_{i \rightarrow \infty} \int_{(0,T_{\max})^2 \times I} J_T(u_2) \partial_s \phi^{(\infty,n,i)}. \end{aligned} \quad (5.16)$$

Set $B_i = \int_{(0,T_{\max})^2 \times I} \phi^{(\infty,n)} \text{sign}_0^+(u_1 - u_2) T(u_1)\mathbf{b}_1 \partial_x \psi^i$. Using integration by parts,

$$B_i = B_i^{(1)} + B_i^{(2)} + B_i^{(3)}, \quad (5.17)$$

where $B_i^{(1)}, B_i^{(2)}$ are interior terms and $B_i^{(3)}$ is a boundary contribution. Following [12], one has: $B_i^{(1)} + B_i^{(2)} \rightarrow 0$ as $i \rightarrow \infty$, $B_i^{(3)}$ converges to the boundary term

$$- \int_{(0,T_{\max})^2 \times \partial I} [\mathbf{b}_1, \nu] \phi^{(\infty,n)} \text{sign}_0^+(u_1 - u_2) T(u_1).$$

Hence,

$$\liminf_{i \rightarrow \infty} B_i \geq \int_{(0, T_{\max})^2 \times \partial I} [\mathbf{b}_1, \nu] \phi^{(\infty, n)} \text{sign}_0^+(u_1 - u_2) T(u_1). \quad (5.18)$$

Passing to the limit $i \rightarrow \infty$, since $\psi^i \rightarrow 1$, $B_i^{2,2} \rightarrow 0$. For the remaining term, using the monotonicity of q , $\liminf_{i \rightarrow \infty} B_i^{1,2} \geq 0$. Hence,

$$\liminf_{i \rightarrow \infty} B_i \geq \int_{(0, T_{\max})^2 \times \partial I} [\mathbf{b}_1(t), \nu] \phi^{(\infty, n)} \text{sign}_0^+(u_1 - u_2) T(u_1) d\mathcal{H}^0 dt ds. \quad (5.19)$$

Similarly, defining

$$C_i = - \int_{(0, T_{\max})^2 \times I} \phi^{(\infty, n)} \text{sign}_0^+(u_1 - u_2) T(u_2) \mathbf{b}_2 \partial_x \psi^i,$$

we obtain

$$\liminf_{i \rightarrow \infty} C_i \geq \int_{(0, T_{\max})^2 \times \partial I} [\mathbf{b}_2(s), \nu] \phi^{(\infty, n)} \text{sign}_0^+(u_1 - u_2) T(u_2) d\mathcal{H}^0 dt ds. \quad (5.20)$$

Let

$$D_i = \int_{(0, T_{\max})^2 \times I} \phi^{(\infty, n)} T(u_2) \mathbf{b}_2 \partial_x \psi^i,$$

we have

$$\liminf_{i \rightarrow \infty} D_i \geq \int_{(0, T_{\max})^2 \times \partial I} [\mathbf{b}_2(s), \nu] \phi^{(\infty, n)} T(u_2) d\mathcal{H}^0 dt ds. \quad (5.21)$$

Combining (5.16), (5.19), (5.20), and (5.21), we obtain

$$\begin{aligned} & - \int_{(0, T_{\max})^2 \times I} \left(J_{T, \text{sign}, u_2}^+(u_1) \partial_t \phi^{(\infty, n)} + J_{T, \text{sign}, u_1}^-(u_2) \partial_s \phi^{(\infty, n)} \right) \\ & + \int_{(0, T_{\max})^2 \times \partial I} [\mathbf{b}_1(t), \nu] \phi^{(\infty, n)} \text{sign}_0^+(u_1 - u_2) T(u_1) \\ & - \int_{(0, T_{\max})^2 \times \partial I} \left(\text{sign}_0^+(u_1 - u_2) - 1 \right) [\mathbf{b}_2(s), \nu] \phi^{(\infty, n)} T(u_2) \\ & \leq \int_{(0, T_{\max})^2 \times I} J_T(u_2) \partial_s \phi^{(\infty, n)}. \end{aligned} \quad (5.22)$$

Using the boundary conditions $u_1 = u_2 = g$ on ∂I , define

$$E = \int_{(0, T_{\max})^2 \times \partial I} \phi^{(\infty, n)} \text{sign}_0^+(u_1 - u_2) T(u_1) \left([\mathbf{b}_1, \nu] - [\mathbf{b}_2, \nu] \right).$$

From [12], $E \leq 0$. Passing to the limit $a \rightarrow 0$, $b \rightarrow 0$ in $T = T_{a,b} - a$, we obtain

$$\begin{aligned} & - \int_{(0, T_{\max})^2 \times I} (u_1 - u_2)^+ \left(\partial_t \phi^{(\infty, n)} + \partial_s \phi^{(\infty, n)} \right) - \int_{(0, T_{\max})^2 \times \partial I} [\mathbf{b}_2(s), \nu] \phi^{(\infty, n)} \\ & - \int_{(0, T_{\max})^2 \times I} J_T(u_2) \partial_s \phi^{(\infty, n)} \leq 0. \end{aligned} \quad (5.23)$$

By Green's formula,

$$\int_{(0,T_{\max})^2 \times I} J_T(u_2) \partial_s \phi^{(\infty,n)} = - \int_{(0,T_{\max})^2 \times \partial I} [\mathbf{b}_2(s), \nu] \phi^{(\infty,n)}.$$

Hence,

$$- \int_{(0,T_{\max})^2 \times I} (u_1 - u_2)^+ (\partial_t \phi^{(\infty,n)} + \partial_s \phi^{(\infty,n)}) \leq 0. \quad (5.24)$$

Since $\partial_t \phi^{(\infty,\infty)} = \partial_s \phi^{(\infty,\infty)} = \xi'(t)$ on the diagonal $t = s$ and vanish elsewhere, letting $n \rightarrow \infty$ in (5.24) yields

$$- 2 \int_{(0,T_{\max}) \times I} (u_1(t, x) - u_2(t, x))^+ \xi'(t) dx dt \leq 0. \quad (5.25)$$

Hence, in the sense of distributions,

$$\frac{d}{dt} \int_I (u_1(t, x) - u_2(t, x))^+ dx \leq 0.$$

Integrating in time, we obtain

$$\int_I (u_1(t, x) - u_2(t, x))^+ dx - \int_I (u_1(0, x) - u_2(0, x))^+ dx \leq 0, \quad \forall t \geq 0. \quad (5.26)$$

Consequently,

$$\int_I (u_1(t, x) - u_2(t, x))^+ dx \leq \int_I (u_1(0, x) - u_2(0, x))^+ dx, \quad \forall t \geq 0.$$

If $u_1(0, x) = u_2(0, x)$ a.e. in I , then

$$(u_1(t, x) - u_2(t, x))^+ = 0 \implies u_1 = u_2 \quad \text{a.e. in } Q.$$

□

REFERENCES

- [1] Andreianov, B., & Gazibo, M. K. (2013). Entropy formulation of degenerate parabolic equations with zero-flux boundary conditions. *Zeitschrift für Angewandte Mathematik und Physik*, 64, 1471–1491. <https://doi.org/10.1007/s00033-012-0297-6>
- [2] Andreianov, B., & Gazibo, M. K. (2016) Explicit formulation for the Dirichlet problem for parabolic-hyperbolic conservation laws. *Networks and Heterogeneous Media*, 11(2): 203–222. <https://doi.org/10.3934/nhm.2016.11.203>
- [3] Andreu, F., Caselles, V., and Mazón, J. M. (2002). Existence and uniqueness of solutions for a parabolic quasilinear problem for linear growth functionals with L^1 data. *Mathematische Annalen*, 322, 139–206. <https://doi.org/10.1007/s002080100270>
- [4] Andreu, F., Caselles, V., and Mazón, J. M. (2005). The Cauchy problem for a strongly degenerate quasilinear equation. *Journal of the European Mathematical Society*, 7, 361–393. <https://doi.org/10.4171/jems/32>
- [5] Andreu, F., Caselles, V. and Mazón, J. M. (2005). A strongly degenerate quasilinear elliptic equation. *Nonlinear Analysis*, 61(4), 637–669. <https://doi.org/10.1016/j.na.2004.11.020>
- [6] Andreu, F., Caselles, V., Mazón, J. M. and Moll, S. (2006). Finite propagation speed for limited flux diffusion equations. *Archive for Rational Mechanics and Analysis*, 182(2), 269–297. <https://doi.org/10.1007/s00205-006-0428-3>

- [7] Andreu, F., Caselles, V., Mazón, J. M. and Moll, S. (2010). The Dirichlet problem associated with the relativistic heat equation. *Archive for Rational Mechanics and Analysis*, 193(1), 135–199. <https://doi.org/10.1007/s00208-009-0428-3>
- [8] Bénéilan, P. and Touré, H. (1995). Sur l'équation générale $u_t = a(\cdot, u, \phi(\cdot, u)_x) + v$ dans L^1 . In *Evolution Equations*, Lecture Notes in Pure and Applied Mathematics, Vol. 168, pp. 35–62.
- [9] Bertsch, M. and Dal Passo, R. (1992). Hyperbolic phenomena in a strongly degenerate parabolic equation. *Archive for Rational Mechanics and Analysis*, 117(4), 349–387. <https://doi.org/10.1007/bf00376188>
- [10] Brenier, Y. (2003). Extended Monge–Kantorovich theory. In *Lecture Notes in Mathematics*, Vol. 1813, pp. 91–122. https://doi.org/10.1007/978-3-540-44857-0_4
- [11] Carrillo, J. A. (1999). Entropy solutions for nonlinear degenerate problems. *Archive for Rational Mechanics and Analysis*, 147(4), 269–361. <https://doi.org/10.1007/s002050050152>
- [12] Carrillo, J. A., Caselles, V. and Moll, S. (2013). On the relativistic heat equation in one space dimension. *Proceedings of the London Mathematical Society*, 107(6), 1395–1423. <https://doi.org/10.1112/plms/pdt015>
- [13] Caselles, V. (2007). Convergence of the relativistic heat equation to the heat equation as $c \rightarrow \infty$. *Publicacions Matemàtiques*, 51(1), 121–142. https://doi.org/10.5565/publmat_51107_06
- [14] Caselles, V. (2011). On the entropy conditions for some flux-limited diffusion equations. *Journal of Differential Equations*, 250(8), 3311–3348. <https://doi.org/10.1016/j.jde.2011.01.027>
- [15] Chertock, A., Kurganov, A. and Rosenau, P. (2003). Formation of discontinuities in flux-saturated degenerate parabolic equations. *Nonlinearity*, 16(5), 1875–1898. <https://doi.org/10.1088/0951-7715/16/6/301>
- [16] Dal Passo, R. (1993). Uniqueness of the entropy solution of a strongly degenerate parabolic equation. *Communications in Partial Differential Equations*, 18(1-2), 265–279. <https://doi.org/10.1080/03605309308820930>
- [17] El Haji, B., El Moumni, M., Talha, A. (2020). Entropy solutions for nonlinear parabolic equations in Musielak Orlicz spaces without Δ_2 -conditions. *Gulf Journal of Mathematics*, 9(1), 1–26. <https://doi.org/10.56947/gjom.v9i1.448>
- [18] Gazibo, M. K. (2024). Degenerate Parabolic Equation with Zero-flux Boundary Condition and its Approximations. *WSEAS Transactions on Mathematics*, 23, 682–705. <https://doi.org/10.37394/23206.2024.23.71>
- [19] Kruzhkov, S. N. (1970). First-order quasilinear equations with several independent variables. *Mathematics of the USSR-Sbornik*, 10(2), 217–243. <https://doi.org/10.1070/sm1970v010n02abeh002156>
- [20] Konate, I., Ouaro, S. (2018). Problèmes multivaleurs non linéaires avec des données de mesure à exposant variable. *Gulf Journal of Mathematics*, 6(2). <https://doi.org/10.56947/gjom.v6i2.130>
- [21] Ouaro, S. and Soma, S. (2011). Weak and entropy solutions to nonlinear Neumann boundary value problems with variable exponents. *Complex Variables and Elliptic Equations*, 56(7-9), 829–851. <https://doi.org/10.1080/17476933.2010.504840>
- [22] Ouaro S. (2009). Uniqueness of Entropy Solutions of Nonlinear Elliptic-Parabolic-Hyperbolic Problems in One Dimension Space. *Revista Matemática Complutense*, 22(1), 7–36. https://doi.org/10.5209/rev_REMA.2009.v22.n1.16303
- [23] Ouaro, S., Sawadogo, N. (2021). Structural stability for nonlinear $p(u)$ -Laplacian problem. *Gulf Journal of Mathematics*, 11(1), 1–37. <https://doi.org/10.56947/gjom.v11i1.665>
- [24] Rosenau, P. (1992). Tempered diffusion. *Physical Review A*, 46(12), 7371–7374. <https://doi.org/10.1103/physreva.46.r7371>

¹ LABORATORY OF MATHEMATICS, COMPUTER SCIENCE AND APPLICATIONS (LAMIA),
NORBERT ZONGO UNIVERSITY, KOUDOUGOU, BURKINA FASO.

Email address: rogerbassole55@gmail.com

² ÉCOLE NORMALE SUPÉRIEURE. DEPARTEMENT DE MATHÉMATIQUES. UNIVERSITÉ AB-
DOU MOUMOUNI. LABORATOIRE DE MATHÉMATIQUES FONDAMENTALES ET APPLICATIONS
(LMFA) BP: 10963 NIAMEY, NIGER

Email address: mohamed.gazibo@uam.edu.ne

Email address: arounaoued2002@yahoo.fr