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CONFORMAL VECTOR FIELDS ON FINSLER SQUARE METRICS

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Abstract. In this paper, we characterize the conformal vector fields on square metric. Further, we obtain the explicit expressions of conformal vector fields on a square metric of weakly isotropic flag curvature with the condition that $b = \|\beta\|_\alpha$ is constant.

1. Introduction

Conformal vector fields are the fundamental objects in differential geometry and physics. It is known that a vector field $V = V^i \frac{\partial}{\partial x^i}$ on Finsler manifold M is a conformal vector field with conformal factor $c = c(x)$ if and only if $X_V(F^2) = 4cF^2$, where $X_V := V^i \frac{\partial}{\partial x^i} + y^i \frac{\partial V^j}{\partial x^i} \frac{\partial}{\partial y^j}$(see[3],[11]). There is a special and important class of Finsler metrics in Finsler geometry namely (α, β)-metrics, given in the following form $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, are defined by a Riemannian metric α, an 1-form β and a smooth function $\phi(s)$ is a C^∞ positive function on an open interval. The simplest non-Riemannian (α, β)-metrics are Randers metrics $F = \alpha + \beta$, which were firstly studied by Randers (see[8]). Beside Randers metrics, there are other interesting (α, β)-metrics constructed by L. Berwald in 1929, if can be expressed as

$$F = \frac{(\alpha + \beta)^2}{\alpha}, \tag{1.1}$$

where,

$$\alpha = \frac{\sqrt{(1 - |x|^2)|y|^2 + \langle x, y\rangle^2}}{(1 - |x|^2)^2}, \quad \beta = \frac{\langle x, y\rangle}{(1 - |x|^2)^2}$$

An (α, β)-metric in the form (1.1) is called the Square metric(see[9],[10]).
The flag curvature in Finsler geometry is a natural analogue of sectional curvature in Riemannian geometry. For a Finsler manifold (M, F), the flag curvature $K = K(p, y)$ of F is a function of $flag\ P \in T_xM$ and $flagpole\ y \in T_xM$ at x with $y \in P$. We say Finsler matric F is of weakly isotropic flag curvature K if $K = \frac{3\theta}{F} + \sigma$, where $\sigma = \sigma(x)$ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on M. The conformal vector fields on a manifold M are vector fields induced by a local

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one-parameter group of conformal transformations of M. Z. Shen and Q. Xia were studied conformal vector fields on Randers manifolds of weakly isotropic flag curvature (see[8]). X. Cheng, Li Yin and T. Li are completely determined conformal vector fields on Kropina manifolds of weakly isotropic flag curvature (see[3]). In this paper, we give several equivalent characterizations for conformal vector fields on a Finsler space and establish the relationships between the curvatures of a given Finsler metric F. Finally, we determine the conformal vector fields on a Finsler manifold with square metric of weakly isotropic flag curvature.

2. **Prelimnaries**

Definition 2.1. [1][5] A Minkowski norm on vector field V is a continous function is denoted by $F : V \longrightarrow [0, \infty)$, such that it satisfies the following properties:

(1) F is smooth on $V\backslash\{0\}$,
(2) F is 1-homogeneous function,
(3) $\forall\ y = y^i e_i \in V\backslash\{0\}$, the symmetric bilinear form $g_y : V \times V \longrightarrow R$ is defind by

$$(u, v) \longmapsto \frac{1}{2}\frac{\partial^2}{\partial s \partial t}(F^2(y + su + tv))/_{t=s=0}$$

is positive definite.

Definition 2.2. [1][4] By a Finsler space, we mean a triple $F^n = (M, D, L)$, where M denotes n-dimentional differential manifolds, D is an open subset of tangent vector bundle TM endowed with the differentiable structure of the manifold TM and $L : D \longrightarrow R$ is a differentiable mapping having the following properties:

(1) $L(x, y) > 0$, for $(x, y) \in D$,
(2) $L(x, \lambda y) = |\lambda| L(x, y)$, for any $(x, y) \in D$ and $\lambda \in R$, such that $(x, \lambda y) \in D$,
(3) The d-tensor field $g_{ij}(x, y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2(x, y)$, $(x, y) \in D$,
where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$, is non-degenerated on D.
The metric tensor and angular metric tensor of Finsler space are given by
$g_{ij} = \dot{\partial}_i\dot{\partial}_j\frac{L^2}{2}$, $\quad g^{ij} = (g_{ij})^{-1}$ and $\quad h_{ij} = g_{ij} - l_i l_j$.
where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$, $\quad l_i = \dot{\partial}_i L$ and $\quad l^i = y^i/L$.

Let (M, F) be an n-dimentional smooth Finsler manifold. Let $\varphi : M \longrightarrow M$ be a diffeomorphism on M and $\varphi_* : T_xM \longrightarrow T_{\varphi(x)}M$ be the tangent map at the point of x. φ is called a conformal transformation if there is a smooth scalar function $\rho = \rho(x)$ on M, such that

$$F(\varphi(x), \varphi_*(y)) = e^{2\rho(x)}F(x, y), \tag{2.1}$$

where $y \in T_xM$. $\rho = \rho(x)$ is called a conformal factor of φ.

A vector field V on a Finsler manifold (M, F) is called a conformal vector field with a conformal factor $\rho = \rho(x)$ if the one-parameter tranformation group φ_t generated by V is a conformal transformation group, i.e,

$$\varphi_t^* F(x, y) = F(\varphi_t(x), (\varphi_t)_*(y)) = e^{2\rho_t(x)}F(x, y), \quad \forall x \in M, y \in T_xM, \tag{2.2}$$

where $\rho_t(x) := \int_0^t \sigma(\varphi_s(x))ds$. In this case, it is easy to see that $\sigma(x) = \frac{d\rho_t(x)}{dt}|_{t=0}$ and $\rho_0(x) = 0$. Let Φ_t be a lifting of φ_t onto the tangent bundel TM, that is,

$$\Phi_t(x, y) := (\varphi_t(x), (\varphi_t)_*(y)). \tag{2.3}$$

Obviously, $\{\Phi_t\}$ is also a one-parameter transformation group on TM. Then (2.2) is equivalent to the following,

$$\Phi_t^* F = e^{2\rho_t(x)} F. \tag{2.4}$$

A vector field V is called a homothetic vector field on M if ρ is constant and V is called a Killing vector field if $\rho = 0$. Recently, Z.Shen and Q.Xia have obtained the equivalent conditions that V is a conformal vector field on a Finsler manifold (M, F).
We need to use the following Lemma proved by Z. Shen, and Q. L. Xia

Lemma 2.3. [8] *Let (M, F) be an n-dimensional smooth Finsler manifold and V be a vector field on M. Then, in local coordinates, the following conditions are equivalent:*

(1) $V = V^i \frac{\partial}{\partial x^i}$ *is a conformal vector field on (M, F) with a conformal factor $\rho = \rho(x)$;*
(2) $\frac{\partial g_{ij}}{\partial x^p} V^p = g_{pj} \frac{\partial V^p}{\partial x^i} + g_{ip} \frac{\partial V^p}{\partial x^j} + 2C_{ijp} \frac{\partial V^p}{\partial x^q} y^q = 4\rho g_{ij}$;
(3) $V_{i|j} + V_{j|i} + 2C^p_{ij} V_{p|q} y^q = 4\rho g_{ij}$;
(4) $\frac{\partial F}{\partial x^i} V^i + \frac{\partial F}{\partial y^i} \frac{\partial V^i}{\partial x^j} y^j = 2\rho F$,

where g_{ij} and C_{ijp} are the coefficient of the fundamental tensor g and Cartan torsion C respectively, $C^p_{ij} = g^{pq} C_{ijq}$. $V_i = g_{ij} V^j$ and "$|$" denotes the covariant derivative with respect to Chern connection of F.

Now, let $X_V = V^i \frac{\partial}{\partial x^i} + y^j \frac{\partial V^i}{\partial x^j} \frac{\partial}{\partial y^i}$ be a vector field on TM. Actually, X_V is the induced tangent vector field of the one-parameter tranformation group $\{\Phi_t\}$ on TM. From (4) in Lemma 2.3, it is easy to see that a vector field $V = V^i \frac{\partial}{\partial x^k}$ on M is a conformal vector field on Finsler manifold (M, F) with conformal factor $\rho = \rho(x)$ if and only if $X_V(F^2) = 4\rho F^2$.
Let us consider a special class of Finsler metrics: (α, β)-metrics. An (α, β)-metric is defined in the form $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = a_{ij}(x) y^i y^j$ is a Riemannian metric on M, $\beta = b_i(x) y^i$ is a 1-form on M and $\phi(s)$ is a C^∞ function on an open interval $(-b_0, b_0)$ satisfying

$$\phi(s) > 0, \quad (\phi - s\phi') + (b^2 - s^2)\phi'' > 0, \tag{2.5}$$

for $|s| \leq b < b_0$, here b is a length of β with respect to α.

3. Conformal vector field on Square metric

In this section we investigate the explicit expression of conformal vector field on square metric.

Theorem 3.1. *Let $F = \frac{(\alpha+\beta)^2}{\alpha}$ be square metric on a Finsler manifold M. Then a vector field V on (M, F) is a conformal vector field with conformal factor $\rho = \rho(x)$*

if and only if V satisfies the following

$$V_{i;j} + V_{j;i} = 4\rho(1+s)^2 a_{ij}, \tag{3.1}$$

$$V^i b_{j;i} + b^i V_{i;j} = \frac{s(\rho(1+s) - (1-s)\lambda)}{b_j}. \tag{3.2}$$

Proof. Firstly, we prove the necessity. Assume that $V = V^i \frac{\partial}{\partial x^i}$ is a conformal vector field on (M, F) with conformal factor $\rho = \rho(x)$. By (4) in Lemma2.3, Vsatisfies

$$\frac{\partial F}{\partial x^i} V^i + \frac{\partial F}{\partial y^i} \frac{\partial V^i}{\partial x^j} y^j = 2\rho F. \tag{3.3}$$

By (3.3), together with $F = \frac{(\alpha+\beta)^2}{\alpha}$, we have

$$(\alpha^2 - \beta^2) 2V_{i;j} y^i y^j + 4\alpha^2(\beta + \alpha)[V^i b_{j;i} + b^i V_{i;j}] y^j = 4\rho\alpha^2(\alpha^2 + \beta^2 + 2\alpha\beta). \tag{3.4}$$

where ";" denotes the covariant derivative with respect to Levi-Civita connection of α.

Fix an any point $x \in M$ and take a local coordinate system at the point x such that $\alpha = \sqrt{\delta_{ij} y^i y^j}$ and $\beta = by^1$. Write $\bar{\alpha} = \sqrt{\Sigma_{a=2}^n (y^a)^2}$. Then

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad y^a = y^a (2 \le a \le n). \tag{3.5}$$

define a local coordinate transformation $\psi : (s, y^a) \longrightarrow (y^1, y^a)$. In the new coordinate system,

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}} \bar{\alpha}. \tag{3.6}$$

Plugging (3.6) into (3.4) yields

$$(1-s^2) 2V_{i;j} y^i y^j + 4\bar{\alpha}(s+1) \left(\frac{b}{\sqrt{b^2 - s^2}} \right) [V^i b_{j;i} + b^i V_{i;j}] y^j = 4\rho\bar{\alpha}^2 \left(\frac{b^2}{b^2 - s^2} \right) (1 + s^2 + 2s).$$

By (3.5) and the irrationality of $\bar{\alpha}$, the above equation is equivalent to

$$[s(1-s^2)(V_{1;a} + V_{a;1}) + 4(s+1)b(V^i b_{a;i} + b^i V_{i;a})] y^a = 0, \tag{3.7}$$

$$2s^2\bar{\alpha}^2(1-s^2)V_{1;1} + 4\bar{\alpha}^2(s+1)bs(V^i b_{1;i} + b^i V_{i;1}) + 2(b^2 - s^2)V_{a;c} y^a y^c = 4\rho\bar{\alpha}^2 b^2 (1 + s^2 + 2s). \tag{3.8}$$

where $1 \le i \le n$ and $2 \le a, c \le n$. Letting $s \longrightarrow 0^+$(that is, letting $s > 0$ and $s \longrightarrow 0$) in (3.7) and (3.8), we obtain

$$V^i b_{a;i} + b^i V_{i;a} = 0, \tag{3.9}$$

$$2b^2 V_{a;c} y^a y^c = 4\rho\bar{\alpha}^2 b^2 F. \tag{3.10}$$

Plugging (3.9) into (3.7) one obtains

$$V_{1;a} + V_{a;1} = 0. \tag{3.11}$$

plugging (3.10) into (3.8) one obtains

$$s\bar{\alpha}^2(1-s^2)V_{1;1} + 2\bar{\alpha}^2(s+1)b(V^i b_{1;i} + b^i V_{i;1}) = sV_{a;c} y^a y^c. \tag{3.12}$$

Plugging (3.12) into (3.10), we have

$$V^i b_{1;i} + b^i V_{i;1} = \frac{s(2\rho F - (1-s^2)V_{1;1})}{2(s+1)b}. \tag{3.13}$$

From this, we can find

$$V_{1;1} = V_{a;a}. \tag{3.14}$$

Now it is reasonable to let $\lambda(x) := \frac{1}{2n}\Sigma_{i=1}^{n} V_{i;i}$. By (3.14), we have $V_{1;1} = V_{a;a} = 2\lambda$. Then, by (3.9) and (3.13), and (3.14), we can get the following

$$V^i b_{j;i} + b V_{1;j} = \frac{s(\rho(1+s) - (1-s)\lambda)\delta_{1j}}{b}.$$

Furthermore, (3.12) is equivalent to

$$[2\rho F \delta_{ab} - V_{a;b}]y^a y^b = 0,$$

which implies the following

$$V_{a;b} + V_{b;a} = 4\rho F \delta_{ab}. \tag{3.15}$$

By (3.11) and (3.15), we obtain

$$V_{i;j} + V_{j;i} = 4\rho(1+s)^2\delta_{ij}.$$

The above identities hold in the local coordinate system (s, y^a) at x. By coming back to the original local coordinate system (y^i) at x, we get (3.1) and (3.2). Conversely, if (3.1) and (3.2) are satisfied, we have (3.4), which is equivalent to (3.3). Hence from Lemma 2.1, V is conformal vector field on (M, F) with conformal factor $\rho = \rho(x)$. □

4. The conformal vector fields on square metrics of weakly isotropic flag curvature

In this section, we will give the explicit expressions of conformal vector fields on a square matrics (M, F) of weakly isotropic flag curvature via solving the system of PDEs (3.1) and (3.2).

Theorem 4.1. *Let $F = \frac{(\alpha+\beta)^2}{\alpha}$ be a square metric on a Finsler manifold M of dimensional $n \geq 3$. If V is a conformal vector field on M with conformal factor $\rho(x)$, $b = \|\beta\|_\alpha$ is constant and F is of weakly isotropic flag curvature $K_F = \frac{3\theta}{F} + \sigma$. Then in the some local co-ordinate system for the local standard expression of F, V is given by one of the following,*

(1) *$V = 2\epsilon x(1+s)^2 + d$, where ϵ is constant and d is non-zero constant vector in R^n with $|d| = b$.*
(2) *$V = Qx + \mu\langle d, x\rangle x + d$, where μ is positive constant, d is non-zero costant vector in R^n with $|d| = b$ and Q is a skew symmetric matrix with $Qd = 0$ and $Q^T + \mu dd^T = \mu b^2 I$, T denote the transpose of matrix and I is an identity matrix. In this case $\rho = \lambda = 0$ and $K = 0$.*

For the above purpose, we first state the following lemma.

Lemma 4.2. *Let (M,α) be a Riemannian manifold and V be a conformal vector field on (M,α) with conformal factor $\rho(x)$ defined by (3.1). Suppose $\hat{b}(x)$ is a general solution of the equation (4.2) and $T_0(x)$ is a special solution of the equation $T_{i;j}+T_{j;i}=4\rho(1+s)^2a_{ij}$. Then the general solution of (3.1) is given by $V=\hat{b}(x)+T_0(x)$, where we use $\alpha=(a_{ij})$ to raise and lower the indices of V, b and T and "; " is the covariant derivative with respect to the Levi-Civita connection of Riemannian metric α.*

The proof is obvious. Assume that V is a conformal vector field on square metric (M,F). By Theorem 3.1, V must satisfy (3.1), that is, $V_{i;j}+V_{j;i}=4\rho(1+s)^2a_{ij}$, where ";" denotes the covariant derivative with respect to Levi-Civita connection of α. When α is of constant sectional curvature, we have the following lemma.

Lemma 4.3. [3]*Let α be a Riemannian metric of constant curvature $K_\alpha=\mu$ and V be a conformal vector field on Riemannian manifold (M,α) of dimension n satisfying $V_{i;j}+V_{j;i}=4\rho a_{ij}$ for some scalar function $\rho=\rho(x)$. Then ρ satisfies the following equations*

$$\rho_{;i;j}+\mu\rho a_{ij}=0,\quad (n>2)$$

$$\Delta\rho+2\mu\rho=0,\qquad (n=2)$$

where $\rho_{;i;j}$ denote the covariant derivatives of ρ with respect to α and $\Delta\rho=a^{ij}\rho a_{;i;j}$ denote the Laplacian of ρ with respect to α.
Now, we prove the main Theorem (4.1) in this section.

Proof. Let $F=\frac{(\alpha+\beta)^2}{\alpha}$ be a square metric on a manifold M of dimensional $n\geq 3$ and $b=\|\beta\|_\alpha$ be a constant. Assume that F is of weakly isotropic flag curvature $K_F=\frac{3\theta}{F}+\sigma(x)$. Then, by Theorem 4.1 in [10], α is of nonnegative constant sectional curvature μ and β is a Killing 1-form, i.e. $b_{i;j}+b_{j;i}=-4\rho a_{ij}$. Hence, there is a local coordinate system, in which α can be expressed by

$$\alpha=\frac{\sqrt{1+\mu|x|^2-\mu\langle y,x\rangle^2}}{1+\mu|x|^2},\tag{4.1}$$

and $\beta=b_iy^i$ is determined by

$$b_{i;j}+b_{j;i}=0.\tag{4.2}$$

where μ is a positive constant and ";" denotes the covariant derivative with respect to Levi-Civita connection of α. In this case, $K_F=\sigma=\frac{1}{4}\mu b^2\geq 0$ and $\theta=0$, and we can get the general solution of (4.2) given by

$$b_i=\frac{q_{ij}x^j+d^i}{1+\mu|x|^2},\tag{4.3}$$

where $Q=(q_{ij})$ is a constant skew symmetric matrix and $d=(d^i)$ is a constant column vector in R^n. Denote $\xi=\xi(x):=\sqrt{1+\mu|x|^2}$. By we have

$$a_{ij}=\xi^{-2}\delta_{ij}-\mu\xi^{-4}x^ix^j,\qquad a^{ij}=\xi^2(\delta^{ij}+\mu x^ix^j)\tag{4.4}$$

and the connection coefficents Γ_{ij}^k of α are given by

$$\Gamma_{ij}^k = -\frac{\mu(x^i\delta_j^k + x^j\delta_i^k)}{1+|x|^2}. \tag{4.5}$$

Furthermore, by (4.3) and (4.5), we get

$$b_{i;j} = \xi^{-2}q_{ij} + \mu\xi^{-4}(x^i q_{jk}x^k - x^j q_{ik}x^k + d^j x^i - d^i x^j), \tag{4.6}$$

where we use δ_{ij} to raise and to lower the indices of the vectors x and d in R^n. Further, by a direct computation [3], we know that

$$b^2 = a^{ij}b_i b_j = \frac{\langle Qx+d, Qx+d\rangle + \mu\langle d,x\rangle^2}{1+\mu|x|^2},$$

which is equivalent to

$$b^2(1+\mu|x|^2) = \langle Qx, Qx\rangle + 2\langle Qx, d\rangle + \mu\langle d,d\rangle^2 + \langle d,d\rangle. \tag{4.7}$$

Replacing x with $-x$ in (4.7) and adding the obtained equation to (4.7) yield

$$\langle Qx, d\rangle = 0, \tag{4.8}$$

$$b^2(1+\mu|x|^2) = \langle Qx, Qx\rangle + \mu\langle d,d\rangle^2 + \langle d,d\rangle. \tag{4.9}$$

Since the above identities hold for any $x \in R^n$, we can get $|d|^2 = \langle d,d\rangle = b^2$ by (4.9). Further, letting $x = d$ in (4.9) yields $Qd = 0$. Now we can rewrite (4.9) as

$$\mu b^2(x^T Ix) = x^T(Q^TQ + \mu dd^T)x,$$

which implies the following

$$Q^TQ + \mu dd^T = \mu b^2 I. \tag{4.10}$$

Here T denotes a transpose of a matrix and I is an identity matrix. By $|d|^2 = \langle d,d\rangle = b^2$, we know that d is a nonzero constant vector.

On the other hand, by the assumption that V is conformal vector field on square matric (M,F), V satisfies (3.1). Note that α is of nonnegative constant sectional curvature μ. By Lemma 4.2, we know that the scalar function ρ satisfies $\rho_{;i;j} + \mu\rho a_{ij} = 0$, and we can get ρ as follows

$$\rho = \frac{\epsilon + \langle a,x\rangle}{\sqrt{1+\mu|x|^2}}, \tag{4.11}$$

where ϵ is a constant and a is a constant column vector in R^n.

In the following, we construct a special solution of equation $T_{i;j} + T_{j;i} = 4\rho(1+s)^2 a_{ij}$. Let

$$T_i = 2\xi^{-2}\rho x^i(1+s)^2 - 2\xi^{-2}(1+\xi)^{-1}|x|^2 a^i.$$

By direct computation, we get

$$T_{i;j} = 2\rho(1+s)^2 a_{ij} + 2\xi^{-4}(a^j x^i - a^i x^j). \tag{4.12}$$

Obviously, T is a special solution of $T_{i;j} + T_{j;i} = 4\rho(1+s)^2 a_{ij}$.

As we mentioned before, (b_i) given in (4.3) is the general solution of (4.2). From Lemma 4.2, $(V_i) = (b_i) + (T_i)$ is the general solution of the equation (3.1). Further, it is easy to get

$$V^i = 2(\epsilon\xi + \langle a,x\rangle)x^i(1+s)^2 - 2(1+\xi)^{-1}|x|^2 a^i + (q_k^i x^k + \mu\langle d,x\rangle x^i + d^i). \tag{4.13}$$

From (4.3) and (4.4), we obtained

$$b^i = a^{ij}b_j = q^i_k x^k + \mu\langle d, x\rangle x^i + d^i, \tag{4.14}$$

where we use δ_{ij} to raise and to lower the indices of the vectors x, d in R^2 and $q^i_k = \delta^{ij}q^{jk}$. Obviously,

$$T^i = 2[\epsilon\xi + \langle a, x\rangle]x^i(1+s)^2 - 2(1+\xi)^{-1}|x|^2a^i. \tag{4.15}$$

In order to determine ϵ, vector a and matrix Q in (4.13), substituting $V_i = b_i + T_i$ and $V^i = b^i + T^i$ into (3.2) yield

$$T^i b_{j;i} + b^i T_{i;j} = \frac{s(\rho(1+s) - (1-s)\lambda)}{b_j}. \tag{4.16}$$

Substituting (4.3), (4.6), (4.12), (4.14) and (4.15) into (4.16), we obtain

$$\begin{aligned} &2(1+\xi)\langle a, Qx + d\rangle x(1+s)^2 + 2\epsilon d\xi^2(1+\xi)(1+s)^2 - 2\xi|x|^2Qa \\ &+ 2(1+\xi)x\epsilon\mu\langle d, x\rangle(1+s)^2 + 2\xi(1+\xi)\langle a, x\rangle d(1+s)^2 - 2\xi(1+\xi)\langle d, x\rangle a(1+s)^2 \\ &- \frac{[s(1+s)(\epsilon\xi + \langle a, x\rangle) - \lambda s(1-s)]\xi^5(1+\xi)}{Qx + d} = 0. \end{aligned} \tag{4.17}$$

Case I: $\mu = 0$. In this case, $\xi = 1$. By (4.7), we have $Q = 0$. Then (4.17) is equivalent to

$$\begin{aligned} &[2(\epsilon + \langle a, x\rangle)(1+s)^2]d + 2\langle a, d\rangle x(1+s)^2 - 2a\langle d, x\rangle(1+s)^2 \\ &- \frac{[s(1+s)(\epsilon + \langle a, x\rangle) - \lambda s(1-s)]}{d} = 0. \end{aligned} \tag{4.18}$$

Multiplying d^2 and d^T on both sides of (4.18), we have

$$2(\epsilon + \langle a, x\rangle)d^2(1+s)^2 - (\epsilon + \langle a, x\rangle)s(1+s) + \lambda s(1-s) = 0.$$

$$\lambda = \frac{(\epsilon + \langle a, x\rangle)s(1+s) - 2(\epsilon + \langle a, x\rangle)d^2(1+s)^2}{s(1-s)}. \tag{4.19}$$

Plugging (4.19) into (4.18) yields

$$[a\langle d, x\rangle - x\langle d, a\rangle]2(1+s)^2 = 0. \tag{4.20}$$

Since the above identity holds for all $x \in R^n$, we assert that a =0. By (4.13), we get

$$V = 2\epsilon x(1+s)^2 + d.$$

In this case, by (4.11) and (4.19), $\lambda = \rho = \epsilon$. Further, we know that $\alpha = |y|$ and β is a constant 1-form.

Case II: $\mu > 0$. In this case, $\xi = \sqrt{1 + \mu|x|^2}$ is an irrational function in x. Then (4.17) can be written as in the following form.

$$A(x)\xi^6 + B(x)\xi^5 + C(x)\xi^3 + D(x)\xi^2 + E(x)\xi + F(x) = 0. \tag{4.21}$$

where,

$$\begin{aligned}
A(x) &= -\frac{[s(1+s)(\epsilon\xi+\langle a,x\rangle)-\lambda s(1-s)]}{Qx+d},\\
B(x) &= -\frac{[s(1+s)(\epsilon\xi+\langle a,x\rangle)-\lambda s(1-s)]}{Qx+d},\\
C(x) &= 2\epsilon d(1+s)^2,\\
D(x) &= 2\epsilon d(1+s)^2+2\langle a,x\rangle d(1+s)^2-2\langle d,x\rangle a(1+s)^2,\\
E(x) &= 2\langle a,Qx+d\rangle x(1+s)^2-2|x|^2Qa+2x\epsilon\mu\langle d,x\rangle(1+s)^2+2\langle a,x\rangle d(1+s)^2-2\langle d,x\rangle a(1+s)^2,\\
F(x) &= 2\langle a,Qx+d\rangle x(1+s)^2+2x\epsilon\mu\langle d,x\rangle(1+s)^2.
\end{aligned}$$

Then (4.21) is equivalent to

$$A(x)\xi^6+C(x)\xi^2+E(x)=0.$$

$$B(x)\xi^4+D(x)\xi^2+F(x)=0.$$

which are equivalent to the following equations, respectively,

$$\begin{aligned}
&-\frac{[s(1+s)(\epsilon\xi+\langle a,x\rangle)-\lambda s(1-s)]}{Qx+d}\xi^6+2\epsilon d(1+s)^2\xi^2+2\langle a,Qx+d\rangle x(1+s)^2\\
&-2|x|^2Qa+2x\epsilon\mu\langle d,x\rangle(1+s)^2+2\langle a,x\rangle d(1+s)^2-2\langle d,x\rangle a(1+s)^2=0.
\end{aligned}\tag{4.22}$$

$$\begin{aligned}
&-\frac{[s(1+s)(\epsilon\xi+\langle a,x\rangle)-\lambda s(1-s)]}{Qx+d}\xi^6+2\epsilon d(1+s)^2\xi^2+2\langle a,x\rangle d(1+s)^2\xi^2\\
&-2\langle d,x\rangle a(1+s)^2\xi^2+2\langle a,Qx+d\rangle x(1+s)^2+2x\epsilon\mu\langle d,x\rangle(1+s)^2=0.
\end{aligned}\tag{4.23}$$

Subtracting (4.22) from (4.23), we have

$$\begin{aligned}
&\left[-\xi^4\frac{[s(1+s)(\epsilon\xi+\langle a,x\rangle)-\lambda s(1-s)]}{Qx+d}-2\langle a,x\rangle d(1+s)^2+2\langle d,x\rangle a(1+s)^2\right](\xi^2-1)\\
&-2|x|^2Qa=0.
\end{aligned}\tag{4.24}$$

Substituting $\xi^2-1=\mu|x|^2$ into (4.22) yields

$$\begin{aligned}
&\mu|x|^2\left[-\xi^4\frac{[s(1+s)(\epsilon\xi+\langle a,x\rangle)-\lambda s(1-s)]}{Qx+d}-2\langle a,x\rangle d(1+s)^2+2\langle d,x\rangle a(1+s)^2\right]\\
&-2|x|^2Qa=0,
\end{aligned}$$

that is,

$$\mu\left[-\xi^4\frac{[s(1+s)(\epsilon\xi+\langle a,x\rangle)-\lambda s(1-s)]}{Qx+d}-2\langle a,x\rangle d(1+s)^2+2\langle d,x\rangle a(1+s)^2\right]-2Qa=0.\tag{4.25}$$

Multiplying x^T on both sides of (4.25) Yields

$$\mu\left[-\xi^4\frac{[s(1+s)(\epsilon\xi+\langle a,x\rangle)-\lambda s(1-s)]}{Qx+d}\right]x^T-2x^TQa=0.\tag{4.26}$$

Letting $x = d$ in (4.26) and by $Qd = 0$, we have

$$\mu\left[-\xi^4\frac{[s(1+s)(\epsilon\xi+\langle a,d\rangle)-\lambda s(1-s)]}{d}\right]d^T = 0.$$

Letting $x = d$ in (4.25) yields

$$2\mu[a|d|^2 - \langle a,d\rangle d](1+s)^2 - 2Qa = 0. \tag{4.27}$$

From (4.27), we can assert that $a = 0$. In fact, if $a \neq 0$, by multiplying a^T on both sides of (4.27), we have the following

$$2\mu[\langle a,d\rangle^2 - |a|^2|d|^2](1+s)^2 = 0.$$

which means that $a = \bar{\rho}d$ by $\mu > 0$ and Cauchy-Schwarz inequality. Substituting it into (4.25) and by $Qd = 0$, we get

$$\mu\left[-\xi^4\frac{[s(1+s)(\epsilon\xi+\langle a,x\rangle)-\lambda s(1-s)]}{Qx+d}\right] = 0. \tag{4.28}$$

We claim that $Qx+d \neq 0$ for all $x \in R^n$. Suppose that there exists some $x_0 \in R^n$, such that $Qx_0 + d = 0$. By (4.10), $Q^TQ + \mu dd^T = \mu b^2 I$, we can get

$$Q^TQx_0 + \mu dd^Tx_0 = \mu b^2 x_0,$$

that is,

$$Q^T(Qx_0) + \mu dd^Tx_0 = \mu b^2 x_0. \tag{4.29}$$

Because $Q^T(Qx_0) = Q^T(-d) = (-Q)(-d) = Qd = 0$, (4.29) is equivalent to $d\langle d, x_0\rangle = b^2x_0$.
(1)If $\langle d, x_0\rangle = 0$, then $x_0 = 0$, which means that $d = -Qx_0 = 0$. This a contradiction because d is a nonzero constant vector.
(2)If $\langle d, x_0\rangle \neq 0$, we have $x_0 = \rho_1 d$ and $\rho_1 \neq 0$. By $d = -Qx_0$ and $x_0^TQx_0 = 0$, we have $x_0^Td = 0$. From $x_0 = \rho_1 d$, we get $d^Td = 0$. This contradicts the fact that d is a nonzero constant vector.
Hence, by (4.28), we have

$$\mu\left[-\xi^4\frac{[s(1+s)(\epsilon\xi+\langle a,x\rangle)-\lambda s(1-s)]}{Qx+d}\right] = 0.$$

By $Qd = 0$ and $a = \bar{\rho}d$, we have $Qa = 0$. Plugging $a = \bar{\rho}d$, $Qa = 0$, $\rho(x) = 0$ into (4.22), one obtains

$$[\epsilon d\xi^2 + x(\langle a,d\rangle + \mu\epsilon\langle d,x\rangle)](1+s)^2 = 0. \tag{4.30}$$

Since the above identity holds for all $x \in R^n$ and $d \neq 0$, $\xi \neq 0$, (4.30) means $\epsilon = 0$ and $\langle a,d\rangle = 0$. By $a = \bar{\rho}d$, we have $\bar{\rho}|d|^2 = 0$. Hence, $a = 0$. This contradicts the assumption that $a \neq 0$.
Now, we can get $\mu\left[-\xi^4\frac{[s(1+s)(\epsilon\xi+\langle a,x\rangle)-\lambda s(1-s)]}{Qx+d}\right] = 0$ from (4.25) because $d+Qx \neq 0$ for all $x \in R^n$. Further, letting $x = 0$ in (4.22) and by $\rho = 0$, we get $\epsilon = 0$. From $a = 0$ and $\epsilon = 0$, we have $\lambda = 0$ by (4.11). By (4.13), the general solution of (3.1) is

$$V = Qx + \mu\langle d,x\rangle x + d.$$

In this case, $\lambda = \rho = 0$, $K = 0$ and Killing vector field. This completes the proof. □

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