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# FIXED-POINT THEOREMS INVOLVING CONVEX TYPES IN QUASI $M$-METRIC SPACES

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Abstract. Some fixed point theorems of continuous functions involving convex contraction of order $r$ and convex type 2 in quasi $M$-metric spaces will be discussed in the paper. The discussion is started by giving some characterizations of the $QM$-Cauchy sequences and $QM$-convergence sequences.

*Keywords.* quasi $M$-metric, convex contraction of order $r$, convex type 2, $QM$-Cauchy sequences, $QM$-convergence sequences.
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## 1. Introduction and preliminaries

1.1. **Introduction.** Fixed point theorems are usually used in proving the uniqueness and existence of solutions of a system of differential equations [11]. One of the famous fixed point theorems used in proving the uniqueness and existence of the system of differential equations is the Banach fixed point theorem [4]. Banach used a Lipschitz condition for a self-mapping on a complete metric space. A special condition of Lipschitz condition is contraction. The Banach fixed point theorem has been widely generalized by following the development of the metric spaces and contraction conditions. The concept of a metric space has been developed to be a quasi-metric space [1], a $b$-metric space [6], a quasi $b$-metric space [10], a strong $b$-metric space [18], a partial metric space [14], an $M$-metric space [2], and a quasi-$M$-metric space [3]. It is a fact that every metric space is an $M$-metric space and every $M$-metric space is a quasi $M$-metric space. The development of the metric space has given several types of fixed point theorems by generalizing or modifying the concept of contraction, such as Kannan contraction [15, 16], paired-Kannan contraction [5], contraction in the small [12], *Ćirić* contraction of type $Z$ [7], fuzzy H-weak contraction [17] convex contraction [13], and convex contractive [9]. Istrăţescu [13] introduced a continuous function that is convex contraction of order $r$ with $r \geq 2$ and convex type 2 in complete metric spaces.

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Cauchy sequences and convergent sequences are closely related in proving fixed-point theorems of contraction functions. Proving fixed-point theorems with contraction functions on a complete space always forms a Cauchy sequence, thus forming a convergent sequence. Recently, Darojad and Indrati gave a relationship between a quasi-$M$-metric space and a metric space constructed by a quasi-$M$-metric, specifically the relationship between sequences in both spaces in the paper [8]. Darojad and Indrati have generalized some fixed point theorems in a quasi-$M$-metric space for continuous convex contraction of order 2 [8].

In this paper, the convex contraction of order 2 will be generalized to be a convex contraction of order $r$ for arbitrary $r \in \mathbb{N}$ in quasi $M$-metric spaces. We begin the discussion with some characteristics of sequences in quasi-$M$-metric spaces. The discussion will be completed by giving some fixed point theorems of continuous functions involving convex contraction of order $r$ with $r \in \mathbb{N}$ and convex type 2 in quasi $M$-metric spaces. The definition and the theorems in the metric spaces can be generalized to a quasi $M$-metric space. The proofs of the theorems are not as simple as those in complete metric spaces. Strong inductions will be used to prove theorems in the main results due to fixed point theorems in quasi $M$-metric spaces. The generalization has the fact that every continuous Kannan contraction is a special case of convex type 2 functions.

1.2. **Preliminaries.** We recall some concepts in quasi $M$-metric spaces that will be used in Main Results. Let $V$ be a non-empty set and $\nu : V \times V \to [0, \infty)$ be a function. For every $x, y \in V$, we define $m_{x,y} = \min\{\nu(x, x), \nu(y, y)\}$ and $M_{x,y} = \max\{\nu(x, x), \nu(y, y)\}$, respectively. The definition of the quasi $M$-metric space in Definition 1.1 is taken from [3].

**Definition 1.1.** Let $V$ be a nonempty set. A function $\nu : V \times V \to [0, \infty)$ is called a quasi $M$-metric on $V$ if for all $x, y, z \in V$,

QM1. $\nu(x, x) = \nu(y, y) = \nu(x, y) = \nu(y, x) \Leftrightarrow x = y$,
QM2. $m_{x,y} \leq \nu(x, y)$,
QM3. $\nu(x, y) - m_{x,y} \leq \nu(x, z) - m_{x,z} + \nu(z, y) - m_{z,y}$.

We call $(V, \nu)$ a quasi $M$-metric space.

The definitions of $QM$-Cauchy sequences, $QM$-convergent sequence, $QM$-complete spaces, and continuous functions follow from the work of Darojad and Indrati in [8].

**Definition 1.2.** Let $(V, \nu)$ be a quasi $M$-metric space, $(x_n)$ be a sequence in $V$, and $x^* \in V$. The sequence $(x_n)$ is said to be $QM$-convergent to $x^*$, written $x_n \to_{QM} x^*$, if for every real number $\varepsilon > 0$, there exists an $N_0 \in \mathbb{N}$ such that $\nu(x_n, x^*) - m_{x_n,x^*} < \varepsilon$, $\nu(x^*, x_n) - m_{x_n,x^*} < \varepsilon$, and $M_{x_n,x^*} - m_{x_n,x^*} < \varepsilon$ for every natural number $n \geq N_0$, which is written

$$\lim_{n\to\infty} (\nu(x_n, x^*) - m_{x_n,x^*}) = \lim_{n\to\infty} (\nu(x^*, x_n) - m_{x_n,x^*}) = \lim_{n\to\infty} (M_{x_n,x^*} - m_{x_n,x^*}) = 0.$$

The sequence $(x_n)$ in $V$ is said to be $QM$-convergent if there exists $x^* \in V$ such that $(x_n)$ $QM$-converges to $x^*$.

**Definition 1.3.** A sequence $(x_n)$ in a quasi $M$-metric space $(V,\nu)$ is called a $QM$-Cauchy sequence if for every real number $\varepsilon > 0$, there exist $N_0 \in \mathbb{N}$ such that $\nu(x_n,x_k) - m_{x_k,x_n} < \varepsilon$ and $M_{x_k,x_n} - m_{x_k,x_n} < \varepsilon$ for all natural numbers $k,n \geq N_0$, which is written

$$\lim_{k,n\to\infty}(\nu(x_n,x_k) - m_{x_k,x_n}) = \lim_{k,n\to\infty}(M_{x_k,x_n} - m_{x_k,x_n}) = 0.$$

**Definition 1.4.** A quasi $M$-metric space $(V,\nu)$ is said to be a $QM$-complete space if for every $QM$-Cauchy sequence in $(V,\nu)$, it is $QM$-convergent in $(V,\nu)$.

The definition of the $QM$-continuous function in the quasi $M$-metric space is as in Definition 1.5.

**Definition 1.5.** Let $g : V \to Y$ be a function, where $(V,\nu)$ and $(Y,\rho)$ are quasi-$M$-metric spaces. The function $g$ is said to be a $QM$-continuous function at $c \in V$, if for every sequence $(x_n)$ converges to $c$ in $(V,\nu)$, this implies that $(g(x_n))$ converges to $g(c)$ in $(Y,\rho)$). The function $g$ is called $QM$-continuous, if $g$ is $QM$-continuous at $c$ for every $c \in V$.

## 2. Main results

In the main results, some sufficient conditions for a function to have a unique fixed point will be provided in quasi $M$-metric spaces based on convex contraction of order $r$ with $r \in \mathbb{N}$ and convex type 2 conditions. We start by giving some characterizations of the $QM$-convergence sequences and the $QM$-Cauchy sequences. Characteristics will be developed based on a fact in the quasi-$M$-metrics as stated in Lemma 2.1.

**Lemma 2.1.** *In a quasi $M$-metric space $(V,\nu)$, for every $c,d,s,t \in V$ we have*

$$\begin{aligned}|\nu(c,d) - \nu(s,t)| \quad &\leq \max\{\nu(s,c) - m_{s,c}, \nu(c,s) - m_{s,c}\} + M_{s,c} - m_{s,c} \\ &\quad + \max\{\nu(t,d) - m_{t,d}, \nu(d,t) - m_{t,d}\} + M_{t,d} - m_{t,d}.\end{aligned}$$

*Proof.* For arbitrarily $c,d,s,t \in V$, put

$$\begin{aligned}z \quad &= \max\{\nu(s,c) - m_{s,c}, \nu(c,s) - m_{s,c}\} + M_{s,c} - m_{s,c} \\ &\quad + \max\{\nu(t,d) - m_{t,d}, \nu(d,t) - m_{d,t}\} + M_{t,d} - m_{t,d},\end{aligned}$$

then

$$\nu(c,d) - m_{c,d} \leq \nu(c,s) - m_{s,c} + \nu(s,t) - m_{s,t} + \nu(t,d) - m_{t,d}$$

and

$$\nu(s,t) - m_{s,t} \leq \nu(s,c) - m_{s,c} + \nu(c,d) - m_{c,d} + \nu(d,t) - m_{t,d}.$$

Consequently,

$$\begin{aligned}\nu(c,d) - \nu(s,t) \quad &\leq \quad \nu(c,s) - m_{s,c} + \nu(t,d) - m_{t,d} + m_{c,d} - m_{s,t} \\ &= \quad \nu(c,s) - m_{s,c} + \nu(t,d) - m_{t,d} \\ &\quad + \frac{1}{2}|\nu(c,c) + \nu(d,d) - (|\nu(c,c) - \nu(d,d)|)| \\ &\quad - \frac{1}{2}|\nu(s,s) + \nu(t,t) - (|\nu(s,s) - \nu(t,t)|)|.\end{aligned}$$

Consider

$$\frac{1}{2}|\nu(c,c)+\nu(d,d)-(|\nu(c,c)-\nu(d,d)|)|-\frac{1}{2}|\nu(s,s)+\nu(t,t)-(|\nu(s,s)-\nu(t,t)|)|$$

$$\begin{aligned}
&\leq \frac{1}{2}|\nu(c,c)+\nu(d,d)-|\nu(c,c)-\nu(d,d)|-\nu(s,s)-\nu(t,t)+(|\nu(s,s)-\nu(t,t)|)|\\
&\leq \frac{1}{2}|\nu(c,c)-\nu(s,s)|+\frac{1}{2}|\nu(d,d)-\nu(t,t)|\\
&\quad +\frac{1}{2}|(|\nu(c,c)-\nu(d,d)|)-(|\nu(s,s)-\nu(t,t)|)|.
\end{aligned}$$

As a corollary,

$$\begin{aligned}
\nu(c,d)-\nu(s,t) &\leq \nu(c,s)-m_{s,c}+\nu(t,d)-m_{t,d}\\
&\quad +|\nu(c,c)-\nu(s,s)|+|\nu(d,d)-\nu(t,t)|\leq z.
\end{aligned}$$

Using a similar way, we have $\nu(s,t)-\nu(c,d)\leq z$.
Since $\nu(s,t)-\nu(c,d)\leq z$ and $\nu(c,d)-\nu(s,t)\leq z$, we have

$$\begin{aligned}
|\nu(s,t)-\nu(c,d)| &\leq z=\max\{\nu(s,c)-m_{s,c},\nu(c,s)-m_{s,c}\}+M_{s,c}-m_{s,c}\\
&\quad +\max\{\nu(t,d)-m_{t,d},\nu(d,t)-m_{t,d}\}+M_{t,d}-m_{t,d}.
\end{aligned}$$

□

Two characteristics of the $QM$-convergence sequence are provided in Theorem 2.2 and Theorem 2.3.

**Theorem 2.2.** *If $(x_n)$ and $(y_n)$ are two sequences in a quasi-$M$ metric space $(V,\nu)$ $QM$-converge to $a$ and $b$, respectively, then $(\nu(x_n,y_n))$ converges to $\nu(a,b)$.*

*Proof.* Let $\varepsilon>0$ be given. There is a natural number $N_0$ such that for every $n\in\mathbb{N}$, with $n\geq N_0$, one has $\nu(x_n,a)-m_{x_n,a}<\frac{\varepsilon}{4}$, $\nu(a,x_n)-m_{x_n,a}<\frac{\varepsilon}{4}$, $M_{x_n,a}-m_{x_n,a}<\frac{\varepsilon}{4}$, $\nu(y_n,b)-m_{y_n,b}<\frac{\varepsilon}{4}$, $\nu(b,y_n)-m_{y_n,b}<\frac{\varepsilon}{4}$, and $M_{y_n,b}-m_{y_n,b}<\frac{\varepsilon}{4}$. By Lemma 2.1, for any $n\in\mathbb{N}$ with $n\geq N_0$,

$$\begin{aligned}
|\nu(x_n,y_n)-\nu(a,b)| &\leq \max\{\nu(x_n,a)-m_{x_n,a},\nu(a,x_n)-m_{x_n,a}\}\\
&\quad +\max\{\nu(y_n,b)-m_{y_n,b},\nu(b,y_n)-m_{y_n,b}\}\\
&\quad +M_{x_n,a}-m_{x_n,a}+M_{y_n,b}-m_{y_n,b}<\varepsilon.
\end{aligned}$$

□

**Theorem 2.3.** *Let $(x_n)$ be a sequence in a quasi $M$-metric space $(V,\nu)$. The sequence $(x_n)$ is a $QM$-Cauchy sequence in $V$ if and only if there is an $a\in[0,\infty)$ such that for every $\varepsilon>0$, there is a natural number $N_0$, such that for all natural numbers $n,k\geq N_0$, we have $|\nu(x_n,x_k)-a|<\varepsilon$, which is written*

$$\lim_{n,k\to\infty}\nu(x_n,x_k)=a.$$

*Proof.* ($\Rightarrow$) Let $\varepsilon>0$ be given. From the hypothesis, there is an $N_1\in\mathbb{N}$ such that $\nu(x_n,x_k)-m_{x_k,x_n}<\varepsilon$, $\nu(x_k,x_n)-m_{x_k,x_n}<\varepsilon$, and $M_{x_k,x_n}-m_{x_k,x_n}<\varepsilon$ for every natural number $n\geq k\geq N_1$. Therefore, for every $n,k\geq N_1$, we have

$$|\nu(x_n,x_n)-\nu(x_k,x_k)|=M_{x_k,x_n}-m_{x_k,x_n}<\varepsilon.$$

Consequently, $(\nu(x_n, x_n))$ is a Cauchy sequence in $\mathbb{R}$, so there is an element $a \in \mathbb{R}$ with $(\nu(x_n, x_n))$ convergent to $a$. We show that $a \geq 0$. Let $\varepsilon > 0$ be given. Since $(\nu(x_n, x_n)$ converges to $a$, there exists $N_2 \in \mathbb{N}$ such that $|\nu(x_n, x_n) - a| < \varepsilon$ for every natural number $n \geq N_2$. Observe that $-\varepsilon < a - \nu(x_n, x_n) \leq a$ for every $n \geq N_2$. Since $-\varepsilon < a$ for all $\varepsilon > 0$, then $a \geq 0$. Let $\varepsilon$ be any positive real number. Since $(x_n)$ is a $QM$-Cauchy sequence in $V$, there is a natural number $N_3$ such that $\nu(x_n, x_k) - m_{x_k,x_n} < \frac{\varepsilon}{3}$, $\nu(x_k, x_n) - m_{x_k,x_n} < \frac{\varepsilon}{3}$, and $M_{x_k,x_n} - m_{x_k,x_n} < \frac{\varepsilon}{3}$ for every $n \geq k \geq N_3$. Since $(\nu(x_n, x_n))$ converges to $a$, there exists a number $N_4 \in \mathbb{N}$ such that $|\nu(x_n, x_n) - a| < \frac{\varepsilon}{3}$ for every $n \geq N_4$. Choose $N_0 = \max\{N_3, N_4\}$. For $n, k \geq N_0$, we have

$$\begin{aligned}
|\nu(x_n, x_k) - a| &\leq |\nu(x_n, x_k) - m_{x_n,x_k}| + |m_{x_n,x_k} - a| = \nu(x_n, x_k) - m_{x_n,x_k} \\
&\quad + \left|\frac{1}{2}\big(\nu(x_n, x_n) + \nu(x_k, x_k) - |\nu(x_n, x_n) - \nu(x_k, x_k)|\big) - a\right| \\
&\leq \nu(x_n, x_k) - m_{x_n,x_k} + \frac{1}{2}|\nu(x_n, x_n) - \nu(x_k, x_k)| \\
&\quad + \frac{1}{2}|\nu(x_n, x_n) - a| + \frac{1}{2}|\nu(x_k, x_k) - a| \\
&= \nu(x_n, x_k) - m_{x_n,x_k} + \frac{1}{2}(M_{x_n,x_k} - m_{x_n,x_k}) \\
&\quad + \frac{1}{2}|\nu(x_n, x_n) - a| + \frac{1}{2}|\nu(x_k, x_k) - a| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{5\varepsilon}{6} < \varepsilon.
\end{aligned}$$

Therefore, $|\nu(x_n, x_k) - a| < \varepsilon$ for $n, k \geq N_0$.
($\Leftarrow$) Let $\varepsilon > 0$ be given. From the hypothesis, there exists a natural number $N_0$ such that $|\nu(x_n, x_k) - a| < \frac{\varepsilon}{3}$ for every $n, k \geq N_0$. Let $n, k \in \mathbb{N}$ be with $k, n \geq N_0$. Assume that $k \geq n$. We observe that

$$\begin{aligned}
|m_{x_n,x_k} - a| &= \left|\frac{1}{2}(\nu(x_n, x_n) + \nu(x_k, x_k) - |\nu(x_n, x_n) - \nu(x_k, x_k)|) - a\right| \\
&\leq \frac{1}{2}|\nu(x_n, x_n) - \nu(x_k, x_k)| + \frac{1}{2}|\nu(x_n, x_n) - a| + \frac{1}{2}|\nu(x_k, x_k) - a| \\
&\leq \frac{1}{2}|\nu(x_n, x_n) - a + a - \nu(x_k, x_k)| + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} \\
&\leq \frac{1}{2}|\nu(x_n, x_n) - a| + \frac{1}{2}|\nu(x_k, x_k) - a| + \frac{\varepsilon}{3} \\
&< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.
\end{aligned}$$

Consequently,

$$\nu(x_n, x_k) - m_{x_n,x_k} \leq |\nu(x_n, x_k) - a| + |m_{x_n,x_k} - a| < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon$$

and

$$\nu(x_k, x_n) - m_{x_n,x_k} \leq |\nu(x_k, x_n) - a| + |m_{x_n,x_k} - a| < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.$$

Observe that

$$M_{x_n,x_k} - m_{x_n,x_k} \leq |\nu(x_n, x_n) - a| + |\nu(x_k, x_k) - a| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} < \varepsilon.$$

That means $(x_n)$ is a $QM$-Cauchy sequence in $V$. □

2.1. **Convex contraction of order $r$.** In this part, a sufficient condition will be given for a function to have a unique fixed point using a convex contraction of order $r$ with $r \in \mathbb{N}$.

**Definition 2.4.** Let $r \in \mathbb{N}$ and $(V, \nu)$ be a quasi $M$-metric space. A $QM$-continuous function $g : V \to V$ is called a convex contraction of order $r$ if there are $a_0, a_1, \cdots, a_{r-1} \in [0, 1)$, $a_0 + a_1 + \cdots + a_{r-1} < 1$, with

$$\nu(g^r(u), g^r(v)) \leq a_0\nu(u, v) + a_1\nu(g(u), g(v)) + \cdots + a_{r-1}\nu(g^{r-1}(u), g^{r-1}(v)),$$

for every $u, v \in V$.

From Definition 2.4, it is clear that a self-contraction function in a metric space is a convex contraction of order 1. An example of a convex contraction function of order 2 is provided in Example 2.5. Moreover, examples of functions that are not convex contraction functions of order $r$, for any $r \in \mathbb{N}$, are provided in Example 2.6.

**Example 2.5.** Let $([0, 1], \nu)$ be a quasi $M$-metric space, where $\nu(x, y) = |x - y|$ for all $x, y \in [0, 1]$. A self-function $g$ on $[0, 1]$ with $g(x) = \frac{1}{2}\big(x^2 + \frac{1}{2}\big)$, is a convex contraction of order 2.

**Example 2.6.** Let $([0, \infty), \nu)$ be a quasi $M$-metric space with $\nu(x, y) = 2x + y$ for every $x, y \in [0, \infty)$. The identity function $I$ on $[0, \infty)$ is $QM$-continuous, but $I$ is not a convex contraction function of order $r$, for any $r \in \mathbb{N}$. The function $g : [0, \infty) \to [0, \infty)$, where $g(x) = \frac{1}{2}\big(x - \lfloor \frac{x}{2} \rfloor\big)$, is not a convex contraction of order $r$, for any $r \in \mathbb{N}$, but $g$ satisfies $\nu(g(x), g(y)) \leq \frac{1}{2}\nu(x, y)$ for all $x, y \in [0, \infty)$.

Theorem 2.7 provides a fixed point theorem for the convex contraction function of order $r \in \mathbb{N}$.

**Theorem 2.7.** *Let $(V, \nu)$ be a $QM$-complete quasi $M$-metric space and $r \in \mathbb{N}$. If $g : V \to V$ is a convex contraction function of order $r$, then $g$ has a unique fixed point in $V$.*

*Proof.* From the hypothesis, there are $a_0, a_1, \cdots, a_{r-1} \in [0, 1)$ with $a_0 + a_1 + \cdots + a_{r-1} < 1$ such that

$$\nu(g^r(x), g^r(y)) \leq a_0\nu(x, y) + a_1\nu(g(x), g(y)) + \cdots + a_{r-1}\nu(g^{r-1}(x), g^{r-1}(y)),$$

for all $x, y \in V$. Fix $x_0 \in V$. Let

$$\begin{aligned} z &= \max\Bigg\{\nu(x_0, g(x_0)) + \sum_{j=1}^{r-1} \nu(g^j(x_0), g^{j+1}(x_0)), \nu(g(x_0), x_0) \\ &\quad + \sum_{j=1}^{r-1} \nu(g^{j+1}(x_0), g^j(x_0)), \nu(x_0, x_0) + \sum_{j=1}^{r-1} \nu(g^j(x_0), g^j(x_0))\Bigg\}. \end{aligned}$$

For every natural number $n$, let us define a sequence $(x_n)$ in $V$, with $x_n = g^n(x_0)$ and $g^0(x_0) = x_0$. We define several sequences in $\mathbb{R}$, namely $(b_n)$, $(c_n)$, and $(d_n)$, with $b_0 = \nu(x_0, g(x_0))$, $c_0 = \nu(g(x_0), x_0)$, and $d_0 = \nu(x_0, x_0)$, and $b_n = \nu(x_n, x_{n+1}), c_n = \nu(x_{n+1}, x_n)$, and $d_n = \nu(x_n, x_n)$ for all $n \in \mathbb{N}$. For $n \geq r$, observe that

$$\begin{aligned} b_n &= \nu(x_n, x_{n+1}) = \nu(g^n(x_0), g^{n+1}(x_0)) = \nu(g^r(g^{n-r}(x_0)), g^r(g^{n+1-r}(x_0))) \\ &\leq a_0\nu(g^{n-r}(x_0), g^{n-r+1}(x_0)) + a_1\nu(g^{n-r+1}(x_0), g^{n-r+2}(x_0)) \\ &\quad + \cdots + a_{r-1}\nu(g^{n-1}(x_0), g^n(x_0)) \\ &= a_0\nu(x_{n-r}, x_{n-r+1}) + a_1\nu(x_{n-r+1}, x_{n-r+2}) + \cdots + a_{r-1}\nu(x_{n-1}, x_n) \\ &= a_0 b_{n-r} + a_1 b_{n-r+1} + \cdots + a_{r-1} b_{n-1}, \end{aligned}$$

Using an identical argument of $b_n$, we obtain the bounds for $c_n$ and $d_n$ as follows.

$$\begin{aligned} c_n &= \nu(x_{n+1}, x_n) \\ &\leq a_0\nu(g^{n-r+1}(x_0), g^{n-r}(x_0)) + a_1\nu(g^{n-r+2}(x_0), g^{n-r+1}(x_0)) \\ &\quad + \cdots + a_{r-1}\nu(g^n(x_0), g^{n-1}(x_0)) \\ &= a_0 c_{n-r} + a_1 c_{n-r+1} + \cdots + a_{r-1} c_{n-1}, \end{aligned}$$

and

$$\begin{aligned} d_n &= \nu(x_n, x_n) \\ &\leq a_0\nu(g^{n-r}(x_0), g^{n-r}(x_0)) + a_1\nu(g^{n-r+1}(x_0), g^{n-r+1}(x_0)) \\ &\quad + \cdots + a_{r-1}\nu(g^{n-1}(x_0), g^{n-1}(x_0)) \\ &= a_0 d_{n-r} + a_1 d_{n-r+1} + \cdots + a_{r-1} d_{n-1}. \end{aligned}$$

Using strong mathematical induction, we get $b_n \leq (a_0 + a_1 + \cdots + a_{r-1})^{\left\lfloor \frac{n}{r} \right\rfloor} z$, $c_n \leq (a_0 + a_1 + \cdots + a_{r-1})^{\left\lfloor \frac{n}{r} \right\rfloor} z$, and $d_n \leq (a_0 + a_1 + \cdots + a_{r-1})^{\left\lfloor \frac{n}{r} \right\rfloor} z$ for $n \geq 0$.

We show that $(x_n)$ is $QM$-convergent. For $n, k \in \mathbb{N}$ with $k \geq n$, we observe that

$$\begin{aligned} 0 &\leq \nu(x_n, x_k) - m_{x_n, x_k} \leq \sum_{i=n}^{k-1} \nu(x_i, x_{i+1}) - m_{x_i, x_{i+1}} \\ &\leq \sum_{i=n}^{k-1} \nu(x_i, x_{i+1}) = \sum_{i=n}^{k-1} b_i \leq \sum_{i=n}^{k-1} (a_0 + a_1 + \cdots + a_{r-1})^{\left\lfloor \frac{i}{r} \right\rfloor} z \\ &\leq \sum_{i=n}^{\infty} (a_0 + a_1 + \cdots + a_{r-1})^{\left\lfloor \frac{i}{r} \right\rfloor} z \leq \frac{(a_0 + a_1 + \cdots + a_{r-1})^{\left\lfloor \frac{n}{r} \right\rfloor} rz}{1 - a_0 - a_1 - \cdots - a_{r-1}}, \end{aligned}$$

$$\begin{aligned}
0 &\leq \nu(x_k,x_n)-m_{x_n,x_k}\leq \sum_{i=n}^{k-1}\nu(x_{i+1},x_i)-m_{x_i,x_{i+1}}\\
&\leq \sum_{i=n}^{k-1}\nu(x_{i+1},x_i)=\sum_{i=n}^{k-1}c_i\leq \sum_{i=n}^{k-1}(a_0+a_1+\cdots+a_{r-1})^{\left\lfloor\frac{i}{r}\right\rfloor}z\\
&\leq \sum_{i=n}^{\infty}(a_0+a_1+\cdots+a_{r-1})^{\left\lfloor\frac{i}{r}\right\rfloor}z\leq \frac{(a_0+a_1+\cdots+a_{r-1})^{\left\lfloor\frac{n}{r}\right\rfloor}rz}{1-a_0-a_1-\cdots-a_{r-1}},
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq M_{x_n,x_k}-m_{x_n,x_k}\leq \sum_{i=n}^{k-1}M_{x_i,x_{i+1}}-m_{x_i,x_{i+1}}\\
&\leq \sum_{i=n}^{k-1}M_{x_i,x_{i+1}}\leq \sum_{i=n}^{k}d_i+d_{i+1}\leq 2\sum_{i=n}^{k-1}(a_0+a_1+\cdots+a_{r-1})^{\left\lfloor\frac{i}{r}\right\rfloor}z\\
&\leq 2\sum_{i=n}^{\infty}(a_0+a_1+\cdots+a_{r-1})^{\left\lfloor\frac{n}{r}\right\rfloor}z\leq \frac{2(a_0+a_1+\cdots+a_{r-1})^{\left\lfloor\frac{i}{r}\right\rfloor}rz}{1-a_0-a_1-\cdots-a_{r-1}}.
\end{aligned}$$

We have $\frac{(a_0+a_1+\cdots+a_{r-1})^{\left\lfloor\frac{n}{r}\right\rfloor}rz}{1-a_0-a_1-\cdots-a_{r-1}}$ tends to 0 as $n\to\infty$. Consequently

$$\lim_{k,n\to\infty}(\nu(x_n,x_k)-m_{x_k,x_n})=\lim_{k,n\to\infty}(M_{x_k,x_n}-m_{x_k,x_n})=0.$$

As a corollary, $(x_n)$ is a $QM$-Cauchy sequence. Therefore, there is $u\in V$ with $(x_n)$ is $QM$-convergent to $u$.

Since $g$ is $QM$-continuous on $V$ and $(x_n)$ $QM$-converges to $u$, then $(g(x_n))$ $QM$-converges to $g(u)$. On the other hand, $g(x_n)=g(g^n(x_0))=g^{n+1}(x_0)=x_{n+1}$ for every $n\in\mathbb{N}$. Since $(x_{n+1})$ is a subsequence of $(x_n)$ and $(x_n)$ $QM$-converges to $u$, then $(x_{n+1})$ $QM$-converges to $u$. In other words, $(g(x_n))$ $QM$-converges to $u$. Since $(g(x_n))$ $QM$-converges to $g(u)$ and $(g(x_n))$ $QM$-converges to $u$, then $g(u)=u$.

Let $w^*\in V$ be with $g(w^*)=w^*$. Since $g(u)=u$ and $g(w^*)=w^*$, then $g^n(u)=u$ and $g^n(w^*)=w^*$ for every $n\in\mathbb{N}$. As a corollary,

$$\begin{aligned}
\nu(u,w^*) &= \nu(g^r(u),g^r(w^*))\\
&\leq a_0\nu(u,w^*)+a_1\nu(g(u),g(w^*))+\cdots+a_{r-1}\nu(g^{r-1}(u),g^{r-1}(w^*))\\
&= a_0\nu(u,w^*)+a_1\nu(u,w^*)+\cdots+a_{r-1}\nu(u,w^*)\\
&= (a_0+a_1+\cdots+a_{r-1})\nu(u,w^*),
\end{aligned}$$

$$\begin{aligned}
\nu(w^*,u) &= \nu(g^r(w^*),g^r(u))\\
&\leq a_0\nu(w^*,u)+a_1\nu(g(w^*),g(u)+\cdots+a_{r-1}\nu(g^{r-1}(w^*),g^{r-1}(u))\\
&= (a_0+a_1+\cdots+a_{r-1})\nu(w^*,u),
\end{aligned}$$

$$\begin{aligned}\nu(u,u) &= \nu(g^r(u),g^r(u))\\ &\leq a_0\nu(u,u)+a_1\nu(g(u),g(u))+\cdots+a_{r-1}\nu(g^{r-1}(u),g^{r-1}(u))\\ &= (a_0+a_1+\cdots+a_{r-1})\nu(u,u),\end{aligned}$$

and

$$\begin{aligned}\nu(w^*,w^*) &= \nu(g^r(w^*),g^r(w^*))\\ &\leq a_0\nu(w^*,w^*)+a_1\nu(g(w^*),g(w^*))+\cdots+a_{r-1}\nu(g^{r-1}(w^*),g^{r-1}(w^*))\\ &= (a_0+a_1+\cdots+a_{r-1})\nu(w^*,w^*).\end{aligned}$$

We get $(1-a_0-a_1-\cdots-a_{r-1})\nu(u,w^*)\leq 0, (1-a_0-a_1-\cdots-a_{r-1})\nu(w^*,u)\leq 0,$

$(1-a_0-a_1-\cdots-a_{r-1})\nu(u,u)\leq 0,$ and $(1-a_0-a_1-\cdots-a_{r-1})\nu(w^*,w^*)\leq 0.$

Since $1-a_0-a_1-\cdots-a_{r-1}>0$, then $\nu(u,w^*)=\nu(w^*,u)=\nu(u,u)=\nu(w^*,w^*)=0$. Since $\nu(u,w^*)=\nu(w^*,u)=\nu(u,u)=\nu(w^*,w^*)$, then $u=w^*$. That proves the theorem. □

2.2. **Convex type 2.** In this part a fixed point theorem will be given using a convex type 2 condition.

**Definition 2.8.** Let $(V,\nu)$ be a quasi $M$-metric space. A $QM$-continuous function $g:V\to V$ is convex type 2 if there exist $p,q,r,s,t,u\in[0,1)$ with $p+q+r+s+t+u<1$ such that for every $x,y\in V$, we have

$$\begin{aligned}\nu(g^2(x),g^2(y)) \leq\ & p\nu(x,y)+q\nu(g(x),g(y))+r\nu(x,g(x))\\ &+s\nu(g(x),g^2(x))+t\nu(y,g(y))+u\nu(g(y),g^2(y)).\end{aligned}$$

In a metric space $(V,d)$, a continuous function $g:V\to V$ is called a Kannan contraction if there exists a $\beta\in[0,\frac{1}{2})$ such that $d(g(u),g(v))\leq\beta d(u,g(u))+\beta d(v,g(v))$, for every $u,v\in V$. Observe that $d(g^2(u),g^2(v))\leq\beta d(g(u),g^2(u))+\beta d(g(v),g^2(v))$, for every $u,v\in V$. Choosing $s=u=\beta$ and $p=q=r=t=0$, then we have $p+q+r+s+t+u=2\beta<1$ and

$$\begin{aligned}\nu(g^2(u),g^2(v)) \leq\ & p\nu(u,v)+q\nu(g(u),g(v))+r\nu(u,g(v))\\ &+s\nu(g(u),g^2(u))+t\nu(v,g(v))+u\nu(g(v),g^2(v))\end{aligned}$$

for every $u,v\in V$. This means that a Kannan contraction function is a special case of convex type 2 functions. We give examples of convex type 2 functions and that are not convex type 2.

**Example 2.9.** A real function $G$, with $G(t)=\frac{t}{2}$ if $t<0$ and $G(t)=\frac{t}{4}$ if $t\geq 0$, is a convex type 2 function on a quasi $M$-metric space $(\mathbb{R},\nu)$, where $\nu(s,t)=|s-t|$ for $s,t\in\mathbb{R}$.

**Example 2.10.** In a quasi $M$-metric space $(V,\nu)$ with $V=[0,1]$ and $\nu(s,t)=|s-t|$ for all $s,t\in V$, a function $g:V\to V$, with $g(t)=\frac{t}{4}$ if $t\in[0,\frac{1}{2})$ and $g(t)=\frac{t}{5}$ if $t\in[\frac{1}{2},1]$, is not a convex type 2 function. The function $g$ satisfies $\nu(g^2(s),g^2(t))\leq\frac{4}{9}\nu(s,g(s))+\frac{4}{9}\nu(t,g(t))$ for all $s,t\in V$.

We give a fixed point theorem for a convex type 2 function in Theorem 2.11.

**Theorem 2.11.** *Let $(V,\nu)$ be a quasi $M$-metric space. If $V$ is $QM$-complete and $g:V\to V$ is a convex type 2 function, then there is a unique fixed point of $g$.*

*Proof.* From the hypothesis, there are $p,q,r,s,t,u\in[0,1)$ with $p+q+r+s+t+u<1$ with

$$\begin{aligned}\nu(g^2(x),g^2(y)) &\leq \nu(x,y)+q\nu(g(x),g(y))+r\nu(x,g(x))\\ &\quad +s\nu(g(x),g^2(x))+t\nu(y,g(y))+u\nu(g(y),g^2(y))\end{aligned}$$

for all $x,y\in V$. Since $p+q+r+s+t+u<1$, then $u(p+q+r+s+t+u)<u$. Therefore, $0<u-u(p+q+r+s+t+u)$. As a corollary, $p+q+r+s+t<p+q+r+s+t+u-u(p+q+r+s+t+u)=(1-u)(p+q+r+s+t+u)$. That means $\dfrac{p+q+r+s+t}{1-u}<p+q+r+s+t+u$.
Fix $x_0\in V$. Let $\gamma=p+q+r+s+t+u$ and

$$\begin{aligned}z_0 &= \max\big\{\nu(x_0,x_0),\nu(g(x_0),g(x_0)),\nu(x_0,g(x_0)),\nu(g(x_0),x_0),\\ &\qquad \nu(g(x_0),g^2(x_0)),\nu(g^2(x_0),g(x_0))\big\}.\end{aligned}$$

In $V$, we define a sequence $(x_n)$ with $x_n=g^n(x_0)$ for every $n\in\mathbb{N}$ and $g^0(x_0)=x_0$. We define several sequences in $\mathbb{R}$, namely $(a_n)$, $(b_n)$, and $(c_n)$, with $a_0=\nu(x_0,g(x_0))$,$b_0=\nu(g(x_0),x_0)$, and $c_0=\nu(x_0,x_0)$, and $a_n=\nu(x_n,x_{n+1})$, $b_n=\nu(x_{n+1},x_n)$, and $c_n=\nu(x_n,x_n)$ for every $n\in\mathbb{N}$. For $n\geq 2$, observe that

$$\begin{aligned}(i)\ a_n &= \nu(x_n,x_{n+1})=\nu(g^2(g^{n-2}(x_0)),g^2(g^{n-1}(x_0)))\\ &\leq p\nu(g^{n-2}(x_0),g^{n-1}(x_0))+q\nu(g^{n-1}(x_0),g^n(x_0))+r\nu(g^{n-2}(x_0),g^{n-1}(x_0))\\ &\quad +s\nu(g^{n-1}(x_0),g^n(x_0))+t\nu(g^{n-1}(x_0),g^n(x_0))+u\nu(g^n(x_0),g^{n+1}(x_0))\\ &= p\nu(x_{n-2},x_{n-1})+q\nu(x_{n-1},x_n)+r\nu(x_{n-2},x_{n-1})\\ &\quad +s\nu(x_{n-1},x_n)+t\nu(x_{n-1},x_n)+u\nu(x_n,x_{n+1})\\ &= pa_{n-2}+qa_{n-1}+ra_{n-2}+sa_{n-1}+ta_{n-1}+ua_n\end{aligned}$$

which is equivalent to $(1-u)a_n\leq(p+r)a_{n-2}+(q+s+t)a_{n-1}$. Therefore,

$$a_n\leq\left(\frac{p+r}{1-u}\right)a_{n-2}+\left(\frac{q+s+t}{1-u}\right)a_{n-1}.$$

Using an identical argument of $a_n$, we obtain the bounds for $b_n$ and $c_n$ as follows.

$(ii)$ $b_n\leq pb_{n-2}+qb_{n-1}+ta_{n-2}+(r+u)a_{n-1}+sa_n$.

$(iii)$ $c_n\leq pc_{n-2}+qc_{n-1}+(r+t)a_{n-2}+(s+u)a_{n-1}$.

Using strong mathematical induction, we will show that

$$a_n\leq\gamma^{\left\lfloor\frac{n}{2}\right\rfloor}z_0,\qquad b_n\leq\gamma^{\left\lfloor\frac{n}{2}\right\rfloor}z_0,\qquad\text{and}\qquad c_n\leq\gamma^{\left\lfloor\frac{n}{2}\right\rfloor}z_0. \tag{2.1}$$

For $n=0$, observe that

$$\begin{aligned}a_0 &= \nu(x_0,x_1)=\nu(x_0,g(x_0))\leq z_0=\gamma^{\left\lfloor\frac{0}{2}\right\rfloor}z_0,\\ b_0 &= \nu(x_1,x_0)=\nu(g(x_0),x_0)\leq z_0=\gamma^{\left\lfloor\frac{0}{2}\right\rfloor}z_0,\end{aligned}$$

and

$$c_0 \;=\; \nu(x_0,x_0) \le z_0 = \gamma^{\left\lfloor \frac{0}{2} \right\rfloor} z_0.$$

For $n=1$, observe that

$$\begin{aligned} a_1 &= \nu(x_1,x_2) = \nu(g(x_0),g^2(x_0)) \le z_0 = \gamma^{\left\lfloor \frac{1}{2} \right\rfloor} z_0, \\ b_1 &= \nu(x_2,x_1) = \nu(g^2(x_0),g(x_0)) \le z_0 = \gamma^{\left\lfloor \frac{1}{2} \right\rfloor} z_0, \text{ and} \end{aligned}$$

$$c_1 = \nu(x_1,x_1) \le z_0 = \gamma^{\left\lfloor \frac{1}{2} \right\rfloor} z_0.$$

Now, let us assume that for every $n \in \{0,1,\cdots,k\}$, we have $a_n, b_n,$, and $c_n$ less than or equal to $\gamma^{\left\lfloor \frac{n}{2} \right\rfloor} z_0$. We will show that

$$a_{k+1} \le \gamma^{\left\lfloor \frac{k+1}{2} \right\rfloor} z_0, \qquad b_{k+1} \le \gamma^{\left\lfloor \frac{k+1}{2} \right\rfloor} z_0, \qquad \text{and} \qquad \mathrm{c_{k+1}} \le \gamma^{\left\lfloor \frac{k+1}{2} \right\rfloor} \mathrm{z_0}. \tag{2.2}$$

In case $k = 2j-1$ for some $j \in \mathbb{N}$, observe that

$$\begin{aligned} a_{k+1} = a_{2j} &\le \left(\frac{p+r}{1-u}\right) a_{2j-2} + \left(\frac{q+s+t}{1-u}\right) a_{2j-1} \\ &\le \left(\frac{p+r}{1-u}\right) \gamma^{\left\lfloor \frac{2j-2}{2} \right\rfloor} z_0 + \left(\frac{q+s+t}{1-u}\right) \gamma^{\left\lfloor \frac{2j-1}{2} \right\rfloor} z_0 \\ &= \left(\frac{p+q+r+s+t}{1-u}\right) \gamma^{j-1} z_0 \le \gamma \gamma^{j-1} z_0 \\ &= \gamma^j z_0 = \gamma^{\left\lfloor \frac{2j}{2} \right\rfloor} z_0 = \gamma^{\left\lfloor \frac{k+1}{2} \right\rfloor} z_0, \end{aligned}$$

$$\begin{aligned} b_{k+1} = b_{2j} &\le p b_{2j-2} + q b_{2j-1} + t a_{2j-2} + (r+u) a_{2j-1} + s a_{2j} \\ &\le p\gamma^{\left\lfloor \frac{2j-2}{2} \right\rfloor} z_0 + q\gamma^{\left\lfloor \frac{2j-1}{2} \right\rfloor} z_0 + t\gamma^{\left\lfloor \frac{2j-2}{2} \right\rfloor} z_0 + (r+u)\gamma^{\left\lfloor \frac{2j-2}{2} \right\rfloor} z_0 \\ &\quad + s\gamma^{\left\lfloor \frac{2j}{2} \right\rfloor} z_0 \\ &= p\gamma^{j-1} z_0 + q\gamma^{j-1} z_0 + t\gamma^{j-1} z_0 + s\gamma^j z_0 + (r+u)\gamma^{j-1} z_0 \\ &\le p\gamma^{j-1} z_0 + q\gamma^{j-1} z_0 + t\gamma^{j-1} z_0 + s\gamma^{j-1} z_0 + (r+u)\gamma^{j-1} z_0 \\ &= \gamma^j z_0 = \gamma^{\left\lfloor \frac{2j}{2} \right\rfloor} z_0 = \gamma^{\left\lfloor \frac{k+1}{2} \right\rfloor} z_0, \end{aligned}$$

and

$$\begin{aligned} c_{k+1} = c_{2j} &\le p c_{2j-2} + q c_{2j-1} + (r+t) a_{2j-2} + (s+u) a_{2j-1} \\ &\le p\gamma^{\left\lfloor \frac{2j-2}{2} \right\rfloor} z_0 + q\gamma^{\left\lfloor \frac{2j-1}{2} \right\rfloor} z_0 + (r+t)\gamma^{\left\lfloor \frac{2j-2}{2} \right\rfloor} z_0 + (s+u)\gamma^{\left\lfloor \frac{2j-1}{2} \right\rfloor} z_0 \\ &= p\gamma^{j-1} z_0 + q\gamma^{j-1} z_0 + (r+t)\gamma^{j-1} z_0 + (s+u)\gamma^{j-1} z_0 \\ &= \gamma\gamma^{j-1} z_0 = \gamma^j z_0 = \gamma^{\left\lfloor \frac{2j}{2} \right\rfloor} z_0 = \gamma^{\left\lfloor \frac{k+1}{2} \right\rfloor} z_0. \end{aligned}$$

In case $k = 2j$ for some $j \in \mathbb{N}$, observe that

$$\begin{aligned} a_{k+1} = a_{2j+1} &\leq \left(\frac{p+r}{1-u}\right) a_{2j-1} + \left(\frac{q+s+t}{1-u}\right) a_{2j} \\ &\leq \left(\frac{p+r}{1-u}\right)\gamma^{\left\lfloor \frac{2j-1}{2} \right\rfloor} z_0 + \left(\frac{q+s+t}{1-u}\right)\gamma^{\left\lfloor \frac{2j}{2} \right\rfloor} z_0 \\ &\leq \gamma\gamma^{j-1} z_0 = \gamma^j z_0 = \gamma^{\left\lfloor \frac{2j+1}{2} \right\rfloor} z_0 = \gamma^{\left\lfloor \frac{k+1}{2} \right\rfloor} z_0, \end{aligned}$$

$$\begin{aligned} b_{k+1} &= b_{2j+1} \leq p b_{2j-1} + q b_{2j} + t a_{2j-1} + (r+u) a_{2j} + s a_{2j+1} \\ &\leq p\gamma^{\left\lfloor \frac{2j-1}{2} \right\rfloor} z_0 + q\gamma^{\left\lfloor \frac{2j}{2} \right\rfloor} z_0 + t\gamma^{\left\lfloor \frac{2j-1}{2} \right\rfloor} z_0 + (r+u)\gamma^{\left\lfloor \frac{2j}{2} \right\rfloor} z_0 + s\gamma^{\left\lfloor \frac{2j+1}{2} \right\rfloor} z_0 \\ &= p\gamma^{j-1} z_0 + q\gamma^j z_0 + t\gamma^{j-1} z_0 + s\gamma^j z_0 + (r+u)\gamma^j z_0 \\ &\leq p\gamma^{j-1} z_0 + q\gamma^{j-1} z_0 + t\gamma^{j-1} z_0 + s\gamma^{j-1} z_0 + (r+u)\gamma^{j-1} z_0 \\ &= \gamma^j z_0 = \gamma^{\left\lfloor \frac{2j+1}{2} \right\rfloor} z_0 = \gamma^{\left\lfloor \frac{k+1}{2} \right\rfloor} z_0, \end{aligned}$$

and

$$\begin{aligned} c_{k+1} &= c_{2j+1} \leq p c_{2j-1} + q c_{2j} + (r+t) a_{2j-1} + (s+u) a_{2j} \\ &\leq p\gamma^{\left\lfloor \frac{2j-1}{2} \right\rfloor} z_0 + q\gamma^{\left\lfloor \frac{2j}{2} \right\rfloor} z_0 + (r+t)\gamma^{\left\lfloor \frac{2j-1}{2} \right\rfloor} z_0 + (s+u)\gamma^{\left\lfloor \frac{2j}{2} \right\rfloor} z_0 \\ &= p\gamma^{j-1} z_0 + q\gamma^j z_0 + (r+t)\gamma^{j-1} z_0 + (s+u)\gamma^j z_0 \\ &\leq \gamma\gamma^{j-1} z_0 = \gamma^j z_0 = \gamma^{\left\lfloor \frac{2j+1}{2} \right\rfloor} z_0 = \gamma^{\left\lfloor \frac{k+1}{2} \right\rfloor} z_0. \end{aligned}$$

From two cases, we have proved (2.2). That means; we have proved (2.1).

Now, it will be shown that $(x_n)$ is $QM$-convergent. For any $n, k \in \mathbb{N}$, $k \geq n$, we have

$$\begin{aligned} 0 &\leq \nu(x_n, x_k) - m_{x_n,x_k} \leq \sum_{j=n}^{k-1} (\nu(x_j, x_{j+1}) - m_{x_j,x_{j+1}}) \leq \sum_{j=n}^{k-1} \nu(x_j, x_{j+1}) = \sum_{j=n}^{k-1} a_j \\ &\leq \sum_{j=n}^{k-1} \gamma^{\left\lfloor \frac{j}{2} \right\rfloor} z_0 \leq \sum_{j=n}^{\infty} \gamma^{\left\lfloor \frac{j}{2} \right\rfloor} z_0 \leq \frac{2\gamma^{\left\lfloor \frac{n}{2} \right\rfloor} z_0}{1-p-q-r-s-t-u}, \end{aligned}$$

$$\begin{aligned} 0 &\leq \nu(x_k, x_n) - m_{x_n,x_k} \leq \sum_{j=n}^{k-1} \nu(x_{j+1}, x_j) - m_{x_j,x_{j+1}} \leq \sum_{j=n}^{k-1} \nu(x_{j+1}, x_j) = \sum_{j=n}^{k-1} b_j \\ &\leq \sum_{j=n}^{k-1} \gamma^{\left\lfloor \frac{j}{2} \right\rfloor} z_0 \leq \sum_{j=n}^{\infty} \gamma^{\left\lfloor \frac{j}{2} \right\rfloor} z_0 \leq \frac{2\gamma^{\left\lfloor \frac{n}{2} \right\rfloor} z_0}{1-p-q-r-s-t-u}, \end{aligned}$$

and

$$\begin{aligned} 0 &\leq M_{x_n,x_k} - m_{x_n,x_k} \leq \sum_{j=n}^{k-1}(M_{x_j,x_{j+1}} - m_{x_j,x_{j+1}}) \leq \sum_{j=n}^{k-1} M_{x_j,x_{j+1}} \\ &\leq \sum_{j=n}^{k}(c_j + c_{j+1}) \leq 2\sum_{j=n}^{k-1}(p+q+r+s+t+u)^{\left\lfloor \frac{j}{2} \right\rfloor} z_0 \\ &\leq 2\sum_{j=n}^{\infty}\gamma^{\left\lfloor \frac{i}{2} \right\rfloor} z_0 \leq \frac{4\gamma^{\left\lfloor \frac{n}{2} \right\rfloor} z_0}{1-p-q-r-s-t-u}. \end{aligned}$$

We have $\frac{\gamma^{\left\lfloor \frac{n}{2} \right\rfloor} z_0}{1-p-q-r-s-t-u}$ tends to 0 as $n \to \infty$. Consequently,

$$\begin{aligned} \lim_{\substack{k,n\to\infty \\ k\geq n}} (\nu(x_n,x_k) - m_{x_k,x_n}) &= \lim_{\substack{k,n\to\infty \\ k\geq n}} (\nu(x_k,x_n) - m_{x_k,x_n}) \\ &= \lim_{k,n\to\infty} (M_{x_k,x_n} - m_{x_k,x_n}) = 0. \end{aligned}$$

Therefore, $(x_n)$ is $QM$-Cauchy, so there is $x \in V$ such that $(x_n)$ $QM$-converges to $x \in V$.

We show that $g(x) = x$. Since the function $g$ is $QM$-continuous on $V$ and $(x_n)$ $QM$-converges to $x$, then $(g(x_n))$ $QM$-converges to $g(x)$. From the construction of $x_n$ and considering $(x_{n+1})$ is a subsequence of $(x_n)$, we obtain $(x_{n+1})$ $QM$-converges to $x$, in other words, $(g(x_n))$ $QM$-converges to $x$. From the uniqueness of the limit, we obtain $g(x) = x$.
Let $v \in V$ be such that $g(v) = v$, then $g^2(v) = g(v) = v$. As a corollary,

$$\begin{aligned} \nu(x,x) = \nu(g^2(x), g^2(x)) &\leq p\nu(x,x) + q\nu(g(x),g(x)) + r\nu(x,g(x)) \\ &\quad + s\nu(g(x),g^2(x)) + t\nu(x,g(x) + u\nu(g(x),g^2(x)) \\ &= \gamma\nu(x,x) \end{aligned}$$

and

$$\begin{aligned} \nu(v,v) = \nu(g^2(v), g^2(v)) &\leq p\nu(v,v) + q\nu(g(v),g(v)) + r\nu(v,g(v)) \\ &\quad + s\nu(g(v),g^2(v)) + t\nu(v,g(v)) + u\nu(g(v),g^2(v)) \\ &= \gamma\nu(v,v). \end{aligned}$$

Therefore, $\nu(x,x) = \nu(v,v) = 0$.
As a corollary, we have the following results:

$$\begin{aligned} \nu(x,v) &= \nu(g^2(x),g^2(v)) \\ &\leq p\nu(x,v) + q\nu(g(x),g(v)) + r\nu(x,g(x)) \\ &\quad + s\nu(g(x),g^2(x)) + t\nu(x,g(v)) + u\nu(g(v),g^2(v)) \\ &= p\nu(x,v) + q\nu(x,v) + r\nu(x,x) + s\nu(x,x) + t\nu(v,v) + u\nu(v,v) \\ &= p\nu(x,v) + q\nu(x,v) \end{aligned}$$

and

$$\begin{aligned}\nu(v,x) &= \nu(g^2(v),g^2(v))\\ &\leq p\nu(v,x)+q\nu(g(v),g(x))+r\nu(v,g(x))\\ &\quad +s\nu(g(v),g^2(v))+t\nu(x,g(x))+u\nu(g(x),g^2(x))\\ &= p\nu(v,x)+q\nu(v,x)+r\nu(v,v)+s\nu(v,v)+t\nu(x,x)+u\nu(x,x)\\ &= p\nu(v,x)+q\nu(v,x).\end{aligned}$$

We get $\nu(x,v)=\nu(v,x)=0$. Since $\nu(x,v)=\nu(v,x)=\nu(x,x)=\nu(v,v)$, then $x=v$. □

## 3. CONCLUSIONS

The Lemma 2.1 is an important tool in proving two characteristics of sequences in quasi $M$-metric space in Theorem 2.2 and Theorem 2.3. The convex contraction of order 2 has been generalized to $r$, $r\in\mathbb{N}$. The generalization has been applied to have a fixed point theorem (Theorem 2.7). Another fixed point theorem has been formulated using a convex type 2 condition in Theorem 2.11.

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