

A STUDY ON CARTAN-EILENBERG GORENSTEIN n -FP-INJECTIVE AND n -FLAT COMPLEXES

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ABSTRACT. This article introduces and studies the concept of Cartan-Eilenberg (CE) Gorenstein n -FP-injective and Cartan-Eilenberg Gorenstein n -flat complexes. We prove that every CE- n -FP-injective complex is CE-injective if and only if R is left Noetherian ring. In addition, we show that every CE- n -FP-injective complex of R -modules is CE-Gorenstein n -FP-injective over a Noetherian ring. Finally, we give the relation between the CE-Gorenstein n -FP-injective complex and the CE-Gorenstein n -flat complex.

Keywords. Cartan-Eilenberg n -FP-injective, Cartan-Eilenberg n -flat, Cartan-Eilenberg Gorenstein n -FP-injective, Cartan-Eilenberg Gorenstein n -flat complexes.

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1. INTRODUCTION

The structural concepts of the classical homological algebra is developed by Projective, injective and flat modules. The definitions of projective, injective and flat resolutions of complexes of modules have been first systematically evolved by using Cartan and Eilenberg in Chapter XVII of Homological Algebra [1], thereby laying the groundwork for the homological study of complexes. This terminology and perspective were subsequently adopted and further refined by Verdier within the framework of derived categories. The concepts of Cartan–Eilenberg injective, projective, and flat complexes had been later introduced and studied in [12], by the motivation of these foundational ideas.

Xiaoyan Yang et al. [15] who had contributed to the evolution of the idea of n -flat and n -FP-injective modules in 2010, providing a foundation for subsequent investigations in this direction. These concepts were later extended to complexes of modules by Selvaraj et al. [11] in 2016. This line of research was further advanced by Bo Lu et al. [7] in 2017, who introduced FP-injective structures within the framework of Cartan–Eilenberg complexes.

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In 2008, Mao and Ding [8] independently presented the concepts of Gorenstein FP-injective and Gorenstein flat modules. These ideas were subsequently generalized to complexes of modules by Xin et al. [14] in 2013. The Gorenstein n -flat and Gorenstein n -FP-injective modules theory were further developed by Selvaraj et al. [10] in 2014 and later extended these ideas to the setting of complexes [9].

Motivated by the above developments, Cartan-Eilenberg n -FP-injective and Cartan-Eilenberg n -flat complexes are the main objective of the existing paper and we establish several fundamental results concerning their Gorenstein counterparts.

In [9, Lemma 3.12], the authors gave the relation between the n -FP-injective complex and the injective complex. Further, the subsequent result is derived by establishing the equivalent condition for a Noetherian ring in relation to the CE- n -FP-injective complex and the CE-injective complex.

Theorem. 3.15 *The following conditions are equivalent for an arbitrary ring R :*

- (1) *R is a Noetherian ring.*
- (2) *If X is Cartan-Eilenberg-FP-injective complex, X is Cartan-Eilenberg-injective.*
- (3) *If X is Cartan-Eilenberg- n -FP-injective complex, X is Cartan-Eilenberg-injective.*

Moreover, we provide equivalent criteria for a Noetherian ring with respect to the connection between the CE- n -FP-injective and the CE-injective complex.

Theorem. 4.10 *Let R be an n -coherent ring, then R is a Noetherian ring if and only if every CE- n -FP-injective complex is CE-Gorenstein n -FP-injective.*

The relation between CE-Gorenstein n -flat complexes and CE-Gorenstein n -FP-injective complexes is established using the character complex has been discussed in the following main theorem.

Theorem. 5.7 *Consider an n -coherent ring R and a complex X , then X is CE-Gorenstein n -flat if and only if X^+ is CE-Gorenstein n -FP-injective complex.*

The remainder of this paper is arranged as described below. Section 2 outlines the essential notation and preliminary results related to Cartan-Eilenberg complexes, which lay the groundwork for the discussions that follow.

Section 3 develops the ideas of Cartan-Eilenberg n -FP-injective and n -flat complexes. Within this section, we demonstrated that R is a Noetherian ring if and only if every Cartan-Eilenberg n -FP-injective complex qualifies as a Cartan-Eilenberg injective. Additionally, in the context of an n -coherent ring the characterizations of Cartan-Eilenberg n -FP-injective and Cartan-Eilenberg n -flat complexes are derived.

In Section 4 the idea of Cartan-Eilenberg Gorenstein n -FP-injective complexes is introduced. A key result established here is that, over a Noetherian ring, any Cartan-Eilenberg n -FP-injective complex can be regarded as a Cartan-Eilenberg Gorenstein n -FP-injective.

Section 5 explains the concept of Cartan-Eilenberg Gorenstein n -flat complexes. The relationship of Cartan-Eilenberg Gorenstein n -flat complexes and Cartan-Eilenberg Gorenstein n -FP-injective complexes is thoroughly examined with the aid of the character complex.

2. PRELIMINARIES

This section recalls some essential concepts and terminology needed for the discussion that follows.

Let R be a ring. Following [6], R is said to be left n -coherent for a positive integers n or $n = \infty$, provided that every finitely generated submodule of a free left R -module that has projective dimension at most $n - 1$ is finitely presented. It is worth noting that every ring is left 1-coherent, since every finitely generated projective submodule of a free module is finitely presented. Moreover, if the global dimension of a ring is $d - 1$, then every coherent ring is readily seen to be d -coherent. In accordance with [2], a ring R is called an n -FC ring if R is both left and right coherent ring satisfying $\text{FP-id}({}_R R)$ and $\text{FP-id}(R_R)$ for a non-negative integer $n \geq 0$. With respect to module-theoretic notions, a left R -module A is referred to as n -FP-injective in the sense of [15], if $\text{Ext}_R^1(Q, A) = 0$ for every finitely presented left R -modules Q with $\text{pd}_R Q \leq n$. A right R -module A is called n -flat if $\text{Tor}_1^R(A, Q) = 0$ for every finitely presented left R -module Q satisfying $\text{pd}_R Q \leq n$.

Consider a complex $X = \cdots \rightarrow X_{m+1} \xrightarrow{\delta_{m+1}^X} X_m \xrightarrow{\delta_m^X} X_{m-1} \rightarrow \cdots$, $\text{Ker}(\delta_m^X)$ and $\text{Im}(\delta_{m+1}^X)$ are the m th cycle and m th boundary of X respectively, denoted by $B_m(X)$ and $Z_m(X)$. $B(X)$ and $Z(X)$ are the subcomplexes of boundaries and cycles of the complex X respectively, while the homology of X is expressed as $H(X) = Z(X)/B(X)$. Let X be a complex, the suspension of X , written as ΣX , and defined by $(\Sigma X)_m = X_{m-1}$ and $\delta_m^{\Sigma X} = -\delta_{m-1}^X$. The complex $\Sigma(\Sigma X)$ is represented by $\Sigma^2 X$ and by induction $\Sigma^m X$ is defined for all $m \in \mathbb{Z}$. Throughout this paper, subscripts are employed to differentiate between complexes. Any R -module A can be identified as a complex $\underline{A}$, so that $\underline{A} = \cdots \rightarrow 0 \rightarrow A \rightarrow 0 \rightarrow \cdots$ with A situated in the degree 0. Similar to this, we denote $\overline{A} = \cdots \rightarrow 0 \rightarrow A \rightarrow 0 \rightarrow \cdots$ by A in 1^{st} and 0^{th} degrees.

Let X and Y be two complexes, and the complex of abelian groups is denoted by $\mathcal{H}om(X, Y)$ where $\mathcal{H}om(X, Y)_n = \prod_{t \in \mathbb{Z}} \text{Hom}(X_t, Y_{n+t})$ and let $f \in \mathcal{H}om(X, Y)_n$, then the differential is given by $(\delta_n(f))_m = \delta_{m+n}^Y f_m - (-1)^n f_{m+1} \delta_m^X$. Morphism is the chain map of degree 0, and the abelian group consisting of all such morphisms from X to Y is denoted by $\text{Hom}(X, Y)$. The right derived functors of Hom are generated from the groups Ext^i for each integer $i \geq 0$. For a complex X and Y the tensor product of $\mathbb{Z}$ -modules is given by $\mathbb{Z}$ -modules $X \otimes Y$, where $(X \otimes Y)^n = \bigoplus_{t \in \mathbb{Z}} X_t \otimes Y_{n-t}$ and the differential is defined as $\delta(X \otimes Y) = \delta_t^X(x) \otimes y + (-1)_t x \otimes \delta_{n-t}^Y(y)$ for $x \in X_t$ and $y \in Y_{n-t}$.

General background information on the complexes can be found in [4]. For the rest of the paper, the abbreviation CE, will be used to refer Cartan-Eilenberg. For a complex X , the notation X^+ represents the character complex $\text{Hom}(-, \mathbb{Q}/\mathbb{Z})$ of X and R denotes an associative ring with identity.

Let n be a non-negative integer. A complex X is said to be *Gorenstein n -FP-injective* [9] if there exists an exact sequence of complexes $\cdots \rightarrow E_{-2} \rightarrow E_{-1} \rightarrow E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \cdots$ in which each E_i is injective for all $i \in \mathbb{Z}$ satisfying $\text{Ker}(E_0 \rightarrow E_1) = X$ and $\text{Hom}(E, -)$ remains exact for any n -FP-injective complex E . Let n be a non-negative integer, and X be a complex of right R -modules. The complex X is defined as *Gorenstein n -flat* [9] if there exists an exact sequence of complexes of the right R -modules $\cdots \rightarrow F_{-1} \rightarrow F_0 \rightarrow F_1 \rightarrow \cdots$ in which each F_i is flat for all $i \in \mathbb{Z}$ such that $\text{Ker}(F_0 \rightarrow F_1) = X$ and $- \otimes E$ remains exact for any n -FP-injective complex E .

A complex P is called *CE projective* [3] if $P, B(P), Z(P)$ and $H(P)$ all consist of complexes of projective modules. Similarly, a complex I is said to be *CE injective* if $I, B(I), Z(I)$ and $H(I)$ all consist of complexes of injective modules. Furthermore, a complex F is defined as *CE flat* if $F, B(F), Z(F)$ and $H(F)$ all consist of complexes of flat modules. A complex X is said to be *CE finitely presented* [7] when X is bounded and each of $X_m, Z_m(X), B_m(X), H_m(X)$ is finitely presented in $R\text{-Mod}$ for all $m \in \mathbb{Z}$. Moreover, a CE exact sequence of complexes $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is said to be *CE-pure* [7] whenever the sequence $0 \rightarrow \text{Hom}(Q, A) \rightarrow \text{Hom}(Q, B) \rightarrow \text{Hom}(Q, C) \rightarrow 0$ is said to be exact for any CE finitely presented complex Q .

Definition 2.1. [3] A sequence of complexes $\cdots \rightarrow X_1 \rightarrow X_0 \rightarrow X_{-1} \rightarrow \cdots$ is said to be *CE-exact* if

- (1) $\cdots \rightarrow X_1 \rightarrow X_0 \rightarrow X_{-1} \rightarrow \cdots$
- (2) $\cdots \rightarrow B(X_1) \rightarrow B(X_0) \rightarrow B(X_{-1}) \rightarrow \cdots$
- (3) $\cdots \rightarrow Z(X_1) \rightarrow Z(X_0) \rightarrow Z(X_{-1}) \rightarrow \cdots$
- (4) $\cdots \rightarrow X_1/B(X_1) \rightarrow X_0/B(X_0) \rightarrow X_{-1}/B(X_{-1}) \rightarrow \cdots$
- (5) $\cdots \rightarrow X_1/Z(X_1) \rightarrow X_0/Z(X_0) \rightarrow X_{-1}/Z(X_{-1}) \rightarrow \cdots$
- (6) $\cdots \rightarrow H(X_1) \rightarrow H(X_0) \rightarrow H(X_{-1}) \rightarrow \cdots$

all are exact.

3. CE- n -FLAT AND CE- n -FP-INJECTIVE COMPLEXES

This section introduces the notion of CE- n -flat and CE- n -FP-injective complexes. We now proceed to introduce the definition as follows.

Definition 3.1. Let X be a complex and n be a non-negative integer. Then X is said to be *CE- n -flat* if for every CE finitely presented complexes Q , $\text{Tor}_1(X, Q) = 0$ with CE-projective dimension $Q \leq n$. A complex X is termed as *CE- n -FP-injective* if, for every CE-finitely presented complexes Q , $\text{Ext}^1(Q, X) = 0$ with CE-projective dimension $Q \leq n$.

Remark 3.2. For an n -FP-injective module A , the complexes $\overline{A}$ and $\underline{A}$ are both CE- n -FP-injective. Each CE-FP-injective complex is necessarily a CE- n -FP-injective complex. In an n -coherent ring R and CE- n -FP-injective complex X , for any CE finitely presented complex Q with CE-pd $Q \leq n$ the relation $\text{Ext}^i(N, C) = 0$ holds for all $i \geq 1$.

We now establish that under several conditions the class of all CE- n -FP-injective complex is closed. Let us denote the class of all CE- n -FP-injective complexes by $\mathcal{CE}\text{-}\mathcal{FP}_n$.

Proposition 3.3. *The class $\mathcal{CE}\text{-}\mathcal{FP}_n$ is preserved under CE pure subcomplexes, direct products, and extensions.*

Proof. The class $\mathcal{CE}\text{-}\mathcal{FP}_n$ is shown to be closed under both extensions and direct products. To prove closure under CE-pure subcomplexes, let X be a CE- n -FP-injective and let J be a CE pure subcomplex of X . Considering the induced short exact sequence of complexes

$$0 \rightarrow J \rightarrow X \rightarrow X/J \rightarrow 0,$$

and applying the functor $\text{Hom}(Q, -)$ to this sequence for any CE finitely presented complex Q satisfying CE-projective dimension of Q at most n , which gives the following exact sequence

$$0 \rightarrow \text{Hom}(Q, J) \rightarrow \text{Hom}(Q, X) \rightarrow \text{Hom}(Q, X/J) \rightarrow 0$$

and thus $\text{Ext}^1(Q, J) = 0$, since $\text{Ext}^1(Q, X) = 0$. Therefore J is an CE- n -FP-injective complex. $\square$

Proposition 3.4. *Over an n -coherent ring R , consider an exact sequence of complexes $0 \rightarrow T \rightarrow U \rightarrow V \rightarrow 0$ with T as CE- n -FP-injective. Then U is CE- n -FP-injective if and only if V is CE- n -FP-injective.*

Proof. Consider an exact sequence of complex $0 \rightarrow T \rightarrow U \rightarrow V \rightarrow 0$ where T and U are CE- n -FP-injective complexes. This yields the exact sequence as follows $0 \rightarrow \text{Ext}(Q, V) \rightarrow \text{Ext}^2(Q, T)$, for any CE finitely presented complex Q satisfying CE-pd $Q \leq n$. Hence, we deduce that $\text{Ext}^2(Q, T) = 0$ by Remark 3.2, and thus V is CE- n -FP-injective. The converse is an immediate consequence of the observation that the class $\mathcal{CE}\text{-}\mathcal{FP}_n$ is closed under extensions. $\square$

Let C represent a class of complexes in $\mathcal{C}$. The class C is termed as injectively resolving if it is closed under all injective complexes and for every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with $A \in C$, the conditions $B \in C$ and $C \in C$ are equivalent. The following result follows directly from the Proposition 3.3 and 3.4.

Lemma 3.5. *Consider an n -coherent ring R , then the class $\mathcal{CE}\text{-}\mathcal{FP}_n$ is injectively resolving.*

Key structural properties of CE- n -FP-injective complexes is described in the following Propositions.

Proposition 3.6. *Let X be an exact complex such that each of its cycle is n -FP-injective R -module, then X is CE- n -FP-injective complex.*

Proof. Assume that X is an exact complex. For each $n \in \mathbb{Z}$, we get an exact sequence of R -modules $0 \rightarrow Z^n(X) \rightarrow X^n \rightarrow Z^{n+1}(X) \rightarrow 0$. We obtain that X^n is an n -FP-injective R -module, as the class of all n -FP-injective R -module is closed under extensions. Hence, the proof is completed by invoking [13, Proposition 2.6 and 2.7]. $\square$

Proposition 3.7. *Let X be a CE- n -FP-injective complex, then X is exact and for all $n \in \mathbb{Z}$, $Z^n(X)$ is an n -FP-injective R -module.*

Proof. It results from the isomorphism $\text{Ext}^1(\sum^k(\underline{L}, W)) = \text{Ext}^1(L, Z_n(W))$ and $\text{Ext}^1(\sum^k(\bar{L}, W)) = \text{Ext}^1(L, W_n)$, with L as module, W as a complex and n as an integer [3, Lemma 9.1 and 9.2]. $\square$

Characterization of n -FP-injective complexes based on the exactness and n -FP-injectivity of their components is established as follows.

Theorem 3.8. *The following are equivalent for a complex X :*

- (1) X is CE- n -FP-injective.
- (2) X^n is an n -FP-injective R -module for all $n \in \mathbb{Z}$ and for each CE finitely presented complex Q , $\text{Hom}(Q, X)$ is exact with CE-projective dimension $Q \leq n$.
- (3) X is exact and for all $n \in \mathbb{Z}$, $Z^n(X)$ is CE- n -FP-injective R -module.

Proof. (1) $\Leftrightarrow$ (2) deduced from [13, Proposition 2.6].

(1) $\Leftrightarrow$ (3) results from Proposition 3.6 and Proposition 3.7. $\square$

We now raise two questions: (1) When is the character complex of a CE- n -flat complex CE- n -injective? (2) When is the character complex of an CE- n -injective complex CE- n -flat? We provide an immediate answer to the first question in the following Proposition, and then we give the answer to the second question over a coherent ring in Proposition 3.10.

Proposition 3.9. *A complex X is CE- n -flat if and only if its character complex X^+ is CE- n -FP-injective.*

Proof. This can be directly deduced from [4, Lemma 5.4.2(b)]. $\square$

We now consider the question whether the character complex of CE- n -FP-injective is CE- n -flat? The next proposition provides an affirmative answer.

Proposition 3.10. *Consider an n -coherent ring R . A complex X is CE- n -FP-injective if and only if its character complex X^+ is CE- n -flat.*

Proof. Assume that X is CE- n -FP-injective. Consider Q be a CE finitely presented complex with CE-projective dimension $Q \leq n$. Then $\text{Tor}_1(X^+, Q) \cong \text{Ext}^1(Q, X)^+ = 0$ which gives that X^+ is CE- n -flat. To prove the converse, assume that X^+ is a CE- n -flat. Consequently, we get $\text{Ext}^1(Q, X)^+ \cong \text{Tor}_1(X^+, Q) = 0$, for any CE finitely presented complex Q with CE-projective dimension $Q \leq n$. Therefore, X is CE- n -FP-injective. $\square$

The following Theorem obtained from [6, Theorem 1]

Theorem 3.11. *A ring R is left n -coherent if and only if, $\text{Ext}_R^2(Q, S) = 0$ for every CE- n -FP-injective left R -module S and every CE-finitely presented left R -module Q of CE-projective dimension at most n .*

We are now ready to answer the question: What properties can be attributed to the quotient of a CE- n -FP-injective complex? The next proposition provides an interpretation of n -coherent rings in terms of such quotients.

Proposition 3.12. *Let R be a ring. Then the following assertions are equivalent:*

- (1) R is an n -coherent ring.
- (2) If X is any CE- n -FP-injective complex, for any CE-pure subcomplex W of X , X/W is CE- n -FP-injective.
- (3) If X is any CE-injective complex, for any CE-pure subcomplex W of X , X/W is CE- n -FP-injective.

Proof. (1) $\Rightarrow$ (2). Consider a CE- n -FP-injective complex X and W a CE-pure subcomplex of X . It follows that we have an exact sequence of complexes $0 \rightarrow W \rightarrow X \rightarrow X/W \rightarrow 0$ for any CE finitely presented complex Q with CE-projective dimension $Q \leq n$, which gives an exact sequence $0 = \text{Ext}^1(Q, X) \rightarrow \text{Ext}^1(Q, X/W) \rightarrow \text{Ext}^2(Q, W)$. By Remark 3.2 $\text{Ext}^2(Q, W) = 0$, therefore we have $\text{Ext}^1(Q, X/W) = 0$. Hence, X/W is CE- n -FP-injective.

(2) $\Rightarrow$ (3) is immediate.

(3) $\Rightarrow$ (1). Assume that X is a CE-injective complex and consider W be any CE-pure subcomplex of X . Hence we obtain $0 \rightarrow W \rightarrow X \rightarrow X/W \rightarrow 0$ an exact sequence that gives the exactness of $\text{Ext}^1(Q, X/W) \rightarrow \text{Ext}^2(Q, W) \rightarrow \text{Ext}^2(Q, X) = 0$ for any CE finitely presented complex Q with CE-projective dimension $Q \leq n$. Since $\text{Ext}^1(Q, X/W) = 0$ by (3) we have $\text{Ext}^2(Q, W) = 0$. Therefore, by Theorem 3.11 R is an n -coherent ring. $\square$

Proposition 3.13. *A complex X is CE- n -flat if and only if for every $i \in \mathbb{Z}$, X^i is an n -flat module and for all CE finitely presented complexes Q , $Q \otimes X$ is exact with CE-projective dimension $Q \leq n$.*

Proof. Assume that X is CE- n -flat. Then, X^+ is CE- n -FP-injective by Proposition 3.9 and by [13, Proposition 2.6] $(X^i)^+$ is an n -FP-injective R -module, for $i \in \mathbb{Z}$. Hence, X^i is n -flat by Proposition 3.9. Next, consider Q to be any CE finitely presented complex with CE-projective dimension $Q \leq n$. Then $(Q \otimes X)^+ \cong \text{Hom}(Q, X^+)$ is exact and therefore $Q \otimes X$ is exact.

For the converse, consider for every $i \in \mathbb{Z}$, each X^i is an n -flat right R -module and $Q \otimes X$ is exact for all CE finitely presented complexes Q with CE-projective dimension $Q \leq n$. Then by [13, Proposition 2.6] $(X^i)^+$ is an n -FP-injective R -module for $i \in \mathbb{Z}$ and $(Q \otimes X)^+$ is exact. Therefore, $\text{Hom}(Q, X^+) \cong (Q \otimes X)^+$ is exact and by Proposition [13, Proposition 2.6], we conclude that X^+ is CE- n -FP-injective. Hence, X is CE- n -flat. $\square$

We now examine the link between the CE- n -FP-injective and its character complex and equivalent condition for n -coherent rings.

Theorem 3.14. *The following are equivalent for a ring R and a complex C :*

- (1) R is an n -coherent ring.
- (2) X is CE- n -FP-injective if and only if X^+ is a CE- n -flat complex of right R -modules.
- (3) X is CE- n -FP-injective if and only if X^{++} is CE- n -FP-injective.

Proof. (1) $\Rightarrow$ (2) is an immediate result of Proposition 3.10.

(2) $\Leftrightarrow$ (3) is immediate from Proposition 3.9.

(2) $\Rightarrow$ (1). Consider CE-injective complex E and let S be a CE-pure subcomplex of CE-injective complex E . Then the CE-pure exact sequence $0 \rightarrow S \rightarrow E \rightarrow E/S \rightarrow 0$ gives $0 \rightarrow (E/S)^+ \rightarrow E^+ \rightarrow S^+ \rightarrow 0$ CE-pure exact sequence. From this we get E^+ is CE- n -flat by (2). Since $(E/S)^+$ is the CE-pure subcomplex of E^+ , by Proposition 3.3 $(E/S)^+$ is CE- n -flat. Hence, by (2), E/S is CE- n -FP-injective. Therefore, R is an n -coherent ring, by Proposition 3.12. $\square$

We now establish the main theorem of this section. We give the equivalent condition for a Noetherian ring in relation to the CE- n -FP-injective complex and the CE-injective complex.

Theorem 3.15. *The following are equivalent for a ring R :*

- (1) R is a Noetherian ring.
- (2) Every CE-FP-injective complex is CE-injective.
- (3) Every CE- n -FP-injective complex is CE-injective.

Proof. (1) $\Rightarrow$ (2). Assume that X is CE-FP-injective complex. Therefore, X is exact and $Z^n(X)$ is a FP-injective R -module. Since R is Noetherian, we have that $Z^n(X)$ is an injective R -module. Hence, X is CE-injective complex.

(2) $\Rightarrow$ (3). Assume that X is CE- n -FP-injective complex. Therefore, X is exact and $Z^n(X)$ is a n -FP-injective left R -module. Since R is the Noetherian ring, by [15, Proposition 3.4] we get that $Z^n(X)$ is a FP-injective R -module. Hence, by hypothesis $Z^n(X)$ is an injective R -module. Thus, X is CE-injective complex.

(3) $\Rightarrow$ (2). It is immediate from Remark 3.2.

(2) $\Rightarrow$ (1). Consider a FP-injective R -module A . Therefore, $\bar{A}$ is CE-FP-injective complex. By hypothesis, $\bar{A}$ is CE-injective complex. Thus A is an injective R -module. Thus, R is a Noetherian ring. $\square$

4. CE-GORENSTEIN n -FP-INJECTIVE COMPLEX

The focus of this section is the CE-Gorenstein n -FP-injective notion, along with various equivalent characterizations of CE-Gorenstein n -FP-injective complexes. To this end, we begin with the definition below.

Definition 4.1. Let X be a complex and n be a non-negative integer. Then X is defined as CE-Gorenstein n -FP-injective if there exists an exact sequence $\cdots \rightarrow E_{-1} \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots$ of CE-injective complexes such that $X \cong \text{Ker}(E_{-1} \rightarrow E_0)$ and $\text{Hom}_R(E, -)$ remains exact for any CE- n -FP-injective complex E .

Remark 4.2. (1) If a complex X is an CE-injective complex, then X is an CE Gorenstein n -FP-injective.

- (2) Suppose X is a CE-Gorenstein n -FP-injective complex, then for all $i \geq 1$, $\text{Ext}^i(E, X) = 0$ with E as CE- n -FP-injective complex.

In the following example, we consider the totally acyclic complex with respect to CE- n -FP-injective complexes. An acyclic complex $I^\bullet$ of CE-injective complexes is called *totally acyclic* if the complex $\text{Hom}(E, I^\bullet)$ remains exact for any CE- n -FP-injective complex E .

Example 4.3. (1) Assume that H is a Gorenstein n -FP-injective R -module. As an immediate consequence, we see that the complex $\overline{H} : \cdots \rightarrow 0 \rightarrow H \rightarrow 0 \rightarrow \cdots$ is CE-Goresntein n -FP-injective.
 (2) Consider a totally acyclic double complex $I^{\bullet\bullet}$ of CE-injective complexes:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & I^{p-1,q+1} & \xrightarrow{d'} & I^{p,q+1} & \xrightarrow{d'} & I^{p+1,q+1} \longrightarrow \cdots \\
 & & \downarrow d'' & & \downarrow d'' & & \downarrow d'' \\
 \cdots & \longrightarrow & I^{p-1,q} & \xrightarrow{d'} & I^{p,q} & \xrightarrow{d'} & I^{p+1,q} \longrightarrow \cdots \\
 & & \downarrow d'' & & \downarrow d'' & & \downarrow d'' \\
 \cdots & \longrightarrow & I^{p-1,q-1} & \xrightarrow{d'} & I^{p,q-1} & \xrightarrow{d'} & I^{p+1,q-1} \longrightarrow \cdots \\
 & & \downarrow d'' & & \downarrow d'' & & \downarrow d'' \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

such that

$$\text{Tot}(I)^n = \bigoplus_{p+q=n} I^{p,q} \quad \text{with filtration} \quad F^r \text{Tot}^n = \bigoplus_{i \geq r} I^{i,n-i}.$$

From the following Cartan-Eilenberg structure diagram

$$\begin{array}{ccc}
 Z^{p,q} & \hookrightarrow & I^{p,q} \\
 \downarrow & & \downarrow \\
 H^{p,q} & \hookrightarrow & I^{p,q}/B^{p,q},
 \end{array}$$

$Z^{p,q}$, $H^{p,q}$ and $I^{p,q}/B^{p,q}$ are clearly n -FP-injective R -modules. Hence, $Z^{\bullet\bullet}$

and $B^{\bullet\bullet}$ are CE-Gorenstein n -FP-injective complexes.

The class of all CE-Gorenstein n -FP-injective complexes is referred and denoted by $\mathcal{CE}\text{-}\mathcal{GFP}_n$.

Proposition 4.4. *The class $\mathcal{CE}\text{-}\mathcal{GFP}_n$ is closed under direct products.*

Proof. This can be deduced from [16, Proposition 2.4] and Remark 4.2(1). $\square$

We now state a theorem that provides a comprehensive description of Cartan-Eilenberg Gorenstein- n -FP-injective complexes.

Proposition 4.5. *The following are equivalent for any complex X :*

- (1) X is CE Gorenstein n -FP-injective.

- (2) (a) For all $i \geq 1$, $\text{Ext}^i(E, X) = 0$ where E is CE- n -FP-injective complex.
 (b) There exists an exact sequence of complexes $\mathbb{E} : \cdots \rightarrow E_2 \rightarrow E_1 \rightarrow X \rightarrow 0$ with E_i CE-injective for $i \geq 1$, and $\text{Hom}(E, \mathbb{E})$ is exact for any CE-FP-injective complex E .
 (3) There exists a short exact sequence of complexes $0 \rightarrow U \rightarrow E \rightarrow X \rightarrow 0$, where E is CE-injective and U is a CE-Gorenstein n -FP-injective.

Proof. The implications (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3) follow from the Definition 4.1.

(3) $\Rightarrow$ (2). Suppose there exists an exact sequence of complexes $0 \rightarrow U \rightarrow E \rightarrow X \rightarrow 0$ where E is CE-injective and U is CE-Gorenstein n -FP-injective. Now, by (1) $\Leftrightarrow$ (2) we have for any CE- n -FP-injective complex E' , $\text{Ext}^i(E', U) = 0$ for all $i \geq 1$. Thus, the following sequence $\cdots \rightarrow \text{Ext}^i(E', E) \rightarrow \text{Ext}^i(E', X) \rightarrow \text{Ext}^{i+1}(E', U) \rightarrow \cdots$ is exact that gives $\text{Ext}^i(E', X) = 0$ for any CE- n -FP-injective complex E' , for all $i \geq 1$. Whereas, consider a CE-Gorenstein n -FP-injective complex U , this yields the following exact sequence $\mathbb{E} : \cdots \rightarrow E_2 \rightarrow E_1 \rightarrow U \rightarrow 0$ such that for any CE- n -FP-injective complex E' , $\text{Hom}(E', \mathbb{E})$ is exact. By merging the exact sequence $\mathbb{E}$ and $0 \rightarrow U \rightarrow E \rightarrow X \rightarrow 0$ we get the following exact sequence

$$\cdots \rightarrow E_2 \rightarrow E_1 \rightarrow E \rightarrow X \rightarrow 0$$

and for any CE-FP-injective complex E' , $\text{Hom}(E', -)$ is exact. $\square$

One may naturally ask: does the CE-Gorenstein n -FP-injective property remain preserved when CE-injective complexes are involved? Here, the next proposition gives the answer.

Proposition 4.6. *Consider an exact sequence $0 \rightarrow C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow 0$ of complexes. Suppose C_1 is CE-injective and C_3 is CE-Gorenstein n -FP-injective, then C_2 is also CE-Gorenstein n -FP-injective.*

Proof. Consider a CE-Gorenstein n -FP-injective complex C_3 . There exists a short exact sequence $0 \rightarrow U \rightarrow E \rightarrow C_3 \rightarrow 0$ of complexes where U is CE-Gorenstein n -FP-injective and E is CE-injective by Proposition 4.5. Let us examine the pullback of $C_2 \rightarrow C_3$ and $E \rightarrow C_3$

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & U & \xlongequal{\quad} & U & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & C_1 & \longrightarrow & E' & \longrightarrow & E \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Consequently, the exact sequence yields $0 \rightarrow C_1 \rightarrow E' \rightarrow E \rightarrow 0$ since E and C_1 are CE-injective, then E' is CE-injective. Thus we have $0 \rightarrow U \rightarrow E' \rightarrow C_2 \rightarrow 0$ a short exact sequence, where U is CE-Gorenstein n -FP-injective and E' is CE-injective. Therefore, by Proposition 4.1 C_2 is CE-Gorenstein n -FP-injective. $\square$

Proposition 4.7. *The following are equivalent for an n -coherent ring R :*

- (1) X is a CE Gorenstein n -FP-injective complex.
- (2) For all $i \geq 1$, $\text{Ext}^i(E, X) = 0$ with E CE- n -FP-injective complexes.

Proof. (1) $\Rightarrow$ (2). This implication results from Remark 4.2.

(2) $\Rightarrow$ (1). This implication is similar to [14, Theorem 3.3]. $\square$

Proposition 4.8. *Consider $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ an exact sequence of complexes. Then*

- (1) *If X' and X'' are CE Gorenstein n -FP-injective, then X is CE Gorenstein n -FP-injective.*
- (2) *If X' and X are CE Gorenstein n -FP-injective, then X'' is CE Gorenstein n -FP-injective.*

Proof. (1). Consider a CE-exact sequence of complexes $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ where X' and X'' are CE-Gorenstein n -FP-injective. There exist an exact sequences of complexes $\mathbf{K} : \cdots \rightarrow K_2 \rightarrow K_1 \rightarrow X'' \rightarrow 0$ and $\mathbf{E} : \cdots \rightarrow E_2 \rightarrow E_1 \rightarrow X' \rightarrow 0$ with E_i and K_i are injective for all $i \geq 1$, such that $\text{Hom}(E, \mathbf{K})$ and $\text{Hom}(E, \mathbf{E})$ are exact for any CE- n -FP-injective complex E by Proposition 4.5. Applying the Horseshoe Lemma, we have the following exact sequence of complexes $\mathbf{L} : \cdots \rightarrow E_2 \oplus K_2 \rightarrow E_1 \oplus K_1 \rightarrow X \rightarrow 0$ such that for all $i \geq 1$, $E_i \oplus K_i$ is injective. For any CE- n -FP-injective complex E , we have the exactness of $\text{Hom}(E, \mathbf{L})$ that comes from the exactness of $\text{Hom}(E, \mathbf{E})$ and $\text{Hom}(E, \mathbf{K})$. However, for all $i \geq 1$, $\text{Ext}^i(E, X) = 0$ since $\text{Ext}^i(E, X') = 0$ and $\text{Ext}^i(E, X'') = 0$ for any CE- n -FP-injective complex E . Therefore, by Proposition 4.5, X is CE-Gorenstein n -FP-injective.

(2). Assume that X' and X are CE-Gorenstein n -FP-injective. There exists a short exact sequence of complexes $0 \rightarrow U \rightarrow E \rightarrow X \rightarrow 0$ where E is CE-injective and U is CE-Gorenstein n -FP-injective by Proposition 4.5. Form the pullback diagram of $X' \rightarrow X$ and $E \rightarrow X$

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & U & \xlongequal{\quad} & U & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & X'' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

This yields the following short exact sequence $0 \rightarrow U \rightarrow E' \rightarrow X' \rightarrow 0$, since X' and U are CE-Gorenstein n -FP-injective, by (1) it is clear that E' is CE-Gorenstein n -FP-injective. Thus we have the following short exact sequence $0 \rightarrow E' \rightarrow E \rightarrow X'' \rightarrow 0$ of complexes, where E is CE-injective and E' is CE-Gorenstein n -FP-injective. Hence by Proposition 4.5, X'' is CE-Gorenstein n -FP-injective. $\square$

Proposition 4.9. *The class $\mathcal{CE}\text{-}\mathcal{GFP}_n$ is injectively resolving and closed under direct summands, over an n -coherent ring R .*

Proof. It can be inferred that the class $\mathcal{CE}\text{-}\mathcal{GFP}_n$ is injectively resolving by Remark 4.2(1) and from Proposition 4.8. However, it is clear that the class $\mathcal{CE}\text{-}\mathcal{GFP}_n$ is closed under direct summands by Proposition 4.4 and [5, Proposition 1.4]. This concludes the proof. $\square$

A natural question that arises in this context concerns the precise conditions under which CE- n -FP-injectivity and CE-Gorenstein n -FP-injectivity coincide.

Theorem 4.10. *For any n -coherent ring R , R is a Noetherian ring if and only if every CE- n -FP-injective complex is CE-Gorenstein n -FP-injective.*

Proof. The direct implication follows from Theorem 3.15. For the converse part, consider a CE- n -FP-injective complex X . There exists an CE exact sequence $\mathbf{B} : 0 \rightarrow X \rightarrow E \rightarrow U \rightarrow 0$ of complexes, where E is CE-injective, since X is CE-Gorenstein n -FP-injective. Then, by Lemma 3.5, U is CE- n -FP-injective. Hence, by hypothesis and Remark 4.2, we obtain $\text{Ext}^1(U, X) = 0$. Thus, the complex $\mathbf{B}$ splits and hence X is CE-injective. Therefore, by Theorem 3.15, R is Noetherian. $\square$

5. CE-GORENSTEIN n -FLAT COMPLEX

We explore the concepts of CE-Gorenstein n -FP-injective and CE-Gorenstein n -flat complexes through the framework of character complexes in this section. We proceed to formulate the definition of the concept under consideration.

Definition 5.1. Let X be a complex and n non-negative integer. Then X is defined as CE-Gorenstein n -flat, if there is an exact sequence $\cdots \rightarrow F_{-1} \rightarrow F_0 \rightarrow F_1 \rightarrow \cdots$ of CE-flat complexes such that $X \cong \text{Ker}(F_0 \rightarrow F_1)$ and $-\otimes_R E$ remains exact for any CE- n -FP-injective complex E .

Remark 5.2. (1) If a complex X is CE-flat, then X is CE-Gorenstein n -flat complex.

(2) Suppose X is a CE-Gorenstein n -flat complex, then for every $i \geq 1$, we have $\text{Tor}_i(X, E) = 0$ with E CE- n -FP-injective complex.

Example 5.3. Let $F^\bullet : \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$ be a totally acyclic complex of CE-flat complexes such that $-\otimes E$ remains exact for all CE-FP-injective complexes E . Since the CE-FP-injective complex is CE- n -FP-injective, it follows that each $Z_n(F^\bullet)$ is Gorenstein n -flat and $B_n(F^\bullet)$ is Goresntein n -flat according to the definition. Since $B_n(F^\bullet) = Z_n(F^\bullet)$, $H_n(F^\bullet) = 0$ for all n . Therefore, the complex $F^\bullet$ is Cartan-Eilenberg Gorenstein n -flat.

Proposition 5.4. *Let R be a Noetherian ring. Then a complex X is CE-Gorenstein n -flat if and only if CE-Gorenstein flat.*

Proof. Every injective R -module is n -FP-injective for any $n \geq 0$. Then the tensor exactness condition required in the definition of CE- n -flat complexes automatically implies the corresponding condition for injective R -modules. Hence, every CE-Gorenstein n -flat complex is CE-Gorenstein flat. The converse part immediately follows from Theorem 3.15. $\square$

The class of all CE-Gorenstein n -flat complexes is denoted by $\mathcal{CE}\text{-}\mathcal{GF}_n$.

Proposition 5.5. *The class $\mathcal{CE}\text{-}\mathcal{GF}_n$ is closed under direct sums.*

Proof. The argument used in [9, Proposition 4.2] extends to yield the following result. $\square$

Lemma 5.6. *For an n -coherent ring R , the following assertions are equivalent:*

- (1) *X is a CE-Gorenstein n -flat complex of right R -modules.*
- (2) *For all $i \geq 1$, $\text{Tor}_i(X, E) = 0$ with E CE- n -FP-injective complex.*

Proof. (1) $\Rightarrow$ (2) This implication is deduced from Remark 5.2.

(2) $\Rightarrow$ (1) This implication proceeds as in [14, Theorem 3.3]. $\square$

The relationship between CE-Gorenstein n -flat complexes and CE-Gorenstein n -FP-injective complexes is established using the character complex has been discussed in the following main theorem

Theorem 5.7. *Consider an n -coherent ring R and a complex X , X is CE-Gorenstein n -flat if and only if X^+ is CE-Gorenstein n -FP-injective.*

Proof. Suppose X is CE-Gorenstein n -flat complex. Then by definition 5.1, there is an exact sequence $\mathbb{F} : \cdots \rightarrow F_{-1} \rightarrow F_0 \rightarrow F_1 \rightarrow \cdots$ of CE-flat complexes of right R -modules with $X = \text{Ker}(F_0 \rightarrow F_1)$ and for any CE- n -FP-injective complex E , $\mathbb{F} \otimes E$ is exact. From this we get the following exact sequence $\mathbb{F}^+ : \cdots \rightarrow F_1^+ \rightarrow F_0^+ \rightarrow F_{-1}^+ \rightarrow \cdots$ of CE-injective complexes with $X^+ = \text{Ker}(F_0^+ \rightarrow F_{-1}^+)$. Let U be a CE- n -FP-injective complex. We now obtain the exact sequence $\cdots \rightarrow F_{-1} \otimes U \rightarrow F_0 \otimes U \rightarrow F_1 \otimes U \rightarrow \cdots$ which induces the exact sequence constructed as follows $\cdots \rightarrow (F_1 \otimes U)^+ \rightarrow (F_0 \otimes U)^+ \rightarrow (F_{-1} \otimes U)^+ \rightarrow \cdots$. Thus, the resulting sequence $\cdots \rightarrow \text{Hom}(U, F_1^+) \rightarrow \text{Hom}(U, F_0^+) \rightarrow \text{Hom}(U, F_{-1}^+) \rightarrow \cdots$ is exact. Therefore, X^+ is CE-Gorenstein n -FP-injective.

As for the converse, assume that X^+ is CE-Gorenstein n -FP-injective. Then for every $i \geq 1$, we have $\text{Ext}^i(E, X^+) = 0$ with CE- n -FP-injective complexes E . Consider a CE-FP-injective complex E . Then for any integer m , $E[m]$ is n -FP-injective. Now, consider an integer m . We get, $\text{Ext}^i(E, X^+[m]) \cong \text{Ext}^i(E[-m], X^+) = 0$, which implies that for all $i \geq 1$, $\text{Ext}^i(E, X^+) = 0$. Thus $\text{Tor}_i(E, X)^+ \cong \text{Ext}^i(E, X^+) = 0$. Hence for all $i \geq 1$, $\text{Tor}_i(E, X) = 0$. Therefore, by Lemma 5.6, X is CE-Gorenstein n -flat. $\square$

Proposition 5.8. *Let R be an n -FC-ring. Then the class $\mathcal{CE}\text{-}\mathcal{GF}_n$ is closed under CE-pure subcomplexes.*

Proof. Assume that U is a CE-pure subcomplex of a CE-Gorenstein n -flat complex V . Now we get the CE-pure exact sequence $0 \rightarrow U \rightarrow V \rightarrow V/U \rightarrow 0$ of complexes. This induces a split exact sequence $0 \rightarrow (V/U)^+ \rightarrow V^+ \rightarrow U^+ \rightarrow 0$. By Theorem 5.7, we obtain that V^+ is CE-Gorenstein n -FP-injective, since V is CE-Gorenstein n -flat. Consequently, that U^+ is CE-Gorenstein n -FP-injective. Therefore, U is CE-Gorenstein n -flat by Theorem 5.7. $\square$

The next theorem gives a description of Cartan-Eilenberg Gorenstein- n -flat complex.

Proposition 5.9. *Let X be any complex. Then the following are equivalent:*

- (1) X is CE-Gorenstein n -flat.
- (2) (a) For all $i \geq 1$, $\text{Tor}_i(X, E) = 0$ with CE- n -FP-injective complexes E .
 (b) There exists an exact sequence of complexes of right R -modules $\mathbb{F} : 0 \rightarrow X \rightarrow F_1 \rightarrow F_2 \rightarrow \cdots$ with F_i flat for $i \geq 1$ such that $\mathbb{F} \otimes E$ is exact for any CE- n -FP-injective complex E .
- (3) There exists a short exact sequence of complexes of right R -modules $0 \rightarrow X \rightarrow F \rightarrow U \rightarrow 0$, where F is CE-flat and U is CE-Gorenstein n -flat.

Proof. The implications (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3) follows from the Definition 5.1.

(3) $\Rightarrow$ (2). Consider an exact sequence $\mathbb{X} : 0 \rightarrow X \rightarrow F \rightarrow C \rightarrow 0$ of complexes with F CE-flat and U CE-Gorenstein n -flat. Then by (1) $\Leftrightarrow$ (2), we have $\text{Tor}_i(U, E) = 0$ with CE- n -FP-injective complex E , for all $i \geq 1$. Thus, we have the following exact sequence $\cdots \rightarrow \text{Tor}_{i+1}(X, E) \rightarrow \text{Tor}_i(X, E) \rightarrow \text{Tor}_i(F, E) \rightarrow \cdots$ that gives $\text{Tor}_i(X, E) = 0$ for all $i \geq 1$ with CE- n -FP-injective complex E . However, with CE-Gorenstein n -flat complex U , we get the following exact sequence $\mathbb{F}' : 0 \rightarrow U \rightarrow F_1 \rightarrow F_2 \rightarrow \cdots$ such that for any CE- n -FP-injective complex E , $\mathbb{F}' \otimes E$ is exact. By merging the exact sequences $\mathbb{F}'$ and $\mathbb{X}$ we get the following exact sequence

$$0 \rightarrow X \rightarrow F \rightarrow F_1 \rightarrow F_2 \rightarrow \cdots$$

such that for any CE- n -FP-injective complex E , $- \otimes E$ is exact. $\square$

Corollary 5.10. *Consider an exact sequences of complexes $0 \rightarrow C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow 0$. If C_1 is CE-Gorenstein n -flat and C_3 is CE-flat, C_2 is CE-Gorenstein n -flat.*

Proof. Consider C_1 is CE-Gorenstein n -flat. There is a short exact sequence of complexes $0 \rightarrow C_1 \rightarrow F \rightarrow C \rightarrow 0$ with F is CE-flat and C is CE-Gorenstein

n -flat by Proposition 5.9. Construct the pushout of $C_1 \rightarrow C_2$ and $C_1 \rightarrow F$:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & F & \longrightarrow & F' & \longrightarrow & C_3 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & C & \xlongequal{\quad} & C & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

This can be deduced from the exact sequence $0 \rightarrow F \rightarrow F' \rightarrow C_3 \rightarrow 0$ where F' is CE-flat, since F and C_3 are CE-flat. Thus we obtain a short exact sequence $0 \rightarrow F \rightarrow F' \rightarrow C_3 \rightarrow 0$ with F' is CE-flat since F and C_3 are CE-flat. Therefore, we get a short exact sequence $0 \rightarrow C_2 \rightarrow F' \rightarrow C \rightarrow 0$ with F' CE-injective and C is CE-Gorenstein n -flat. Hence, C_2 is CE-Gorenstein n -flat, by Theorem 5.7. $\square$

The result given below shows that CE-Gorenstein n -flatness is preserved under the tensor product with CE-flat complexes over a commutative ring.

Proposition 5.11. *Consider a commutative ring R and a CE-flat complex F . If X is a CE-Gorenstein n -flat complex, $X \otimes F$ is also a CE-Gorenstein n -flat complex.*

Proof. Suppose X is CE-Gorenstein n -flat. Then there exists an exact sequence of flat complexes $\cdots \rightarrow F_{-1} \rightarrow F_0 \rightarrow F_1 \rightarrow \cdots$ with $X = \text{Ker}(F_0 \rightarrow F_1)$ such that $-\otimes E$ leaves the sequence exact for any CE- n -FP-injective complex E . Since F is CE-flat complex and since the tensor product of the CE-flat complexes is CE-flat, we have the following exact sequence of complexes $\cdots \rightarrow F \otimes F_{-1} \rightarrow F \otimes F_0 \rightarrow F \otimes F_1 \rightarrow \cdots$ with $F \otimes F_i$ CE-flat for all $i \in \mathbb{Z}$ and $F \otimes X = \text{Ker}(F \otimes F_0 \rightarrow F \otimes F_1)$. Let E be any CE- n -FP-injective complex. Then by [4, Proposition 4.2.1(3)], we have the following commutative diagram:

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & (F \otimes F_{-1}) \otimes E & \longrightarrow & (F \otimes F_0) \otimes E & \longrightarrow & (F \otimes F_1) \otimes E \longrightarrow \cdots \\
 & & \parallel & & \parallel & & \parallel \\
 \cdots & \longrightarrow & F \otimes (F_{-1} \otimes E) & \longrightarrow & F \otimes (F_0 \otimes E) & \longrightarrow & F \otimes (F_1 \otimes E) \longrightarrow \cdots
 \end{array}$$

As the lower sequence is exact by hypothesis, it follows that the upper sequence is also exact. Therefore, $X \otimes F$ is a CE-Gorenstein n -flat complex. $\square$

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REFERENCES

1. Cartan, H., and Eilenberg, S. (1999). Homological algebra (Vol. 19). Princeton university press.
2. Ding, N., and Chen, J. (1996). Coherent rings with finite self-FP-injective dimension. *Comm. Algebra*, 24(9), 2963–2980. <https://doi.org/10.1080/00927879608825724>
3. Enochs, E. E. (2011). Cartan–Eilenberg complexes and resolutions, *J. Algebra*, 342(1), 16–39. <https://doi.org/10.1016/j.jalgebra.2011.05.011>
4. Garcia Rozas, J. R. (1999). Covers and Envelopes in the Category of Complexes of Modules, CRC Press, Boca Raton, FL. <https://doi.org/10.1201/9781315140438>
5. Holm, H. (2004). Gorenstein homological dimensions, *J. Pure Appl. Algebra* 189, 542–558. <https://doi.org/10.1016/j.jpaa.2003.11.007>
6. Lee, S. B. (2002). n -coherent rings, *Comm. Algebra*, 30 (3), 1119–1126. <https://doi.org/10.1080/00927870209342374>
7. Lu, B., & Liu, Z. (2017). Cartan Eilenberg FP-injective complexes, *J. Aus. Math. Soc.*, 103(3), 387–401. <https://doi.org/10.1017/S1446788716000586>
8. Mao, L., & Ding, N. (2008). Gorenstein FP-injective and Gorenstein flat modules. *J. Algebra Appl.* 7(4), 491–506. <https://doi.org/10.1142/S0219498808002953>
9. Selvaraj, C., & Saravanan, R. (2018). Gorenstein n -FP-injective and Gorenstein n -flat complexes. *Boll Unione Mat Ital*, 11, 147–161. <https://doi.org/10.1007/s40574-016-0113-x>
10. Selvaraj, C., Udhayakumar, R., & Umamaheswaran, A. (2014). Gorenstein n -flat modules and their covers. *Asian Eur. J. Math.* 7 (3), 13 p. <https://doi.org/10.1142/S179355711450051X>
11. Selvaraj, C., & Saravanan, R. (2016). n -FP-injective and n -flat complexes. *Gulf J. Math.* 4(1), 86–97. <https://doi.org/10.56947/gjom.v4i1.59>
12. Verdier, J. L. (1996). Des catégories dérivées des catégories abéliennes, *Astérisque*, 239, 269p. https://www.numdam.org/item/AST_1996__239__R1_0/
13. Wang, Z. & Liu, Z. (2011). FP-injective complexes and FP-injective dimension of complexes, *J. Aust. Math. Soc.*, 91, 163–187. <https://doi.org/10.1017/S1446788711001364>
14. Xin, D., Chen, J. & Zhang, X. (2013). On Gorenstein FP-injective complexes and Gorenstein flat complexes. *Comm. Algebra*, 41 (4), 1247–1267. <https://doi.org/10.1080/00927872.2012.660260>
15. Yang, X., & Liu, Z. (2011). n -flat and n -FP-injective modules, *Czechoslovak Math. J.* 61(136), 359–369. <https://doi.org/10.1007/s10587-011-0080-4>
16. Zhai, X., & Zhang, C. (2016). Cartan-Eilenberg projective, injective and flat complexes, *Czechoslovak Math. J.*, 66 (141), 151–167. <https://doi.org/10.1007/s10587-016-0247-0>.

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