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ON CERTAIN BI-UNIVALENT FUNCTIONS DEFINED VIA THE STRUVE FUNCTION AND SYMMETRIC POINT STRUCTURE

SARA FALIH MAKTOOF[1], SARAH JALAWI ABD[2] and ADEL SALIM TAYYAH[3*]

Abstract. The theory of bi-univalent functions has attracted considerable attention due to its rich geometric structure and its connections with various special functions. However, the interaction between bi-univalent function theory, symmetric point structures, and the Struve function has not been sufficiently explored in the existing literature. Motivated by this gap, we introduce a new class of bi-univalent functions associated with symmetric points through an operator involving the Struve function. For functions belonging to this class, we establish estimates for the initial Taylor–Maclaurin coefficients and examine the corresponding Fekete–Szegö inequality. Furthermore, several consequences and special cases of the main results are discussed, which illustrate the effectiveness of the proposed approach and lead to improved bounds for the first coefficients. The results obtained here provide an additional link between geometric function theory and special functions and may stimulate further developments in this direction.

Keywords. Struve function, Bi-univalent functions, starlike functions, symmetric starlike functions.
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1. Introduction

Let $\mathbb{A}$ denote the set of all functions that are analytic within the open unit disk $\mathbb{O} = \{\delta \in \mathbf{C} : |\delta| < 1\}$. Within this family, we define the subfamily $\mathbb{S}$ as consisting of functions that are univalent in $\mathbb{O}$ and satisfy the standard normalization conditions

$$\chi(0) = 0 \quad \text{and} \quad \chi'(0) = 1,$$

with the series representation

$$\chi(\delta) = \delta + \sum_{n=2}^{\infty} a_n \delta^n. \tag{1.1}$$

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* Corresponding author
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A function χ belonging to the family $\mathbb{A}$ is called *subordinate* to an analytic function ξ defined on $\mathbb{O}$, and we write $\chi \prec \xi$, if there exists an analytic mapping $\omega : \mathbb{O} \to \mathbb{O}$ such that $\omega(0) = 0$ and $|\omega(\delta)| < 1$ for all $\delta \in \mathbb{O}$, with the property that

$$\chi(\delta) = \xi(\omega(\delta)).$$

According to the classical Schwarz lemma, this notion of subordination is equivalent to the conditions

$$\chi(0) = \xi(0) \quad \text{and} \quad \chi(\mathbb{O}) \subseteq \xi(\mathbb{O}),$$

holding for every $\delta \in \mathbb{O}$ (see [7, 15] for further details). We now recall several significant subfamilies of $\mathbb{S}$ that are frequently studied in geometric function theory. Let $\chi \in \mathbb{A}$, and consider the subordinations

$$\frac{\delta\chi'(\delta)}{\chi(\delta)} \prec p(\delta) \quad \text{and} \quad 1 + \frac{\delta\chi''(\delta)}{\chi'(\delta)} \prec p(\delta),$$

where

$$p(\delta) = \frac{1+\delta}{1-\delta}.$$

In this context, the function χ is referred to as *starlike* if it satisfies the first subordination, and as *convex* if it satisfies the second. The corresponding families of functions are the classical families denoted by $\mathbb{S}^*$ and $\mathbb{C}$, respectively.The family $\mathbb{L}_\varrho(\varepsilon)$, consisting of ϱ-pseudo-starlike functions of order ε with $0 \leq \varepsilon < 1$, was introduced and analyzed by Babalola [2]. The defining geometric condition for these functions is given by

$$\mathrm{Re}\left\{\frac{\delta\big(\chi'(\delta)\big)^{\varrho}}{\chi(\delta)}\right\} > \varepsilon, \quad \varrho > 0.$$

Babalola established that every pseudo-starlike function belongs to the family of Bazilevič functions of type $\left(1 - \frac{1}{\varrho}\right)$ and order $\frac{1}{\varrho}$, and moreover, all such functions are univalent in the open unit disk $\mathbb{O}$.

When $\varrho = 1$, the family $\mathbb{L}_\varrho(\varepsilon)$ reduces to the classical starlike functions of order ε, which in this framework correspond to the 1-pseudo-starlike functions of the same order.

A function $\chi \in \mathbb{A}$ is said to be *starlike with respect to symmetric points* in the unit disk $\mathbb{O}$ if, for each radius ρ sufficiently close to 1 ($\rho < 1$) and for every point δ_0 on the circle $|\delta| = \rho$, the angular velocity of $\chi(\delta)$ about $\chi(-\delta_0)$ is positive at $\delta = \delta_0$ as δ moves along $|\delta| = \rho$ in the counter-clockwise direction.

This family was introduced and investigated by Sakaguchi [19] and is formally defined as

$$\mathbb{S}_s^* = \left\{\chi(\delta) \in \mathbb{S} : \mathrm{Re}\left\{\frac{2\delta\chi'(\delta)}{\chi(\delta) - \chi(-\delta)}\right\} > 0, \quad \delta \in \mathbb{O}\right\}.$$

In a subsequent study, Wang et al. [25] introduced the family of *convex functions with respect to symmetric points*, defined by

$$\mathbb{K}_s = \left\{ \chi(\delta) \in \mathbb{S} : \mathrm{Re}\left\{ \frac{2\big(\delta\chi'(\delta)\big)'}{\big(\chi(\delta) - \chi(-\delta)\big)'} \right\} > 0, \quad \delta \in \mathbb{O} \right\}.$$

It follows directly that if $\chi(\delta) \in \mathbb{K}_s$, then $\delta\chi'(\delta) \in \mathbb{S}_s^*$. For further insight into related recent works, we refer the reader to [11, 14, 16, 20].

The classical Koebe one-quarter theorem [7] asserts that the image of the unit disk $\mathbb{O}$ under any function $\chi \in \mathbb{S}$ contains a disk of radius $\frac{1}{4}$. Consequently, every $\chi \in \mathbb{S}$ possesses an inverse χ^{-1} satisfying

$$\chi^{-1}(\chi(\delta)) = \delta, \quad \delta \in \mathbb{O}, \qquad \text{and} \qquad \chi(\chi^{-1}(\varsigma)) = \varsigma, \quad |\varsigma| < \rho_0(\chi), \quad \rho_0(\chi) \geq \frac{1}{4}.$$

Let Σ denote the family of *bi-univalent functions* defined on $\mathbb{O}$, that is, functions $\chi \in \mathbb{A}$ for which both χ and its inverse χ^{-1} are univalent in $\mathbb{O}$. For a function $\chi \in \Sigma$ with the Maclaurin series given in (1.1), it can be verified that its inverse $\xi = \chi^{-1}$ admits the expansion

$$\xi(\varsigma) = \chi^{-1}(\varsigma) = \varsigma - a_2\varsigma^2 + \big(2a_2^2 - a_3\big)\varsigma^3 + \cdots . \tag{1.2}$$

It is worth noting that the family Σ is non-empty. Typical examples include

$$\delta, \quad \frac{\delta}{1-\delta}, \quad -\log(1-\delta), \quad \text{and} \quad \frac{1}{2}\log\frac{1+\delta}{1-\delta}.$$

On the other hand, the Koebe function does not belong to Σ, and neither do the functions

$$\delta - \frac{\delta^2}{2} \quad \text{and} \quad \frac{\delta}{1-\delta^2}.$$

In recent years, the study of analytic and bi-univalent functions has been revitalized, notably by Srivastava et al. [21], followed by subsequent developments as reported in [1, 3, 4, 6, 8, 12, 13, 18, 23, 24, 26].

In mathematics, the *Struve function* $H_\tau(x)$ is defined as a solution $y(x)$ of the non-homogeneous Bessel differential equation [22]:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \tau^2)y = \frac{4\left(\frac{x}{2}\right)^{\tau+1}}{\sqrt{\pi}\,\Gamma\left(\tau + \frac{1}{2}\right)}, \tag{1.3}$$

where $\tau \in \mathbf{C}$ denotes the order of the Struve function, which is often chosen to be an integer (see Figure 1).

The second-kind Struve function, commonly denoted by $K_\tau(x)$, is defined as

$$K_\tau(x) = H_\tau(x) - Y_\tau(x), \tag{1.4}$$

where $Y_\tau(x)$ represents the Neumann function. Because the underlying differential equation is non-homogeneous, its general solution can be expressed as the sum of a particular solution (the Struve function) and the general solution of the associated homogeneous Bessel equation.

Moreover, the Struve functions $H_\tau(\delta)$ admit the power series expansion

$$H_\tau(\delta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+3/2)\,\Gamma(n+\tau+3/2)} \left(\frac{z}{2}\right)^{2n+\tau+1}. \tag{1.5}$$

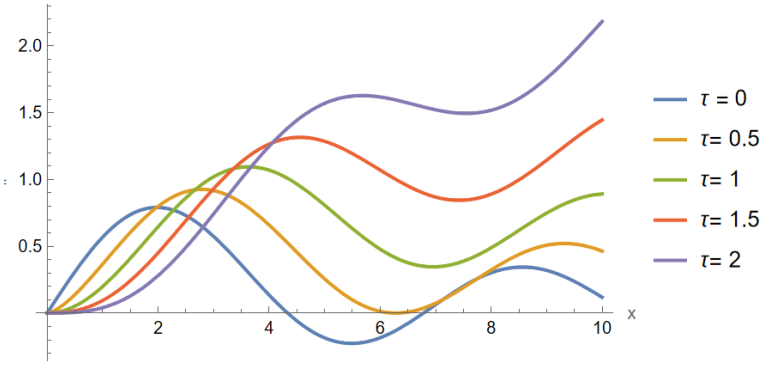


FIGURE 1. Plot of the Struve function $H_\tau(x)$ for $\tau = 0, 0.5, 1, 1.5, 2$.

For half-integer orders, i.e., when $\tau = n + 1/2$ with n an integer, Struve functions can be represented using elementary functions. Specifically, for non-negative integers $n \geq 0$, one has

$$H_{-n-1/2}(\delta) = (-1)^n J_{n+1/2}(\delta), \tag{1.6}$$

where $J_{n+1/2}(\delta)$ denotes the spherical Bessel function of order $n + 1/2$.

More generally, Struve functions can be expressed in terms of the generalized hypergeometric function ${}_1F_2$ as follows:

$$H_\tau(\delta) = \frac{\delta^{\tau+1}}{2^\tau \sqrt{\pi}\,\Gamma(\tau+3/2)}\, {}_1F_2\left(1; \frac{3}{2}, \tau+\frac{3}{2}; -\frac{\delta^2}{4}\right). \tag{1.7}$$

Struve functions play a significant role in various applications in physics and engineering. They appear, for instance, in beamforming problems and in modeling the influence of confining interfaces on the Brownian motion of colloidal particles under low Reynolds number conditions [5, 10].

To construct a version of the Struve function suitable for analytic function theory in the unit disk, we define a normalized Struve-type function $\mathbb{H}_\tau(\delta)$ by multiplying $H_\tau(\sqrt{\delta})$ with an appropriate factor such that its Maclaurin series starts with 1. More precisely, we set

$$\mathbb{H}_\tau(\delta) := \frac{\Gamma(3/2)\,\Gamma(\tau+3/2)\,2^{\tau+1}}{\delta^{(\tau+1)/2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+3/2)\,\Gamma(n+\tau+3/2)} \left(\frac{\sqrt{\delta}}{2}\right)^{2n+\tau+1}. \tag{1.8}$$

With this normalization, the series expansion of $\mathbb{H}_\tau(\delta)$ around $\delta = 0$ begins as

$$\mathbb{H}_\tau(\delta) = 1 + A_1\delta + A_2\delta^2 + A_3\delta^3 + \cdots,$$

where $A_1, A_2, \ldots$ are constants depending on τ. This form allows $\mathbb{H}_\tau(\delta)$ to be directly employed in the study of analytic and bi-univalent functions in the unit

disk (see Figure 2). Accordingly, we obtain the series:

$$\begin{aligned}\mathbb{H}_{\tau}(\delta) &= \sum_{n=0}^{\infty}(-1)^{n}\frac{\Gamma(3/2)\,\Gamma(\tau+3/2)}{2^{2n}\,\Gamma(n+3/2)\,\Gamma(n+\tau+3/2)}\delta^{n},\\ &= 1-\frac{1}{3(3+2\tau)}\delta+\frac{1}{15(15+16\tau+4\tau^{2})}\delta^{2}\\ &\quad-\frac{1}{105\,(105+142\tau+60\tau^{2}+8\tau^{3})}\delta^{3}+\cdots\end{aligned} \tag{1.9}$$

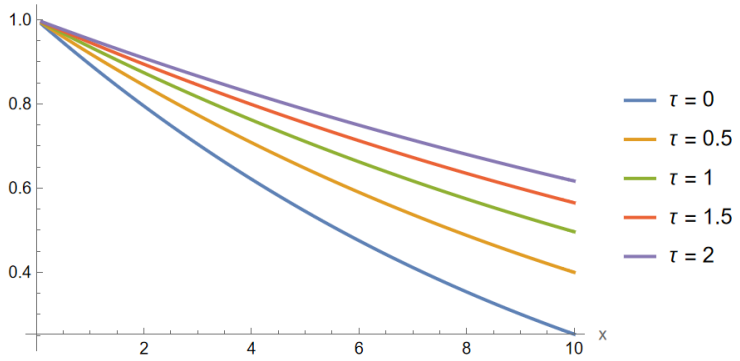


FIGURE 2. Plot of the normalized Struve function $\mathbb{H}_{\tau}(x)$ for $\tau = 0, 0.5, 1, 1.5, 2$.

2. Motivation

Recent investigations in geometric function theory have shown a growing interest in subclasses of bi-univalent functions generated via special functions and symmetric-point conditions. Such approaches often lead to new coefficient bounds and structural properties for analytic functions.

Inspired by this direction, the present work aims to study subclasses of bi-univalent functions associated with symmetric points that are subordinated to the Struve function. Since the Struve function represents an important class of special functions with significant analytical properties, employing it in the framework of subordination provides a natural setting for constructing new subclasses and deriving related results.

3. Bi-Univalent Function Family $\mathbb{BST}^{\tau}_{s,\varrho}$ Associated with the Struve Function

In this section, we introduce a novel subfamily of the family Σ defined via the normalized Struve function $\mathbb{H}_{\tau}(\delta)$. By selecting the parameters appropriately, one can obtain a variety of interesting subfamilies as particular instances. This newly proposed family provides a fresh link between the theory of bi-univalent functions and Struve functions, a connection that, to the best of our knowledge, has not yet been explored in the existing literature.

Definition 3.1. Let $0 \leq \varepsilon \leq 1$, and $\varrho > 0$. A function $\chi \in \Sigma$ of the form (1.1) is said to belong to the family $\mathbb{BST}^{\tau}_{s,\varrho}(\varepsilon)$ associated with the normalized Struve function $\mathbb{H}_{\tau}(\delta)$ if the following subordinations hold:

$$\varepsilon\left(\frac{2\delta\,(\chi'(\delta))^{\varrho}}{\chi(\delta)-\chi(-\delta)}\right)+(1-\varepsilon)\left(\frac{2\,[(\delta\chi'(\delta))']^{\varrho}}{[\chi(\delta)-\chi(-\delta)]'}\right)\prec \mathbb{H}_{\tau}(\delta),\quad \delta\in\mathbb{O}, \tag{3.1}$$

and

$$\varepsilon\left(\frac{2\varsigma\,(\xi'(\varsigma))^{\varrho}}{\xi(\varsigma)-\xi(-\varsigma)}\right)+(1-\varepsilon)\left(\frac{2\,[(\varsigma\xi'(\varsigma))']^{\varrho}}{[\xi(\varsigma)-\xi(-\varsigma)]'}\right)\prec \mathbb{H}_{\tau}(\varsigma),\quad \varsigma\in\mathbb{O}, \tag{3.2}$$

where the function $\xi=\chi^{-1}$ is given by (1.2).

We further denote

$$\mathbb{PST}^{\tau}_{s,\varrho}\equiv\mathbb{BST}^{\tau}_{s,\varrho}(1)\quad\text{and}\quad\mathbb{PKT}^{\tau}_{s,\varrho}\equiv\mathbb{BST}^{\tau}_{s,\varrho}(0),$$

which represent the subfamilies of Babalola-type pseudo-starlike and pseudo-convex bi-univalent functions with respect to symmetric points associated with the normalized Struve function $\mathbb{H}_{\tau}(\delta)$, respectively.

Furthermore, we define

$$\mathbb{PST}^{\tau}_{s,1}\equiv\mathbb{S}^{*,\tau}_{s}\quad\text{and}\quad\mathbb{PKT}^{\tau}_{s,1}\equiv\mathbb{K}^{\tau}_{s},$$

which correspond to the subfamilies of Sakaguchi-type bi-starlike and bi-convex functions with respect to symmetric points associated with the normalized Struve function $\mathbb{H}_{\tau}(\delta)$, respectively.

It is worth noting that the above-mentioned subfamilies arise as special cases of the newly defined family and, to the best of our knowledge, have not been previously studied in connection with the normalized Struve function. The corresponding coefficient estimates for these subfamilies are presented in the form of corollaries.

Example 3.2. Let

$$\chi_{\beta}(\delta)=\frac{\delta}{1-\beta\delta},\qquad |\beta|<1.$$

Then a direct computation gives

$$\begin{aligned}&\varepsilon\left(\frac{2\delta\,\chi'_{\beta}(\delta)}{\chi_{\beta}(\delta)-\chi_{\beta}(-\delta)}\right)+(1-\varepsilon)\left(\frac{2\,[(\delta\,\chi'_{\beta}(\delta))']}{[\chi_{\beta}(\delta)-\chi_{\beta}(-\delta)]'}\right)\\&=-\frac{(1+\beta\delta)\,(1-2(-1+\varepsilon)\beta\delta+\beta^{2}\delta^{2})}{(-1+\beta\delta)\,(1+\beta^{2}\delta^{2})}\\&:=\Xi^{\varepsilon}_{\beta}(\delta).\end{aligned}$$

In particular: If $\varepsilon=1$, then

$$\frac{2\delta\chi'_{\beta}(\delta)}{\chi_{\beta}(\delta)-\chi_{\beta}(-\delta)}=\frac{1+\beta\delta}{1-\beta\delta}:=\Xi^{1}_{\beta}(\delta). \tag{3.3}$$

Therefore,

$$\Xi^{1}_{\beta}(\delta)\prec\mathbb{H}_{0}(\delta)\quad\text{whenever}\quad|\beta|<0.05.$$

Hence,

$$\chi_\beta \in \mathbb{S}_s^{*,0} \quad \text{whenever} \quad |\beta| < 0.05$$

If $\varepsilon = \frac{1}{2}$, then

$$\begin{aligned}
&\frac{1}{2}\left(\frac{2\delta\,\chi'_\beta(\delta)}{\chi_\beta(\delta)-\chi_\beta(-\delta)}\right)+\frac{1}{2}\left(\frac{2\left[(\delta\,\chi'_\beta(\delta))'\right]}{[\chi_\beta(\delta)-\chi_\beta(-\delta)]'}\right)\\
&=-\frac{(1+\delta\beta)\,(1+\delta\beta+\delta^2\beta^2)}{(-1+\delta\beta)\,(1+\delta^2\beta^2)}\\
&:=\Xi_\beta^{\frac{1}{2}}(\delta). \qquad (3.4)
\end{aligned}$$

Therefore,

$$\Xi_\beta^{\frac{1}{2}}(\delta) \prec \mathbb{H}_0(\delta) \quad \text{whenever} \quad |\beta| < 0.03.$$

Hence,

$$\chi_\beta \in \mathbb{BST}_{s,1}^0\left(\frac{1}{2}\right) \quad \text{whenever} \quad |\beta| < 0.03.$$

If $\varepsilon = 0$, then

$$\frac{2\left[(\delta\chi'_\beta(\delta))'\right]}{[\chi_\beta(\delta)-\chi_\beta(-\delta)]'} = -\frac{(1+\beta\delta)^3}{(-1+\beta\delta)(1+\beta^2\delta^2)} := \Xi_\beta^0(\delta). \qquad (3.5)$$

Therefore,

$$\Xi_\beta^0(\delta) \prec \mathbb{H}_0(\delta) \quad \text{whenever} \quad |\beta| < 0.02.$$

Hence,

$$\chi_\beta \in \mathbb{K}_s^0 \quad \text{whenever} \quad |\beta| < 0.02.$$

Figure 3 clearly illustrates the inclusion cases under which χ_β belongs to the family $\mathbb{BST}_{s,\varrho}^\tau(\varepsilon)$.

4. Initial Taylor Coefficients for the Family $\mathbb{BST}_{s,\varrho}^\tau$

Let $\mathbb{P}(\beta)$, $0 \le \beta < 1$, denote the family of analytic functions p in the unit disk $\mathbb{O}$ satisfying the normalization condition $p(0) = 1$ and the inequality

$$Re\{p(\delta)\} > \beta, \quad \delta \in \mathbb{O}.$$

In particular, we write $\mathbb{P} := \mathbb{P}(0)$. We now present the following lemma, which will be useful in establishing the coefficient estimates for functions belonging to the family $\mathbb{BST}_{s,\varrho}^\tau$.

Lemma 4.1 ([17]). *Let $p \in \mathbb{P}$ be of the form*

$$p(\delta) = 1 + c_1\delta + c_2\delta^2 + \cdots, \quad \delta \in \mathbb{O}.$$

Then

$$|c_n| \le 2, \quad n \ge 1. \qquad (4.1)$$

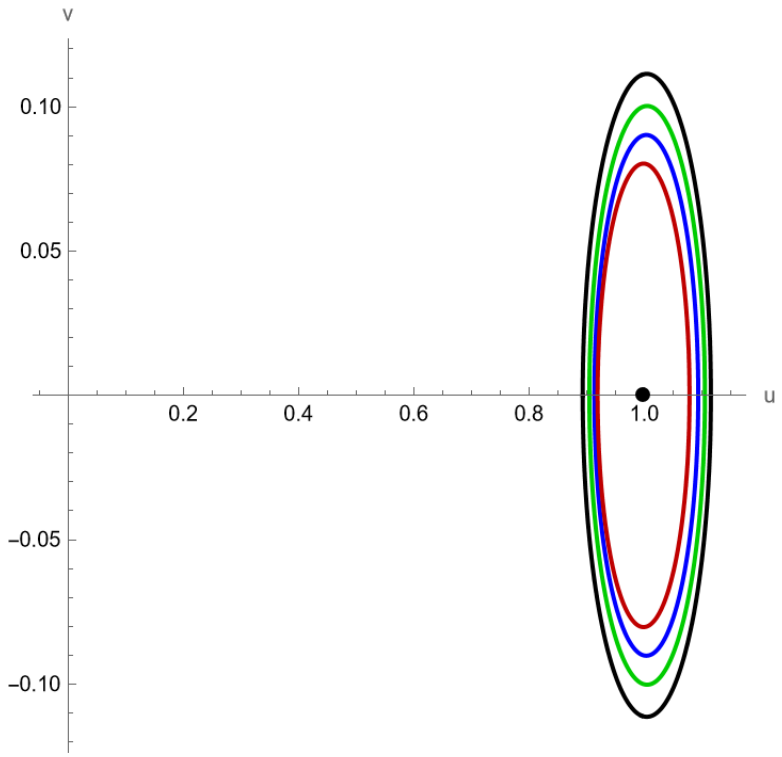


FIGURE 3. Plots of $\Xi^{1}_{0.05}(\delta)$, $\Xi^{1/2}_{0.03}(\delta)$, $\Xi^{0}_{0.02}(\delta)$, and $\mathbb{H}_0(\delta)$ shown in green, blue, red, and black, respectively.

Firstly, let

$$p(\delta) = 1 + p_1\delta + p_2\delta^2 + \cdots,$$

and suppose that $p \prec \mathbb{H}_\tau$. Then there exists an analytic function u such that

$$|u(\delta)| < 1, \quad \delta \in \mathbb{O},$$

and

$$p(\delta) = \mathbb{H}_\tau(u(\delta)).$$

Therefore, the function

$$T(\delta) = \frac{1+u(\delta)}{1-u(\delta)} = 1 + c_1\delta + c_2\delta^2 + \cdots \tag{4.2}$$

belongs to the family $\mathbb{P}$. It follows that

$$u(\delta) = \frac{c_1}{2}\delta + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\delta^2 + \frac{1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)\delta^3 + \cdots \tag{4.3}$$

and

$$\mathbb{H}_\tau(u(\delta)) = 1 - \frac{c_1}{6(3+2\tau)}\,\delta + \left(\frac{-c_2}{6(3+2\tau)} + \frac{c_1^2(13+5\tau)}{30(3+2\tau)(5+2\tau)}\right)\delta^2 + \left(\frac{-c_3}{6(3+2\tau)} + \frac{14c_1c_2(13+5\tau)}{210(3+2\tau)(5+2\tau)} + \frac{-c_1^3\big(331+217\tau+35\tau^2\big)}{210(3+2\tau)(5+2\tau)(7+2\tau)}\right)\delta^3 + \cdots. \tag{4.4}$$

Similarly, there exists an analytic function v such that

$$|v(\varsigma)| < 1, \quad \varsigma \in \mathbb{O},$$

and

$$p(\varsigma) = \mathbb{H}_\tau(v(\varsigma)).$$

Therefore, the function

$$k(\varsigma) = \frac{1+v(\varsigma)}{1-v(\varsigma)} = 1 + d_1\varsigma + d_2\varsigma^2 + \cdots \tag{4.5}$$

belongs to the family $\mathbb{P}$. It follows that

$$v(\varsigma) = \frac{d_1}{2}\varsigma + \frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right)\varsigma^2 + \frac{1}{2}\left(d_3 - d_1d_2 + \frac{d_1^3}{4}\right)\varsigma^3 + \cdots \tag{4.6}$$

and

$$\mathbb{H}_\tau(v(\varsigma)) = 1 - \frac{d_1}{6(3+2\tau)}\,\varsigma + \left(\frac{-d_2}{6(3+2\tau)} + \frac{d_1^2(13+5\tau)}{30(3+2\tau)(5+2\tau)}\right)\varsigma^2 + \left(\frac{-d_3}{6(3+2\tau)} + \frac{14d_1d_2(13+5\tau)}{210(3+2\tau)(5+2\tau)} + \frac{-d_1^3\big(331+217\tau+35\tau^2\big)}{210(3+2\tau)(5+2\tau)(7+2\tau)}\right)\varsigma^3 + \cdots. \tag{4.7}$$

Theorem 4.2. *Let χ be given by (1.1), $\tau \neq -\frac{3}{2}$, and suppose that $\chi \in \mathbb{BST}^{\tau}_{s,\varrho}(\varepsilon)$. Then*

$$a_2 \leq \frac{\sqrt{5|5+2\tau|}}{\sqrt{3\left|\begin{array}{l}(3+2\tau)\big[5(-3-7\varrho+16\varrho^2+\varepsilon(2+6\varrho-12\varrho^2))(5+2\tau)\\ -24(-2+\varepsilon)^2\varrho^2(3+2\tau)(13+5\tau)\big]\end{array}\right|}}. \tag{4.8}$$

and

$$|a_3| \leq \frac{1}{3|3+2\tau||-3+2\varepsilon+9\varrho-6\varepsilon\varrho|} + \frac{5|5+2\tau|}{3\left|\begin{array}{l}(3+2\tau)\big[5(-3-7\varrho+16\varrho^2+\varepsilon(2+6\varrho-12\varrho^2))(5+2\tau)\\ -24(-2+\varepsilon)^2\varrho^2(3+2\tau)(13+5\tau)\big]\end{array}\right|} \tag{4.9}$$

Proof. Let $\chi \in \mathbb{BST}^{\tau}_{s,\varrho}(\varepsilon)$ and let $\xi = \chi^{-1}$. Considering (3.1) and (3.2), we obtain

$$\varepsilon\left(\frac{2\delta\big(\chi'(\delta)\big)^{\varrho}}{\chi(\delta)-\chi(-\delta)}\right)+(1-\varepsilon)\left(\frac{2\big((\delta\chi'(\delta)\big)'\big)^{\varrho}}{\big[\chi(\delta)-\chi(-\delta)\big]'}\right)=\mathbb{H}_{\tau}\big(u(\delta)\big) \tag{4.10}$$

and

$$\varepsilon\left(\frac{2\varsigma\big(\xi'(\varsigma)\big)^{\varrho}}{\xi(\varsigma)-\xi(-\varsigma)}\right)+(1-\varepsilon)\left(\frac{2\big((\varsigma\xi'(\varsigma)\big)'\big)^{\varrho}}{\big[\xi(\varsigma)-\xi(-\varsigma)\big]'}\right)=\mathbb{H}_{\tau}\big(v(\varsigma)\big). \tag{4.11}$$

for some Schwarz functions u and v, where $\delta,\varsigma \in \mathbb{O}$ and ξ is given by (1.2). Since

$$\begin{aligned}&\varepsilon\left(\frac{2\delta\big(\chi'(\delta)\big)^{\varrho}}{\chi(\delta)-\chi(-\delta)}\right)+(1-\varepsilon)\left(\frac{2\big((\delta\chi'(\delta)\big)'\big)^{\varrho}}{\big[\chi(\delta)-\chi(-\delta)\big]'}\right)=1-2\left(-2+\varepsilon\right)\varrho a_2\delta\\&+\left((-8\varrho+6\varepsilon\varrho+8\varrho^2-6\varepsilon\varrho^2)a_2^2+(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)a_3\right)\delta^2+\cdots\end{aligned}$$

and

$$\begin{aligned}&\varepsilon\left(\frac{2\varsigma\big(\xi'(\varsigma)\big)^{\varrho}}{\xi(\varsigma)-\xi(-\varsigma)}\right)+(1-\varepsilon)\left(\frac{2\big((\varsigma\xi'(\varsigma)\big)'\big)^{\varrho}}{\big[\xi(\varsigma)-\xi(-\varsigma)\big]'}\right)=1+2\left(-2+\varepsilon\right)\varrho a_2\varsigma\\&+\left((-8\varrho+6\varepsilon\varrho+8\varrho^2-6\varepsilon\varrho^2)a_2^2+(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)(2a_2^2-a_3)\right)\varsigma^2+\cdots\end{aligned}$$

we have

$$\begin{aligned}&1-2(-2+\varepsilon)\varrho a_2\delta+\Big[(-8\varrho+6\varepsilon\varrho+8\varrho^2-6\varepsilon\varrho^2)a_2^2+(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)a_3\Big]\delta^2+\cdots\\&=1-\frac{c_1}{6(3+2\tau)}\,\delta+\left(\frac{-c_2}{6(3+2\tau)}+\frac{c_1^2(13+5\tau)}{30(3+2\tau)(5+2\tau)}\right)\delta^2+\cdots\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}&1+2\left(-2+\varepsilon\right)\varrho a_2\varsigma+\left((-8\varrho+6\varepsilon\varrho+8\varrho^2-6\varepsilon\varrho^2)a_2^2+(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)(2a_2^2-a_3)\right)\varsigma^2+\cdots\\&=1-\frac{d_1}{6(3+2\tau)}\,\varsigma+\left(\frac{-d_2}{6(3+2\tau)}+\frac{d_1^2(13+5\tau)}{30(3+2\tau)(5+2\tau)}\right)\varsigma^2+\cdots\end{aligned} \tag{4.13}$$

It follows from (1.9), (4.12) and (4.13) that

$$2(-2+\varepsilon)\varrho a_2=\frac{c_1}{6(3+2\tau)}, \tag{4.14}$$

$$(-8\varrho+6\varepsilon\varrho+8\varrho^2-6\varepsilon\varrho^2)a_2^2+(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)a_3=\frac{-c_2}{6(3+2\tau)}+\frac{c_1^2(13+5\tau)}{30(3+2\tau)(5+2\tau)}. \tag{4.15}$$

and

$$2\left(-2+\varepsilon\right)\varrho a_2=-\frac{d_1}{6(3+2\tau)}, \tag{4.16}$$

$$(-8\varrho+6\varepsilon\varrho+8\varrho^2-6\varepsilon\varrho^2)a_2^2+(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)(2a_2^2-a_3)=\frac{-d_2}{6(3+2\tau)}+\frac{d_1^2(13+5\tau)}{30(3+2\tau)(5+2\tau)}. \tag{4.17}$$

From (4.14) and (4.16), we have

$$c_1=-d_1, \tag{4.18}$$

and

$$a_2^2=\frac{c_1^2+d_1^2}{288\varrho^2(-2+\varepsilon)^2(3+2\tau)^2}. \tag{4.19}$$

Hence

$$|a_2|\leq\frac{1}{6\varrho(2-\varepsilon)|3+2\tau|}. \tag{4.20}$$

Now, by summing (4.15) and (4.17), we obtain

$$(-3-7\varrho+16\varrho^2+\varepsilon\left(2+6\varrho-12\varrho^2\right))2a_2^2=\frac{-c_2-d_2}{6(3+2\tau)}+\frac{(c_2^2+d_1^2)(13+5\tau)}{30(3+2\tau)(5+2\tau)}. \tag{4.21}$$

By substituting (4.19) into (4.21), we have

$$a_2^2=\frac{5(-c_2-d_2)(5+2\tau)}{12(3+2\tau)\begin{bmatrix}5\big(-3-7\varrho+16\varrho^2+\varepsilon(2+6\varrho-12\varrho^2)\big)(5+2\tau)\\-24(-2+\varepsilon)^2\varrho^2(3+2\tau)(13+5\tau)\end{bmatrix}}. \tag{4.22}$$

Therefore, using Lemma 4.1, we obtain

$$a_2\leq\frac{\sqrt{5|5+2\tau|}}{\sqrt{3\left|\begin{matrix}(3+2\tau)\big[5(-3-7\varrho+16\varrho^2+\varepsilon(2+6\varrho-12\varrho^2))(5+2\tau)\\-24(-2+\varepsilon)^2\varrho^2(3+2\tau)(13+5\tau)\big]\end{matrix}\right|}}. \tag{4.23}$$

Now, in order to find $|a_3|$, we subtract (4.17) from (4.15). Then we obtain

$$(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)(-2a_2^2+2a_3)=\frac{d_2-c_2}{6(3+2\tau)} \tag{4.24}$$

Hence, we get

$$|a_3|\leq\frac{1}{3|3+2\tau||-3+2\varepsilon+9\varrho-6\varepsilon\varrho|}+|a_2|^2. \tag{4.25}$$

Then, in view of (4.23), we obtain

$$|a_3| \leq \frac{1}{3|3+2\tau||-3+2\varepsilon+9\varrho-6\varepsilon\varrho|} + \frac{5|5+2\tau|}{3\left|\begin{array}{l}(3+2\tau)\big[5(-3-7\varrho+16\varrho^2+\varepsilon(2+6\varrho-12\varrho^2))(5+2\tau)\\ -24(-2+\varepsilon)^2\varrho^2(3+2\tau)(13+5\tau)\big]\end{array}\right|} \tag{4.26}$$

□

For special choices of the parameters ϱ and ε in Theorem 4.2, we obtain improved bounds for the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function families $\mathbb{PST}^{\tau}_{s,\varrho}$, $\mathbb{PKT}^{\tau}_{s,\varrho}$, $\mathbb{S}^{*,\tau}_{s}$ and $\mathbb{K}^{\tau}_{s}$, respectively, as follows.

Corollary 4.3. *Let χ be given by (1.1), $\tau \neq -\frac{3}{2}$, $\varrho \neq -\frac{1}{3}$, and suppose that $\chi \in \mathbb{PST}^{\tau}_{s,\varrho}$. Then*

$$a_2 \leq \frac{\sqrt{\frac{5}{3}}\sqrt{|5+2\tau|}}{\sqrt{\left|(3+2\tau)\big(5(-1-\varrho+4\varrho^2)(5+2\tau)-24\varrho^2(3+2\tau)(13+5\tau)\big)\right|}}. \tag{4.27}$$

and

$$|a_3| \leq \frac{1}{3}\left(\frac{1}{|1-3\varrho|\,|3+2\tau|} + \frac{5\,|5+2\tau|}{\left|(3+2\tau)\Big(5(-1-\varrho+4\varrho^2)(5+2\tau)-24\varrho^2(3+2\tau)(13+5\tau)\Big)\right|}\right) \tag{4.28}$$

Corollary 4.4. *Let χ be given by (1.1), $\tau \neq -\frac{3}{2}$, $\varrho \neq -\frac{1}{3}$, and suppose that $\chi \in \mathbb{PKT}^{\tau}_{s,\varrho}$. Then*

$$|a_2| \leq \frac{\sqrt{\frac{5}{3}}\sqrt{|5+2\tau|}}{\sqrt{\left|(3+2\tau)\Big[5(-3-7\varrho+16\varrho^2)(5+2\tau)-96\varrho^2(3+2\tau)(13+5\tau)\Big]\right|}}, \tag{4.29}$$

$$|a_3| \leq \frac{1}{3}\left(\frac{1}{|3-9\varrho|\,|3+2\tau|} + \frac{5\,|5+2\tau|}{\left|(3+2\tau)\Big(5(-3-7\varrho+16\varrho^2)(5+2\tau)-96\varrho^2(3+2\tau)(13+5\tau)\Big)\right|}\right) \tag{4.30}$$

Corollary 4.5. *Let χ be given by (1.1), $\tau \neq -\frac{3}{2}$, and suppose that $\chi \in \mathbb{S}^{*,\tau}_{s}$. Then*

$$a_2 \leq \frac{\sqrt{\frac{5}{6}}\sqrt{|5+2\tau|}}{\sqrt{\left|(3+2\tau)\big(443+482\tau+120\tau^2\big)\right|}}. \tag{4.31}$$

and

$$|a_3| \leq \frac{1}{3}\left(\frac{1}{2|3+2\tau|}+\frac{5|5+2\tau|}{|(3+2\tau)\,(10(5+2\tau)-24(3+2\tau)(13+5\tau))|}\right). \tag{4.32}$$

Corollary 4.6. *Let χ be given by (1.1), $\tau \neq -\frac{3}{2}$, and suppose that $\chi \in \mathbb{K}_s^{\tau}$. Then*

$$|a_2| \leq \frac{\sqrt{\frac{5}{2}}\sqrt{|5+2\tau|}}{3\sqrt{|(3+2\tau)(599+646\tau+160\tau^2)|}}, \tag{4.33}$$

$$|a_3| \leq \frac{1}{18}\left(\frac{1}{|3+2\tau|}+\frac{5\,|5+2\tau|}{|(3+2\tau)(599+646\tau+160\tau^2)|}\right) \tag{4.34}$$

Corollary 4.7. *Let χ be given by (1.1) and suppose that $\chi \in \mathbb{S}_s^{*,0}$. Then $|a_2| \leq 0.056$ and $|a_3| \leq 0.0587$.*

Corollary 4.8. *Let χ be given by (1.1) and suppose that $\chi \in \mathbb{K}_s^{0}$. Then $|a_2| \leq 0.028$ and $|a_3| \leq 0.0193$.*

5. Fekete–Szegö Inequality for the Function Family $\mathbb{BST}_{s,\varrho}^{\tau}(\varepsilon)$

Fekete and Szegö [9] introduced the generalized functional $|a_3 - \mathcal{Y}a_2^2|$, where $\mathcal{Y}$ is a real parameter. Motivated by the result of Zaprawa [27], we determine in the following theorem the Fekete–Szegö functional for functions belonging to the family $\chi \in \mathbb{BST}_{s,\varrho}^{\tau}(\varepsilon)$.

Theorem 5.1. *Let $\varrho > 0$ ($\varrho \neq \frac{1}{3}$) and $\mathcal{Y} \in \mathbb{R}$. If the function χ, given by (1.1), belongs to the family $\mathbb{BST}_{s,\varrho}^{\tau}(\varepsilon)$, then*

$$|a_3 - \mathcal{Y}a_2^2| \leq \begin{cases} \dfrac{1}{12(3+2\tau)(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)}, \\ 0 \leq |T(\mathcal{Y})| \leq \dfrac{1}{12(3+2\tau)(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)}, \\ 4|T(\mathcal{Y})|, \\ |T(\mathcal{Y})| \geq \dfrac{1}{12(3+2\tau)(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)}. \end{cases} \tag{5.1}$$

where

$$T(\mathcal{Y}) = \frac{5(1-\mathcal{Y})(5+2\tau)}{12(3+2\tau)\begin{bmatrix}5\big(-3-7\varrho+16\varrho^2+\varepsilon(2+6\varrho-12\varrho^2)\big)(5+2\tau)\\ -24(-2+\varepsilon)^2\varrho^2(3+2\tau)(13+5\tau)\end{bmatrix}}. \tag{5.2}$$

Proof. From (4.22) and (4.24), we obtain

$$
\begin{aligned}
a_3 - \mathcal{Y}a_2^2 &= (1-\mathcal{Y})a_2^2 + (a_3 - a_2^2) \\
&= \frac{5(1-\mathcal{Y})(-c_2-d_2)(5+2\tau)}{12(3+2\tau)\begin{bmatrix}5\big(-3-7\varrho+16\varrho^2+\varepsilon(2+6\varrho-12\varrho^2)\big)(5+2\tau) \\ -24(-2+\varepsilon)^2\varrho^2(3+2\tau)(13+5\tau)\end{bmatrix}} \\
&\quad + \frac{(d_2-c_2)}{12(3+2\tau)(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)}
\end{aligned}
$$

$$
\begin{aligned}
&= -\Bigg[\frac{5(1-\mathcal{Y})(5+2\tau)}{12(3+2\tau)\begin{bmatrix}5\big(-3-7\varrho+16\varrho^2+\varepsilon(2+6\varrho-12\varrho^2)\big)(5+2\tau) \\ -24(-2+\varepsilon)^2\varrho^2(3+2\tau)(13+5\tau)\end{bmatrix}} \\
&+ \frac{1}{12(3+2\tau)(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)}\Bigg]c_2 \\
&- \Bigg[\frac{5(1-\mathcal{Y})(5+2\tau)}{12(3+2\tau)\begin{bmatrix}5\big(-3-7\varrho+16\varrho^2+\varepsilon(2+6\varrho-12\varrho^2)\big)(5+2\tau) \\ -24(-2+\varepsilon)^2\varrho^2(3+2\tau)(13+5\tau)\end{bmatrix}} \\
&- \frac{1}{12(3+2\tau)(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)}\Bigg]d_2.
\end{aligned}
$$

So, we have

$$
\begin{aligned}
a_3 - \mathcal{Y}a_2^2 = &-\left[T(\mathcal{Y}) + \frac{1}{12(3+2\tau)(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)}\right]c_2 \\
&-\left[T(\mathcal{Y}) - \frac{1}{12(3+2\tau)(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)}\right]d_2, \qquad (5.3)
\end{aligned}
$$

where

$$
T(\mathcal{Y}) = \frac{5(1-\mathcal{Y})(5+2\tau)}{12(3+2\tau)\begin{bmatrix}5\big(-3-7\varrho+16\varrho^2+\varepsilon(2+6\varrho-12\varrho^2)\big)(5+2\tau) \\ -24(-2+\varepsilon)^2\varrho^2(3+2\tau)(13+5\tau)\end{bmatrix}}. \qquad (5.4)
$$

Then, by taking the modulus of (5.3), and using the fact that $|c_2| \leq 2$ and $|d_2| \leq 2$, we conclude that

$$\left|a_3 - \mathcal{Y}a_2^2\right| \leq \begin{cases} \dfrac{1}{12(3+2\tau)(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)}, \\ 0 \leq |T(\mathcal{Y})| \leq \dfrac{1}{12(3+2\tau)(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)}, \\ 4|T(\mathcal{Y})|, \\ |T(\mathcal{Y})| \geq \dfrac{1}{12(3+2\tau)(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)}. \end{cases} \tag{5.5}$$

□

By taking $\mathcal{Y} = 1$ in Theorem 5.1, we obtain the following corollary.

Corollary 5.2. *If $\chi \in \mathbb{BST}^{\tau}_{s,\varrho}(\varepsilon)$, then*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{12(3+2\tau)(-3+2\varepsilon+9\varrho-6\varepsilon\varrho)} \tag{5.6}$$

Further, by fixing the parameters ϱ and ε in Theorem 5.1, we obtain the Fekete–Szegö inequalities for the function families $\mathbb{PST}^{\tau}_{s,\varrho}$,$\mathbb{PKT}^{\tau}_{s,\varrho}$, $\mathbb{S}^{*,\tau}_{s}$, and $\mathbb{K}^{\tau}_{s}$, respectively.

Corollary 5.3. *Let $\chi \in \mathbb{PST}^{\tau}_{s,\varrho}$ be given by (1.1) and $\mathcal{Y} \in \mathbb{R}$. Then*

$$\left|a_3 - \mathcal{Y}a_2^2\right| \leq \begin{cases} \dfrac{1}{12(3+2\tau)(-1+3\varrho)}, & 0 \leq |T(\mathcal{Y})| \leq \dfrac{1}{12(3+2\tau)(-1+3\varrho)}, \\ 4|T(\mathcal{Y})|, & |T(\mathcal{Y})| \geq \dfrac{1}{12(3+2\tau)(-1+3\varrho)}. \end{cases} \tag{5.7}$$

where

$$T(\mathcal{Y}) = \frac{5(1-\mathcal{Y})(5+2\tau)}{12(3+2\tau)\begin{bmatrix} 5\big(-1-\varrho+4\varrho^2\big)(5+2\tau) \\ -24\varrho^2(3+2\tau)(13+5\tau) \end{bmatrix}}. \tag{5.8}$$

Corollary 5.4. *Let $\chi \in \mathbb{PKT}^{\tau}_{s,\varrho}$ be given by (1.1) and $\mathcal{Y} \in \mathbb{R}$. Then*

$$\left|a_3 - \mathcal{Y}a_2^2\right| \leq \begin{cases} \dfrac{1}{12(3+2\tau)(-3+9\varrho)}, & 0 \leq |T(\mathcal{Y})| \leq \dfrac{1}{12(3+2\tau)(-3+9\varrho)}, \\ 4|T(\mathcal{Y})|, & |T(\mathcal{Y})| \geq \dfrac{1}{12(3+2\tau)(-3+9\varrho)}. \end{cases} \tag{5.9}$$

where

$$T(\mathcal{Y}) = \frac{5(1-\mathcal{Y})(5+2\tau)}{12(3+2\tau)\begin{bmatrix}5\big(-3-7\varrho+16\varrho^2\big)(5+2\tau)\\ -96\varrho^2(3+2\tau)(13+5\tau)\end{bmatrix}}. \tag{5.10}$$

Corollary 5.5. *Let $\chi \in \mathbb{S}_s^{*,\tau}$ be given by (1.1) and $\mathcal{Y} \in \mathbb{R}$. Then*

$$\left|a_3 - \mathcal{Y}a_2^2\right| \leq \begin{cases} \dfrac{1}{24(3+2\tau)}, & 0 \leq |T(\mathcal{Y})| \leq \dfrac{1}{24(3+2\tau)}, \\ 4|T(\mathcal{Y})|, & |T(\mathcal{Y})| \geq \dfrac{1}{24(3+2\tau)}. \end{cases} \tag{5.11}$$

where

$$T(\mathcal{Y}) = \frac{5(1-\mathcal{Y})(5+2\tau)}{12(3+2\tau)\left[10(5+2\tau)-24(3+2\tau)(13+5\tau)\right]}. \tag{5.12}$$

Corollary 5.6. *Let $\chi \in \mathbb{K}_s^{\tau}$ be given by (1.1) and $\mathcal{Y} \in \mathbb{R}$. Then*

$$\left|a_3 - \mathcal{Y}a_2^2\right| \leq \begin{cases} \dfrac{1}{72(3+2\tau)}, & 0 \leq |T(\mathcal{Y})| \leq \dfrac{1}{72(3+2\tau)}, \\ 4|T(\mathcal{Y})|, & |T(\mathcal{Y})| \geq \dfrac{1}{72(3+2\tau)}. \end{cases} \tag{5.13}$$

where

$$T(\mathcal{Y}) = \frac{5(1-\mathcal{Y})(5+2\tau)}{12(3+2\tau)\left[30(5+2\tau)-96(3+2\tau)(13+5\tau)\right]}. \tag{5.14}$$

Corollary 5.7. *Let $\chi \in \mathbb{S}_s^{*,0}$ be given by (1.1) and $\mathcal{Y} \in \mathbb{R}$. Then*

$$\left|a_3 - \mathcal{Y}a_2^2\right| \leq \begin{cases} \dfrac{1}{72}, & 0 \leq |\mathcal{Y}-1| \leq \dfrac{443}{25}, \\ \frac{25}{7974}|\mathcal{Y}-1|, & |\mathcal{Y}-1| \geq \dfrac{443}{25}. \end{cases} \tag{5.15}$$

Corollary 5.8. *Let $\chi \in \mathbb{K}_s^{0}$ be given by (1.1) and $\mathcal{Y} \in \mathbb{R}$. Then*

$$\left|a_3 - \mathcal{Y}a_2^2\right| \leq \begin{cases} \dfrac{1}{216}, & 0 \leq |\mathcal{Y}-1| \leq \dfrac{599}{25}, \\ \frac{25}{32346}|\mathcal{Y}-1|, & |\mathcal{Y}-1| \geq \dfrac{599}{25}. \end{cases} \tag{5.16}$$

6. Conclusions

In this work, a new subfamily of bi-univalent functions associated with symmetric points and connected with the Struve function has been introduced and investigated. For functions belonging to this family, bounds for the initial Taylor–Maclaurin coefficients were obtained. Moreover, the corresponding Fekete–Szegö

functional was studied and several coefficient estimates were derived. Some particular cases of the main results were also presented, illustrating the novelty of the proposed family and yielding improved bounds for the initial Taylor–Maclaurin coefficients.

The results obtained in this paper may serve as a starting point for further investigations in the theory of bi-univalent functions. Possible directions for future research include the determination of higher-order coefficient estimates, the study of Hankel determinants associated with this subfamily, and the extension of the present approach to other subfamilies defined via different integral or differential operators and special functions.

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[1] Department of Mathematics, Faculty of Education for Girls, University of Kufa, Najaf 54001, Iraq.
Email address: saraf.alkhazai@uokufa.edu.iq

[2] Department of Mathematics, College of Education, Al-Mustansiriyah University, Baghdad, Iraq.
Email address: sarahjalawi7@uomustansiriyah.edu.iq

[3] Department of Cybersecurity, College of Computer Science and Information Technology, University of Al-Qadisiyah, Diwaniyah 58002, Iraq.
Email address: adel.tayh@qu.edu.iq