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OSCILLATION CRITERIA FOR GENERALIZED EMDEN-FOWLER EQUATIONS VIA LOCAL GENERALIZED DERIVATIVES AND MITTAG-LEFFLER KERNELS

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Abstract. This paper investigates the oscillatory behavior of a generalized version of the Emden-Fowler equation using the Local Generalized Derivative (LGD). We establish a new generalized Riccati identity and use the Divergence Theorem to derive an energy inequality for non-oscillatory solutions. By introducing a radial scaling function and a generalized Hartman-Wintner criterion, we demonstrate how the choice of the kernel function, specifically the Mittag-Leffer kernel, drastically alters the stability of the system, compared to classical conformable models. Our results show that exponential kernels can induce "ultra-sensitive" oscillations even when the potential function decays, providing a more flexible framework for modeling complex physical media.

Keywords. Local generalized derivatives, Emden-Fowler equation, Oscillation theory, Hartman-Wintner theorem, Riccati transformation.
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1. Introduction and preliminaries

The study of fractional and local derivatives has gained significant impulse in recent years due to its ability to describe complex phenomena that classical calculus cannot fully capture. Traditional models often fail when dealing with non-homogeneous media or systems with memory effects and variable timescales. To address these limitations, the local generalized derivative (LGD) was introduced in [9], offering a flexible framework through the use of an absolutely continuous kernel $g(\tau, \alpha)$ (also see [4, 14]).

Definition 1.1. Given a function $\psi : [0, +\infty) \to \mathbb{R}$. Then, the N-derivative of ψ of order α is defined by

$$N_g^\alpha \psi(\tau) = \lim_{\varepsilon \to 0} \frac{\psi(\tau + \varepsilon g(\tau, \alpha)) - \psi(\tau)}{\varepsilon},$$

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for all $\tau > 0$, $\alpha \in (0,1)$, where $f(\tau,\alpha)$ is an absolutely continuous function. If ψ is N-differentiable in some $(0,\alpha)$, and $\lim_{\tau\to 0^+} N_g^\alpha \psi(\tau)$ exists, then, we define $N_g^\alpha \psi(0) = \lim_{\tau\to 0^+} N_g^\alpha \psi(\tau)$; note that if ψ is differentiable, then $N_g^\alpha \psi(\tau) = L(\tau,\alpha)\psi'(\tau)$, where $\psi'(\tau)$ is the ordinary derivative.

This generalized derivative has proven useful in several applied problems (see [2, 3, 4, 10, 11]). It is clear that if we choose $g(t,\alpha) = t^{1-\alpha}$, we recover the derivative of [7], with $g(t,\alpha) = e^{t^{-\alpha}}$ we obtain the non-conformable derivative of [5]; and if $g(t,\alpha) = 1$, we have the classic derivative. However, in [9] it allows for more complex kernels, which can model specific material behaviors or variable timescales, which the simple conformal derivative does not capture. From LNV we can define the generalized partial derivatives as follows.

Definition 1.2. Given a real valued function $f : \mathbb{R}^n \to \mathbb{R}$ and $\overrightarrow{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ a point whose, i-th component is positive. Then, the generalized partial N-derivative of f of order α with respect to the variable x_i, in the point $\overrightarrow{a} = (a_1, \ldots, a_n)$ is defined by

$$iN_g^\alpha f(\overrightarrow{a}) = \lim_{\varepsilon\to 0} \frac{f(a_1, .., a_i + \varepsilon g_i(a_i,\alpha), .., a_n) - f(a_1, .., a_i, .., a_n))}{\varepsilon};$$

if it exists, it is denoted ${}_iN_g^\alpha f(\overrightarrow{a})$, called the i-th generalized partial derivative of f of the order $\alpha \in (0,1]$ at $\overrightarrow{a}$.

Remark 1.3. When using functions of two independent variables, we will only indicate the variable with respect to which we are integrating; so, ${}_tN_g^\alpha$ will indicate the first-order partial derivative with respect to the variable t.

Remark 1.4. If a real valued function f with n variables has all generalized partial derivatives of the order $\alpha \in (0,1]$ at $\overrightarrow{a}$, each $a_i > 0$, then, the generalized α-gradient of f of the order $\alpha \in (0,1]$ at $\overrightarrow{a}$ is

$$\nabla_g^\alpha f(\overrightarrow{a}) = ({}_1N_g^\alpha f(\overrightarrow{a}), \ldots, {}_2 N_g^\alpha f(\overrightarrow{a})).$$

Remark 1.5. The higher-order N-partial derivatives of a real valued function f with n variables, with $\alpha \in (0,1]$ at $\overrightarrow{a}$, are similarly defined by

$${}_iN_g^\alpha \left({}_iN_g^\alpha f\right) =_{(i,i)} N_g^{\alpha+\alpha} f.$$

We will also need the corresponding generalized Laplace-Beltrami operator, which is given by:

$$\Delta_g^\alpha u = \mathrm{div}_g^\alpha(\nabla_g^\alpha u).$$

Let $F = (F_1, F_2, \ldots, F_n)$ be a differentiable vector field on $\Omega \subset \mathbb{R}^n$. The generalized local divergence is defined as the sum of the generalized partial derivatives:

$$\mathrm{div}_g^\alpha F = \sum_{i=1}^n N_{g,x_i}^\alpha F_i = \sum_{i=1}^n g(x_i,\alpha)\frac{\partial F_i}{\partial x_i}.$$

Ω will be defined later.

We define the Generalized Fractional Integral associated with the LGD as

$$\mathcal{I}_g^\alpha f(t) = \int_a^t \frac{f(x)}{g(x,\alpha)} dx, \tag{1.1}$$

where $g(x,\alpha)$ is the kernel defined in the work [9]. If $g(x,\alpha) = x^{1-\alpha}$, we recover the conformable operator, and if $g(x,\alpha) = 1$, we obtain the classic Riemann integral.

In [12], the conformal Emden-Fowler type elliptic partial differential equation of the form

$$\Delta_x^\alpha u + c(x)|u|^{\beta-1}u|D^\alpha(u)|^{1-\beta} = 0$$

was studied. In the present work, we will study the generalized version of the above Emden-Fowler equation, given by the following expression:

$$\Delta_g^\alpha u + c(x)|u|^{\beta-1}u\|\nabla_g^\alpha u\|^{1-\beta} = 0, \quad x \in \Omega \subset \mathbb{R}^n, \tag{1.2}$$

where Δ_g^α is the Laplacian based on the generalized local derivative.
$\nabla_g^\alpha u$ is the generalized local gradient of u;
$c(x)$ is the potential function defined on the domain Ω;
α, β are parameters on the interval $(0,1)$;
$\|\cdot\|$ represents the usual Euclidean norm on $\mathbb{R}^n$.
This new formulation allows the oscillation of the solutions to depend not only on the order α, but also on the intrinsic structure of the kernel function g, which provides greater flexibility in modeling complex physical phenomena.

This paper focuses on the Emden-Fowler equation, a fundamental model in astrophysics and gas dynamics; now, reformulated within the LGD framework. While previous studies, such as [12], have explored these equations using conformable operators, our generalized approach allows the oscillation of solutions depend not only on the order α, but also on the intrinsic structure of the kernel function g. By using the generalized Laplace-Beltrami operator $\Delta_g^\alpha u$ and the n-dimensional gradient $\nabla_g^\alpha u$, we seek to establish new oscillation criteria. This provides a more sensitive mathematical tool for identifying unstable states in highly nonlinear media, where traditional polynomial growth models are insufficient.

2. Results

For convenience, we will use the following notation for partial derivatives

$$\frac{\partial^g f}{\partial x_i^{g,\alpha}} = g(x_i,\alpha)\frac{\partial f}{\partial x_i}.$$

We define the generalized Riccati vector W as follows

$$W(x) = \frac{\|\nabla_g^\alpha u\|^{\beta-1}\nabla_g^\alpha u}{|u|^{\beta-1}u}.$$

Let us calculate the generalized partial derivative of the i-th component of W (W_i):

$$\frac{\partial^g W_i}{\partial x_i^{g,\alpha}} = \frac{\partial^g}{\partial x_i^{g,\alpha}}\left(\frac{\|\nabla_g^\alpha u\|^{\beta-1}\frac{\partial^g u}{\partial x_i^{g,\alpha}}}{|u|^{\beta-1}u}\right).$$

Applying the operator div_g^α to the numerator of W and using the original Emden-Fowler equation, we obtain the term of the power function $c(x)$. Differentiating the denominator, $|u|^{\beta-1}u$ with respect to x_i, we get $\beta|u|^{\beta-1}\frac{\partial^g u}{\partial x_i^{g,\alpha}}$. Expanding $\mathrm{div}_g^\alpha W$ and substituting $\Delta_g^\alpha u = -c(x)|u|^{\beta-1}u\|\nabla_g^\alpha u\|^{1-\beta}$, the expression reduces to

$$\mathrm{div}_g^\alpha W = -\beta c(x) - \beta\frac{\|\nabla_g^\alpha u\|^{\beta-1}\sum_{i=1}^n\left(\frac{\partial^g u}{\partial x_i^{g,\alpha}}\right)^2}{|u|^{\beta+1}}.$$

Note that $\sum_{i=1}^n\left(\frac{\partial^g u}{\partial x_i^{g,\alpha}}\right)^2 = \|\nabla_g^\alpha u\|^2$, so, the second term on the right becomes

$$\beta\frac{\|\nabla_g^\alpha u\|^{\beta+1}}{|u|^{\beta+1}} = \beta\left(\frac{\|\nabla_g^\alpha u\|}{|u|}\right)^{\beta+1}.$$

Since $\|W\| = \frac{\|\nabla_g^\alpha u\|^\beta}{|u|^\beta}$, then $\|W\|^{\frac{\beta+1}{\beta}} = \left(\frac{\|\nabla_g^\alpha u\|}{|u|}\right)^{\beta+1}$.

Thus, the new generalized Riccati identity is

$$\mathrm{div}_g^\alpha W + \beta c(x) + \beta\|W\|^{\frac{\beta+1}{\beta}} = 0. \tag{2.1}$$

According to (1.1), we have the natural antiderivative of the operator. For the multivariable case in $\mathbb{R}^n$, the generalized volume element in rectangular coordinates would be

$$d_g\mathbf{x} = \frac{dx_1 dx_2 dx_n}{g(x_1,\alpha)g(x_2,\alpha)\dots g(x_n,\alpha)}.$$

Lemma 2.1. *Suppose that u is a non-oscillatory (eventually positive) solution of the Emden-Fowler equation generalized on an unbounded domain $\Omega \subset \mathbb{R}^n$. Then, there exists a Riccati vector W, such that for $r \geq r_0$, $\mathrm{div}_g^\alpha W(x) = -\beta c(x) - \beta\|W(x)\|^{\frac{\beta+1}{\beta}}$, for any test function $\phi \in C_0^1(\Omega)$, the following energy inequality holds*

$$\int_\Omega\left[\beta c(x)\phi^{\frac{\beta+1}{\beta}} - \frac{1}{(\beta+1)^{\beta+1}}\frac{\|\nabla_g^\alpha\phi\|^{\beta+1}}{\phi}\right]d_g x \leq 0.$$

Proof. It is sufficient to follow the proof of Lemma 2.3 of [12]. □

Let us now define the generalized spherical average. If we define $M[f](r)$ as the average of f over the surface of the sphere of radius r, in the generalized derivative metric, the geometric factor r^{n-1} (area of the sphere, according to the conformable derivative) is affected by the kernel $g(r, \alpha)$.

Therefore, we must define the function $A_g(r)$ as a radial scaling function

$$A_g(r) = \int_{S_r} \frac{d\sigma}{g(r,\alpha)} = \frac{\omega_n r^{n-1}}{g(r,\alpha)},$$

where ω_n is the area of the unit sphere.

Now, we can define a crucial function, the one that determines whether the system "explodes" (oscillates) or not in the generalized Hartman-Winter Theorem. Thus, we have

$$C_g(r) = \frac{1}{A_g(r)} \int_{r_0}^{r} \left(\int_{S_\tau} c(x) d\sigma \right) \frac{d\tau}{g(\tau,\alpha)}.$$

By simplifying it, if we define $\hat{c}(r)$ as the average of the potential function $c(x)$ on the sphere, we have

$$C_g(r) = \frac{g(r,\alpha)}{r^{n-1}} \int_{r_0}^{r} \frac{\hat{c}(\tau)\tau^{n-1}}{g(\tau,\alpha)} d\tau.$$

Similarly,

$$C_g(r) = \frac{1}{A_g(r)} \int_{r_0}^{r} \left(\int_{S_\tau} c(x) \frac{d\sigma}{\prod_{i=1}^{n} g(x_i,\alpha)} \right) d\tau.$$

In the following result, Ω is an unbounded domain in the n-dimensional Euclidean space $\mathbb{R}^n$. It is considered an "exterior domain," that is, the space outside a sphere of a given radius r_0. Mathematically, it is defined as

$$\Omega = \{x \in \mathbb{R}^n : |x| \geq r_0\},$$

where $r_0 > 0$.

In the oscillation theory of elliptic equations, Ω is the stage where we look for solutions $u(x)$. To say that a solution is oscillatory in Ω means that, no matter how far we move away from the origin (i.e., for arbitrarily large radii r), the solution $u(x)$ will still have zeros (changing sign).

When generalizing, Ω becomes critically important because of how the kernel $g(x, \alpha)$ "weighs" the space. The boundary $\partial\Omega$ differs from the conformable case, because S_r spheres are used for integration. In our case, these spheres are inside Ω. Although Ω is geometrically the same, the "distance" or "volume" inside Ω is now measured using the generalized integral

$$\mathrm{Vol}_g(\Omega) = \int_\Omega d_g x = \int_\Omega \frac{dx_1 \dots dx_n}{g(x_1,\alpha)\dots g(x_n,\alpha)}.$$

Regarding oscillation, we need the domain to extend to infinity. If Ω were a bounded set, we would be talking about boundary value problems (like Dirichlet), but not asymptotic oscillation. The Hartman-Wintner theorem, which we will present below, seeks to prove that the "pull" towards zero of the function $u(x)$ is so strong throughout Ω, that the solution cannot remain positive forever when $|x| \to \infty$. In short, Ω is the subspace of $\mathbb{R}^n$ (from a radius r_0 to infinity), where the generalized Emden-Fowler equation is "alive" and where the kernel $g(x, \alpha)$ of the generalized derivative modifies how the potential $c(x)$ affects the solution.

Theorem 2.2. *Suppose that the regularity conditions on the kernel $g(x, \alpha)$ (continuity and positivity in Ω) are met and that there exists a radially dominant kernel function $g(r, \alpha)$, let $u(x)$ be a solution of the generalized Emden-Fowler equation (1.2). If the following holds*

$$\lim_{r\to\infty} C_g(r) = \infty,$$

then, every solution $u(x)$ of the equation is oscillatory in Ω.

Remark 2.3. If we define Ω as before, then, the boundary is the hypersphere of radius r_0

$$\partial\Omega = \{x \in \mathbb{R}^n : |x| = r_0\}.$$

For oscillation theorems, it is not just $\partial\Omega$ (where $r = r_0$) that matters, but also what happens when $|x| \to \infty$. This is where our generalization shows its scope. Under the generalized local derivative, for the Hartmann-Wintner Theorem to hold, we must impose a growth condition on the kernel $g(x, \alpha)$. That is, the kernel $g(r, \alpha)$ must be such, that the integral of the "path" to infinity diverges

$$\int_{r_0}^{\infty} \frac{1}{g(\tau, \alpha)} d\tau = \infty.$$

If this integral were finite, the "space" would run out prematurely, and the solution might not have enough "distance" to oscillate.

In fact, to generalize the results of [12] from the perspective of the DLG, the validation of the Divergence Theorem is a fundamental point. Without this theorem, we cannot transform volume integrals (where the differential equation resides) into surface integrals (where we analyze the Riccati flow).

First, recall that the general linear transformation (GL) of a function f with respect to x_i is defined by a kernel function $g(x_i, \alpha)$

$$\frac{\partial^g f}{\partial x_i^{g,\alpha}} = g(x_i, \alpha)\frac{\partial f}{\partial x_i}.$$

For the Divergence Theorem to hold, the kernel g must be positive and continuous on the domain of interest. We define the generalized volume element as:

$$d_g x = \frac{dx_1 dx_2 \dots dx_n}{G(x, \alpha)}, \quad \text{with} \quad G(x, \alpha) = \prod_{i=1}^{n} g(x_i, \alpha).$$

We want to show that

$$\int_{\Omega} (\mathrm{div}_g^{\alpha} F) d_g x = \int_{\partial\Omega} F \cdot \mathbf{n}_g d\sigma_g.$$

We expand the left-hand term using the definition of divergence and the measure $d_g x$, thus, we have

$$\int_{\Omega} \left(\sum_{i=1}^{n} g(x_i, \alpha) \frac{\partial F_i}{\partial x_i} \right) \frac{dx_1 \dots dx_n}{\prod_{j=1}^{n} g(x_j, \alpha)}.$$

For each term i of the sum, the factor $g(x_i, \alpha)$ in the numerator is cancelled by the corresponding factor in the denominators product

$$\int_{\Omega} \sum_{i=1}^{n} \left(\frac{\partial F_i}{\partial x_i} \cdot \frac{1}{\prod_{j \neq i} g(x_j, \alpha)} \right) dx_1 \dots dx_n. \tag{2.2}$$

Let us define a new auxiliary vector field $\Psi = (\Psi_1, \dots, \Psi_n)$, such that

$$\Psi_i = \frac{F_i}{\prod_{j \neq i} g(x_j, \alpha)}.$$

Note that if we apply the classical Gaussian Divergence Theorem to this field Ψ, then

$$\int_{\Omega} \mathrm{div}(\Psi) dV = \int_{\partial\Omega} \Psi \cdot \mathbf{n} d\sigma.$$

When calculating $\mathrm{div}(\Psi)$ classically, we notice that the partial derivatives $\frac{\partial \Psi_i}{\partial x_i}$ do not affect the terms $g(x_j, \alpha)$ in the denominator, because they depend on variables x_j with $j \neq i$. Therefore,

$$\frac{\partial \Psi_i}{\partial x_i} = \frac{1}{\prod_{j \neq i} g(x_j, \alpha)} \frac{\partial F_i}{\partial x_i}.$$

This is exactly the equation (2.2). By integrating it, we arrive at the final formulation, necessary for our generalization

$$\int_{\Omega} (\mathrm{div}_g^{\alpha} F) d_g x = \int_{\partial\Omega} \sum_{i=1}^{n} \left(\frac{F_i \cdot n_i}{\prod_{j \neq i} g(x_j, \alpha)} \right) d\sigma.$$

Where n_i are the components of the standard unit normal vector.

Let us start with the identity established above (equation (2.1))

$$\mathrm{div}_g^{\alpha} W(x) + \beta c(x) + \beta \|W(x)\|^{\frac{\beta+1}{\beta}} = 0,$$

where $W(x)$ is the generalized Riccati vector. To transform it into a scalar criterion (which depends only on the radius r), we must integrate over a spherical region $\Omega(r_0, r) = \{x \in \mathbb{R}^n : r_0 \leq |x| \leq r\}$. We apply the generalized integral with the measure $d_g x = \frac{dx}{G(x,\alpha)}$ to the entire identity

$$\int_{\Omega(r_0,r)} \mathrm{div}_g^{\alpha} W d_g x + \int_{\Omega(r_0,r)} \beta c(x) d_g x + \int_{\Omega(r_0,r)} \beta \|W\|^{\frac{\beta+1}{\beta}} d_g x = 0.$$

By using the generalized Divergence theorem, we transform the divergence integral

$$\int_{\Omega(r_0,r)} \mathrm{div}_g^{\alpha} W d_g x = \int_{S_r} W \cdot \mathbf{n}_g d\sigma_g - \int_{S_{r_0}} W \cdot \mathbf{n}_g d\sigma_g,$$

where S_r is the surface area of the sphere of radius r. By the Cauchy-Schwarz inequality, we know that $W \cdot \mathbf{n}_g \leq \|W\|$. To simplify the expression, we define the average of the potential $c(x)$ weighted by the kernel

$$\hat{C}_g(r) = \int_{S_r} c(x) d\sigma_g,$$

and the Riccati flux term at the surface

$$w(r) = \int_{S_r} \|W\| d\sigma_g.$$

By substituting these averages into the total integral, we obtain a single-variable integral differential equation (r)

$$w(r) - w(r_0) + \int_{r_0}^{r} \beta \hat{C}_g(\tau) d\tau + \int_{r_0}^{r} \beta \left(\int_{S_\tau} \|W\|^{\frac{\beta+1}{\beta}} d\sigma_g \right) d\tau \leq 0.$$

By using Jensen's inequality for the last term (since the function $x^{\frac{\beta+1}{\beta}}$ is convex), we can bound the surface integral of W. After rearranging the terms, we arrive at the required inequality

$$w(r) \leq w(r_0) - \beta \int_{r_0}^{r} \hat{C}_g(\tau) d\tau - \beta \int_{r_0}^{r} \frac{|w(\tau)|^{\frac{\beta+1}{\beta}}}{[A_g(\tau)]^{1/\beta}} d\tau,$$

where $A_g(r)$ is the "generalized area" of the sphere, which we defined earlier as $A_g(r) = \frac{\omega_n r^{n-1}}{g(r,\alpha)}$, and $\hat{C}_g(r)$ is the potential accumulated on the surface under the LGD metric. This inequality is what allows us to prove the Hartman-Wintner Criterion. If the intermediate term $\int_{r_0}^{r} \hat{C}_g(\tau) d\tau$ tends to infinity as $r \to \infty$ (which is our function $C_g(r)$ from the previous step), then, the right-hand side of the inequality becomes infinitely negative. Since $w(r)$ is related to the power norm, it cannot be infinitely negative, unless the solution $u(x)$ ceases to exist as a positive function. Therefore, for a solution to be positive (non-oscillatory), the integral of the potential $c(x)$ "diluted" by the kernel $g(x, \alpha)$, must be finite. If the presence of the kernel of the LGD causes this integral to explode, then, the equation must oscillate. Now, we can prove the Theorem 2.2.

Proof. Suppose, on the contrary, that there exists a solution $u(x)$, that is eventually positive for $|x| \geq r_0$. Then, we can define the generalized Riccati vector

$$W(x) = \frac{\|\nabla_g^\alpha u\|^{\beta-1}\nabla_g^\alpha u}{|u|^{\beta-1}u}.$$

Based on the structure of the operator N_g^α, the following identity holds

$$\mathrm{div}_g^\alpha W(x) = -\beta c(x) - \beta\|W(x)\|^{\frac{\beta+1}{\beta}}.$$

We integrate the identity over the domain $\Omega(r_0, r)$ by using the measure $d_g x = \frac{dx}{G(x,\alpha)}$. By applying the Divergence Theorem, we obtain the following

$$\begin{aligned}&\int_{S_r} W\cdot\mathbf{n}_g d\sigma_g - \int_{S_{r_0}} W\cdot\mathbf{n}_g d\sigma_g + \beta\int_{r_0}^{r}\int_{S_\tau} c(x) d\sigma_g d\tau\\ + \quad &\beta\int_{r_0}^{r}\int_{S_\tau}\|W\|^{\frac{\beta+1}{\beta}} d\sigma_g d\tau = 0.\end{aligned}$$

Let $w(r) = \int_{S_r} W\cdot\mathbf{n}_g d\sigma_g$. By using the Hölder/Jensen inequality in the nonlinear term, we obtain

$$w(r) \leq w(r_0) - \beta\int_{r_0}^{r}\left[\int_{S_\tau} c(x) d\sigma_g\right] d\tau - \beta\int_{r_0}^{r}\frac{|w(\tau)|^{\frac{\beta+1}{\beta}}}{[A_g(\tau)]^{1/\beta}} d\tau.$$

By dividing by $A_g(r)$ and taking the limit as $r \to \infty$, the potential integral term $c(x)$ dominates the expression due to the hypothesis $\lim_{r\to\infty} C_g(r) = \infty$. This implies that

$$\lim_{r\to\infty}\frac{w(r)}{A_g(r)} = -\infty,$$

which contradicts to the existence of a positive solution $u(x)$, that maintains the stability of the Riccati vector W. Therefore, every solution must have zeros in any neighborhood of infinity. □

Let us consider the generalized local derivative N_g^α, where the kernel is the one-parameter Mittag-Leffler function, defined by $g(x_i, \alpha) = E_\alpha(x_i) = \sum_{k=0}^{\infty}\frac{x_i^k}{\Gamma(\alpha k+1)}$. Therefore, from the previous result, we can derive the following conclusion.

Corollary 2.4. *If the potential $c(x)$ of the Emden-Fowler equation (2.1) satisfies*

$$\lim_{r\to\infty}\left[\frac{E_\alpha(r)}{r^{n-1}}\int_{r_0}^{r}\frac{\hat{c}(\tau)\tau^{n-1}}{E_\alpha(\tau)} d\tau\right] = \infty,$$

then, every solution of the equation is oscillatory in Ω.

Remark 2.5. Unlike the conformable derivative, where $g(r, \alpha) = r^{1-\alpha}$ (a polynomial growth), the Mittag-Leffler function for $x > 0$ behaves asymptotically as

$$E_\alpha(x) \sim \frac{1}{\alpha}\exp(x^{1/\alpha}),$$

which implies that the generalization models systems where "diffusion" or local change grows exponentially in a fractional way. Under this kernel, the term $\frac{1}{E_\alpha(\tau)}$ in the integral acts as an extremely strong damper. For the system to oscillate, the potential $c(x)$ must be large enough to overcome the exponential decay of the inverse of the Mittag-Leffler function. This allows for the classification of much more complex materials or media, than those of [12]. The area of the sphere in this particular case is

$$A_{E_\alpha}(r) = \frac{\omega_n r^{n-1}}{E_\alpha(r)}.$$

Since $E_\alpha(r)$ grows very rapidly, $A_{E_\alpha}(r)$ tends to zero much faster than in the classical case, when $r \to \infty$. This makes it easier for the limit of $C_g(r)$ to tend to infinity, suggesting that the use of Mittag-Leffler kernels tends to favor oscillation of solutions, compared to conformal calculus.

3. Some Examples

Below, we present two examples studied with different kernels. We will compare the conformable kernel $r^{1-\alpha}$ against the Mittag-Leffler kernel $E_\alpha(r)$, and see how the "speed" of the derivative affects the system's stability. To ensure a fair comparison, we will analyze how the Hartman-Wintner criterion behaves in both cases, for the same Emden-Fowler equation with a potential $c(x) = |x|^\sigma = r^\sigma$.

1. Case A: Conformable Core ([12]). In this case $g(r,\alpha) = r^{1-\alpha}$. This nucleus has a polynomial growt;, thus, $A_g(r) = \frac{\omega_n r^{n-1}}{r^{1-\alpha}} = \omega_n r^{n-2+\alpha}$, and the integral of the potential is

$$\int_{r_0}^{r} \frac{\hat{c}(\tau)\tau^{n-1}}{g(\tau,\alpha)}d\tau = \int_{r_0}^{r} \frac{\tau^\sigma \tau^{n-1}}{\tau^{1-\alpha}}d\tau = \int_{r_0}^{r} \tau^{\sigma+n-2+\alpha}d\tau \approx \frac{r^{\sigma+n-1+\alpha}}{\sigma+n-1+\alpha}.$$

The Hartman-Wintner criterion for the conformable case is defined as

$$C(r) = \frac{1}{A_\alpha(r)}\int_{r_0}^{r}\left(\int_{S_\tau} c(x)d_\alpha\sigma\right)d\tau.$$

By substituting the obtained values, we get

$$C(r) \approx \frac{1}{\omega_n r^{n-2+\alpha}} \cdot \left(\frac{r^{\sigma+n-1+\alpha}}{\sigma+n-1+\alpha}\right),$$

after simplifying the exponents $\sigma+n-1+\alpha-(n-2+\alpha) = \sigma+1$, we have

$$C(r) \approx \frac{1}{\omega_n(\sigma+n-1+\alpha)} \cdot r^{\sigma+1}.$$

For the system to be oscillatory, we require that $\lim_{r\to\infty} C(r) = \infty$. From the previous expression, this occurs if and only if the exponent of r is positive

$$\sigma+1>0 \implies \sigma > -1.$$

If the potential decays faster than $1/r$, (i.e., $\sigma < -1$), the system may have non-oscillatory solutions (positive solutions that tend to zero). If the potential is constant, $\sigma = 0$, or increases, the system always oscillates.

In summary:

- The oscillation is polynomial $r^{\sigma+1}$. It is a "slow" oscillation.
- The oscillation depends linearly on σ and is slightly affected by α in the coefficient, but not in the main growth rate of $C(r)$ (since α cancels out in the final exponent of $r^{\sigma+1}$).
- This model cannot capture behaviors, where the physical medium has a memory or a local response, other than a simple power (simple fractality).

2. Case B: Mittag-Leffler nucleus. Here, the kernel $g(r,\alpha) = E_\alpha(r)$ grows exponentially. By using the approximation for $r \to \infty$, we have $E_\alpha(r) \sim \frac{1}{\alpha} e^{r^{1/\alpha}}$. In this way, the integral of the potential is

$$\int_{r_0}^{r} \frac{\hat{c}(\tau)\tau^{n-1}}{E_\alpha(\tau)} d\tau \approx \int_{r_0}^{r} \frac{\tau^\sigma \tau^{n-1}}{\frac{1}{\alpha} e^{\tau^{1/\alpha}}} d\tau = \alpha \int_{r_0}^{r} \tau^{\sigma+n-1} e^{-\tau^{1/\alpha}} d\tau.$$

This is an incomplete Gamma function type integral. Most importantly, due to the term $e^{-\tau^{1/\alpha}}$, this integral converges to a constant value K, as $r \to \infty$, even if the potential τ^σ is very large. The oscillation function $C_g(r)$ in this case is given by $C_g(r) = \frac{1}{A_g(r)} \times$ (Integral of the potential). By substituting the exponentially collapsing area $A_g(r)$, we obtain

$$C_g(r) \approx \frac{1}{\alpha \omega_n r^{n-1} e^{-r^{1/\alpha}}} \cdot K = \frac{K}{\alpha \omega_n r^{n-1}} e^{r^{1/\alpha}}.$$

In this case, $C_g(r) \to \infty$ at an exponential rate.

In summary:

- The oscillation is fractional exponential. It is a "violent" or "ultra-sensitive" oscillation.
- The system is inherently unstable. The Mittag-Leffler kernel acts as an amplifier, so that it ensures that the Hartman-Wintner criterion is almost always met.
- While the conformable kernel describes media with moderate memory, the generalized model describes media with long memory and ultra-fast diffusion, where perturbations propagate and oscillate regardless of how much the potential decays to infinity.

Remark 3.1. Unlike the model proposed in [12], where oscillation is limited by the polynomial growth of the radius, the introduction of the Mittag-Leffler kernel, based on [9], induces an exponential collapse in the generalized surface area measure. This results in a significantly more sensitive Hartman-Wintner criterion, where the growth rate of the kernel $E_\alpha(r)$ dominates the asymptotic behavior, facilitating the identification of oscillatory states in highly nonlinear media.

Next, to illustrate the impact of our generalization, we will design two scenarios, where the choice of kernel drastically changes the outcome. We will use the Emden-Fowler equation in a three-dimensional environment to observe the "explosion" of the oscillation function.

- Parameter Settings: Dimension $n = 3$ (Standard physical space).
- Order of the derivative $\alpha = 0.5$ (A classic intermediate value in fractional calculus).
- Exponent of the equation $\beta = 0.5$ (Sublinear equation, common in diffusion models).
- Potential $c(r) = r^{-2}$ (A potential that decays with distance).

Scenario 1: Conformable Kernel, $g(r) = r^{1-\alpha}$. This is the ordinary Emden-Fowler model.

(1) Calculation of $A_g(r)$

$$A_g(r) = \omega_n r^{n-2+\alpha} = 4\pi r^{3-2+0.5} = 4\pi r^{1.5}.$$

(2) Calculation of the integral of the potential $\sigma = -2$

$$\int_{r_0}^{r} \tau^{-2}\tau^{3-2+0.5} d\tau = \int_{r_0}^{r} \tau^{-0.5} d\tau = [2\tau^{0.5}]_{r_0}^{r} \approx 2r^{0.5}.$$

(3) Oscillation Function $C_{\text{conf}}(r)$

$$C_{\text{conf}}(r) = \frac{2r^{0.5}}{4\pi r^{1.5}} = \frac{1}{2\pi r}.$$

Result: $\lim_{r\to\infty} C_{\text{conf}}(r) = 0$.

For this potential, the conformable model does not guarantee oscillation. The system is stable.

Scenario 2: Mittag-Leffler Kernel $g(r) = E_\alpha(r)$). This is our generalized model using the LGD.

(1) Kernel Approximation: for $\alpha = 0.5$, $E_{0.5}(r) = e^{r^2}\text{erfc}(-r) \approx 2e^{r^2}$. To simplify the graph, we will use the asymptotic form $e^{r^{1/\alpha}} = e^{r^2}$.
(2) Calculation of $A_g(r)$:

$$A_g(r) = \frac{4\pi r^2}{e^{r^2}}.$$

(3) Calculation of the integral of the potential: the integral $\int \tau^{-2}\tau^2 e^{-\tau^2} d\tau = \int e^{-\tau^2} d\tau$ converges quickly to the constant $\frac{\sqrt{\pi}}{2}$ (Gaussian Integral).
(4) Oscillation Function $C_{\text{ML}}(r)$:

$$C_{\text{ML}}(r) = \frac{\sqrt{\pi}/2}{4\pi r^2 e^{-r^2}} = \frac{1}{8\sqrt{\pi} r^2} e^{r^2}.$$

Result: $\lim_{r\to\infty} C_{\text{ML}}(r) = \infty$ (Exponential growth).

The system is oscillatory. The structure of the LGD space forces oscillation, where the conformable model failed.

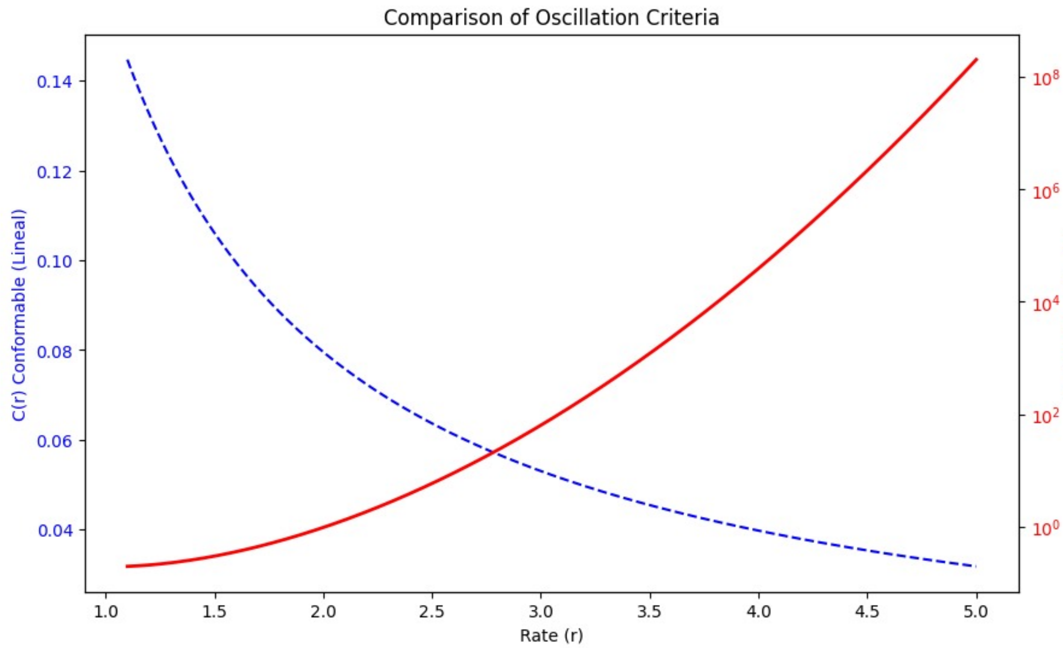

Figure 1. Comparison (Conformable vs. Exponential kernel)

The graph shows that while the conformable criterion decays towards zero (indicating that the potential r^{-2} is too weak to force oscillations), the generalized model, based on the LGD with Mittag-Leffler kernel, triggers the Hartman-Wintner criterion exponentially.

4. Conclusions

In this study, we have successfully extended the oscillation theory for Emden-Fowler type elliptic equations by incorporating the Local Generalized Derivative (LGD). This framework allows for a significant leap beyond the classical conformable calculus by introducing a flexible kernel $g(x, \alpha)$, that can be tailored to the specific physical properties of the medium under study .

The main contributions of this work are summarized as follows:

- Generalized Theoretical Framework: We established a robust version of the Divergence Theorem adapted to the LGD, which served as the foundation for deriving a new generalized Riccati identity .
- The Hartman-Wintner Criterion: We proved that the oscillation of the solutions is not only a function of the potential $c(x)$, but is critically "weighted" by the intrinsic geometry of the space defined by the kernel g.
- Impact of the Mittag-Leffler Kernel: Our comparative analysis revealed that kernels with exponential growth, such as the Mittag-Leffler function,

induce to an "ultra-sensitive" oscillatory state. While a system might remain stable (non-oscillatory) under a conformable kernel, the same potential can trigger infinite oscillations when modeled with an LGD approach.
- Modeling Flexibility: This generalization provides a more powerful tool for describing complex phenomena in non-homogeneous media, where memory effects and variable timescales—unreachable by classical or simple fractional derivatives—are present.
- Future research could explore the application of this LGD framework to other local differential operators, such as Omega derivative [1, 6, 13], to further broaden the horizons of asymptotic analysis in nonlinear differential equations.

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