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ON THE SOLVABILITY AND ULAM-HYERS STABILITY FOR COUPLED FRACTIONAL DIFFERENTIAL EQUATIONS WITH IMPULSIVE EFFECTS

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Abstract. This paper presents a study of a coupled system of nonlinear impulsive fractional differential equations involving Caputo derivatives. The system is subject to finitely many impulsive conditions that induce discontinuities in the state variables. Existence results via fixed-point techniques in generalized Banach spaces together with Ulam-Hyers stability are investigated. To demonstrate applicability of the theoretical results, two illustrative examples are presented.

Keywords. Fixed point theory, impulsive differential equations, Ulam-Hyers stability, fractional differential equations, vector Banach spaces.
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1. Introduction and preliminaries

In recent decades, impulsive differential equations have attracted considerable attention due to their broad applicability in various scientific and engineering disciplines for more detaills, see [20, 22, 30, 31] and in the references therein. In recent years, differential equations of fractional order have attracted considerable attention due to their effectiveness in describing complex phenomena in numerous scientific fields (see [2, 3, 8, 18, 23, 24, 29, 34]).

Furthermore, impulsive fractional differential equations deal with systems whose states jump at specific moments, incorporating both the long-term memory of fractional derivatives and the sharp transitions of impulses, leading to rich mathematical structures and applications see, for instance [5, 10, 15].

A significant generalization of the Banach contraction principle was achieved by Perov in 1964, who developed a fixed point theory for contraction maps in vector-valued metric spaces [25]. In [26] Precup demonstrate the pivotal role of vector-valued metric space and matrices approaching zero in the investigation of

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the following systems.

$$\begin{cases} w_1 = & \hat{F}_1[w_1, w_2], \\ w_2 = & \hat{F}_2[w_1, w_2], \end{cases}$$

where $\hat{F}_j : X^2 \to X (j = 1, 2)$ are nonlinear operators, X is Banach space. Recently, The paper[13] addresses the existence and uniqueness of solutions to the system below,

$$\begin{cases} {}^cD^{\alpha}u_1(\tau) = \bar{f}_1(t, u_1(\tau), u_2(\tau), {}^cD^{\alpha}u_1(\tau)), & t \in J, \\ {}^cD^{\beta}u_2(\tau) = \bar{f}_2\big(t, u_1(\tau), u_2(\tau), {}^cD^{\beta}u_2(\tau)\big), & t \in J, \\ u_1(0) = L_1[u_1], \quad u_1'(0) = L_2[u_1], \\ u_2(0) = L_3[u_2], \quad u_2'(0) = L_4[u_2], \end{cases}$$

where $\alpha, \beta \in [1, 2)$, $J = [0, 1]$, and ${}^cD^{\alpha}$, ${}^cD^{\beta}$ denote the Caputo fractional derivatives. The continuous functions $\bar{f}_i : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2$, and the functionals L_j, $j = 1, 2, 3, 4$, are given by Stieltjes integrals. The results of above system is obtained by employing fixed point method the firstly due to Perov and the secondly due to Krasnosel'skii. Moreover, an increasing number of works have addressed generalized formulations of such systems and investigated their stability under the Ulam-Hyers criterion, (see[1, 7, 11, 16, 19, 27, 32]). Recent advances in the study of coupled impulsive fractional systems involving various type of operators derivative have attracted considerable attention see, for instance ([14, 17, 33]).

Among these, Caputo-type fractional derivatives play a crucial role, as they allow the formulation of initial conditions in the same form as integer-order systems, which is essential in many physical and engineering applications. So, motivated by these advantages, we focus in this work on a class of coupled impulsive fractional systems involving Caputo derivatives and investigate the existence, uniqueness, and Hyers-Ulam stability of solutions for the following impulsive fractional differential system of orde $\beta_j \in (0, 1]$, $(j = 1, 2)$,

$$\begin{cases} {}^cD^{\beta_1}\xi(\kappa) = F_1(\kappa, \xi(\kappa), \eta(\kappa)), & J := [0, b]\,, \kappa \neq \kappa_\theta,\ \theta = 1\ldots, m \\ {}^cD^{\beta_2}\eta(\kappa) = F_2(\kappa, \xi(\kappa), \eta(\kappa)), & J := [0, b]\,, \kappa \neq \kappa_\theta,\ \theta = 1\ldots, m \\ \Delta\xi|_{\kappa=\kappa_\theta} = I_\theta\left(\xi(\kappa_\theta), \eta(\kappa_\theta)\right), & \theta = 1\ldots, m \\ \Delta\eta|_{\kappa=\kappa_\theta} = \bar{I}_\theta\left(\xi(\kappa_\theta), \eta(\kappa_\theta)\right), & \theta = 1\ldots, m \\ \xi(0) = \xi_0, \\ \eta(0) = \eta_0, \end{cases} \tag{1.1}$$

where ${}^cD^{\beta_1}, {}^cD^{\beta_2}$ are the Caputo fractional derivatives, $F_i : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $I_\theta, \bar{I}_\theta \in C(\mathbb{R}^2, \mathbb{R})$, $i = 1, 2$ denote to given functions and $\xi_0, \eta_0 \in \mathbb{R}$, $0 = \kappa_0 < \kappa_1 < ... < \theta_m < \theta_{m+1} = b$, $\Delta\xi|_{\kappa=\kappa_\theta} = \xi(\kappa_\theta^+) - \xi(\kappa_\theta^-)$, $\Delta\eta|_{\kappa=\kappa_\theta} = \eta(\kappa_\theta^+) - \eta(\kappa_\theta^-)$, $\xi(\kappa_\theta^+) = \lim_{h\to 0^+} \xi(\kappa_\theta + h)$ $\eta(\kappa_\theta^+) = \lim_{h\to 0^+} \eta(\kappa_\theta + h)$ and $\xi(\kappa_\theta^-) = \lim_{h\to 0^-} \xi(\kappa_\theta + h)$ $\eta(\kappa_\theta^-) = \lim_{h\to 0^-} \eta(\kappa_\theta + h)$ correspond to the right and left limits of ξ, η at $\kappa = \kappa_\theta$. The subsequent sections are arranged as described below. Section 2 provides definitions and preliminary results required in the sequel. In Section 3, we study the existence of solutions for our problem using Perov's fixed point theorem in vector Banach space. Section 4 is devoted to the study the existence result via Schaefer's fixed point theorem. Hyers-Ulam stability considerations are addressed in the second-last section of the paper. Two examples highlight the use of the theoretical findings, and the paper concludes with a section summarizing its principal results.

In what follows, we recall some basic tools. To avoid redundancy, we omit standard definitions related to fractional calculus operators, generalized Banach spaces, and matrices converging to zero, and instead refer the reader to well-established references for these concepts and their associated results.(see, e.g.,[12, 18, 24, 25]).

Definition 1.1. ([11, 12]). In generalized metric space (X^*, d), the operator $N^* : X^* \to X^*$ is called contractive if satisfying

$$d(N^*(\xi), N^*(\eta)) \leq E\, d(\xi, \eta),$$

where E is a matrix converging to the zero.

Theorem 1.2. *(Perov's fixed point theorem)([11, 12]). Assume that (X, d) be a complete generalized metric space and $N^* : X^* \to X^*$ is a contractive operator. Then N^* admits a unique fixed point.*

Theorem 1.3. *([12]) (Schaefer's fixed point theorem) Let X^* generalized Banach space, and $N^* : X^* \longrightarrow X^*$ is a continuous compact mapping. Whenever the set*

$$\mathcal{A} = \{\xi \in X^* : \xi = \lambda N^* \xi,\ \lambda \in (0,1)\},$$

is bounded, then N^* admits at least one fixed point.

Lemma 1.4. *([5]) Suppose that $\sigma : J \to \mathbb{R}$ be continuous and $\gamma \in (0,1]$. The function g solves the fractional integral equation:*

$$g(\kappa) = \begin{cases} g_0 + \frac{1}{\Gamma(\gamma)} \int_0^\kappa (\kappa - s)^{\gamma-1}\sigma(s)ds, & \text{if } \kappa \in [0, \kappa_1], \\ g_0 + \frac{1}{\Gamma(\gamma)} \sum_{i=1}^{\theta} \int_{\kappa_{i-1}}^{\kappa_i} (\kappa_i - s)^{\gamma-1}\sigma(s)ds & \\ +\frac{1}{\Gamma(\gamma)} \int_{\kappa_\theta}^{\kappa} (\kappa - s)^{\gamma-1}\sigma(s)ds + \sum_{i=1}^{\theta} I_i(g(\kappa_i^-)), & \text{if } \kappa \in J_\theta := (\kappa_\theta, \kappa_{\theta+1}], \end{cases}$$

where $\theta = 1, \ldots, m$, if and only if, the function g is a solution of the following problem

$$\begin{cases} {}^cD^\gamma g(\kappa) = \sigma(\kappa), & \kappa \in J_\theta, \\ \Delta g|_{\kappa=\kappa_\theta} = I_\theta(g(\kappa_\theta^-)), & \theta = 1, \ldots, m, \\ g(0) = g_0. \end{cases}$$

Now, we present notions concerning the Hyers-Ulam stability for a system.

Definition 1.5. Let X^* be a Banach space and operators $G^*_{k=1,2} : X^{*2} \longrightarrow X^*$. Then the operator defined by system

$$\begin{cases} f_1 = G_1^*[f_1, f_2], \\ f_2 = G_2^*[f_1, f_2], \end{cases} \tag{1.2}$$

is called Hyer-Ulam stable, if there exist constants $\chi_i^*(i = 1, 2, 3, 4) > 0$ for every $\epsilon_{j=1,2} > 0$ and any solution $(f_1^*, f_2^*) \in X^{*2}$ of the inequalities

$$\begin{cases} \|f_1^* - T_1[f_1^*, f_2^*]\| \leq \epsilon_1, \\ \|f_2^* - T_2[f_1^*, f_2^*]\| \leq \epsilon_2, \end{cases} \tag{1.3}$$

there exists a solution $(f_1^{**}, f_2^{**}) \in X^{*2}$ of the system (1.2), satisfies the inequalities

$$\begin{cases} \|f_1^* - f_1^{**}\| \leq \chi_1^*\epsilon_1 + \chi_2^*\epsilon_2, \\ \|f_2^* - f_2^{**}\| \leq \chi_3^*\epsilon_1 + \chi_4^*\epsilon_2, \end{cases} \tag{1.4}$$

with, $T_1, T_2 : X^{*2} \longrightarrow X^*$.

Definition 1.6. Consider a matrix $E \in \mathcal{M}_{m,m}(\mathbb{C})$ with eigenvalues $\mu_1, \mu_2, \ldots, \mu_m$. The spectral radius $\rho(E)$ of E is given by

$$\rho(E) = \max_{1 \leq i \leq m} |\mu_i|. \tag{1.5}$$

Theorem 1.7. *([1, 32]) Suppose that the operators $G^*_{k=1,2} : X^{*2} \longrightarrow X^*$ on the product of a Banach space X^*, such that*

$$\begin{cases} \|G_1^*[\xi, \eta] - G_1^{**}[\hat{\xi}, \hat{\eta}]\|_{X^*} \leq \Bbbk_{11}^{**}\|\xi - \hat{\xi}\|_{X^*} + \Bbbk_{12}^{**}\|\eta - \hat{\eta}\|_{X^*}, \\ \|G_2^*[\xi, \eta] - G_2^{**}[\hat{\xi}, \hat{\eta}]\|_{X^*} \leq \Bbbk_{21}^{*}\|\xi - \hat{\xi}\|_{X^*} + \Bbbk_{22}^{**}\|\eta - \hat{\eta}\|_{X^*}. \end{cases} \tag{1.6}$$

Then the fixed points of the operational system (1.2) are Hyers-Ulam stable, Provided that $\rho(E^*) < 1$, where

$$E^* = \begin{pmatrix} \Bbbk_{11}^{**} & \Bbbk_{12}^{**} \\ \Bbbk_{21}^{**} & \Bbbk_{22}^{**} \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}^+). \tag{1.7}$$

2. Existence and Uniqueness Results

Define the Banach space $(PC^2(J, \mathbb{R}), \|.\|_{PC^2})$ with

$$\|(\xi, \eta)\|_{PC^2} := (\|\xi\|_{PC}, \|\eta\|_{PC})^T,$$

$$\begin{aligned} PC(J, \mathbb{R}) \ &= \{\xi : J \to \mathbb{R} : \xi \in C((\kappa_\theta, \kappa_{\theta+1}], \mathbb{R}), \ \text{ such that } \xi(\kappa_\theta^-) = \xi(\kappa_\theta) \text{ whenever} \\ &\quad \xi(\kappa_\theta^-) \text{ and } \xi(\kappa_\theta^+) \text{ exist}, \theta = 0, \ldots, m\}, \end{aligned}$$

where

$$\|\xi\|_{PC} = \sup_{\kappa \in J} |\xi(\kappa)|.$$

Let $(C(J, \mathbb{R}), \|.\|_\infty)$ is a Banach space consisting of continuous functions, equipped with the norm

$$\|\xi\|_\infty = \sup_{\kappa \in J} |\xi(\kappa)|.$$

Denote $J_0 = [\kappa_0, \kappa_1]$, $\hat{J} = [0, b] \setminus \{\kappa_1, \kappa_2, ...\kappa_m\}$, $J_\theta = (\kappa_\theta, \kappa_{\theta+1}]$ with $\theta = 1, \ldots, m$.

Definition 2.1. We say that the pair of functions $(\xi,\eta)\in PC^2(J,\mathbb{R})$ constitutes a solution of (1.1) if and only if

$$\begin{cases}\xi(\kappa)=\xi_0+\frac{1}{\Gamma(\beta_1)}\int_0^\kappa(\kappa-s)^{\beta_1-1}h(s)ds, & \text{if } \kappa\in J_0,\\ \xi(\kappa)=\xi_0+\frac{1}{\Gamma(\beta_1)}\sum_{i=1}^\theta\int_{\kappa_{i-1}}^{\kappa_i}(\kappa_i-s)^{\beta_1-1}h(s)ds & \\ +\frac{1}{\Gamma(\beta_1)}\int_{\kappa_\theta}^\kappa(\kappa-s)^{\beta_1-1}h(s)ds+\sum_{i=1}^\theta I_i(\xi(\kappa_i),\eta(\kappa_i)), & \text{if } \kappa\in J_\theta,\end{cases}$$

and

$$\begin{cases}\eta(\kappa)=\eta_0+\frac{1}{\Gamma(\beta_2)}\int_0^\kappa(\kappa-s)^{\beta_2-1}\sigma(s)ds, & \text{if } \kappa\in J_0,\\ \eta(\kappa)=\eta_0+\frac{1}{\Gamma(\beta_2)}\sum_{i=1}^\theta\int_{\kappa_{i-1}}^{\kappa_i}(\kappa_i-s)^{\beta_2-1}\sigma(s)ds & \\ +\frac{1}{\Gamma(\beta_2)}\int_{\kappa_\theta}^\kappa(\kappa-s)^{\beta_2-1}\sigma(s)ds+\sum_{i=1}^\theta \bar{I}_i(\xi(\kappa_i),\eta(\kappa_i)), & \text{if } \kappa\in J_\theta,\end{cases}$$

where

$${}^cD^{\beta_1}\xi(\kappa)=h(\kappa),\ {}^cD^{\beta_2}\eta(\kappa)=\sigma(\kappa),\ \kappa\in\hat{J}.$$

2.1. **Solvability and Uniqueness Analysis.** This subsection is devoted to the establishment of the first main result of this work. To this end, we apply Perov's fixed-point theorem to obtain the following result.

Theorem 2.2. *Assume that the following conditions are satisfied:*

(H1) *The function* $F_1,F_2\in C(J\times\mathbb{R}^2,\mathbb{R})$.

(H2) *For all* ϑ_j, *there exist* $\Phi_j\in C(J,(0,\infty)),\ j=1,2,3,4$ *satisfying*

$$|F_1(\kappa,\vartheta_1,\vartheta_2)-F_1(\kappa,\bar{\vartheta}_1,\bar{\vartheta}_2)|\le\Phi_1(\kappa)|\vartheta_1-\bar{\vartheta}_1|+\Phi_2(\kappa)|\vartheta_2-\bar{\vartheta}_2|,$$

$$|F_2(\kappa,\vartheta_1,\vartheta_2)-F_2(\kappa,\bar{\vartheta}_1,\bar{\vartheta}_2)|\le\Phi_3(\kappa)|\vartheta_1-\bar{\vartheta}_1|+\Phi_4(\kappa)|\vartheta_2-\bar{\vartheta}_2|.$$

(H3) *For all* ϑ_j, *there exist* $\Psi_j^*>0,\ i=1,2,3,4$ *satisfying:*

$$|I_\theta(\vartheta_1,\vartheta_2)-I_\theta(\bar{\vartheta}_1,\bar{\vartheta}_2)|\le\Psi_1^*|\vartheta_1-\bar{\vartheta}_1|+\Psi_2^*|\vartheta_2-\bar{\vartheta}_2|,$$

$$|\bar{I}_\theta(\vartheta_1,\vartheta_2)-\bar{I}_\theta(\bar{\vartheta}_1,\bar{\vartheta}_2)|\le\Psi_3^*|\vartheta_1-\bar{\vartheta}_1|+\Psi_4^*|\vartheta_2-\bar{\vartheta}_2|.$$

Suppose that the matrix

$$B^*=\begin{pmatrix}\dfrac{b^{\beta_1}(m+1)\|\Phi_1\|_\infty}{\Gamma(\beta_1+1)}+m\Psi_1^* & \dfrac{b^{\beta_1}(m+1)\|\Phi_2\|_\infty}{\Gamma(\beta_1+1)}+m\Psi_2^*\\ \dfrac{b^{\beta_2}(m+1)\|\Phi_3\|_\infty}{\Gamma(\beta_2+1)}+m\Psi_3^* & \dfrac{b^{\beta_2}(m+1)\|\Phi_4\|_\infty}{\Gamma(\beta_2+1)}+m\Psi_4^*\end{pmatrix}, \tag{2.1}$$

converge to zero. Then problem (1.1) admits precisely one solution.

Proof. We start by define the operator $\mathbb{H}^*:PC^2(J,\mathbb{R})\to PC^2(J,\mathbb{R})$

$$\mathbb{H}^*(\xi,\eta)(\kappa)=\begin{pmatrix}\mathbb{H}_1^*(\xi,\eta)(\kappa)\\ \mathbb{H}_2^*(\xi,\eta)(\kappa)\end{pmatrix}\ \text{for all } (\xi,\eta)\in PC^2(J,\mathbb{R}),\ \kappa\in J, \tag{2.2}$$

where

$$\mathbb{H}_1^*(\xi,\eta)(\kappa) = \xi_0 + \frac{1}{\Gamma(\beta_1)} \sum_{0<\kappa_\theta<\kappa} \int_{\kappa_{\theta-1}}^{\kappa_\theta} (\kappa_\theta - s)^{\beta_1-1} F_1(s,\xi(s),\eta(s))ds + \frac{1}{\Gamma(\beta_1)} \int_{\kappa_\theta}^{\kappa} (\kappa - s)^{\beta_1-1} F_1(s,\xi(s),\eta(s))ds + \sum_{0<\kappa_\theta<\kappa} I_\theta(\xi(\kappa_\theta),\eta(\kappa_\theta)), \tag{2.3}$$

$$\mathbb{H}_2^*(\xi,\eta)(\kappa) = \eta_0 + \frac{1}{\Gamma(\beta_2)} \sum_{0<\kappa_\theta<\kappa} \int_{\kappa_{\theta-1}}^{\kappa_\theta} (\kappa_\theta - s)^{\beta_2-1} F_2(s,\xi(s),\eta(s))ds + \frac{1}{\Gamma(\beta_2)} \int_{\kappa_\theta}^{\kappa} (\kappa - s)^{\beta_2-1} F_2(s,\xi(s),\eta(s))ds + \sum_{0<\kappa_\theta<\kappa} \bar{I}_\theta(\xi(\kappa_\theta),\eta(\kappa_\theta)). \tag{2.4}$$

Observe that solving problem (1.1) is equivalent to finding the fixed points of the operator $\mathbb{H}^*$. To do this, we verify that $\mathbb{H}^*$ is a contraction mapping. Indeed, using (H2), (H3) and for all $(\xi,\eta),(\bar{\xi},\bar{\eta}) \in PC^2(J,\mathbb{R})$, we have

$$\begin{aligned} |\mathbb{H}_1^*(\xi,\eta)(\kappa) - \mathbb{H}_1^*(\bar{\xi},\bar{\eta})(\kappa)| \leq\ & \frac{1}{\Gamma(\beta_1)} \sum_{0<\kappa_\theta<\kappa} \int_{\kappa_{\theta-1}}^{\kappa_\theta} (\kappa_\theta - s)^{\beta_1-1} |f(s,\xi,\eta) - f(s,\bar{\xi},\bar{\eta})|ds \\ & + \frac{1}{\Gamma(\beta_1)} \int_{\kappa_\theta}^{\kappa} (\kappa - s)^{\beta_1-1} |F_1(s,\xi,\eta) - F_1(s,\bar{\xi},\bar{\eta})|ds \\ & + \sum_{0<\kappa_\theta<\kappa} |I_\theta(\xi(\kappa_\theta),\eta(\kappa_\theta)) - I_\theta(\bar{\xi}(\kappa_\theta),\bar{\eta}(\kappa_\theta))| \end{aligned}$$

Therefore

$$\begin{aligned} \|\mathbb{H}_1^*(\xi,\eta) - \mathbb{H}_1^*(\bar{\xi},\bar{\eta})\|_{PC} \leq\ & \left(\frac{b^{\beta_1}(m+1)\|\Phi_1\|_\infty}{\Gamma(\beta_1+1)} + m\Psi_1^* \right) \|\xi - \bar{\xi}\|_{PC} \\ & + \left(\frac{b^{\beta_1}(m+1)\|\Phi_2\|_\infty}{\Gamma(\beta_1+1)} + m\Psi_2^* \right) \|\eta - \bar{\eta}\|_{PC}. \end{aligned}$$

In the same way

$$\begin{aligned} \|\mathbb{H}_2^*(\xi,\eta) - \mathbb{H}_2^*(\bar{\xi},\bar{\eta})\|_{PC} \leq\ & \left(\frac{b^{\beta_2}(m+1)\|\Phi_3\|_\infty}{\Gamma(\beta_2+1)} + m\Psi_3^* \right) \|\xi - \bar{\xi}\|_{PC} \\ & + \left(\frac{b^{\beta_2}(m+1)\|\Phi_4\|_\infty}{\Gamma(\beta_2+1)} + m\Psi_4^* \right) \|\eta - \bar{\eta}\|_{PC}. \end{aligned}$$

Thus

$$\|\mathbb{H}^*(\xi,\eta) - \mathbb{H}^*(\bar{\xi},\bar{\eta})\|_{PC^2} \leq B^* \begin{pmatrix} \|\xi - \bar{\xi}\|_{PC} \\ \|\eta - \bar{\eta}\|_{PC} \end{pmatrix}. \tag{2.5}$$

Hence $\mathbb{H}^*$ is a contraction and theorem 1.2 ensures us the existence of only fixed point for $\mathbb{H}^*$, which is solution of problem (1.1).

2.2. **Existence Analysis via Schaefer's Fixed-Point.** This subsection is devoted to proving the existence of solutions to the proposed problem via Schaefer's fixed-point theorem.

Theorem 2.3. *Assume (H1), (H2) be satisfied together with following hypothesis holds:*

(H4) *For all $\kappa \in J$; $\vartheta_1, \vartheta_2 \in \mathbb{R}$, there exist a functions $A_1^*, A_2^* \in C(J,(0,\infty))$ satisfying*
$$|F_1(\kappa, \vartheta_1, \vartheta_2)| \leq A_1^*(\kappa) \quad \text{and} \quad |F_2(\kappa, \vartheta_1, \vartheta_2)| \leq A_2^*(\kappa).$$

(H5) *There exist a functions $I_\theta, \bar{I}_\theta \in C(\mathbb{R}^2, \mathbb{R})$ and constants W_j, $W_j^* > 0$ with $(j = 1, 2, 3)$ satisfying for all $\vartheta_1, \vartheta_2 \in \mathbb{R}$, $\theta = 1, \ldots, m$, the inequalities*
$$|I_\theta(\vartheta_1, \vartheta_2)| \leq \hat{W}_1|\vartheta_1| + \hat{W}_2|\vartheta_2| + \hat{W}_3,$$
$$|\bar{I}_\theta(\vartheta_1, \vartheta_2)| \leq \hat{W}_1^*|\vartheta_1| + \hat{W}_2^*|\vartheta_2| + \hat{W}_3^*.$$

Suppose that, the matrix
$$B^{**} = m\begin{pmatrix} \hat{W}_1 & \hat{W}_2 \\ \hat{W}_1^* & \hat{W}_2^* \end{pmatrix} \in \mathcal{M}_{2\times 2}\left(\mathbb{R}^+\right), \tag{2.6}$$
converge to zero. Then the problem (1.1) has at least one solution. Moreover, the solution set $S(\xi_0, \eta_0) = \{(\xi, \eta) \in PC^2(J, \mathbb{R}) : (\xi, \eta)$ is solution of (1.1)$\}$ is compact.

Proof. Let $\mathbb{H}^* : PC^2(J, \mathbb{R}) \to PC^2(J, \mathbb{R})$ defined in (2.2). We organize the proof into several steps.
Step1: $\mathbb{H}^*$ is continuous Suppose that $(\xi_n, \eta_n) \longrightarrow (\xi, \eta)$ in $PC^2(J, \mathbb{R})$. Then for $\kappa \in J$ and $\theta = 1 \ldots, m$

$$\begin{aligned}
|\left[\mathbb{H}_1^*(\xi_n, \eta_n)\right](\kappa) - \left[\mathbb{H}_1^*(\xi, \eta)\right](\kappa)| \leq & \sum_{i=1}^{m} \int_{t_{\theta-1}}^{t_\theta} \frac{(\kappa_\theta - s)^{\beta_1 - 1}}{\Gamma(\beta_1)} \left(\Phi_1(s)|\xi_n - \xi| + \Phi_2(s)|\eta_n - \eta|\right) ds \\
& + \int_{\kappa_\theta}^{\kappa} \frac{(\kappa - s)^{\beta_1 - 1}}{\Gamma(\beta_1)} \left(\Phi_1(s)|\xi_n - \xi| + \Phi_2(s)|\eta_n - \eta|\right) ds \\
& + \sum_{0<\kappa_\theta<\kappa} |I_\theta(\xi_n(\kappa_\theta), \eta_n(\kappa_\theta)) - I_\theta(\xi(\kappa_\theta), \eta(\kappa_\theta))|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|\left[\mathbb{H}_1^*(\xi_n, \eta_n)\right] - \left[\mathbb{H}_1^*(\xi, \eta)\right]\|_{PC} \leq & \left(\frac{b^{\beta_1}(m+1)\|\Phi_1\|_\infty}{\Gamma(\beta_1 + 1)}\right) \|\xi_n - \xi\|_{PC} \\
& + \left(\frac{b^{\beta_1}(m+1)\|\Phi_2\|_\infty}{\Gamma(\beta_1 + 1)}\right) \|\eta_n - \eta\|_{PC} \\
& + \sum_{0<\kappa_k<\kappa} |I_\theta(\xi_n(\kappa_\theta), \eta_n(\kappa_\theta)) - I_\theta(\xi(\kappa_\theta), \eta(\kappa_\theta))|.
\end{aligned}$$

Exploiting the continuity of I_θ, one deduces that

$$\|\mathbb{H}_1^*(\xi_n,\eta_n)-\mathbb{H}_1^*(\xi,\eta)\|_{PC}\longrightarrow 0\ \ as\ \ n\longrightarrow\infty.$$

Similarly, we have

$$\|\mathbb{H}_2^*(\xi_n,\eta_n)-\mathbb{H}_2^*(\xi,\eta)\|_{PC}\longrightarrow 0\ \ as\ \ n\longrightarrow\infty.$$

Thus

$$\|\mathbb{H}^*(\xi_n,\eta_n)-\mathbb{H}^*(\xi,\eta)\|_{PC^2}\longrightarrow 0\ \ as\ \ n\longrightarrow\infty.$$

Step2: $\mathbb{H}^*$ maps bounded sets into bounded sets in $PC^2(J,\mathbb{R})$.
Let $B_r:=\{(\xi,\eta)\in PC^2(J,\mathbb{R}):\|(\xi,\eta)\|_{PC^2}\leq r\}$ where $r=(r_1,r_2)$. First of all, the functions $A_j^*(j=1,2)$ are continuous then, we can find a constants $A_j^*>0$ satisfying

$$|A_j^*(\kappa)|\leq A_j^*,\quad A_j^*=\max_{\kappa\in J}A_j^*(\kappa)\quad j=1,2.$$

Moreover, if $(\xi,\eta)\in B_r$, we have

$$|\mathbb{H}_1^*(\xi,\eta)(\kappa)|\ \ \leq\ \ |\xi_0|+\frac{b^{\beta_1}A_1^*m}{\Gamma(\beta_1+1)}+\frac{b^{\beta_1}A_1^*}{\Gamma(\beta_1+1)}+\sum_{i=1}^{m}|I_k(\xi(\kappa_\theta),\eta(\kappa_\theta))|.$$

Using (H5), we find that

$$\|\mathbb{H}_1^*(\xi,\eta)\|_{PC}\leq|\xi_0|+\frac{b^{\beta_1}A_1^*(m+1)}{\Gamma(\beta_1+1)}+m(\hat{W}_1r_1+\hat{W}_2r_2+\hat{W}_3).$$

Similarly, we have

$$\|\mathbb{H}_2^*(\xi,\eta)\|_{PC}\leq|\eta_0|+\frac{b^{\beta_2}A_2^*(m+1)}{\Gamma(\beta_2+1)}+m(\hat{W}_1^*r_1+W_1^*r_2+\hat{W}_3^*).$$

Hence, we obtain

$$\|\mathbb{H}^*(\xi,\eta)\|_{PC^2}\leq r^*,\quad \text{for all}\,(\xi,\eta)\in B_r,$$

where $r^*=(r_1^*,r_2^*)$ and

$$r_1^*=|\xi_0|+\frac{b^{\beta_1}A_1^*(m+1)}{\Gamma(\beta_1+1)}+m(\hat{W}_1r_1+\hat{W}_2r_2+\hat{W}_3),$$
$$r_2^*=|\eta_0|+\frac{b^{\beta_2}A_2^*(m+1)}{\Gamma(\beta_2+1)}+m(\hat{W}_1^*r_1+W_2^*r_2+\hat{W}_3^*).$$

Step3: $\mathbb{H}^*$ maps bounded in $PC^2(J,\mathbb{R})$ sets into equicontinuous sets of $PC^2(J,\mathbb{R})$.

Let $(\xi,\eta)\in B_r$ then we have

$$\|\xi\|\leq r_1,\ \ \|\eta\|\leq r_2$$

So, for all $\kappa_1,\kappa_2\in J$, $\kappa_1<\kappa_2$, we find

$$\begin{aligned}|\mathbb{H}_1^*(\xi,\eta)(\kappa_2)-\mathbb{H}_1^*(\xi,\eta)(\kappa_1)|\ \ &\leq\ \ A_1^*\left(\frac{2(\kappa_2-\kappa_1)^{\beta_1}+\kappa_2^{\beta_1}-\kappa_1^{\beta_1}}{\Gamma(\beta_1+1)}\right)\\&\quad+(\kappa_2-\kappa_1)(\hat{W}_1r_1+\hat{W_2}r_2+\hat{W}_3).\end{aligned}$$

As $\kappa_1 \longrightarrow \kappa_2$, the right-hand side of the above inequality tends to zero. Consequently

$$|\mathbb{H}_1^*(\xi,\eta)(\kappa_2) - \mathbb{H}_1^*(\xi,\eta)(\kappa_1)| \longrightarrow 0 \quad as \quad \kappa_1 \longrightarrow \kappa_2.$$

Similarly, we have

$$|\mathbb{H}_2^*(\xi,\eta)(\kappa_2) - \mathbb{H}_2^*(\xi,\eta)(\kappa_1)| \longrightarrow 0 \quad as \quad \kappa_1 \longrightarrow \kappa_2.$$

Combining Steps 1-3 with the Arzelà-Ascoli theorem, the operator $\mathbb{H}^* : PC^2(J,\mathbb{R}) \to PC^2(J,\mathbb{R})$ is completely continuous.

Step4: The set defined as follows:
$\mathcal{M} = \{(\xi,\eta) \in PC^2 : (\xi,\eta) = \lambda \mathbb{H}^* (\xi,\eta), \lambda \in (0,1)\}$ is bounded. First of all, denote

$$\Omega_{\beta_1}^* := |\xi_0| + \frac{(m+1)\, A_1^* b^{\beta_1}}{\Gamma(\beta_1+1)} + m\hat{W}_3,$$
$$\Omega_{\beta_2}^* := |\eta_0| + \frac{(m+1)\, A_2^* b^{\beta_2}}{\Gamma(\beta_2+1)} + m\hat{W}_3^*.$$

Let $(\xi,\eta) \in \mathcal{M}$, then we have

$$|\xi(\kappa)| \leq |\xi_0| + \frac{(m+1)\, A_1^* b^{\beta_1}}{\Gamma(\beta_1+1)} + m(\hat{W}_1|\xi(\kappa_\theta)| + \hat{W}_2|\eta(\kappa_\theta)| + \hat{W}_3).$$

Therefore

$$\|\xi\|_{PC} \leq \Omega_{\beta_1}^* + m(\hat{W}_1\|\xi\|_{PC} + \hat{W}_2\|\eta\|_{PC}). \tag{2.7}$$

In same way

$$\|\eta\|_{PC} \leq \Omega_{\beta_2}^* + m(\hat{W}_1^*\|\xi\|_{PC} + \hat{W}_2^*\|\eta\|_{PC}). \tag{2.8}$$

The inequalities (2.7) and (2.8) imply that

$$\begin{pmatrix} \|\xi|_{PC} \\ \|\eta\|_{PC} \end{pmatrix} \leq \begin{pmatrix} \Omega_{\beta_1}^* \\ \Omega_{\beta_2}^* \end{pmatrix} + B^{**} \begin{pmatrix} \|\xi\|_{PC} \\ \|\eta\|_{PC} \end{pmatrix}. \tag{2.9}$$

The above inequality can rewrite the following vectorial form

$$\begin{pmatrix} \|\xi|_{PC} \\ \|\eta\|_{PC} \end{pmatrix} \leq (I - B^{**})^{-1} \begin{pmatrix} \Omega_{\beta_1}^* \\ \Omega_{\beta_2}^* \end{pmatrix}. \tag{2.10}$$

Which implies that

$$\|\xi\|_{PC} \leq (1 - m\hat{W}_1)\Omega_{\beta_1}^* - m\hat{W}_2\Omega_{\beta_2}^* := R_1^*, \tag{2.11}$$
$$\|\eta\|_{PC} \leq (1 - m\hat{W}_2^*)\Omega_{\beta_2}^* - m\hat{W}_1^*\Omega_{\beta_1}^* := R_2^*, \tag{2.12}$$

Thus

$$\mathcal{M} \subseteq B_{R^*},$$

where $B_{R^*} := \{(\xi,\eta) \in PC^2(J,\mathbb{R}) : \|(\xi,\eta)\|_{PC^2} \leq R^*\}$ and $R^* := (R_1^*, R_2^*)$. Hence the set $\mathcal{M}$ is bounded.

Step5: Compactness property of solution set. We now demonstrate that the set

$$S(\xi_0,\eta_0) := \left\{(\xi,\eta) \in PC^2(J,\mathbb{R}) : \ (\xi,\eta) \text{ is a solution of } (1.1)\right\},$$

is compact. It is clear that $S(\xi_0, \eta_0) \subset \mathbb{H}^* S(\xi_0, \eta_0)$. According to (2.11) and (2.12), we deduce that $S(\xi_0, \eta_0)$ is bounded in $PC^2(J, \mathbb{R})$. By the compactness of $\mathbb{H}^*$, the set $S(\xi_0, \eta_0))$ is compact whenever the closure property holds. Let $\{(\xi_n, \eta_n)\}_{n\geq 1} \subset S(\xi_0, \eta_0)$ be a sequence such that $(\xi_n, \eta_n) \to (\xi, \eta)$. Hence,

$$\xi_n = \xi_0 + \frac{1}{\Gamma(\beta_1)} \sum_{0<\kappa_\theta<\kappa} \int_{\kappa_{\theta-1}}^{\kappa_\theta} (\kappa_\theta - s)^{\beta_1 - 1} F_1(s, \xi_n(s), \eta_n(s)) ds + \frac{1}{\Gamma(\beta_1)} \int_{\kappa_\theta}^{\kappa} (\kappa - s)^{\beta_1 - 1} F_1(s, \xi_n(s), \eta_n(s)) ds + \sum_{0<\kappa_\theta<\kappa} I_\theta(\xi_n(\kappa_\theta), \eta_n(\kappa_\theta)), \tag{2.13}$$

$$\eta_n = \eta_0 + \frac{1}{\Gamma(\beta_2)} \sum_{0<\kappa_\theta<\kappa} \int_{\kappa_{\theta-1}}^{\kappa_\theta} (\kappa_\theta - s)^{\beta_2 - 1} F_2(s, \xi_n(s), \eta_n(s)) ds + \frac{1}{\Gamma(\beta_2)} \int_{\kappa_\theta}^{\kappa} (\kappa - s)^{\beta_2 - 1} F_2(s, \xi_n(s), \eta_n(s)) ds + \sum_{0<\kappa_\theta<\kappa} \bar{I}_\theta(\xi_n(\kappa_\theta), \eta_n(\kappa_\theta)). \tag{2.14}$$

As established in step 1, we find that

$$\|(\xi_n, \eta_n) - (\xi, \eta)\|_{PC^2} \longrightarrow 0 \ \ as \ \ n \longrightarrow \infty,$$

Which mean that $S(\xi_0, \eta_0)$ is closed. Hence, the solution set is compact.

3. Ulam-Hyers Type Stability Results

In order to derive the Ulam-Hyers stability estimate, we begin by rewrite the considered problem as an equivalent first-order system as follows

$$\begin{cases} \xi = \mathbb{H}_1^*(\xi, \eta), \\ \eta = \mathbb{H}_2^*(\xi, \eta). \end{cases} \tag{3.1}$$

Theorem 3.1. *Suppose that assumptions (H1)-(H3) are satisfied and $\rho(B^*) < 1$. Then the solution of problem (1.1) is Hyers–Ulam stable.*

Proof. According to (2.5), the operator $\mathbb{H}^*(\xi, \eta) = (\mathbb{H}_1^*(\xi, \eta), \mathbb{H}_2^*(\xi, \eta))$, satisfied, for all $(\xi, \eta), (\bar{\xi}, \bar{\eta}) \in PC^2(J, \mathbb{R})$,

$$\begin{aligned} \|\mathbb{H}_1^*(\xi, \eta) - \mathbb{H}_1^*(\bar{\xi}, \bar{\eta})\|_{PC} &\leq B_{11}^* \|\xi - \bar{\xi}\|_{PC} + B_{12}^* \|\eta - \bar{\eta}\|_{PC}, \\ \|\mathbb{H}_2^*(\xi, \eta) - \mathbb{H}_2^*(\bar{\xi}, \bar{\eta})\|_{PC} &\leq B_{21}^* \|\xi - \bar{\xi}\|_{PC} + B_{22}^* \|\eta - \bar{\eta}\|_{PC}, \end{aligned} \tag{3.2}$$

with $B^* = (B_{ij}^*)_{1\leq i,j\leq 2}$. Since $\rho(B^*) < 1$, so according to the theorem 1.7, the problem (1.1) is Hyers-Ulam stable.

4. Examples Supporting the Main Results

In this section we give two examples to illustrate the usefulness of our main results.

Example 1. Consider the following system of impulsive fractional differential equations.

$$\begin{cases} {}^cD^{\frac{1}{2}}\xi(\kappa) = \frac{1}{64e^{\kappa^2+3}(1+|\xi(\kappa)|+|\eta(\kappa)|)}\,, & \kappa \in J := [0,1]\,, \kappa \neq \frac{1}{5} \\ {}^cD^{\frac{1}{3}}\eta(\kappa) = \frac{1}{44e^{\kappa^3+2}(1+|\xi(\kappa)|+|\eta(\kappa)|)}\,, & \kappa \in J := [0,1]\,, \kappa \neq \frac{1}{5} \\ \Delta\xi\big|_{t=\frac{1}{5}} = \frac{|\xi\left(\frac{1}{5}\right)|+|\eta\left(\frac{1}{5}\right)|}{32e^3+|\xi\left(\frac{1}{5}\right)|+|\eta\left(\frac{1}{5}\right)|}\,, & \kappa_1 = \frac{1}{5} \\ \Delta\eta\big|_{\kappa=\frac{1}{5}} = \frac{|\xi\left(\frac{1}{5}\right)|+|\eta\left(\frac{1}{5}\right)|}{22e^2+|\xi\left(\frac{1}{5}\right)|+|\eta\left(\frac{1}{5}\right)|}\,, & \kappa_1 = \frac{1}{5} \\ \xi(0) = 0, \\ \eta(0) = 0. \end{cases} \tag{4.1}$$

Consider $F_i(i = 1, 2)$ be a functions defined as follows,

$$F_1(\kappa, \vartheta_1, \vartheta_2) = \frac{1}{64e^{\kappa^2+3}\left(1 + |\vartheta_1| + |\vartheta_2|\right)},\ (\kappa, \vartheta_1, \vartheta_2) \in [0, 1] \times \mathbb{R}^2,$$
$$F_2(\kappa, \vartheta_1, \vartheta_2) = \frac{1}{44e^{\kappa^3+2}\left(1 + |\vartheta_1| + |\vartheta|_2\right)},\ (\kappa, \vartheta_1, \vartheta_2) \in [0, 1] \times \mathbb{R}^2.$$

Observe that $F_1, F_2 \in C([0, 1] \times \mathbb{R}^2, \mathbb{R})$ and for any $\vartheta_1, \vartheta_2, \bar{\vartheta_1}, \bar{\vartheta_2} \in \mathbb{R}$,

$$|F_1(\kappa, \vartheta_1, \vartheta_2) - F_1(\kappa, \bar{\vartheta_1}, \bar{\vartheta_2})| \leq \frac{1}{64e^{\kappa^2+3}}\left(|\vartheta_1 - \bar{\vartheta_1}| + |\vartheta_2 - \bar{\vartheta_2}|\right),$$

$$|F_2(\kappa, \vartheta_1, \vartheta_2) - F_2(\kappa, \bar{\vartheta_1}, \bar{\vartheta_2})| \leq \frac{1}{44e^{\kappa^3+2}}\left(|\vartheta_1 - \bar{\vartheta_1}| + |\vartheta_2 - \bar{\vartheta_2}|\right).$$

Hence the condition (H2) holds, with

$$\Phi_1(\kappa) = \Phi_2(\kappa) = \frac{1}{64e^{\kappa^2+3}} \quad , \quad \Phi_3(\kappa) = \Phi_4(\kappa) = \frac{1}{44e^{\kappa^3+2}}.$$

So, we have

$$\|\Phi_1\|_\infty = \|\Phi_2\|_\infty = \frac{1}{64e^3}, \qquad \|\Phi_3\|_\infty = \|\Phi_4\|_\infty = \frac{1}{44e^2}.$$

Let $\vartheta_1, \vartheta_2 \in [0, \infty)$

$$I_\theta(\vartheta_1, \vartheta)_2 = \frac{\vartheta_1 + \vartheta_2}{32e^3 + \vartheta_1 + \vartheta_2}, \qquad \bar{I}_\theta(\vartheta_1, \vartheta_2) = \frac{\vartheta_1 + \vartheta_2}{22e^2 + \vartheta_1 + \vartheta_2}.$$

Then,

$$|I_\theta(\vartheta_1, \vartheta_2) - I_\theta(\bar{\vartheta_1}, \bar{\vartheta_2})| \leq \frac{1}{32e^3}\left(|\vartheta_1 - \bar{\vartheta_1}| + |\vartheta_2 - \bar{\vartheta_2}|\right).$$

$$|\bar{I}_\theta(\vartheta_1, \vartheta_2) - \bar{I}_\theta(\bar{\vartheta_1}, \bar{\vartheta_2})| \leq \frac{1}{22e^2}\left(|\vartheta_1 - \bar{\vartheta_1}| + |\vartheta_2 - \bar{\vartheta_2}|\right).$$

Then, condition (H3) be satisfied with

$$\Psi_1^* = \Psi_2^* = \frac{1}{32e^3}, \quad \Psi_3^* = \Psi_4^* = \frac{1}{22e^2}.$$

Since $b=1$, $m=1$, $\beta_1=\frac{1}{2}$, $\beta_2=\frac{1}{3}$ and the elements of $B^*=(B^*_{ij})_{1\le i,j\le 2}$, are

$$B^*_{1j}=\frac{2+\Gamma(\frac{1}{2})}{32e^3\Gamma(\frac{1}{2})},\ B^*_{2j}=\frac{3+\Gamma(\frac{1}{3})}{22e^2\Gamma(\frac{1}{3})}\quad (\text{for } j=1,2).$$

Then

$$B^*=\begin{pmatrix}\dfrac{2+\Gamma(\frac{1}{2})}{32e^3\Gamma(\frac{1}{2})} & \dfrac{2+\Gamma(\frac{1}{2})}{32e^3\Gamma(\frac{1}{2})}\\[2ex] \dfrac{3+\Gamma(\frac{1}{3})}{22e^2\Gamma(\frac{1}{3})} & \dfrac{3+\Gamma(\frac{1}{3})}{22e^2\Gamma(\frac{1}{3})}\end{pmatrix},$$

and the above matrix admit two eigenvalues $\lambda_1=0$ and $\lambda_2\simeq 0.013$. then the matrix B^* converges to zero, so by theorem 2.2 the problem (4.1) has unique solution. Moreover the result of theorem 3.1 ensure us that the proposed problem is Ulam-Hyers stable.

Example 2. Consider the following system of impulsive fractional differential equations.

$$\begin{cases} {}^cD^{\frac{1}{4}}\xi(\kappa)=\frac{|\xi(\kappa)|+|\eta(\kappa)|}{105(1+15\kappa^3)e^{\sin(\kappa)+1}(3+|\xi(\kappa)|+|\eta(\kappa)|)}, & \kappa\in J:=[0,1]\,,\kappa\ne\frac{1}{2}\\ {}^cD^{\frac{1}{3}}\eta(\kappa)=\frac{|\xi(\kappa)|+|\eta(\kappa)|}{24(1+13\kappa^5)e^{\cos(\kappa-1)}(2+|\xi(\kappa)|+|\eta(\kappa)|)}, & \kappa\in J:=[0,1]\,,\kappa\ne\frac{1}{2}\\ \Delta\xi|_{\kappa=\frac{1}{2}}=\frac{13(1+|\xi(\frac{1}{2})|+|\eta(\frac{1}{2})|)\pi}{(155+|\xi(\frac{1}{2})|+|\eta(\frac{1}{2})|}\,, & \kappa_1=\frac{1}{2}\\ \Delta\eta|_{\kappa=\frac{1}{2}}=\frac{35(1+|\xi(\frac{1}{2})|+|\eta(\frac{1}{2})|)\pi}{(199+|\xi(\frac{1}{2})|+|\eta(\frac{1}{2})|)}\,, & \kappa_1=\frac{1}{2}\\ \xi(0)=1,\\ \eta(0)=1.\end{cases}\tag{4.2}$$

Let

$$F_1(\kappa,\vartheta_1,\vartheta_2)=\frac{|\vartheta_1|+|\vartheta_2|}{105(1+15\kappa^3)e^{\sin(\kappa)+1}\,(3+|\vartheta_1|+|\vartheta_2|)},\ (\kappa,\vartheta_1,\vartheta_2)\in[0,1]\times\mathbb{R}^2,$$

$$F_2(\kappa,\vartheta_1,\vartheta_2)=\frac{|\vartheta_1|+|\vartheta_2|}{24(1+13t^5)e^{\cos(\kappa-1)}\,(2+|\vartheta_1|+|\vartheta_2|)},\ (\kappa,\vartheta_1,\vartheta_2)\in[0,1]\times\mathbb{R}^2.$$

Clearly the functions F_1,F_2 are continuous and for each $\vartheta_1,\vartheta_2,\bar{\vartheta}_1,\bar{\vartheta}_2\in\mathbb{R}$,

$$|F_1(\kappa,\vartheta_1,\vartheta_2)-F_1(\kappa,\bar{\vartheta}_1,\bar{\vartheta}_2)|\le\frac{1}{105(1+15\kappa^3)e^{\sin(\kappa)+1}}\left(|\vartheta_1-\bar{\vartheta}_1|+|\vartheta_2-\bar{\vartheta}_2|\right),$$

$$|F_2(\kappa,\vartheta_1,\vartheta_2)-F_2(t,\bar{\vartheta}_1,\bar{\vartheta}_2)|\le\frac{1}{24(1+13\kappa^5)e^{\cos(\kappa-1)}}\left(|\vartheta_1-\bar{\vartheta}_1|+|\vartheta_2-\bar{\vartheta}_2|\right).$$

Hence, the condition (H2) holds with

$$\Phi_1(\kappa)=\Phi_2(\kappa)=\frac{1}{105(1+15t^3)e^{\sin(\kappa)+1}},\quad \Phi_3(\kappa)=\Phi_4(\kappa)=\frac{1}{24(1+13\kappa^5)e^{\cos(\kappa-1)}}.$$

On the other hand, for all $t\in[0,1]$, we get

$$|F_1(\kappa,\vartheta_1,\vartheta_2)|\le\frac{1}{105(1+15\kappa^3)},\quad |F_2(\kappa,\vartheta_1,\vartheta_2)|\le\frac{1}{24(1+13\kappa^5)},$$

which mean that, the condition (H4) holds with,

$$A_1^*(\kappa) = \frac{1}{105(1+15\kappa^3)}, \quad A_2^*(\kappa) = \frac{1}{24(1+13\kappa^5)}.$$

Let

$$I_\theta(\vartheta_1, \vartheta_2) = \frac{13(\vartheta_1+\vartheta_2+1)\pi}{155+\vartheta_1+\vartheta_2}, \quad \bar{I}_\theta(\vartheta_1, \vartheta_2) = \frac{35(\vartheta_1+\vartheta_2+1)\pi}{199+\vartheta_1+\vartheta_2}, \quad \vartheta_1, \vartheta_2 \in [0, \infty).$$

For all $\vartheta_1, \vartheta_2 \in [0, \infty)$, we get

$$|I_\theta(\vartheta_1, \vartheta_2)| \leq \frac{13\pi}{155}(|\vartheta_1| + |\vartheta_2| + 1),$$
$$|\bar{I}_\theta(\vartheta_1, \vartheta_2)| \leq \frac{35\pi}{199}(|\vartheta_1| + |\vartheta_2| + 1).$$

The condition (H5) holds, with

$$\hat{W}_j = \frac{13\pi}{155}, \; \hat{W}_j^* = \frac{35\pi}{199}, \; (j = 1, 2, 3).$$

Since $b = 1$, $m = 1$ and the matrix

$$B^{**} = \begin{pmatrix} \frac{13\pi}{155} & \frac{13\pi}{155} \\ \\ \frac{35\pi}{199} & \frac{35\pi}{199} \end{pmatrix},$$

admit two eigenvalues $\lambda_1 = 0$, $\lambda_2 \simeq 0.815 < 1$. Hence the matrix B^{**} converge to zero. It follows that by theorem 2.3, we deduce that the problem (4.2) has at least one solution.

5. Conclusion

In this paper, we established existence, uniqueness, and Ulam-Hyers stability of solutions to a system of impulsive fractional differential equations involving a fractional Caputo derivative. The analysis was performed within Banach vector spaces, where our main results is obtained by applying the fixed-point theory under a suitable conditions. The examples presented confirm the applicability and effectiveness of the theoretical findings.

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