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LIMIT THEOREMS FOR HILBERT-VALUED RANDOM VARIABLES WITH MARTINGALE PERTURBATIONS

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Abstract. This paper studies limit theorems for perturbed Hilbert-valued random variables under martingale-type errors. The observed sequence is modeled as an original sequence plus a martingale difference perturbation and an asymptotically negligible remainder term. We establish weak and strong laws of large numbers and central limit theorems under suitable moment and negligibility assumptions. In the negligible perturbation regime, the Gaussian limit of the original sequence is preserved. When the martingale perturbation contributes at the central limit scale, we obtain a Gaussian limit with a modified covariance operator. These results provide a perturbation framework for limit theorems in Hilbert spaces.

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1. Introduction

Limit theorems for random variables in infinite dimensional spaces play a fundamental role in probability theory and have many applications in statistics, functional data analysis, and stochastic approximation. In particular, laws of large numbers and central limit theorems in Banach and Hilbert spaces provide basic tools for studying the asymptotic behavior of random functions and other infinite dimensional data; see, for example, [6,9,12]. For dependent Hilbert-valued sequences, a substantial literature is also available; see [1–3, 13–15]. Related results on stochastic equations and stochastic processes in infinite-dimensional or Hilbert-type settings can also be found in [4, 10, 11].

In many situations, however, the sequence of interest is not observed directly. Instead, one observes perturbed data of the form

$$Y_k = X_k + \varepsilon_k, \qquad k \geq 1,$$

where $\{\varepsilon_k\}_{k\geq 1}$ represents an error term. Such perturbations may arise from measurement errors, numerical approximation, preprocessing, or recursive procedures,

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and they naturally occur in sequential and functional data settings; see, for example, [7–9].

A recent paper studied perturbed limit theorems in Hilbert spaces for the model

$$Y_k = X_k + e_k$$

and established several laws of large numbers and central limit theorems under suitable assumptions on the perturbation; see [16]. Motivated by this work, we consider a more structured perturbation of the form

$$Y_k = X_k + \varepsilon_k, \qquad \varepsilon_k = \xi_k + r_k, \qquad k \geq 1,$$

where $\{\xi_k\}_{k\geq 1}$ is a martingale difference sequence with respect to a filtration $\{\mathcal{F}_k\}_{k\geq 0}$, while $\{r_k\}_{k\geq 1}$ is an additional error term which is asymptotically negligible under the normalization under consideration.

This decomposition is natural in situations where one separates a principal random fluctuation from a lower order error term. In particular, martingale difference noise often appears in stochastic approximation and recursive schemes, whereas the remainder term may represent approximation, discretization, or preprocessing errors. This leads to the following general question: to what extent are the classical limit theorems for the original sequence $\{X_k\}_{k\geq 1}$ preserved under the structured perturbation $\varepsilon_k = \xi_k + r_k$?

The aim of this paper is to develop a perturbation framework for laws of large numbers and central limit theorems for Hilbert-valued random variables under this decomposition. We first establish weak and strong laws of large numbers, showing that the asymptotic mean behavior of the original sequence is preserved whenever the martingale perturbation is negligible after averaging and the additional error term vanishes under the same averaging scheme. We then study central limit theorems for the perturbed sequence. If the normalized total perturbation converges to zero in probability, then the Gaussian limit of the original sequence is preserved. If the martingale perturbation contributes at the central limit scale, then, under suitable martingale central limit assumptions, we obtain a Gaussian limit with a modified covariance operator. In this way, the paper distinguishes between two regimes: stability of the original Gaussian limit and Gaussian correction produced by the martingale perturbation.

The paper is organized as follows. In Section 2, we collect some preliminaries on probability in Hilbert spaces and martingale difference sequences. Section 3 is devoted to laws of large numbers. In Section 4, we study central limit theorems when the normalized perturbation is negligible. Section 5 contains a Gaussian central limit theorem for the case where the martingale perturbation contributes at the limit scale.

2. Preliminaries

Throughout the paper, let $(\Omega, \mathcal{F}, P)$ be a probability space, and let H be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote by $\mathcal{B}(H)$ the Borel σ-algebra on H. Unless otherwise stated, all random variables considered in this paper are assumed to be H-valued and $\mathcal{F}/\mathcal{B}(H)$-measurable.

Let $\{v_\ell\}_{\ell\geq 1}$ be a fixed orthonormal basis of H. For each $N \in \mathbb{N}$, define the orthogonal projection $\Pi_N : H \to H$ by

$$\Pi_N x = \sum_{\ell=1}^{N} \langle x, v_\ell\rangle v_\ell, \qquad x \in H,$$

and set

$$Q_N = I - \Pi_N,$$

where I denotes the identity operator on H.

If X is Bochner integrable, its expectation is denoted by

$$EX = \int_\Omega X\,dP.$$

It is well known that for every $h \in H$,

$$\langle EX, h\rangle = E\langle X, h\rangle.$$

If $E\|X\|^2 < \infty$, then the covariance operator of X is the bounded linear operator on H defined by

$$\langle \Gamma_X h, g\rangle = E\big[\langle X - EX, h\rangle\langle X - EX, g\rangle\big], \qquad h, g \in H.$$

In particular, Γ_X is symmetric, nonnegative, and of trace class; see, for instance, [9, 12]. Moreover,

$$E\|X - EX\|^2 = \mathrm{tr}(\Gamma_X).$$

An H-valued random variable X is called Gaussian if, for every $h \in H$, the real-valued random variable $\langle X, h\rangle$ is Gaussian. If X has mean $\mu \in H$ and covariance operator Σ, we write

$$X \sim N_H(\mu, \Sigma).$$

For a sequence $\{X_n\}_{n\geq 1}$ of H-valued random variables and an H-valued random variable X, we use the standard notations

$$X_n \xrightarrow{P} X, \qquad X_n \xrightarrow{a.s.} X, \qquad X_n \xrightarrow{d} X$$

for convergence in probability, almost sure convergence, and convergence in distribution, respectively.

The characteristic functional of an H-valued random variable X is defined by

$$\phi_X(t) = E\exp(i\langle t, X\rangle), \qquad t \in H.$$

If $X \sim N_H(\mu, \Sigma)$, then

$$\phi_X(t) = \exp\big(i\langle \mu, t\rangle\big)\exp\left(-\frac{1}{2}\langle \Sigma t, t\rangle\right), \qquad t \in H.$$

We shall also use the following well-known Slutsky type result in Hilbert spaces.

Lemma 2.1 (Slutsky's theorem). *Let $\{X_n\}_{n\geq 1}$ and $\{Z_n\}_{n\geq 1}$ be sequences of H valued random variables. If*

$$X_n \xrightarrow{d} X \qquad \text{and} \qquad Z_n \xrightarrow{P} a$$

for some H valued random variable X and some $a \in H$, then

$$X_n + Z_n \xrightarrow{d} X + a.$$

We shall further use a standard criterion based on finite dimensional projections.

Lemma 2.2 (Finite dimensional projection criterion)**.** *Let $\{X_n\}_{n\geq 1}$ be a sequence of H valued random variables and let X be an H valued random variable. Assume that*

(i) for every $N \in \mathbb{N}$,

$$\Pi_N X_n \xrightarrow{d} \Pi_N X \qquad \text{as } n \to \infty;$$

(ii) for every $\delta > 0$,

$$\lim_{N\to\infty} \limsup_{n\to\infty} P\big(\|Q_N X_n\| > \delta\big) = 0.$$

Then

$$X_n \xrightarrow{d} X \qquad \text{as } n \to \infty.$$

Lemmas 2.1 and 2.2 are standard; see, for example, [9, 12].

Let $\{\mathcal{F}_k\}_{k\geq 0}$ be a filtration on $(\Omega, \mathcal{F}, P)$. A sequence $\{\xi_k\}_{k\geq 1}$ of H valued random variables is called a martingale difference sequence with respect to $\{\mathcal{F}_k\}_{k\geq 0}$ if, for every $k \geq 1$,

(i) ξ_k is $\mathcal{F}_k$ measurable,
(ii) $E\|\xi_k\| < \infty$,
(iii) $E(\xi_k \mid \mathcal{F}_{k-1}) = 0$ a.s.

If we define

$$M_n = \sum_{k=1}^n \xi_k, \qquad n \geq 1,$$

then $\{M_n, \mathcal{F}_n\}_{n\geq 1}$ is an H valued martingale.

The following orthogonality identity [9] will be used repeatedly.

Lemma 2.3 (Isometry)**.** *Let $\{\xi_k\}_{k\geq 1}$ be an H valued martingale difference sequence with respect to $\{\mathcal{F}_k\}_{k\geq 0}$, and assume that $E\|\xi_k\|^2 < \infty$ for all $k \geq 1$. Then, for every $n \geq 1$,*

$$E\left\|\sum_{k=1}^n \xi_k\right\|^2 = \sum_{k=1}^n E\|\xi_k\|^2.$$

As in the introduction, the perturbation is modeled as

$$\varepsilon_k = \xi_k + r_k, \qquad k \geq 1,$$

where $\{\xi_k\}_{k\geq 1}$ is the principal martingale difference perturbation and $\{r_k\}_{k\geq 1}$ is an additional error term which is asymptotically negligible under the normalization under consideration. Thus the observed sequence takes the form

$$Y_k = X_k + \varepsilon_k = X_k + \xi_k + r_k, \qquad k \geq 1.$$

Depending on the asymptotic regime, we shall impose negligibility assumptions of one of the following types:

$$\frac{1}{n}\sum_{k=1}^n r_k \xrightarrow{P} 0, \qquad \frac{1}{n}\sum_{k=1}^n r_k \xrightarrow{a.s.} 0, \qquad \frac{1}{B_n}\sum_{k=1}^n r_k \xrightarrow{P} 0,$$

where $\{B_n\}_{n\geq 1}$ is a positive normalization sequence.

We shall frequently use the Hilbert space version of Chebyshev's inequality [9, 12].

Lemma 2.4 (Chebyshev's inequality). *Let X be an H valued random variable with $E\|X\|^2 < \infty$. Then for every $\delta > 0$,*

$$P\big(\|X - EX\| > \delta\big) \leq \frac{E\|X - EX\|^2}{\delta^2}.$$

In particular, if $EX = 0$, then

$$P\big(\|X\| > \delta\big) \leq \frac{E\|X\|^2}{\delta^2}.$$

For the central limit part, we shall appeal to a martingale central limit theorem in Hilbert spaces [2, 5, 13]. Rather than proving it from scratch, we use it as a standard tool. The precise version needed in this paper can be formulated as follows.

Theorem 2.5 (Martingale central limit theorem). *Let $\{\xi_k\}_{k\geq 1}$ be an H-valued martingale difference sequence with respect to $\{\mathcal{F}_k\}_{k\geq 0}$, and let*

$$M_n = \sum_{k=1}^{n} \xi_k.$$

Let $\{B_n\}_{n\geq 1}$ be a sequence of positive numbers such that $B_n \to \infty$. Assume that there exists a symmetric nonnegative trace class operator $\Lambda : H \to H$ such that, for every $h, g \in H$,

$$\frac{1}{B_n^2} \sum_{k=1}^{n} E\big[\langle \xi_k, h\rangle \langle \xi_k, g\rangle \mid \mathcal{F}_{k-1}\big] \xrightarrow{P} \langle \Lambda h, g\rangle,$$

and that, for every $\varepsilon > 0$,

$$\frac{1}{B_n^2} \sum_{k=1}^{n} E\big[\|\xi_k\|^2 1_{\{\|\xi_k\| > \varepsilon B_n\}} \mid \mathcal{F}_{k-1}\big] \xrightarrow{P} 0.$$

Then

$$\frac{M_n}{B_n} \xrightarrow{d} N_H(0, \Lambda).$$

The above theorem is used as a standard martingale central limit theorem in Hilbert spaces; see, for example, [2, 5, 13].

We also record a simple criterion that will be used in the proof of the weak law of large numbers for martingale perturbations. If $\{\xi_k\}_{k\geq 1}$ is an H-valued martingale difference sequence satisfying

$$\frac{1}{n^2} \sum_{k=1}^{n} E\|\xi_k\|^2 \to 0 \qquad \text{as } n \to \infty,$$

then, by Lemmas 2.3 and 2.4,

$$\frac{1}{n} \sum_{k=1}^{n} \xi_k \xrightarrow{P} 0.$$

Likewise, if

$$\sum_{n=1}^{\infty} \frac{E\|\xi_n\|^2}{n^2} < \infty,$$

then the averages of the martingale perturbation are asymptotically negligible almost surely, as will be shown in the next section.

The material collected above will be used throughout the sequel to establish laws of large numbers and central limit theorems for the perturbed sequence

$$Y_k = X_k + \xi_k + r_k.$$

Lemma 2.6 (Kronecker's lemma [5])**.** *Let $\{a_n\}_{n\geq 1}$ be a sequence in a Banach space such that the series*

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

converges. Then

$$\frac{1}{n}\sum_{k=1}^{n} a_k \to 0 \qquad \text{as } n \to \infty.$$

3. Laws of large numbers under martingale perturbations

In this section we establish weak and strong laws of large numbers for the perturbed sequence

$$Y_k = X_k + \varepsilon_k = X_k + \xi_k + r_k, \qquad k \geq 1,$$

where $\{X_k\}_{k\geq 1}$ is the original sequence, $\{\xi_k\}_{k\geq 1}$ is an H valued martingale difference sequence, and $\{r_k\}_{k\geq 1}$ is an additional error term which is negligible after averaging.

We first show that the weak law of large numbers for the original sequence is stable under martingale perturbations with sufficiently small second moments and under an averagely negligible error term.

Theorem 3.1 (Weak law of large numbers)**.** *Let $\{X_k\}_{k\geq 1}$, $\{\xi_k\}_{k\geq 1}$, and $\{r_k\}_{k\geq 1}$ be sequences of H-valued random variables, and set $Y_k = X_k + \xi_k + r_k$ for $k \geq 1$. Assume that*

(i) $\frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{P} \mu$ as $n \to \infty$ for some $\mu \in H$;

(ii) $\{\xi_k\}_{k\geq 1}$ is an H-valued martingale difference sequence with respect to a filtration $\{\mathcal{F}_k\}_{k\geq 0}$, and

$$\frac{1}{n^2}\sum_{k=1}^{n} E\|\xi_k\|^2 \to 0 \qquad \text{as } n \to \infty;$$

(iii) $\frac{1}{n}\sum_{k=1}^{n} r_k \xrightarrow{P} 0$ as $n \to \infty$.

Then $\frac{1}{n}\sum_{k=1}^{n} Y_k \xrightarrow{P} \mu$ as $n \to \infty$.

Proof. We decompose

$$\frac{1}{n}\sum_{k=1}^{n}(Y_k-\mu)=\frac{1}{n}\sum_{k=1}^{n}(X_k-\mu)+\frac{1}{n}\sum_{k=1}^{n}\xi_k+\frac{1}{n}\sum_{k=1}^{n}r_k.$$

Hence, for every $\delta>0$,

$$P\left(\left\|\frac{1}{n}\sum_{k=1}^{n}(Y_k-\mu)\right\|>\delta\right)\le P\left(\left\|\frac{1}{n}\sum_{k=1}^{n}(X_k-\mu)\right\|>\frac{\delta}{3}\right)$$
$$+P\left(\left\|\frac{1}{n}\sum_{k=1}^{n}\xi_k\right\|>\frac{\delta}{3}\right)+P\left(\left\|\frac{1}{n}\sum_{k=1}^{n}r_k\right\|>\frac{\delta}{3}\right).$$

By (i) and (iii), the first and third terms converge to 0. It remains to prove that $\frac{1}{n}\sum_{k=1}^{n}\xi_k \xrightarrow{P} 0$.

Let $M_n=\sum_{k=1}^{n}\xi_k$. Since $\{\xi_k\}$ is a martingale difference sequence, Lemma 2.3 gives

$$E\|M_n\|^2=\sum_{k=1}^{n}E\|\xi_k\|^2.$$

Thus, by Lemma 2.4,

$$P\left(\left\|\frac{M_n}{n}\right\|>\frac{\delta}{3}\right)\le\frac{9}{\delta^2 n^2}E\|M_n\|^2=\frac{9}{\delta^2 n^2}\sum_{k=1}^{n}E\|\xi_k\|^2\to 0$$

as $n\to\infty$ by (ii). Therefore $\frac{1}{n}\sum_{k=1}^{n}\xi_k=\frac{M_n}{n}\xrightarrow{P}0$, and the proof is complete. $\square$

Remark 3.2. Theorem 3.1 shows that the weak law of large numbers for the original sequence is preserved under martingale perturbations whenever the averaged martingale term is negligible in L^2 scale, namely,

$$\frac{1}{n^2}\sum_{k=1}^{n}E\|\xi_k\|^2\to 0,$$

and the additional error term is negligible after averaging.

We next turn to the strong law of large numbers. In this case, the negligibility of the martingale perturbation is obtained under a summability condition on the second moments.

Theorem 3.3 (Strong law of large numbers)**.** *Let $\{X_k\}_{k\ge1}$, $\{\xi_k\}_{k\ge1}$, and $\{r_k\}_{k\ge1}$ be sequences of H-valued random variables, and set $Y_k=X_k+\xi_k+r_k$ for $k\ge1$. Assume that*

(i) $\frac{1}{n}\sum_{k=1}^{n}X_k\xrightarrow{a.s.}\mu$ *as* $n\to\infty$ *for some* $\mu\in H$*;*
(ii) $\{\xi_k\}_{k\ge1}$ *is an H-valued martingale difference sequence with respect to a filtration* $\{\mathcal{F}_k\}_{k\ge0}$*, and*

$$\sum_{n=1}^{\infty}\frac{E\|\xi_n\|^2}{n^2}<\infty;$$

(iii) $\frac{1}{n}\sum_{k=1}^{n}r_k\xrightarrow{a.s.}0$ *as* $n\to\infty$.

Then $\frac{1}{n}\sum_{k=1}^{n} Y_k \xrightarrow{a.s.} \mu$ *as* $n \to \infty$.

Proof. Since

$$\frac{1}{n}\sum_{k=1}^{n}(Y_k - \mu) = \frac{1}{n}\sum_{k=1}^{n}(X_k - \mu) + \frac{1}{n}\sum_{k=1}^{n}\xi_k + \frac{1}{n}\sum_{k=1}^{n} r_k,$$

assumptions (i) and (iii) reduce the proof to showing that

$$\frac{1}{n}\sum_{k=1}^{n}\xi_k \xrightarrow{a.s.} 0.$$

Let $M_n = \sum_{k=1}^{n}\xi_k$ and define

$$Z_n = \sum_{k=1}^{n}\frac{\xi_k}{k}, \qquad n \geq 1.$$

Then $\{Z_n, \mathcal{F}_n\}_{n\geq 1}$ is an H-valued martingale. By Lemma 2.3,

$$E\|Z_n\|^2 = \sum_{k=1}^{n}\frac{E\|\xi_k\|^2}{k^2} \leq \sum_{k=1}^{\infty}\frac{E\|\xi_k\|^2}{k^2} < \infty.$$

Hence $\{Z_n\}$ is bounded in $L^2(H)$. The martingale convergence theorem therefore yields an H-valued random variable $Z \in L^2(H)$ such that

$$Z_n \xrightarrow{a.s.} Z \qquad \text{and} \qquad Z_n \xrightarrow{L^2(H)} Z$$

as $n \to \infty$. In particular, the series $\sum_{k=1}^{\infty}\xi_k/k$ converges almost surely in H. Applying Kronecker's lemma in H, we obtain

$$\frac{1}{n}\sum_{k=1}^{n}\xi_k = \frac{M_n}{n} \xrightarrow{a.s.} 0.$$

Therefore,

$$\frac{1}{n}\sum_{k=1}^{n} Y_k = \frac{1}{n}\sum_{k=1}^{n} X_k + \frac{1}{n}\sum_{k=1}^{n}\xi_k + \frac{1}{n}\sum_{k=1}^{n} r_k \xrightarrow{a.s.} \mu,$$

which completes the proof. □

Remark 3.4. The condition

$$\sum_{n=1}^{\infty}\frac{E\|\xi_n\|^2}{n^2} < \infty$$

is standard in strong laws for martingale perturbations. It guarantees that the averaged martingale term converges to zero almost surely, so the perturbation does not affect the almost sure limiting mean.

The results of this section show that laws of large numbers for the original sequence are stable under perturbations of the form

$$\varepsilon_k = \xi_k + r_k,$$

provided that the martingale part is asymptotically negligible after averaging and the additional error term vanishes under the same averaging scheme. In the next section, we turn to central limit theorems and study the effect of perturbations at the normalization scale.

4. Central limit theorems with asymptotically negligible perturbations

In this section we study central limit theorems for the perturbed sequence

$$Y_k = X_k + \varepsilon_k = X_k + \xi_k + r_k, \qquad k \geq 1,$$

in the regime where the total perturbation is asymptotically negligible under the normalization. In this case, the Gaussian limit of the original sequence is preserved.

Let

$$S_n = \sum_{k=1}^{n} X_k, \qquad M_n = \sum_{k=1}^{n} \xi_k, \qquad R_n = \sum_{k=1}^{n} r_k, \qquad T_n = \sum_{k=1}^{n} Y_k.$$

Then

$$T_n = S_n + M_n + R_n.$$

Throughout this section, $\{B_n\}_{n\geq 1}$ denotes a sequence of positive numbers such that $B_n \to \infty$ as $n \to \infty$.

We begin with a general stability result: if the normalized perturbation converges to zero in probability, then the central limit theorem for the original sequence is inherited by the perturbed sequence.

Theorem 4.1. *Assume that*

$$\frac{S_n - ES_n}{B_n} \xrightarrow{d} N_H(0, \Sigma)$$

for some symmetric nonnegative trace class operator Σ *on* H*, and that*

$$\frac{M_n + R_n}{B_n} \xrightarrow{P} 0.$$

Then

$$\frac{T_n - ES_n}{B_n} \xrightarrow{d} N_H(0, \Sigma).$$

Proof. Since

$$\frac{T_n - ES_n}{B_n} = \frac{S_n - ES_n}{B_n} + \frac{M_n + R_n}{B_n},$$

by assumption,

$$\frac{S_n - ES_n}{B_n} \xrightarrow{d} N_H(0, \Sigma) \quad \text{and} \quad \frac{M_n + R_n}{B_n} \xrightarrow{P} 0.$$

Therefore, Lemma 2.1 yields

$$\frac{T_n - ES_n}{B_n} \xrightarrow{d} N_H(0, \Sigma).$$

This proves the result. $\square$

Remark 4.2. Theorem 4.1 expresses a stability property of central limit theorems in Hilbert spaces: if the total perturbation is negligible at the normalization scale, then the perturbed sequence has the same Gaussian limit as the original sequence.

Proposition 4.3. *Assume that*

$$\frac{S_n - ES_n}{B_n} \xrightarrow{d} N_H(0, \Sigma) \qquad \text{as } n \to \infty$$

for some symmetric nonnegative trace class operator Σ on H. Assume also that

(i) $\{\xi_k\}_{k\ge1}$ *is an H valued martingale difference sequence with respect to a filtration* $\{\mathcal{F}_k\}_{k\ge0}$, *and*

$$\frac{1}{B_n^2}\sum_{k=1}^{n} E\|\xi_k\|^2 \to 0 \qquad \text{as } n \to \infty;$$

(ii)

$$\frac{R_n}{B_n} \xrightarrow{P} 0 \qquad \text{as } n \to \infty.$$

Then

$$\frac{T_n - ES_n}{B_n} \xrightarrow{d} N_H(0, \Sigma) \qquad \text{as } n \to \infty.$$

Proof. By Theorem 4.1, it is enough to prove that

$$\frac{M_n + R_n}{B_n} \xrightarrow{P} 0.$$

Let $\delta > 0$. Then

$$P\left(\left\|\frac{M_n + R_n}{B_n}\right\| > \delta\right) \le P\left(\left\|\frac{M_n}{B_n}\right\| > \frac{\delta}{2}\right) + P\left(\left\|\frac{R_n}{B_n}\right\| > \frac{\delta}{2}\right).$$

By assumption *(ii)*, the second term tends to zero as $n \to \infty$. For the first term, Lemmas 2.3 and 2.4 yield

$$P\left(\left\|\frac{M_n}{B_n}\right\| > \frac{\delta}{2}\right) \le \frac{4}{\delta^2 B_n^2} E\|M_n\|^2 = \frac{4}{\delta^2 B_n^2}\sum_{k=1}^{n} E\|\xi_k\|^2.$$

Hence, by assumption *(i)*,

$$P\left(\left\|\frac{M_n}{B_n}\right\| > \frac{\delta}{2}\right) \to 0 \qquad \text{as } n \to \infty.$$

Therefore

$$\frac{M_n + R_n}{B_n} \xrightarrow{P} 0.$$

The conclusion follows from Theorem 4.1. □

Corollary 4.4. *Assume that*

$$\frac{S_n - ES_n}{B_n} \xrightarrow{d} N_H(0, \Sigma) \qquad \text{as } n \to \infty,$$

and that $\{\xi_k\}_{k\geq 1}$ *is an* H *valued martingale difference sequence satisfying*

$$\frac{1}{B_n^2}\sum_{k=1}^{n} E\|\xi_k\|^2 \to 0 \qquad \text{as } n \to \infty.$$

If $r_k = 0$ *for all* $k \geq 1$, *then*

$$\frac{T_n - ES_n}{B_n} \xrightarrow{d} N_H(0,\Sigma) \qquad \text{as } n \to \infty.$$

Proof. Since $r_k = 0$ for all $k \geq 1$, we have $R_n = 0$ for every $n \geq 1$. Hence

$$\frac{R_n}{B_n} \xrightarrow{P} 0.$$

The conclusion now follows from Proposition 4.3. □

In some situations it is convenient to allow the additional error term to be not necessarily centered. The following formulation is useful in such a case.

Theorem 4.5. *Let* $\{r_k\}_{k\geq 1}$ *be a sequence of Bochner integrable* H*-valued random variables and define*

$$m_k = Er_k, \qquad k \geq 1.$$

Assume that

$$\frac{S_n - ES_n}{B_n} \xrightarrow{d} N_H(0,\Sigma) \qquad \text{as } n \to \infty,$$

that $\{\xi_k\}_{k\geq 1}$ *is an* H*-valued martingale difference sequence satisfying*

$$\frac{1}{B_n^2}\sum_{k=1}^{n} E\|\xi_k\|^2 \to 0,$$

and that

$$\frac{1}{B_n}\sum_{k=1}^{n}(r_k - m_k) \xrightarrow{P} 0, \qquad \frac{1}{B_n}\sum_{k=1}^{n} m_k \to 0.$$

Then

$$\frac{T_n - ES_n}{B_n} \xrightarrow{d} N_H(0,\Sigma) \qquad \text{as } n \to \infty.$$

Proof. Set

$$M_n = \sum_{k=1}^{n}\xi_k, \qquad R_n = \sum_{k=1}^{n} r_k, \qquad T_n = \sum_{k=1}^{n} Y_k.$$

Since $Y_k = X_k + \xi_k + r_k$, we have

$$T_n - ES_n = (S_n - ES_n) + M_n + R_n,$$

and hence

$$\frac{T_n - ES_n}{B_n} = \frac{S_n - ES_n}{B_n} + \frac{M_n + R_n}{B_n}.$$

By Theorem 4.1, it is enough to prove that

$$\frac{M_n + R_n}{B_n} \xrightarrow{P} 0.$$

By definition,

$$R_n = \sum_{k=1}^{n} r_k = \sum_{k=1}^{n} (r_k - m_k) + \sum_{k=1}^{n} m_k,$$

where $m_k = E r_k$. Hence

$$\frac{R_n}{B_n} = \frac{1}{B_n} \sum_{k=1}^{n} (r_k - m_k) + \frac{1}{B_n} \sum_{k=1}^{n} m_k.$$

By assumption,

$$\frac{1}{B_n} \sum_{k=1}^{n} (r_k - m_k) \xrightarrow{P} 0, \qquad \frac{1}{B_n} \sum_{k=1}^{n} m_k \to 0.$$

Since deterministic convergence implies convergence in probability,

$$\frac{1}{B_n} \sum_{k=1}^{n} m_k \xrightarrow{P} 0.$$

Therefore,

$$\frac{R_n}{B_n} \xrightarrow{P} 0.$$

Next, let $\delta > 0$. Since $\{\xi_k\}_{k \geq 1}$ is an H-valued martingale difference sequence, Lemma 2.3 gives

$$E\|M_n\|^2 = \sum_{k=1}^{n} E\|\xi_k\|^2.$$

Hence, by Lemma 2.4,

$$P\left(\left\|\frac{M_n}{B_n}\right\| > \delta\right) \leq \frac{1}{\delta^2 B_n^2} E\|M_n\|^2 = \frac{1}{\delta^2 B_n^2} \sum_{k=1}^{n} E\|\xi_k\|^2.$$

By assumption,

$$\frac{1}{B_n^2} \sum_{k=1}^{n} E\|\xi_k\|^2 \to 0,$$

and therefore

$$P\left(\left\|\frac{M_n}{B_n}\right\| > \delta\right) \to 0 \qquad \text{as } n \to \infty.$$

Thus

$$\frac{M_n}{B_n} \xrightarrow{P} 0.$$

Finally, for every $\delta > 0$,

$$P\left(\left\|\frac{M_n + R_n}{B_n}\right\| > \delta\right) \leq P\left(\left\|\frac{M_n}{B_n}\right\| > \frac{\delta}{2}\right) + P\left(\left\|\frac{R_n}{B_n}\right\| > \frac{\delta}{2}\right).$$

Since both terms on the right-hand side tend to 0, we conclude that

$$\frac{M_n + R_n}{B_n} \xrightarrow{P} 0.$$

Applying Theorem 4.1, we obtain

$$\frac{T_n - ES_n}{B_n} \xrightarrow{d} N_H(0, \Sigma).$$

This completes the proof. □

Remark 4.6. Theorem 4.5 separates the asymptotically negligible random fluctuation of the additional error term from its deterministic bias contribution. This form is sometimes convenient when the remainder term is produced by an approximation or preprocessing procedure.

The results of this section show that when the perturbation does not contribute at the central limit scale, the Gaussian limit of the original sequence remains unchanged. This is the central limit analogue of the stability property established in Section 3 for laws of large numbers. In the next section we turn to the genuinely nontrivial case where the martingale perturbation contributes at the normalization scale and produces an additional Gaussian component in the limit.

5. Gaussian central limit theorems for martingale perturbations

In this section we study the case where the martingale perturbation contributes at the central limit scale. In contrast with Section 4, the normalized perturbation is no longer negligible. Instead, it produces an additional Gaussian term in the limit.

Recall that

$$Y_k = X_k + \varepsilon_k = X_k + \xi_k + r_k, \qquad k \geq 1,$$

and define

$$S_n = \sum_{k=1}^{n} X_k, \qquad M_n = \sum_{k=1}^{n} \xi_k, \qquad R_n = \sum_{k=1}^{n} r_k, \qquad T_n = \sum_{k=1}^{n} Y_k.$$

Then

$$T_n = S_n + M_n + R_n.$$

Throughout this section, let $\{B_n\}_{n\geq 1}$ be a sequence of positive numbers such that $B_n \to \infty$ as $n \to \infty$.

We first record a standard fact on sums of independent random variables.

Lemma 5.1. *Let $\{U_n\}_{n\geq 1}$ and $\{V_n\}_{n\geq 1}$ be sequences of H valued random variables such that, for every $n \geq 1$, U_n and V_n are independent. Assume that*

$$U_n \xrightarrow{d} U \qquad \text{and} \qquad V_n \xrightarrow{d} V$$

for some H valued random variables U and V. Then

$$U_n + V_n \xrightarrow{d} U + V,$$

where U and V are independent.

Theorem 5.2. *Let $\{X_k\}_{k\geq 1}$, $\{\xi_k\}_{k\geq 1}$, and $\{r_k\}_{k\geq 1}$ be sequences of H valued random variables, and define*

$$Y_k = X_k + \xi_k + r_k, \qquad k \geq 1.$$

Assume that the following conditions hold.

(i) The signal part satisfies

$$\frac{S_n - ES_n}{B_n} \xrightarrow{d} N_H(0, \Sigma) \qquad \text{as } n \to \infty,$$

for some symmetric nonnegative trace class operator Σ on H.

(ii) The perturbation part $\{\xi_k\}_{k\geq 1}$ is an H valued martingale difference sequence with respect to a filtration $\{\mathcal{F}_k\}_{k\geq 0}$, and there exists a symmetric nonnegative trace class operator Λ on H such that, for every $h, g \in H$,

$$\frac{1}{B_n^2} \sum_{k=1}^{n} E\big[\langle \xi_k, h\rangle \langle \xi_k, g\rangle \mid \mathcal{F}_{k-1}\big] \xrightarrow{P} \langle \Lambda h, g\rangle.$$

(iii) For every $\varepsilon > 0$,

$$\frac{1}{B_n^2} \sum_{k=1}^{n} E\big[\|\xi_k\|^2 1_{\{\|\xi_k\| > \varepsilon B_n\}} \mid \mathcal{F}_{k-1}\big] \xrightarrow{P} 0.$$

(iv) The additional error term is negligible after normalization, namely,

$$\frac{R_n}{B_n} \xrightarrow{P} 0 \qquad \text{as } n \to \infty.$$

(v) For every $n \geq 1$, the sigma algebra generated by $\{X_1, \ldots, X_n\}$ is independent of the σ algebra generated by $\{\xi_1, \ldots, \xi_n, r_1, \ldots, r_n\}$.

Then

$$\frac{T_n - ES_n}{B_n} \xrightarrow{d} N_H(0, \Sigma + \Lambda) \qquad \text{as } n \to \infty.$$

Proof. Set

$$U_n := \frac{S_n - ES_n}{B_n}, \qquad V_n := \frac{M_n}{B_n}, \qquad W_n := \frac{R_n}{B_n}.$$

Then

$$\frac{T_n - ES_n}{B_n} = U_n + V_n + W_n.$$

By assumption *(i)*,

$$U_n \xrightarrow{d} N_H(0, \Sigma).$$

By assumptions *(ii)* and *(iii)*, Theorem 2.5 yields

$$V_n \xrightarrow{d} N_H(0, \Lambda).$$

Next, we verify the independence needed to combine the two limits. For each $n \geq 1$, the random variable U_n is measurable with respect to $\sigma(X_1, \ldots, X_n)$, while V_n is measurable with respect to $\sigma(\xi_1, \ldots, \xi_n)$. By assumption *(v)*,

$$\sigma(X_1, \ldots, X_n) \quad \text{is independent of} \quad \sigma(\xi_1, \ldots, \xi_n, r_1, \ldots, r_n).$$

Hence, in particular, $\sigma(X_1, \ldots, X_n)$ is independent of $\sigma(\xi_1, \ldots, \xi_n)$, and therefore U_n and V_n are independent for every $n \geq 1$.

It follows from Lemma 5.1 that

$$U_n + V_n \xrightarrow{d} U + V,$$

where
$$U \sim N_H(0,\Sigma), \qquad V \sim N_H(0,\Lambda),$$
and U and V are independent.

We now identify the law of $U+V$. For any $h \in H$, using the independence of U and V, we obtain
$$E\exp\big(i\langle h, U+V\rangle\big) = E\exp(i\langle h,U\rangle)\, E\exp(i\langle h,V\rangle).$$
Since $U \sim N_H(0,\Sigma)$ and $V \sim N_H(0,\Lambda)$, their characteristic functionals are
$$E\exp(i\langle h,U\rangle) = \exp\left(-\frac{1}{2}\langle \Sigma h, h\rangle\right),$$
and
$$E\exp(i\langle h,V\rangle) = \exp\left(-\frac{1}{2}\langle \Lambda h, h\rangle\right).$$
Therefore,
$$E\exp\big(i\langle h, U+V\rangle\big) = \exp\left(-\frac{1}{2}\langle (\Sigma+\Lambda) h, h\rangle\right).$$
Hence
$$U+V \sim N_H(0,\Sigma+\Lambda),$$
and consequently
$$U_n + V_n \xrightarrow{d} N_H(0,\Sigma+\Lambda).$$

Finally, assumption *(iv)* gives
$$W_n = \frac{R_n}{B_n} \xrightarrow{P} 0.$$
Applying Lemma 2.1, we conclude that
$$U_n + V_n + W_n \xrightarrow{d} N_H(0,\Sigma+\Lambda).$$
Equivalently,
$$\frac{T_n - ES_n}{B_n} \xrightarrow{d} N_H(0,\Sigma+\Lambda).$$
This completes the proof. □

Remark 5.3. Theorem 5.2 shows that, at the central limit scale, the martingale perturbation contributes an additional Gaussian term with covariance operator Λ, while the extra error term r_k remains negligible. Thus the limit covariance of the perturbed sequence is obtained by adding the covariance of the signal part and that of the martingale perturbation.

The negligibility of the additional error term can be verified in several simple ways. We record one direct consequence.

Corollary 5.4. *Under the assumptions of Theorem 5.2, condition* (iv) *is satisfied if*
$$\frac{1}{B_n^2} E\left\|\sum_{k=1}^{n} r_k\right\|^2 \to 0 \qquad \text{as } n \to \infty.$$

Consequently,

$$\frac{T_n - ES_n}{B_n} \xrightarrow{d} N_H(0, \Sigma + \Lambda).$$

Proof. By Lemma 2.4,

$$P\left(\left\|\frac{R_n}{B_n}\right\| > \delta\right) \le \frac{1}{\delta^2 B_n^2} E\|R_n\|^2, \qquad \delta > 0.$$

Hence

$$\frac{R_n}{B_n} \xrightarrow{P} 0.$$

The conclusion follows from Theorem 5.2. □

The case without the additional error term is immediate.

Corollary 5.5. *Assume that*

$$Y_k = X_k + \xi_k, \qquad k \ge 1,$$

and that assumptions (i), (ii), (iii), *and* (v) *of Theorem 5.2 hold. Then*

$$\frac{T_n - ES_n}{B_n} \xrightarrow{d} N_H(0, \Sigma + \Lambda) \qquad \text{as } n \to \infty.$$

Proof. This is just Theorem 5.2 with $r_k = 0$ for all $k \ge 1$. □

The results of this section complement those of Section 4. When the normalized perturbation is negligible, the Gaussian limit of the original sequence is unchanged. When the martingale perturbation contributes at the central limit scale, it produces an additional Gaussian correction through the covariance operator Λ.

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