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ENTROPY SOLUTIONS FOR NONLINEAR PARABOLIC EQUATIONS IN MUSIELAK ORLICZ SPACES WITHOUT Δ_2-CONDITION

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ABSTRACT. In this paper, we are interested in results concerning entropy solutions for nonlinear parabolic equations in Musielak Orlicz spaces without Δ_2-condition.

1. Introduction and Basic assumptions

In this paper, we study the nonlinear parabolic problem whose model is

$$(\mathcal{P})\begin{cases} \frac{\partial u}{\partial t}-\operatorname{div}(a(x,t,u,\nabla u)+\Phi(x,t,u))+g(x,t,u,\nabla u)=f-\operatorname{div}(F) \quad \text{in } Q,\\ u=0 \quad \text{in } \partial\Omega\times(0,T),\\ u(x,0)=u_0 \quad \text{in} \quad \Omega,\end{cases}$$

where Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and $T>0$, we denote $Q=\Omega\times[0,T]$, and let φ and γ be two Musielak-Orlicz functions such that φ is locally integrable and $\gamma\prec\prec\varphi$ Let $A: D(A)\subset W_0^{1,x}L_\varphi(Q)\longrightarrow W^{-1,x}L_\psi(Q)$ be a mapping given by

$$A(u)=-\operatorname{div}(a(x,t,u,\nabla u)),$$

with $a: a(x,t,s,\xi):\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}^N$ is a Carathodory function satisfying, for a.e $(x,t)\in Q$ and for all $s\in\mathbb{R}$ and all $\xi,\xi'\in\mathbb{R}^N,\xi\neq\xi'$:

$$|a(x,t,s,\xi)|\leq\beta\left(c(x,t)+\psi_x^{-1}\gamma(x,\nu|s|)+\psi_x^{-1}\varphi(x,\nu|s|)\right), \tag{1.1}$$

$$\left(a(x,t,s,\xi)-a\left(x,t,s,\xi'\right)\right)\left(\xi-\xi'\right)>0, \tag{1.2}$$

$$a(x,t,s,\xi)\cdot\xi\geq\alpha\varphi(x,|\xi|), \tag{1.3}$$

where $c(x,t)$ a positive function, $c(x,t)\in E_\psi(Q)$ and ν,β,α are positive constants.

Let $g:\Omega\times[0,t]\times\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}^N$ be a Caratheodory function satisfying for a.e. $(x,t)\in\Omega\times[0,t]$ and $\forall s\in\mathbb{R},\xi\in\mathbb{R}^N$:

$$|g(x,t,s,\xi)|\leq b(|s|)\left(c_2(x,t)+\varphi(x,|\xi|)\right), \tag{1.4}$$

$$g(x,t,s,\xi)s\geq 0, \tag{1.5}$$

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where $c_2(x,t) \in L^1(Q)$ and $b : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a continuous and nondecreasing function. Furthermore the function Φ is a Carathodory function which satisfies the following growth condition for a.e. $(x,t) \in Q$ and for all $\forall s \in \mathbb{R}$

$$|\Phi(x,t,s)| \leq P(x,t)\overline{\gamma_x}^{-1}\gamma_x(|s|), \tag{1.6}$$

where

$$P(x,t) \in L^\infty(Q).$$

$$f \in L^1(Q) \quad and \quad F \in (E_\psi(Q))^N, \tag{1.7}$$

$$u_0 \in L^1(\Omega). \tag{1.8}$$

A several work have treated the same problem ,we can't recite all examples but i will just choose some of them, for instance :

In the case where A is a Leary-Lions operator defined on $L^p(0,T;W^{1,p}(\Omega))$ Porretta [38] proved the existence of solutions for the problem $(\mathcal{P})$, where g is a nonlinearity with "natural" growth condition and which satisfies the classical sign condition $g(x,t,s,\xi)s \geq 0$.

In the case where $\Phi \equiv \mathrm{div}(F) \equiv 0$, the existence of entropy solutions for parabolic problems of the form $(\mathcal{P})$ in the setting of Orlicz spaces has been proved in A. Elmahi and D. Meskine [29] in the case where f belongs to $L^1(Q)$ and g be a carathodory function satisfying

$$|g(x,t,s,\xi)| \leq b(|s|)(c(x,t) + M(|\xi|));$$
$$g(x,t,s,\xi)s \geq 0.$$

in [33] the authors have been studied the existence of solutions for the prpblem $(\mathcal{P})$ in variable exponent sobolev spaces where $\Phi \equiv \mathrm{div}(F) \equiv g \equiv 0$ and the second membe f is in $W^{-1,x}L^{p'(x)}(Q)$.

On the other hand, the existence of a weak solution for the problem $(\mathcal{P})$ has been studied in the framwork of Musielak spaces where $\Phi \equiv \mathrm{div}(F) \equiv 0$, (see [1]) and recently, in the same fremwork, Talha. A, Benkirane.A and Elemine Vall M.S.B. were proved the existence of entropy solutions for the problem $(\mathcal{P})$ where $\Phi \equiv 0$, (see [36]).

A large number of research deals with existence solutions of elliptic and parabolic problems under different assumptions in order to get a classical results see [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 25, 26, 27, 28, 31, 37, 42, 43] for more details.

The goal of this paper is to investigate the existence of entropy solutions for problem $(\mathcal{P})$ in the Musielak Orlicz spaces without Δ_2 condition, when the term nonlinear $g(x,t,u,\nabla u)$ satisfying the natural growth and the classical signe conditions. Note that the second member is in L^1 and no hypothesis of coercivity is assumed on Φ.

The paper is organized as follows: In Section 2 , we present some preliminaries and background. Section 3 concern some technical lemmas which will be needed later. In the final section 4, we give our main result and state the prove of an existence of solution .

2. Background

Here we give some definitions and properties that concern Musielak-Orlicz spaces (see [39]).

2.1. Musielak-Orlicz functions.

Let Ω be an open subset of $\mathbb{R}^n$.

A Musielak-Orlicz function φ is a real-valued function defined in $\Omega \times \mathbb{R}_+$ such that
a) $\varphi(x,t)$ is an N-function i.e. convex, nondecreasing, continuous, $\varphi(x,0) = 0, \varphi(x,t) > 0$ for all $t > 0$ and

$$\lim_{t\to 0} \sup_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0, \qquad \lim_{t\to\infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0.$$

b) $\varphi(\cdot,t)$ is a Lebesgue measurable function.

Now, let $\varphi_x(t) = \varphi(x,t)$ and let φ_x^{-1} be the non-negative reciprocal function with respect to t, i.e the function that satisfies

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi\left(x, \varphi_x^{-1}(t)\right) = t.$$

The Musielak-orlicz function φ is said to satisfy the Δ_2 -condition if for some $k > 0$, and a non negative function h, integrable in Ω, we have

$$\varphi(x,2t) \le k\varphi(x,t) + h(x) \text{ for all } x \in \Omega \text{ and } t \ge 0. \tag{2.1}$$

When 2.1 holds only for $t \ge t_0 > 0$, then φ is said to satisfy the Δ_2 -condition near infinity. Let φ and γ be two Musielak-orlicz functions, we say that φ dominate γ and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for almost all $x \in \Omega$

$$\gamma(x,t) \le \varphi(x,ct) \ \text{ for all } t \ge t_0, \quad (\text{ resp. for all } t \ge 0 \text{ i.e. } t_0 = 0).$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity) and we write $\gamma \prec\prec \varphi$ if for every positive constant c we have

$$\lim_{t\to 0}\left(\sup_{x\in\Omega} \frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0, \quad \left(\text{ resp. } \lim_{t\to\infty}\left(\sup_{x\in\Omega} \frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0\right).$$

Remark 2.1. (see [27]) If $\gamma \prec\prec \varphi$ near infinity, then $\forall \varepsilon > 0$ there exists a non-negative integrable function h, such that

$$\gamma(x,t) \le \varphi(x,\varepsilon t) + h(x). \text{ for all } t \ge 0 \text{ and for a. e. } x \in \Omega. \tag{2.2}$$

2.2. Musielak-Orlicz-Sobolev spaces.

For a Musielak-Orlicz function φ and a measurable function $u : \Omega \longrightarrow \mathbb{R}$, we define the functional

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx.$$

The set $K_\varphi(\Omega) = \{u : \Omega \longrightarrow \mathbb{R} \text{ measurable } / \rho_{\varphi,\Omega}(u) < \infty\}$ is called the Musielak-Orlicz class (or generalized Orlicz class). The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$

is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega)=\left\{u:\Omega\longrightarrow\mathbb{R}\text{ measurable }/\rho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right)<\infty,\text{ for some }\lambda>0\right\}.$$

For a Musielak-Orlicz function φ we put: $\psi(x,s)=\sup_{t>0}\{st-\varphi(x,t)\}$, ψ is the Musielak-Orlicz function complementary to φ (or conjugate of φ) in the sens of Young with respect to the variable s In the space $L_\varphi(\Omega)$ we define the following two norms:

$$\|u\|_{\varphi,\Omega}=\inf\left\{\lambda>0/\int_\Omega\varphi\left(x,\frac{|u(x)|}{\lambda}\right)dx\leq 1\right\},$$

which is called the Luxemburg norm and the so-called Orlicz norm by:

$$|||u|||_{\varphi,\Omega}=\sup_{\|v\|_\psi\leq 1}\int_\Omega|u(x)v(x)|dx,$$

where ψ is the Musielak Orlicz function complementary to φ. These two norms are equivalent (see [39])

We will also use the space $E_\varphi(\Omega)$ defined by

$$E_\varphi(\Omega)=\left\{u:\Omega\longrightarrow\mathbb{R}\text{ measurable }/\rho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right)<\infty,\text{ for all }\lambda>0\right\}.$$

A Musielak function φ is called locally integrable on Ω if $\rho_\varphi\left(t\chi_D\right)<\infty$ for all $t>0$ and all measurable $D\subset\Omega$ with meas $(D)<\infty$ Let φ a Musielak function which is locally integrable. Then $E_\varphi(\Omega)$ is separable (see [39], Theorem 7.10) .

We say that sequence of functions $u_n\in L_\varphi(\Omega)$ is modular convergent to $u\in L_\varphi(\Omega)$ if there exists a constant $\lambda>0$ such that

$$\lim_{n\to\infty}\rho_{\varphi,\Omega}\left(\frac{u_n-u}{\lambda}\right)=0.$$

For any fixed nonnegative integer m we define

$$W^mL_\varphi(\Omega)=\left\{u\in L_\varphi(\Omega):\forall|\alpha|\leq m, D^\alpha u\in L_\varphi(\Omega)\right\},$$

and

$$W^mE_\varphi(\Omega)=\left\{u\in E_\varphi(\Omega):\forall|\alpha|\leq m, D^\alpha u\in E_\varphi(\Omega)\right\},$$

where $\alpha=(\alpha_1,\ldots,\alpha_n)$ with nonnegative integers $\alpha_i, |\alpha|=|\alpha_1|+\ldots+|\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives.
The space $W^mL_\varphi(\Omega)$ is called the Musielak Orlicz Sobolev space.

Let

$$\bar{\rho}_{\varphi,\Omega}(u)=\sum_{|\alpha|\leq m}\rho_{\varphi,\Omega}\left(D^\alpha u\right)\text{ and }\|u\|^m_{\varphi,\Omega}=\inf\left\{\lambda>0:\bar{\rho}_{\varphi,\Omega}\left(\frac{u}{\lambda}\right)\leq 1\right\}$$

for $u\in W^mL_\varphi(\Omega)$.
These functionals are a convex modular and a norm on $W^mL_\varphi(\Omega)$, respectively, and the pair $\left(W^mL_\varphi(\Omega),\|\|^m_{\varphi,\Omega}\right)$ is a Banach space if φ satisfies the following condition (see[39]):

$$\text{there exist a constant } c_0 > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x,1) \geq c_0. \tag{2.3}$$

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$ closed.

The space $W_0^m L_\varphi(\Omega)$ is defined as the $\sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$. and the space $W_0^m E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W_0^m L_\varphi(\Omega)$ be the $\sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$, the following spaces of distributions will also be used:

$$W^{-m} L_\psi(\Omega) = \left\{ f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega) \right\},$$

and

$$W^{-m} E_\psi(\Omega) = \left\{ f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega) \right\}.$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \bar{\rho}_{\varphi,\Omega}\left(\frac{u_n - u}{k}\right) = 0.$$

For φ and her complementary function ψ, the following inequality is called the Young inequality (see[39]):

$$ts \leq \varphi(x,t) + \psi(x,s), \quad \forall t, s \geq 0, x \in \Omega, \tag{2.4}$$

this inequality implies that

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) + 1. \tag{2.5}$$

In $L_\varphi(\Omega)$ we have the relation between the norm and the modular

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} > 1, \tag{2.6}$$

$$\|u\|_{\varphi,\Omega} \geq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} \leq 1. \tag{2.7}$$

For two complementary Musielak Orlicz functions φ and ψ, let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$, then we have the Holder inequality (see[39]):

$$\left| \int_\Omega u(x)v(x)dx \right| \leq \|u\|_{\varphi,\Omega} \||v|\|_{\psi,\Omega}. \tag{2.8}$$

2.3. **Inhomogeneous Musielak-Orlicz-Sobolev spaces.** Let Ω a bounded open subset of $\mathbb{R}^N$ and let $Q = \Omega\times]0,T[$ with some given $T>0$. Let φ and ψ be two complementary Musielak-Orlicz functions. For each $\alpha\in\mathbb{N}^N$ denote by D_x^α the distributional derivative on Q of order α with respect to the variable $x\in\mathbb{R}^N$. The inhomogeneous Musielak-Orlicz-Sobolev spaces of order 1 are defined as follows.

$$W^{1,x}L_\varphi(Q)=\{u\in L_\varphi(Q):\forall|\alpha|\leq 1 D_x^\alpha u\in L_\varphi(Q)\}$$

et

$$W^{1,x}E_\varphi(Q)=\{u\in E_\varphi(Q):\forall|\alpha|\leq 1 D_x^\alpha u\in E_\varphi(Q)\}.$$

This second space is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\|=\sum_{|\alpha|\leq 1}\|D_x^\alpha u\|_{\varphi,Q}$$

These spaces constitute a complementary system since Ω satisfies the segment property.These spaces are considered as subspaces of the product space $\Pi L_\varphi(Q)$ which has $(N+1)$ copies.

We shall also consider the weak topologies $\sigma(\Pi L_\varphi,\Pi E_\psi)$ and $\sigma(\Pi L_\varphi,\Pi L_\psi)$ If $u\in W^{1,x}L_\varphi(Q)$ then the function $t\to u(t)=u(\cdot,t)$ is defined on $[0,T]$ with values in $W^1L_\varphi(\Omega)$. If $u\in W^{1,x}E_\varphi(Q)$, then $u\in W^1E_\varphi(\Omega)$ and it is strongly measurable. Furthermore, the imbedding $W^{1,x}E_\varphi(Q)\subset L^1(0,T,W^1E_\varphi(\Omega))$ holds. The space $W^{1,x}L_\varphi(Q)$ is not in general separable, for $u\in W^{1,x}L_\varphi(Q)$ we cannot conclude that the function $u(t)$ is measurable on $[0,T]$.

However, the scalar function $t\to\|u(t)\|_{\varphi,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_\varphi(Q)$ is defined as the norm closure of $\mathcal{D}(Q)$ in $W^{1,x}E_\varphi(Q)$. We can easily show as in [34] that when Ω has the segment property, then each element u of the closure of $\mathcal{D}(Q)$ with respect of the weak $*$ topology $\sigma(\Pi L_\varphi,\Pi E_\psi)$ is a limit in $W^{1,x}L_\varphi(Q)$ of some subsequence $(v_j)\in\mathcal{D}(Q)$ for the modular convergence, i.e. there exists $\lambda>0$ such that for all $|\alpha|\leq 1$

$$\int_Q\varphi\left(x,\left(\frac{D_x^\alpha v_j-D_x^\alpha u}{\lambda}\right)\right)dxdt\to 0 \text{ as } j\to\infty,$$

this implies that (v_j) converges to u in $W^{1,x}L_\varphi(Q)$ for the weak topology $\sigma(\Pi L_\varphi,\Pi L_\psi)$ Consequently

$$\overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi,\Pi E_\psi)}=\overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi,\Pi L_\psi)}$$

The space of functions satisfying such a property will be denoted by $W_0^{1,x}L_\psi(Q)$ Furthermore, $W_0^{1,x}E_\varphi(Q)=W_0^{1,x}L_\varphi(Q)\cap\Pi E_\varphi(Q)$. Thus, both sides of the last inequality are equivalent norms on $W_0^{1,x}L_\varphi(Q)$. We then have the following complementary system:

$$\begin{pmatrix} W_0^{1,x}L_\varphi(Q) & F\\ W_0^{1,x}E_\varphi(Q) & F_0\end{pmatrix}$$

where F states for the dual space of $W_0^{1,x}E_\varphi(Q)$. and can be defined, except for an isomorphism, as the quotient of ΠL_ψ by the polar set $W_0^{1,x}E_\varphi(Q)^\perp$. It will be denoted by $F = W^{-1,x}L_\psi(Q)$, where

$$W^{-1,x}L_\psi(Q) = \left\{ f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in L_\psi(Q) \right\}$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha|\le 1} \|f_\alpha\|_{\psi,Q}$$

where the infimum is taken over all possible decompositions

$$f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha, \quad f_\alpha \in L_\psi(Q)$$

The space F_0 is then given by

$$F_0 = \left\{ f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in E_\psi(Q) \right\}$$

and is denoted by $F_0 = W^{-1,x}E_\psi(Q)$.

Theorem 2.2. [1] *Let φ be a Musielak-Orlicz function satisfies the assumption 3.1. If $u \in W^{1,x}L_\varphi(Q) \cap L^2(Q)$ $\big($ respectively $u \in W_0^{1,x}L_\varphi(Q) \cap L^2(Q)\big)$ and $\frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(Q) + L^2(Q)$, then there exists a sequence $(v_j) \in D(\bar{Q})$ (respectively $D(\bar{I}, D(\Omega))$) such that $v_j \longrightarrow u$ in $W^{1,x}L_\varphi(Q) \cap L^2(Q)$ and $\frac{\partial v_j}{\partial t} \longrightarrow \frac{\partial u}{\partial t}$ in $W^{-1,x}L_\psi(Q) + L^2(Q)$ for the modular convergence.*

Lemma 2.3. [1] *Let $a < b \in \mathbb{R}$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$.*

$$\left\{ u \in W_0^{1,x}L_\varphi(\Omega\times]a,b[) : \frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(\Omega\times]a,b[) + L^1(\Omega\times]a,b[) \right\}$$

is a subset of $\mathcal{C}(]a,b\,[, L^1(\Omega))$.

In order to show the existence theorem we need the follwing lemmas .

3. Some technical Lemmas

Lemma 3.1. [27]

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:

i)There exist a constant $c > 0$ such that $\inf_{x\in\Omega}\varphi(x,1) \ge c$.

ii) There exist a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x - y| \le \frac{1}{2}$ we have

$$\frac{\varphi(x,t)}{\varphi(y,t)} \le c.t^{\left(\frac{A}{\log\left(\frac{1}{|x-y|}\right)}\right)}, \quad \forall t \ge 1. \tag{3.1}$$

iii)

$$\text{If } D \subset \Omega \text{ is a bounded measurable set, then } \int_D \varphi(x,1)dx < \infty. \tag{3.2}$$

iv) There exist a constant $C > 0$ such that $\psi(x, 1) \leq C$ a.e in Ω.

Under this assumptions, $\mathcal{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1 L_\varphi(\Omega)$ for the modular convergence and $\mathcal{D}(\bar{\Omega})$ is dense in $W^1 L_\varphi(\Omega)$ the modular convergence.

Consequently, the action of a distribution S in $W^{-1}L_\psi(\Omega)$ on an element u of $W_0^1 L_\varphi(\Omega)$ is well defined. It will be denoted by $< S, u >$.

Lemma 3.2. *([38]) Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak-Orlicz function and let $u \in W_0^1 L_\varphi(\Omega)$. Then $F(u) \in W_0^1 L_\varphi(\Omega)$ Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e\ in \{x \in \Omega : u(x) \in D\} \\ 0 & a.e\ in\ \{x \in \Omega : u(x) \notin D\}. \end{cases}$$

Lemma 3.3. [4] *(Poincare inequality). Let φ a Musielak Orlicz function which satisfies the assumptions of lemma 3.1, suppose that $\varphi(x, t)$ decreases with respect of one of coordinate of x Then, there exists a constant $c > 0$ depends only of Ω such that*

$$\int_\Omega \varphi(x, |u(x)|)dx \leq \int_\Omega \varphi(x, c|\nabla u(x)|)dx, \quad \forall u \in W_0^1 L_\varphi(\Omega).$$

Lemma 3.4. [35] *Suppose that Ω satisfies the segment property and let $u \in W_0^1 L_\varphi(\Omega)$. Then, there exists a sequence $(u_n) \subset \mathcal{D}(\Omega)$ such that*

$$u_n \to u \text{ for modular convergence in } W_0^1 L_\varphi(\Omega)$$

Furthermore, if $u \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ then $\|u_n\|_\infty \leq (N + 1)\|u\|_\infty$.

Lemma 3.5. *Let $(f_n), f \in L^1(\Omega)$ such that*
i) $f_n \geq 0$ a.e in Ω
ii) $f_n \longrightarrow f$ a.e in Ω
iii) $\displaystyle\int_\Omega f_n(x)dx \longrightarrow \int_\Omega f(x)dx$
then $f_n \longrightarrow f$ strongly in $L^1(\Omega)$.

Lemma 3.6. *(Jensen inequality).* [40] *Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ a convex function and $g : \Omega \longrightarrow \mathbb{R}$ is function measurable, then*

$$\varphi\left(\int_\Omega g d\mu\right) \leq \int_\Omega \varphi \circ g d\mu.$$

Lemma 3.7. *(The Nemytskii Operator)*[2] *. Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak-Orlicz functions. Let $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:*

$$|f(x, s)| \leq c(x) + k_1 \psi_x^{-1} \varphi(x, k_2|s|). \tag{3.3}$$

where k_1 and k_2 are real positives constants and $c(.) \in E_\psi(\Omega)$. Then the Nemytskii Operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is continuous from

$$\mathcal{P}\left(E_\varphi(\Omega), \frac{1}{k_2}\right) = \left\{u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega)) < \frac{1}{k_2}\right\}.$$

into $L_\psi(\Omega)$.
Furthermore if $c(\cdot) \in E_\gamma(\Omega)$ *and* $\gamma \prec\prec \psi$ *then* N_f *is strongly continuous from* $\mathcal{P}\Big(E_\varphi(\Omega), \frac{1}{k_2}\Big)$ *to* $E_\gamma(\Omega)$.

4. **Existence result**

For $k > 0$ we define the truncation function $T_k : \mathbb{R} \longrightarrow \mathbb{R}$ by:

$$T_k(s) = \left\{ \begin{array}{ll} s & \text{if } |s| \leq k, \\ k\frac{s}{|s|} & \text{if } |s| > k, \end{array} \right.$$

and let us define the following function

$$S_k(r) = \int_0^r T_k(\sigma)d\sigma = \left\{ \begin{array}{ll} \frac{r^2}{2} & \text{if } |r| \leq k, \\ k|r| - \frac{r^2}{2} & \text{if } |r| > k, \end{array} \right.$$

and

$$T_0^{1,\varphi}(Q) = \left\{ u : \Omega \longrightarrow \mathbb{R} \text{ measurable such that } T_k(u) \in W_0^{1,x}L_\varphi(Q) \forall k > 0 \right\}.$$

Our aim now is to prove the following existence theorem.

Theorem 4.1. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^N, \varphi$ *and* ψ *be two complementary Musielak-Orlicz functions satisfying the assumptions of Lemma 3.1 and* $\varphi(x,t)$ *decreases with respect to one of coordinate of* x*, we assume also that* (1.1)*–*(1.7) *and* (1.8) *hold true. Then the problem* $(\mathcal{P})$ *has at least one entropy solution of the following sense*

$$\left\{ \begin{array}{l} u \in T_0^{1,\varphi}(Q) \cap W_0^{1,x}L_\varphi(Q), S_k(u) \in L^1(Q), g(.,u,\nabla u) \in L^1(Q), \\ \displaystyle\int_\Omega S_k(u(T) - v(T))dx + \left\langle \frac{\partial v}{\partial t}, T_k(u-v) \right\rangle + \int_Q a(x,t,u,\nabla u)\cdot \nabla T_k(u-v)dxdt \\ \displaystyle + \int_Q \Phi(x,t,u)\cdot \nabla T_k(u-v)dxdt + \int_Q g(x,t,u,\nabla u)T_k(u-v)dxdt \\ \displaystyle \leq \int_Q fT_k(u-v)dxdt + \int_Q F\cdot \nabla T_k(u-v)dxdt + \int_\Omega S_k\left(u_0 - v(0)\right)dx, \\ \forall v \in W_0^{1,x}L_\varphi(Q) \cap L^\infty(Q) \textit{ such that } \frac{\partial v}{\partial t} \in W^{-1,x}L_\psi(Q) + L^1(Q). \end{array} \right.$$

4.1. **Proof.**

4.1.1. The approximate problems :
Let us define the approximation

$$g_n(x,t,s,\xi) = T_n(g(x,t,s,\xi)).$$

Consider the nonlinear approximate problems

$$(\mathcal{P}_n) \left\{ \begin{array}{c} u_n \in W_0^{1,x}L_\varphi(Q), \quad u_n(.,0) = u_{0n} \text{ in } \partial Q = \partial\Omega \times [0,T], \\ \frac{\partial u_n}{\partial t} - \text{div}\left(a\left(x,t,u_n,\nabla u_n\right) + \Phi_n\left(x,t,u_n\right)\right) + g_n\left(x,t,u_n,\nabla u_n\right) = f_n - \text{div}(F). \end{array} \right.$$

We have, the sequence $(f_n) \subset \mathcal{D}(Q)$ is such that

$$f_n \longrightarrow f \text{ strongly in } L^1(Q),$$

and $\|f_n\|_{L^1(Q)} \leq \|f\|_{L^1(Q)}$ and $(u_{0n}) \subset \mathcal{D}(\Omega)$ is such that

$$u_{0n} \longrightarrow u_0 \text{ strongly in } L^1(\Omega) \text{ and } \|u_{0n}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}.$$

Now by theorem 5.1 of [1], the problem $(\mathcal{P}_n)$ has at least one solution u_n .

4.1.2. A priori estimates:

In this step we define $c_i, i = 1, 2, \ldots$ a constants not depends on k and n. For $k > 0$, by taking the function $T_k(u_n)$ as test in $(\mathcal{P}_n)$, we get

$$\begin{aligned}
&\int_Q \frac{\partial u_n}{\partial t} T_k(u_n)\,dxdt + \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n)) \cdot \nabla T_k(u_n)\,dxdt \\
&\qquad + \int_Q g_n(x,t,u_n,\nabla u_n)\,T_k(u_n)\,dxdt \\
&= \int_Q f_n T_k(u_n)\,dxdt + \int_Q F \cdot \nabla T_k(u_n)\,dxdt + \int_Q \Phi_n(x,t,u_n) \cdot \nabla T_k(u_n)\,dxdt \\
&\leq \|f\|_{L^1(Q)} k + \int_Q F \cdot \nabla T_k(u_n)\,dxdt + \int_{Q_\tau} \Phi_n(x,t,u_n) \cdot \nabla T_k(u_n)\,dxdt. \qquad (4.1)
\end{aligned}$$

Furthermore, let $0 < p < \min(\alpha, 1)$, with α is the constant of (1.3), and thinks to the Young's inequality, we obtain

$$\begin{aligned}
\int_Q F \cdot \nabla T_k(u_n)\,dxdt &= \int_Q \frac{1}{p} F \cdot p \nabla T_k(u_n)\,dxdt \\
&\leq \int_Q \psi\left(x, \frac{1}{p}|F|\right) dxdt + p \int_Q \varphi(x, |\nabla T_k(u_n)|)\,dxdt. \qquad (4.2)
\end{aligned}$$

Now, note that if $\gamma \prec\prec \varphi$, we get, for all $\varepsilon > 0$ there exists a constant $d_\varepsilon > 0$ depending on $\varepsilon > 0$ such that for almost all $x \in \Omega$

$$\gamma(x,t) \leq \varphi(x, \varepsilon t) + d_\varepsilon, \quad \text{for all } t \geq 0$$

By assuming that $\varepsilon = \frac{\alpha - p}{(\alpha + C_p)(\lambda + 1)}$, (where α is the constant of (1.3). and thinks to (1.6) we obtain

$$\int_Q \Phi_n(x,t,u_n) \nabla T_k(u_n)\,dxdt \leq \int_Q P(x,t) \overline{\gamma_x}^{-1} \gamma_x(|T_k(u_n)|) \nabla T_k(u_n)\,dxdt. \qquad (4.3)$$

Remember that $\gamma \prec\prec \varphi \Longleftrightarrow \bar{\varphi} = \psi \prec\prec \bar{\gamma}$, by Young inequality and taking into account that $P \in L^\infty(Q)$, we get

$$\begin{aligned}
\int_Q \Phi_n(x,t,u_n) \nabla T_k(u_n)\,dxdt \leq & C_p \int_Q \varphi\left(x, \frac{\varepsilon\lambda |T_k(u_n)|}{\lambda}\right) + 2d_\varepsilon \operatorname{meas}(Q) \\
& + \varepsilon C_p \int_Q \varphi(x, |\nabla T_k(u_n)|)\,dxdt.
\end{aligned}$$

Thinks to (3.3) and using the convexity of φ where $\lambda\varepsilon \leq 1$, we obtain

$$\int_Q \Phi_n(x,t,u_n) \nabla T_k(u_n)\,dxdt \leq (\varepsilon C_p + \varepsilon\lambda C_p) \int_Q \varphi(x, |\nabla T_k(u_n)|)\,dxdt + 2d_\varepsilon \operatorname{meas}(Q). \qquad (4.4)$$

According to (4.1),(4.2) and (4.4) ,one has

$$\int_Q \frac{\partial u_n}{\partial t} T_k(u_n)\,dxdt + \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n)) \cdot \nabla T_k(u_n)\,dxdt$$
$$+ \int_Q \Phi_n(x,t,u_n) \cdot \nabla T_k(u_n)\,dxdt + \int_Q g_n(x,t,u_n,\nabla u_n) T_k(u_n)\,dxdt$$
$$\leq c_1 k + c_2 + (p + \varepsilon C_p + \varepsilon\lambda C_p) \int_Q \varphi(x,|\nabla T_k(u_n)|)\,dxdt + c_3. \tag{4.5}$$

Thinks to (1.5) and (1.3), we can obtain

$$\int_Q \frac{\partial u_n}{\partial t} T_k(u_n)\,dxdt + \frac{\alpha - (p + \varepsilon C_p + \varepsilon\lambda C_p)}{\alpha} \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n)) \cdot \nabla T_k(u_n)\,dxdt \leq c_1 k + c_2. \tag{4.6}$$

The first term of the left hand side of (4.6) , translates as

$$\int_Q \frac{\partial u_n}{\partial t} T_k(u_n)\,dxdt = \int_\Omega S_k(u_n(T))\,dx - \int_\Omega S_k(u_{0n})\,dx.$$

Consequently

$$\int_\Omega S_k(u_n(T))\,dx + \frac{\alpha - (p + \varepsilon C_p + \varepsilon\lambda C_p)}{\alpha} \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n)) \cdot \nabla T_k(u_n)\,dxdt$$
$$\leq c_1 k + c_2 + c_3 + \int_\Omega S_k(u_{0n})\,dx.$$

By taking that $S_k(\sigma) > 0$, $|S_k(u_{0n})| \leq k\,|u_{0n}|$, Thus 4.6 becomes

$$\frac{\alpha - (p + \varepsilon C_p + \varepsilon\lambda C_p)}{\alpha} \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n)) \cdot \nabla T_k(u_n)\,dxdt \leq c_4 k + c_2 + c_3.$$

Hence

$$\frac{\alpha - (p + \varepsilon C_p + \varepsilon\lambda C_p)}{\alpha} \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n)) \cdot \nabla T_k(u_n)\,dxdt \leq c_4 k + c_5. \tag{4.7}$$

Then thinks to (1.3), we obtain

$$\int_Q \varphi(x,|\nabla T_k(u_n)|)\,dxdt \leq c_6 k + c_7$$

by applying the Lemma (3.3) we get

$$\int_Q \varphi\left(x, \frac{|T_k(u_n)|}{c}\right) dx \leq \int_Q \varphi(x,|\nabla T_k(u_n)|)\,dx \leq c_6 k + c_7, \tag{4.8}$$

where c is the constant of Lemma (3.3) .

Then

$$(T_k(u_n))_n \text{ and } (\nabla T_k(u_n))_n \text{ bounded in } L_\varphi(\Omega),$$

hence

$$(T_k(u_n))_n \text{ is bounded in } W_0^1 L_\varphi(\Omega),$$

there exist some $v_k \in W_0^1 L_\varphi(\Omega)$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup v_k \text{ weakly in } W_0^1 L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \\ T_k(u_n) \longrightarrow v_k \text{ strongly in } E_\varphi(\Omega). \end{cases} \tag{4.9}$$

4.1.3. In this part we prove that

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,x} L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi),$$

and

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in } E_\varphi(Q).$$

Let $k > 0$ large enough, by (4.8) we get

$$\begin{aligned} \text{meas}\{|u_n| > k\} &\leq \frac{1}{\inf_{x\in\Omega} \varphi\left(x, \frac{k}{\lambda}\right)} \int_{\{|u_n|>k\}} \varphi\left(x, \frac{k}{\lambda}\right) dxdt \\ &\leq \frac{1}{\inf_{x\in\Omega} \varphi\left(x, \frac{k}{\lambda}\right)} \int_Q \varphi\left(x, \frac{1}{\lambda}|T_k(u_n)|\right) dxdt \\ &\leq \frac{c_6 k + c_7}{\inf_{x\in\Omega} \varphi\left(x, \frac{k}{\lambda}\right)} \quad \forall n, \quad \forall k \geq 0 \end{aligned}$$

where c_5 is a constant not dependent on k, consequently

$$\text{meas}\{|u_n| > k\} \leq \tfrac{c_6 k + c_7}{\inf_{x\in\Omega} \varphi\left(x, \frac{k}{\lambda}\right)} \longrightarrow 0 \text{ as } k \longrightarrow \infty$$

For every $\lambda > 0$ we obtain

$$\begin{aligned} \text{meas}\{|u_n - u_m| > \lambda\} \leq\ & \text{meas}\{|u_n| > k\} \\ &+ \text{meas}\{|u_m| > k\} \\ &+ \text{meas}\{|T_k(u_n) - T_k(u_m)| > \lambda\}. \end{aligned} \tag{4.10}$$

Hence, thinks to (4.8) we suppose that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Q. Let $\varepsilon > 0$, thus by (4.10) there exists some $k = k(\varepsilon) > 0$ such that

$$\text{meas}\ \{|u_n - u_m| > \lambda\} < \varepsilon, \qquad \text{for all } n, m \geq h_0(k(\varepsilon), \lambda).$$

which gives that $(u_n)_n$ is a Cauchy sequence in measure in Q, then converge almost every where to some measurable functions u. hence

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,x} L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \\ \qquad T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in } E_\varphi(Q). \end{cases} \tag{4.11}$$

4.1.4. In this subsubsection we show that

$$a(x, t, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k \text{ weakly} * \text{ in } (L_\psi(\Omega))^N \text{ for } \sigma(\Pi L_\psi, \Pi E\varphi).$$

Let $w \in E_\varphi(Q)^N$ such that $\|w\|_{\varphi,\Omega} \leq 1$,
taking account to (1.2) we get .

$$\left(a(x, t, T_k(u_n), \nabla T_k(u_n)) - a\left(x, t, T_k(u_n), \frac{w}{\nu}\right)\right)\left(\nabla T_k(u_n) - \frac{w}{\nu}\right) > 0$$

consequently

$$\int_\Omega a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\frac{w}{\nu}dxdt \le \int_\Omega a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\nabla T_k\left(u_n\right)dxdt$$
$$-\int_\Omega a\left(x,t,T_k\left(u_n\right),\frac{w}{\nu}\right)\left(\nabla T_k\left(u_n\right)-\frac{w}{\nu}\right)dxdt \tag{4.12}$$

by (4.7), we obtain

$$\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\nabla T_k\left(u_n\right)dxdt \le c_4k+c_2+c_3$$

Now, for λ large enough ($\lambda>\beta$), by according to (1.1) we get

$$\begin{aligned}
&\int_\Omega \psi_x\left(\left|\frac{a\left(x,t,T_k\left(u_n\right),\frac{w}{\nu}\right)}{3\lambda}\right|\right)dxdt\\
&\le \int_\Omega \psi_x\left(\frac{\beta\left(d(x)+\psi_x^{-1}\left(\gamma\left(x,\nu\left|T_k\left(u_n\right)\right|\right)\right)+\psi_x^{-1}(\varphi(x,|w|))\right)}{3\lambda}\right)dxdt\\
&\le \frac{\beta}{\lambda}\int_\Omega \psi_x\left(\frac{h_1(x,t)+\psi_x^{-1}\left(\gamma\left(x,\nu\left|T_k\left(u_n\right)\right|\right)\right)+\psi_x^{-1}(\varphi(x,|w|))}{3}\right)dxdt\\
&\le \frac{\beta}{3\lambda}\left(\int_\Omega \psi_x\left(h_1(x,t)\right)dxdt+\int_\Omega \gamma\left(x,\nu\left|T_k\left(u_n\right)\right|\right)dxdt+\int_\Omega \varphi(x,|w|)dxdt\right)\\
&\le \frac{\beta}{3\lambda}\left(\int_\Omega \psi_x\left(h_1(x,t)\right)dxdt+\int_\Omega \gamma(x,\nu k)dxdt+\int_\Omega \varphi(x,|w|)dxdt\right).
\end{aligned}$$

Finally, since γ grows essentially less rapidly than φ near infinity and thinks to Remark2.1 there exists $r(k)>0$ such that $\gamma(x,\nu k)\le r(k)\varphi(x,1)$ then we obtain

$$\begin{aligned}
&\int_\Omega \psi_x\left(\frac{a\left(x,t,T_k\left(u_n\right),\frac{w}{\nu}\right)}{3\lambda}\right)dxdt\\
&\le \frac{\beta}{3\lambda}\left(\int_\Omega \psi_x\left(h_1(x,t)\right)dxdt+r(k)\int_\Omega \varphi(x,1)dxdt+\int_\Omega \varphi(x,|w|)dxdt\right).
\end{aligned}$$

Conseqently

$$a\left(x,t,T_k\left(u_n\right),\frac{w}{\nu}\right)\ \text{ is bounded in }\ \left(L_\psi(\Omega)\right)^N.$$

this implies that second term of the right hand side of (4.12) is bounded, hence we obtain

$$\int_\Omega a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)wdxdt \le c_8(k),\qquad \text{for all } w\in\left(L^\varphi(\Omega)\right)^N \text{ where } \|w\|_{\varphi,\Omega}\le 1.$$

Then by applying the theorem of Banach Steinhous one has,

$$\text{the sequence }\ \left(a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\right)_n\ \text{ bounded in }\ \left(L_\psi(\Omega)\right)^N.$$

Which implies that, for all $k>0$ there exists a function $h_k \in (L_\psi(\Omega))^N$ such that

$$a\left(x, t, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \rightharpoonup h_k \text{ weakly}* \text{ in } \left(L_\psi(\Omega)\right)^N \text{ for } \sigma\left(\Pi L_\psi, \Pi E\varphi\right). \quad (4.13)$$

4.1.5. In this step we show that

$$\nabla T_k\left(u_n\right) \longrightarrow \nabla T_k(u) \text{ as } n \longrightarrow +\infty \text{ for the modular convergence,}$$

We will write only $\varepsilon(n, j, \mu, s)$ to mean all quantities (possibly different) such that

$$\lim_{n\to+\infty}\lim_{j\to+\infty}\lim_{\mu\to+\infty}\lim_{s\to+\infty}\varepsilon(n, j, \mu, s)=0,$$

since $T_k(u) \in W_0^{1,x}L_\varphi(\Omega)$ then there exists a sequence $\left(\alpha_k^j\right) \subset D(\Omega)$ such that $\left(\alpha_k^j\right) \longrightarrow T_k(u)$ for the modular convergence in $W_0^{1,x}L_\varphi(\Omega)$.

For the remaining of this article, χ_s and $\chi_{j,s}$ will denoted respectively the characteristic functions of the sets $Q_s = \{(x,t) \in \Omega : |\nabla T_k(u(x,t))| \leq s\}$ and $\Omega_{j,s} = \left\{(x,t) \in \Omega \ \left|\nabla T_k\left(\alpha_k^j(x,t)\right)\right| \leq s\right\}$.

By taking $T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)$ as test function in $(\mathcal{P}_n)$ we get

$$\begin{aligned}
&\int_Q \frac{\partial u_n}{\partial t} T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\
&+\int_Q a\left(x, t, u_n, \nabla u_n\right) \cdot \nabla T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\
&+\int_Q g_n\left(x, t, u_n, \nabla u_n\right) T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\
&\leq \|f\|_1 \eta + \int_{\left\{\left|T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right| < \eta\right\}} F \cdot \nabla T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\
&-\int_Q \Phi_n\left(x, t, u_n\right) \cdot \nabla T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt.
\end{aligned}$$

Let $0<p<\min(1,\alpha)$, by Young's inequality, we have

$$\begin{aligned}
&\int_Q \frac{\partial u_n}{\partial t} T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\
&+\int_Q a\left(x, t, u_n, \nabla u_n\right) \cdot \nabla T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\
&+\int_Q g_n\left(x, t, u_n, \nabla u_n\right) T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\
&\leq \|f\|_1 \eta + \int_{\left\{\left|T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right| < \eta\right\}} \psi\left(x, \frac{|F|}{p}\right) dxdt + p\int_Q \varphi\left(x, \left|\nabla T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right|\right) dxdt. \\
&-\int_Q \Phi_n\left(x, t, u_n\right) \cdot \nabla T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt.
\end{aligned} \quad (4.14)$$

On the other hand while $\gamma \prec\prec \varphi$, we have, for all $\varepsilon > 0$ there exists a constant $d_\varepsilon > 0$ depending on $\varepsilon > 0$ such that for almost all $x \in \Omega$

$$\gamma(x,t) \leq \varphi(x,\varepsilon t) + d_\varepsilon, \qquad \text{for all } t \geq 0$$

Without loss of generality, we can assume that $\varepsilon = \frac{\alpha - p}{(\alpha + C_p)(\lambda + 1)}$, (with α is the constant of 1.3).

Using 1.6 we get

$$\int_Q \Phi_n(x,t,u_n) \nabla T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt$$

$$\leq \int_Q P(x,t)\overline{\gamma_x}^{-1}\gamma_x\left(\left|T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right|\right) \nabla T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt. \quad (4.15)$$

Recall that $\gamma \prec\prec \varphi \Longleftrightarrow \bar{\varphi} = \psi \prec\prec \bar{\gamma}$ then, with Young inequality and bearing in mind that $P \in L^\infty(Q)$, we obtain

$$\int_Q \Phi_n(x,t,u_n) \nabla T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \leq C_p \int_Q \varphi\left(x, \frac{\varepsilon\lambda\left|T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right|}{\lambda}\right) + 2d_\varepsilon \operatorname{meas}(Q)$$
$$+ \varepsilon C_p \int_Q \varphi\left(x, \left|\nabla T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right|\right) dxdt$$

by Lemma (3.3) and the convexity of φ with $\lambda\varepsilon \leq 1$, we get

$$\int_Q \Phi_n(x,t,u_n) \nabla T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt$$

$$\leq (\varepsilon C_p + \varepsilon\lambda C_p) \int_Q \varphi\left(x, \left|\nabla T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right|\right) dxdt + 2d_\varepsilon \operatorname{meas}(Q) \quad (4.16)$$

Combining and (4.14) ,(4.16) we obtain

$$\begin{aligned}
&\int_Q \frac{\partial u_n}{\partial t} T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\
&+ \int_Q a(x,t,u_n,\nabla u_n) \cdot \nabla T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\
&+ \int_Q g_n(x,t,u_n,\nabla u_n) T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\
&\leq \|f\|_1 \eta + \int_{\left\{\left|T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right| < \eta\right\}} \psi\left(x, \frac{|F|}{p}\right) dxdt \\
&+ p\int_Q \varphi\left(x, \left|\nabla T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right|\right) dxdt. \\
&- (\varepsilon C_p + \varepsilon\lambda C_p) \int_Q \varphi\left(x, \left|\nabla T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right|\right) dxdt - 2d_\varepsilon \operatorname{meas}(Q)
\end{aligned}$$

i.e.

$$\begin{aligned}
&\int_Q \frac{\partial u_n}{\partial t} T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\
&+\int_Q a\left(x,t,u_n,\nabla u_n\right)\cdot \nabla T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\
&+\int_Q g_n\left(x,t,u_n,\nabla u_n\right) T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\
&\le \|f\|_1 \eta + \int_{\left\{\left|T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right|<\eta\right\}} \psi\left(x, \frac{|F|}{p}\right) dxdt \\
&+c_9 + \left(p-\varepsilon C_p - \varepsilon\lambda C_p\right)\int_Q \varphi\left(x, \left|\nabla T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right|\right) dxdt.
\end{aligned}$$

Using now (1.3) on the last term of the last inequality, we get

$$\begin{aligned}
&\int_Q \frac{\partial u_n}{\partial t} T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\
&+\int_Q a\left(x,t,u_n,\nabla u_n\right)\cdot \nabla T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\
&+\int_Q g_n\left(x,t,u_n,\nabla u_n\right) T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\
&\le \|f\|_1 \eta + \int_{\left\{\left|T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right|<\eta\right\}} \psi\left(x, \frac{|F|}{p}\right) dxdt \\
&+c_9 + \frac{\left(p-\varepsilon C_p - \varepsilon\lambda C_p\right)}{\alpha}\int_Q a\left(x,t,T_{k+\eta}\left(u_n\right), \nabla T_{k+\eta}\left(u_n\right)\right)\nabla u_n dxdt.
\end{aligned}$$

Which implies that,

$$\begin{aligned}
&\int_Q \frac{\partial u_n}{\partial t} T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \qquad (4.17)\\
&+\frac{\alpha-\left(p-\varepsilon C_p - \varepsilon\lambda C_p\right)}{\alpha}\int_Q a\left(x,t,T_{k+\eta}\left(u_n\right), \nabla T_{k+\eta}\left(u_n\right)\right)\nabla u_n dxdt \\
&+\int_Q g_n\left(x,t,u_n,\nabla u_n\right) T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\
&\le c_1\eta + c_9 + \int_{\left\{\left|T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right|<\eta\right\}} \psi\left(x, \frac{|F|}{p}\right) dxdt. \qquad (4.18)
\end{aligned}$$

The first term of the left hand side of the last equality reads as

$$\begin{aligned}
\int_Q \frac{\partial u_n}{\partial t} T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt &= \int_Q \left(\frac{\partial u_n}{\partial t} - \frac{\partial T_k\left(\alpha_k^j\right)_\mu}{\partial t}\right) T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\
&+\int_Q \frac{\partial T_k\left(\alpha_k^j\right)_\mu}{\partial t} T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt.
\end{aligned}$$

The second term of the last equality can be easily to see that is positive and the third term can be written as

$$\int_Q \frac{\partial T_k\left(\alpha_k^j\right)_\mu}{\partial t} T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt$$

$$= \mu \int_Q \left(T_k\left(\alpha_k^j\right) - T_k\left(\alpha_k^j\right)_\mu\right) T_{eta}\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt,$$

thus by letting $n, j \longrightarrow +\infty$, and since $\left(\alpha_k^j\right) \longrightarrow T_k(u)$ a.e. in Q and by using Lebesgue Theorem,

$$\int_Q \left(T_k\left(\alpha_k^j\right) - T_k\left(\alpha_k^j\right)_\mu\right) T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt = \int_Q \left(T_k(u) - T_k(u)_\mu\right) \cdots$$
$$\cdots T_\eta\left(u - T_k(u)_\mu\right) dxdt + \varepsilon(n, j)$$

Consequently

$$\int_Q \frac{\partial T_k\left(\alpha_k^j\right)_\mu}{\partial t} T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \geq \varepsilon(n, j).$$

Then,(4.17) can be write as

$$\begin{aligned} &\frac{\alpha - (p - \varepsilon C_p - \varepsilon\lambda C_p)}{\alpha} \int_Q a\left(x, t, u_n, \nabla u_n\right) T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\ &+ \int_Q g_n\left(x, t, u_n, \nabla u_n\right) T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \leq c_1\eta + c_9 \\ &+ \int_{\left\{\left|T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right| < \eta\right\}} \psi\left(x, \frac{|F|}{p}\right) dxdt + \varepsilon(n, j). \end{aligned} \tag{4.19}$$

On the other hand,

$$\begin{aligned} &\int_Q a\left(x, t, u_n, \nabla u_n\right) T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right) dxdt \\ &= \int_{\left\{\left|u_n - T_k\left(\alpha_k^j\right)_\mu\right| < \eta\right\}} a\left(x, t, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\left(\nabla T_k\left(u_n\right) - \nabla T_k\left(\alpha_k^j\right)_\mu \chi_{j,s}\right) dxdt \\ &+ \int_{\left\{\left|u_n - T_k\left(\alpha_k^j\right)_\mu\right| < \eta\right\}} a\left(x, t, u_n, \nabla u_n\right) \cdot \nabla u_n dxdt \\ &- \int_{\{|u_n| > k\} \cap \left\{\left|u_n - T_k\left(\alpha_k^j\right)_\mu\right| < \eta\right\}} a\left(x, t, u_n, \nabla u_n\right) \cdot \nabla T_k\left(\alpha_j^k\right)_\mu \chi_{\left\{\left|\nabla T_k\left(\alpha_j^k\right)\right| > s\right\}} dxdt \\ &- \int_{\{|u_n| > k\} \cap \left\{\left|u_n - T_k\left(\alpha_k^j\right)_\mu\right| < \eta\right\}} a\left(x, t, u_n, \nabla u_n\right) \cdot \nabla T_k\left(\alpha_j^k\right)_\mu \chi_{\left\{\left|\nabla T_k\left(\alpha_j^k\right)\right| > s\right\}} dxdt. \end{aligned}$$

Thus, by using the fact that

$$\int_{\{|u_n| > k\} \cap \left\{\left|u_n - T_k\left(\alpha_k^j\right)_\mu\right| < \eta\right\}} a\left(x, t, u_n, \nabla u_n\right) \cdot \nabla u_n dxdt \geq 0$$

We obtain

$$\frac{\alpha - (p - \varepsilon C_p - \varepsilon\lambda C_p)}{\alpha} \int_{\left\{\left|u_n - T_k\left(\alpha_k^j\right)_\mu\right| < \eta\right\}} a\left(x, t, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)$$
$$\times \left(\nabla T_k\left(u_n\right) - \nabla T_k\left(\alpha_k^j\right)_\mu \chi_{j,s}\right) dxdt$$

$$+\int_Q g_n(x,t,u_n,\nabla u_n)\,T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)dxdt$$

$$\leq c_1\eta + c_9 + \int_{\left\{\left|T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right|<\eta\right\}} \psi\left(x, \frac{|F|}{p}\right)dxdt$$

$$+\frac{\alpha-(p-\varepsilon C_p-\varepsilon\lambda C_p)}{\alpha}\int_{\{|u_n|>k\}\cap\left\{\left|u_n-T_k\left(\alpha_k^j\right)_\mu\right|<\eta\right\}} a(x,t,u_n,\nabla u_n)\cdot\nabla T_k\left(\alpha_j^k\right)_\mu \chi_{\left\{\left|\nabla T_k\left(\alpha_j^k\right)\right|>s\right\}}dxdt$$

$$+\varepsilon(n,j). \tag{4.20}$$

By using (1.5) and the fact that $T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)$ has the same sign of u_n on the set $\{|u_n|>k\}$, we get

$$\frac{\alpha-(p-\varepsilon C_p-\varepsilon\lambda C_p)}{\alpha}\int_{\left\{\left|u_n-T_k\left(\alpha_k^j\right)_\mu\right|<\eta\right\}} a(x,t,T_k(u_n),\nabla T_k(u_n))$$

$$\times\left(\nabla T_k(u_n) - \nabla T_k\left(\alpha_k^j\right)_\mu \chi_{j,s}\right)dxdt$$

$$+\int_{\{|u_n|\leq k\}} g_n(x,t,u_n,\nabla u_n)\,T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)dxdt$$

$$\leq c_1\eta + c_9 + \int_{\left\{\left|T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right|<\eta\right\}} \psi\left(x, \frac{|F|}{p}\right)dxdt$$

$$+\frac{\alpha-(p-\varepsilon C_p-\varepsilon\lambda C_p)}{\alpha}\int_{\{|u_n|>k\}\cap\left\{\left|u_n-T_k\left(\alpha_k^j\right)_\mu\right|<\eta\right\}} a(x,t,u_n,\nabla u_n)\cdot\nabla T_k\left(\alpha_j^k\right)_\mu \chi_{\left\{\left|\nabla T_k\left(\alpha_j^k\right)\right|>s\right\}}dxdt$$

$$+\varepsilon(n,j). \tag{4.21}$$

by using (1.4), we get

$$\frac{\alpha-(p-\varepsilon C_p-\varepsilon\lambda C_p)}{\alpha}\int_{\left\{\left|u_n-T_k\left(\alpha_k^j\right)_\mu\right|<\eta\right\}} a(x,t,T_k(u_n),\nabla T_k(u_n))\left(\nabla T_k(u_n) - \nabla T_k\left(\alpha_k^j\right)_\mu \chi_{j,s}\right)dxdt$$

$$\leq c_1\eta + c_9 + \int_{\left\{\left|T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right|<\eta\right\}} \psi\left(x, \frac{|F|}{p}\right)dxdt$$

$$+\frac{\alpha-(p-\varepsilon C_p-\varepsilon\lambda C_p)}{\alpha}\int_{\{|u_n|>k\}\cap\left\{\left|u_n-T_k\left(\alpha_k^j\right)_\mu\right|<\eta\right\}} a(x,t,u_n,\nabla \eta_n)\cdot\nabla T_k\left(\alpha_j^k\right)_\mu \chi_{\left\{\left|\nabla T_k\left(\alpha_j^k\right)\right|>s\right\}}dxdt$$

$$+\varepsilon(n,j) + \int_{\{|u_n|\leq k\}} b_k\left(h_2(x,t) + \varphi(x,|\nabla T_k(u_n)|)\right)\left|T_\eta\left(u_n - T_k\left(\alpha_k^j\right)_\mu\right)\right|dxdt \tag{4.22}$$

where $b_k = \sup\{b(s) : |s| \leq k\}$.
Using now (4.8) there exists a constant $c_{10} > 0$ depends on k such that

$$\begin{aligned}
&\int_{\left\{\left|u_n-T_k\left(\alpha_k^j\right)_k\right|<\eta\right\}} a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(\alpha_k^j\right)_\mu\chi_{j,s}\right)dxdt\\
&\leq c_{10}\eta+c_9+\int_{\left\{\left|T_\eta\left(u_n-T_k\left(\alpha_k^j\right)_\mu\right)\right|<\eta\right\}}\psi\left(x,\frac{|F|}{p}\right)dxdt\\
&+\int_{\{|u_n|>k\}\cap\left\{\left|u_n-T_k\left(\alpha_k^j\right)_\mu\right|<\eta\right\}} a\left(x,t,u_n,\nabla T_k\left(\alpha_j^k\right)_\mu\chi_{\left\{\left|\nabla T_k\left(\alpha_j^k\right)\right|>s\right\}}dxdt\\
&+\varepsilon(n,j).
\end{aligned}\tag{4.23}$$

since $a\left(x,t,T_{k+\eta}\left(u_n\right),\nabla T_{k+\eta}\left(u_n\right)\right)\rightharpoonup h_{k+\eta}$ weakly-star in $\left(L_\psi(Q)\right)^N$ for $\sigma\left(\Pi L_\psi,\Pi E_\varphi\right)$. Then

$$\begin{aligned}
&\int_{\{|u_n|>k\}\cap\left\{\left|u_n-T_k\left(\alpha_k^j\right)_u\right|<\eta\right\}} a\left(x,t,u_n,\nabla u_n\right)\cdot\nabla T_k\left(\alpha_j^k\right)_\mu\chi_{\left\{\left|\nabla T_k\left(\alpha_j^k\right)\right|>s\right\}}dxdt\\
&\quad=\int_{\{|u|>k\}\cap\left\{\left|u-T_k\left(\alpha_k^j\right)_u\right|<\eta\right\}} h_{k+\eta}\cdot\nabla T_k\left(\alpha_j^k\right)_\mu\chi_{\left\{\left|\nabla T_k\left(\alpha_j^k\right)\right|>s\right\}}dxdt+\varepsilon(n).
\end{aligned}$$

Now, letting j to infinity, we obtain

$$\begin{aligned}
&\int_{\{|u_n|>k\}\cap\left\{\left|u_n-T_k\left(\alpha_k^j\right)_u\right|<\eta\right\}} a\left(x,t,u_n,\nabla u_n\right)\cdot\nabla T_k\left(\alpha_j^k\right)_\mu\chi_{\left\{\left|\nabla T_k\left(\alpha_j^k\right)\right|>s\right\}}dxdt\\
&=\int_{\{|u|>k\}\cap\{|u-T_k(u)_\mu|<\eta\}} h_{k+\eta}\cdot\nabla T_k(u)_\mu\chi_{\{|\nabla T_k(u)|>s\}}dxdt+\varepsilon(n,j).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&\int_{\{|u_n|>k\}\cap\left\{\left|u_n-T_k\left(\alpha_k^j\right)_\mu\right|<\eta\right\}} a\left(x,t,u_n,\nabla u_n\right)\cdot\nabla T_k\left(\alpha_j^k\right)_\mu\chi_{\left\{\left|\nabla T_k\left(\alpha_j^k\right)\right|>s\right\}}dxdt\\
&=\int_{\{|u|>k\}\cap\{|u-T_k(u)|<\eta\}} h_{k+\eta}\cdot\nabla T_k(u)\chi_{\{|\nabla T_k(u)|>s\}}dxdt+\varepsilon(n,j,\mu)\\
&=\varepsilon(n,j,\mu,s).
\end{aligned}$$

Then (4.23) becomes

$$\int_{\left\{\left|u_n-T_k\left(\alpha_k^j\right)_\mu\right|<\eta\right\}} a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(\alpha_k^j\right)_\mu\chi_{j,s}\right)dxdt\tag{4.24}$$

$\leq c_{10}\eta+c_9+\int_{\left\{\left|T_\eta\left(u_n-T_k\left(\alpha_k^j\right)_\mu\right)\right|<\eta\right\}}\psi\left(x,\frac{|F|}{p}\right)dxdt+\varepsilon(n,j,\mu,s)$. On the other hand, remark that

$$\begin{aligned}
&\int_{\left\{\left|u_n-T_k\left(\alpha_k^j\right)_\mu\right|<\eta\right\}} a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(\alpha_k^j\right)_\mu\chi_{j,s}\right)dxdt\\
&=\int_{\left\{\left|u_n-T_k\left(\alpha_k^j\right)_\mu\right|<\eta\right\}} a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(\alpha_k^j\right)\chi_{j,s}\right)dxdt
\end{aligned}$$

$$+\int_{\left\{\left|u_n-T_k\left(\alpha_k^j\right)_\mu\right|<\eta\right\}} a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdots\cdots\left(\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}-\nabla T_k\left(\alpha_k^j\right)_\mu\chi_{j,s}\right)dxdt \tag{4.25}$$

for the second term of the last inequality, we have obviously that

$$\int_{\left\{\left|u_n-T_k\left(\alpha_k^j\right)_\mu\right|<\eta\right\}} a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\left(\nabla T_k\left(\alpha_j^k\right)-\nabla T_k\left(\alpha_k^j\right)_\mu\chi_{j,s}\right)dxdt$$
$$=\varepsilon(n,j,\mu,s).$$

Then (4.24) becomes

$$\int_{\left\{\left|u_n-T_k\left(\alpha_k^j\right)_\mu\right|<\eta\right\}} a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(\alpha_k^j\right)\chi_{j,s}\right)dxdt$$

$$\leq c_{10}\eta+c_9+\int_{\left\{\left|T_\eta\left(u_n-T_k\left(\alpha_k^j\right)_\mu\right)\right|<\eta\right\}}\psi\left(x,\frac{|F|}{p}\right)dxdt+\varepsilon(n,j,\mu,s). \tag{4.26}$$

Hence by letting η to zero, we get

$$\int_{\left\{\left|u_n-T_k\left(\alpha_k^j\right)_\mu\right|<\eta\right\}} a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(\alpha_k^j\right)\chi_{j,s}\right)dxdt$$
$$\leq\varepsilon(n,j,\mu,s,\eta)+c_9. \tag{4.27}$$

Now, let $0<\theta<1$, by applying the Young's inequality with $p=\frac{1}{\theta}$ and $\frac{1}{1-\theta}$ $y_n=\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right), y=\left(x,t,T_k\left(u_n\right),\nabla T_k(u)\right)$, we get

$$\int_{Q\cap\left\{\left|T_k(u_n)-T_k\left(\alpha_j^k\right)_u\right|<\eta\right\}}\left(\left[a\left(y_n\right)-a(y)\right]\times\left[\nabla T_k\left(u_n\right)-\nabla T_k(u)\right]\right)^\theta dxdt$$
$$=\int_Q\left(\left[a\left(y_n\right)-a(y)\right]\times\left[\nabla T_k\left(u_n\right)-\nabla T_k(u)\right]\right)^\theta\chi_{\left\{\left|T_k(u_n)-T_k\left(\alpha_j^k\right)_\mu\right|<\eta\right\}}dxdt$$
$$\leq c\,\mathrm{meas}\left\{\left|T_k\left(u_n\right)-T_k\left(\alpha_j^k\right)_\mu\right|<\eta\right\}^{\frac{1}{1-\theta}}$$
$$+c\left(\int_{Q\cap\left\{\left|T_k(u_n)-T_k\left(\alpha_j^k\right)_\mu\right|<\eta\right\}}\left[a\left(y_n\right)-a(y)\right]\times\left[\nabla T_k\left(u_n\right)-\nabla T_k(u)\right]dxdt\right)^\theta. \tag{4.28}$$

But we have for $s>0, y_\chi=\left(x,t,T_k\left(u_n\right),\nabla T_k(u)\chi_s\right)$ and $y_\alpha=\left(x,t,T_k\left(u_n\right)\right.$ $\left.\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right)$, we have

$$
\begin{aligned}
&\int_{Q\cap\left\{\left|T_k(u_n)-T_k\left(\alpha_j^k\right)_\mu\right|<\eta\right\}}\left[a\left(y_n\right)-a(y)\right]\times\left[\nabla T_k\left(u_n\right)-\nabla T_k(u)\right]dxdt\\
&\leq\int_{\left\{\left|T_k(u_n)-T_k\left(\alpha_j^k\right)_\mu\right|<\eta\right\}}\left[a\left(y_n\right)-a\left(y_\chi\right)\right]\times\left[\nabla T_k\left(u_n\right)-\nabla T_k(u)\chi_s\right]dxdt\\
&\leq\int_{\left\{\left|T_k(u_n)-T_k\left(\alpha_j^k\right)_\mu\right|<\eta\right\}}\left[a\left(y_n\right)-a\left(y_\alpha\right)\right]\times\left[\nabla T_k\left(u_n\right)-\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right]dxdt\\
&+\int_{\left\{\left|T_k(u_n)-T_k\left(\alpha_j^k\right)_\mu\right|<\eta\right\}}a\left(y_n\right)\left[\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}-\nabla T_k(u)\chi_s\right]dxdt.\\
&+\int_{\left\{\left|T_k(u_n)-T_k\left(\alpha_j^k\right)_\mu\right|<\eta\right\}}\left[a\left(y_\alpha\right)-a\left(y_\chi\right)\right]\nabla T_k\left(u_n\right)dxdt\\
&-\int_{\left\{\left|T_k(u_n)-T_k\left(\alpha_j^k\right)_\mu\right|<\eta\right\}}a\left(y_\alpha\right)\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}dxdt\\
&+\int_{\left\{\left|T_k(u_n)-T_k\left(\alpha_j^k\right)_\mu\right|<\eta\right\}}a\left(x,t,T_k\left(u_n\right),\nabla T_k(u)\chi_s\right)\nabla T_k(u)\chi_s dxdt\\
&=J_1+J_2+J_3+J_4+J_5.
\end{aligned}
\tag{4.29}
$$

We shall go to limit as n,j,μ and s to infinity in the last fifth integrals of the last side. Starting by J_1, one has

$$
J_1\leq c_9+\varepsilon(n,j,\mu,\eta)-\int_{\left\{\left|T_k(u_n)-T_k\left(\alpha_j^k\right)_\mu\right|<\eta\right\}}a\left(y_\alpha\right)\left[\nabla T_k\left(u_n\right)-\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right]dxdt.
$$

Since $a\left(y_\alpha\right)$ converge strongly to $a\left(x,t,T_k(u),\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right)$ in $\left(E_\psi(Q)\right)^N$ and $\nabla T_k\left(u_n\right)\rightharpoonup\nabla T_k(u)$ weakly in $\left(L_\varphi(Q)\right)^N$, then

$$
\begin{aligned}
&\int_{\left\{\left|T_k(u_n)-T_k\left(\alpha_j^k\right)_\mu\right|<\eta\right\}}a\left(y_\alpha\right)\left[\nabla T_k\left(u_n\right)-\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right]dxdt\\
&=\int_{\left\{\left|T_k(u)-T_k\left(\alpha_j^k\right)_\mu\right|<\eta\right\}}a\left(x,t,T_k(u),\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right)\left[\nabla T_k(u)-\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right]dxdt\\
&+\varepsilon(n)
\end{aligned}
$$

which gives by letting $j\longrightarrow\infty,\mu\longrightarrow\infty$ and $s\longrightarrow\infty$ respectively

$$
\begin{aligned}
&\int_{\left\{\left|T_k(u_n)-T_k\left(\alpha_j^k\right)_\mu\right|<\eta\right\}}a\left(y_\alpha\right)\left[\nabla T_k\left(u_n\right)-\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right]dxdt\\
&=\int_{\left\{\left|T_k(u)-T_k\left(\alpha_j^k\right)_\mu\right|<\eta\right\}}a\left(x,t,T_k(u),\nabla T_k(u)\chi_s\right)\left[\nabla T_k(u)-\nabla T_k(u)\chi_s\right]dxdt\\
&\quad+\varepsilon(n,j)\\
&=\int_Q a\left(x,t,T_k(u),\nabla T_k(u)\chi_s\right)\left[\nabla T_k(u)-\nabla T_k(u)\chi_s\right]dxdt+\varepsilon(n,j,\mu)\\
&=\varepsilon(n,j,\mu,s).
\end{aligned}
$$

Finally, we get

$$J_1 = \varepsilon(n, j, \mu, s, \eta). \tag{4.30}$$

Similarly, we get

$$J_2 = J_3 = J_4 = J_5 = \varepsilon(n, j, \mu, s, \eta). \tag{4.31}$$

Combining (4.30)-(4.31), we get

$$\lim_{n \to +\infty} \int_Q \left([a(y_n) - a(y)] \times [\nabla T_k(u_n) - \nabla T_k(u)]\right)^\theta dxdt = 0.$$

and, like a same argument in [4] we have

$$\nabla T_k(u_n) \longrightarrow \nabla T_k(u) \text{ as } n \longrightarrow +\infty \text{ for the modular convergence,} \tag{4.32}$$

4.1.6. In this step, We shall prove that

$$g_n(x, t, u_n, \nabla u_n) \longrightarrow g(x, t, u, \nabla u) \text{ strongly in } L^1(Q); \tag{4.33}$$

thinks to (4.32) we obtain

$$g_n(x, t, u_n, \nabla u_n) \longrightarrow g(x, t, u, \nabla u) \text{ a.e. in } Q. \tag{4.34}$$

Let E be measurable subset of Q and let $m > 0$. by (1.3) and (1.3) one has

$$\begin{aligned}
&\int_E [g_n(x, t, u_n, \nabla u_n)| dxdt \\
&= \int_{E \cap \{|u_n| \le m\}} |g_n(x, t, u_n, \nabla u_n)| dxdt + \int_{E \cap \{|u_n| > m\}} |g_n(x, t, u_n, \nabla u_n)| dxdt \\
&\le b(m) \int_E h_2(x, t) dxdt + b(m) \int_E a(x, t, T_m(u_n), \nabla T_m(u_n)) \cdot \nabla T_m(u_n) dxdt \\
&+ \frac{1}{m} \int_E g_n(x, t, u_n, \nabla u_n) u_n dxdt.
\end{aligned}$$

Now we choose u_n as a test function in $(\mathcal{P}_n)$ and by analogy to what is done in step 2 there exists a constant $c > 0$ such that

$$\int_E g_n(x, t, u_n, \nabla u_n) u_n dxdt \le c$$

Thus, we get

$$\lim_{m \to +\infty} \frac{1}{m} \int_E g_n(x, t, u_n, \nabla u_n) u_n dxdt = 0$$

in view of (4.32) the sequence $\left(a(x, t, T_m(u_n), \nabla T_m(u_n)) \cdot T_m(u_n)\right)_n$ is equi-integrable, this implies that

$$\lim_{|E| \to 0} \sup_n \int_E a(x, t, T_m(u_n), \nabla T_m(u_n)) \cdot \nabla T_m(u_n) dxdt = 0$$

This prove that $g_n(x, t, u_n, \nabla u_n)$ is equi-integrable. Therefore, Vitali's theorem gives $g(x, t, u, \nabla u) \in L^1(Q)$ and

$$g_n(x, t, u_n, \nabla u_n) \longrightarrow g(x, t, u, \nabla u) \text{ strongly in } L^1(Q).$$

4.1.7. Let $v \in W_0^{1,x}L_\varphi(Q)$ such that $\frac{\partial v}{\partial t} \in W^{-1,x}L_\psi(Q) + L^1(Q)$ There exists a prolongation $\bar{v}$ of v such that (see the proof of Lemma 2.3 and Theorem 4.6. in [1])

$$\begin{cases} \bar{v} = v \quad \text{on } Q \\ \bar{v} \in W_0^{1,x}L_\varphi(\Omega\times\mathbb{R}) \cap L^1(\Omega\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R}) \\ \text{and } \frac{\partial \bar{v}}{\partial t} \in W^{-1,x}L_\psi(\Omega\times\mathbb{R}) + L^1(\Omega\times\mathbb{R}). \end{cases}$$

In view of Lemma 2.3, there exists a sequence $(w_j)_j$ in $D(\Omega\times\mathbb{R})$ such that $w_j \longrightarrow \bar{v}$ in $W_0^{1,x}L_\varphi(\Omega\times\mathbb{R})$ and $\frac{\partial w_j}{dt} \longrightarrow \frac{\partial \bar{v}}{\partial t}$ in $W^{-1,x}L_\psi(\Omega\times\mathbb{R}) + L^1(\Omega\times\mathbb{R})$ for the modular convergence and $\|w_j\|_{\infty,Q} \leq (N+2)\|v\|_{\infty,Q}$.
Now, we take $T_k(u_n - w_j)\chi_{[0,\tau]}$ for every $\tau \in [0,T]$, as a test function in $(\mathcal{P}_n)$ we get

$$\begin{aligned} &\int_{Q_\tau} \frac{\partial u_n}{\partial t} T_k(u_n - w_j)\,dxdt + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n - w_j)\,dxdt \\ &\qquad + \int_{Q_\tau} \Phi_n(x,t,u_n)\cdot\nabla T_k(u_n - w_j)\,dxdt \\ &\qquad + \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)\,T_k(u_n - w_j)\,dxdt \\ &\leq \int_{Q_\tau} f_n T_k(u_n - w_j)\,dxdt + \int_{Q_\tau} F\cdot\nabla T_k(u_n - w_j)\,ddxdt. \end{aligned} \tag{4.35}$$

Concerning the first term of (4.35), we obtain

$$\begin{aligned} \int_{Q_\tau} \frac{\partial u_n}{\partial t} T_k(u_n - w_j)\,dxdt &= \left[\int_\Omega T_k(u_n - w_j)\,dx\right]_0^\tau + \int_{Q_\tau} \frac{\partial w_j}{\partial t} T_k(u_n - w_j)\,dxdt. \\ &= \left[\int_\Omega T_k(u - w_j)\,dx\right]_0^\tau + \int_{Q_\tau} \frac{\partial w_j}{\partial t} T_k(u - w_j)\,dxdt + \varepsilon(n) \\ &= \int_{Q_\tau} \frac{\partial u}{\partial t} T_k(u - w_j)\,dxdt. \end{aligned}$$

On the others hand,for the second term of (4.35) we get if $|u_n| > \lambda$ then $|u_n - w_j| \geq |u_n| - \|w_j\|_\infty > k$, thus $\{|u_n - w_j| \leq k\} \subseteq \{|u_n| \leq k + (N+2)\|v\|_\infty\}$, which translates that,

$$\begin{aligned} &\liminf_{n\to+\infty} \int_Q a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - w_j)\,dxdt \\ &\geq \int_Q a\left(x,t,T_{k+(N+2)\|v\|_\infty}(u), \nabla T_{k+(N+2)\|v\|_\infty}(u)\right) \\ &\quad \geq \int_Q a\left(x,t,T_{k+(N)\|_\infty}(u) - \nabla w_j\right)\chi_{\{|u-v|\leq k\}}dxdt \\ &= \int_Q a(x,t,u,\nabla u)(\nabla u - \nabla w_j)\,\chi_{\{|u-w_j|\leq k\}}dxdt \\ &= \int_Q a(x,t,u,\nabla u)\nabla T_k(u - w_j)\,dxdt \end{aligned} \tag{4.36}$$

as $\nabla T_k(u_n - w_j) \to \nabla T_k(u - w_j)$ in $L_\varphi(Q)$ as $n \to +\infty$, we have (as $n \to +\infty$)

$$\int_{Q_\tau} \Phi_n(x,t,u_n) \cdot \nabla T_k(u_n - w_j)\,dxdt \to \int_{Q_\tau} \Phi(x,t,u) \cdot \nabla T_k(u - w_j)\,dxdt$$

Thus, the strong convergence of $(g_n(x,t,u_n,\nabla u_n))_n$ and $((f_n))_n$, implies that

$$\begin{aligned}\int_{Q_\tau} \frac{\partial u}{\partial t} T_k(u - w_j)\,dxdt &+ \int_{Q_\tau} a(x,t,u,\nabla u) \cdot \nabla T_k(u - w_j)\,dxdt \\ + \int_{Q_\tau} \Phi(x,t,u) \cdot \nabla T_k(u - w_j)\,ddxdt &+ \int_{Q_\tau} g(x,t,u,\nabla u) T_k(u - w_j)\,dxdt \\ \le \int_{Q_\tau} f T_k(u - w_j)\,dxdt &+ \int_{Q_\tau} F \cdot \nabla T_k(u - w_j)\,ddxdt.\end{aligned} \tag{4.37}$$

Therefore , by using the modular convergence of j , we have the result desired in this step. As a conclusion of Step 1 to Step 7, the proof of our existence result is acheived.

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