Gulf Journal of Mathematics
Vol 23, Issue 1 (2026) 1-15

ISSN: 2309-4966
https://doi.org/10.56947/1sdev540

EXPONENTIAL BOUNDARY STABILIZATION OF A ONE-DIMENSIONAL ANTI-DAMPED WAVE EQUATION VIA INTEGRAL TRANSFORMATION

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Abstract. This work investigates the boundary stabilization of a one-dimensional wave equation subject to anti-damping effects. The objective is to derive an explicit boundary control that guarantees exponential dissipation of the system energy within a suitable functional framework. The approach is based on a Volterra integral transformation, which converts the original unstable dynamics into a target system exhibiting favorable stability properties. The associated kernels are characterized as solutions of a coupled hyperbolic system posed on a triangular domain. The well-posedness of this target model is established using semigroup theory. A Lyapunov functional is then constructed to design a stabilizing boundary feedback for the transformed system. By combining this control law with the inverse transformation, an explicit feedback controller for the original system is obtained. In addition, a similarity relationship between the semigroups of the original and target systems is derived, allowing the exponential stability of the latter to be transferred to the closed-loop dynamics of the former. Finally, numerical experiments are provided to illustrate the performance and effectiveness of the proposed control methodology.

Keywords. PDEs, Feedback stabilization, Integral transformation, Semigroup and its generator, Lyapunov function.
2020 Mathematics Subject Classification. Primary 93C05, 93C20; Secondary 47D06

1. Introduction

The analysis and control of systems governed by hyperbolic partial differential equations constitute a central topic in infinite-dimensional systems theory. Among these, the one-dimensional wave equation plays a fundamental role in modeling vibrating structures, transmission lines, and propagation phenomena. Its boundary control properties have been extensively studied using spectral methods [14, 10], semigroup techniques [9], multiplier methods [12], and Lyapunov-based approaches [17]. General operator-theoretic frameworks for such systems can be found in [3, 16].

Date: Received: Mar 20, 2026; Revised: Apr 15, 2026; Accepted: Apr 29, 2026.
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In the presence of damping, stability is typically ensured through dissipativity arguments. However, when destabilizing mechanisms such as anti-damping are present, the system energy grows exponentially in open loop, and classical approaches fail to provide stabilization. Anti-damped wave equations arise naturally in applications involving active structures or feedback interconnections that inject energy into the system.

A major advance in the constructive stabilization of PDEs has been achieved through the backstepping methodology, developed by Krstic and collaborators. This approach is based on the design of Volterra-type integral transformations that map unstable systems into stable target models. It provides explicit feedback laws and reduces the control design to the solution of kernel equations ensuring cancellation of destabilizing terms; see [11] for a comprehensive treatment.

For hyperbolic systems, boundary stabilization of unstable wave equations using such transformations has been investigated in several contributions. In particular, Smyshlyaev, Cerpa, and Krstic [15] established exponential stabilization results by constructing suitable kernels and boundary feedback laws. Further developments addressing stabilization of hyperbolic systems, including nonlinear settings, can be found in [2, 1].

An additional layer of difficulty arises when dynamic boundary conditions are present. For instance, the attachment of a tip mass at one endpoint leads to a boundary equation involving acceleration, resulting in a coupled PDE–ODE structure. Such systems have been analyzed within the framework of boundary control of flexible structures; see [13].

In this work, we consider the exponential stabilization of the one-dimensional anti-damped wave equation

$$
\begin{cases}
\mathcal{Y}_{tt}(t,x) = \mathcal{Y}_{xx}(t,x) - 2a\,\mathcal{Y}_t(t,x) - a^2\mathcal{Y}(t,x), & (t,x) \in (0,\infty)\times(0,1),\\
\mathcal{Y}_{tt}(t,0) = \mathcal{Y}_x(t,0) - a\,\mathcal{Y}_t(t,0), & t > 0,\\
\mathcal{Y}(t,1) = \mathcal{U}(t), & t > 0,\\
\mathcal{Y}(0,x) = \mathcal{Y}_0(x), \quad \mathcal{Y}_t(0,x) = \mathcal{Y}_1(x), & x \in (0,1),
\end{cases}
\tag{1.1}
$$

where $a < 0$ and $\mathcal{U}(t)$ is a boundary control input.

The system combines two sources of instability: interior anti-damping, induced by the terms $-2a\,\mathcal{Y}_t - a^2\mathcal{Y}$, and dynamic boundary effects arising from the tip-mass condition at $x = 0$. This coupling produces a non-dissipative system that falls outside the scope of standard stabilization techniques.

The contribution of this paper is twofold. First, we construct an explicit boundary feedback law that guarantees exponential stability of the closed-loop system in a suitable energy space. Second, we introduce a modified Volterra-type transformation that incorporates an additional integral term depending on the boundary trace. Compared to existing backstepping designs, this modification eliminates mixed space–time boundary terms in the kernel equations, thereby simplifying their structure while preserving invertibility of the transformation. This feature leads to a more tractable analysis in the presence of dynamic boundary conditions.

The kernel functions are obtained as solutions of a coupled hyperbolic system posed on a triangular domain. Once the existence and uniqueness of the associated target model are established within the semigroup framework, a Lyapunov-based approach is employed to derive a boundary feedback that stabilizes the auxiliary dynamics. The control law for the original system is subsequently reconstructed through the inverse transformation.

In addition, a similarity relationship between the semigroups corresponding to the original and transformed systems is proved. This property enables the exponential stability achieved for the auxiliary model to be transmitted to the closed-loop behavior of the initial system, providing a rigorous interpretation of the stabilization mechanism.

The remainder of the paper is organized as follows. Section 2 presents the Volterra integral transformation that converts system (1.1) into a suitable auxiliary model. The associated kernels are shown to satisfy a system of hyperbolic partial differential equations defined on a triangular domain, and the well-posedness of the resulting system is established using semigroup theory.

Section 3 focuses on the derivation of a stabilizing boundary feedback. By constructing an appropriate Lyapunov functional, a control law is first obtained for the auxiliary system. This result is then transferred to the original model through the inverse transformation, yielding an explicit boundary controller.

Section 5 provides numerical experiments that illustrate the performance of the proposed approach. Finally, Section 6 summarizes the main findings and discusses the key outcomes of the study.

2. Target system

2.1. A Volterra backstepping transformation. The analysis of system (1.1) presents two intrinsic structural difficulties. First, the lower-order contributions $-2a\mathcal{Y}_t - a^2\mathcal{Y}$ act as a distributed anti-damping mechanism, inducing exponential growth of the open-loop energy. Second, the boundary condition at $\tau = 0$ is of dynamic type, coupling boundary acceleration with the interior evolution. These two features obstruct the direct use of classical dissipativity or multiplier-based arguments.

To overcome these difficulties, we introduce a Volterra backstepping transformation mapping (1.1) into a suitably chosen target system. The key idea is to redistribute the unstable distributed terms toward the actuated boundary, where they can be compensated through feedback.

Let $\mathcal{Y}$ solve (1.1). We define

$$\begin{aligned}\kappa(t,x) = \mathcal{B}(\mathcal{Y}(t))(x) :=\ & \cosh(ax)\,\mathcal{Y}(t,x) - \int_0^x p(x,\tau)\mathcal{Y}(t,\tau)\,d\tau - \int_0^x q(x,\tau)\mathcal{Y}_t(t,\tau)\,d\tau \\ & + a\int_0^t \mathcal{Y}(r,0)\,dr. \end{aligned} \tag{2.1}$$

where the kernels p and q are to be determined. The operator $\mathcal{B}$ is triangular in x and therefore invertible under standard regularity assumptions. The kernels are chosen so that κ satisfies a homogeneous wave equation with dissipative boundary structure.

A transformation of the form (2.1), without the integral term $a\int_0^t \mathcal{Y}(r,0)\,dr$, has been previously investigated in [15, 8]. The present work introduces this additional term, which leads to a substantial simplification in the structure of the associated kernel equations.

Differentiating (2.1), integrating by parts over $(0,x)$, and substituting the dynamics from (1.1), one obtains

$$\begin{aligned}\kappa_{tt}-\kappa_{xx} &= \big[-2a\cosh(ax)+2q_x(x,x)\big]\mathcal{Y}_t+\big[-2a\sinh(ax)+2aq(x,x)\big]\mathcal{Y}_x\\ &\quad+\big[2p_x(x,x)-2a^2\cosh(ax)-2aq_\tau(x,x)\big]\mathcal{Y}\\ &\quad+\int_0^x\big[p_{xx}-p_{\tau\tau}+2aq_{\tau\tau}+a^2p-2a^3q\big]\mathcal{Y}(t,\tau)d\tau\\ &\quad+\int_0^x\big[q_{xx}-q_{\tau\tau}+2ap-3a^2q\big]\mathcal{Y}_t(\tau)d\tau\\ &\quad+[-q_\tau(x,0)+a]\mathcal{Y}_t(t,0)+[-p_\tau(x,0)+2aq_\tau(x,0)]\mathcal{Y}(t,0)\\ &\quad+q(x,0)\mathcal{Y}_{tx}(t,0)+[p(x,0)-2aq(x,0)]\mathcal{Y}_x(t,0).\end{aligned}$$

Hence $\kappa_{tt}=\kappa_{xx}$ if and only if all distributed and boundary residual terms vanish. This requirement leads to a coupled hyperbolic kernel system posed on the triangular domain

$$\triangle=\{(x,\tau):0\le\tau\le x\le 1\}.$$

The interior equations are

$$p_{xx}-p_{\tau\tau}=-2aq_{\tau\tau}-a^2p+2a^3q, \tag{2.2}$$

$$q_{xx}-q_{\tau\tau}=3a^2q-2ap. \tag{2.3}$$

The diagonal compatibility conditions are

$$p_x(x,x)=a^2\cosh(ax)-aq_\tau(x,x), \tag{2.4}$$

$$q_x(x,x)=a\cosh(ax), \tag{2.5}$$

$$q(x,x)=\sinh(ax). \tag{2.6}$$

The boundary conditions at $\tau=0$ read

$$p(x,0)=0,\quad p_\tau(x,0)=2aq_\tau(x,0), \tag{2.7}$$

$$q(x,0)=0,\quad q_\tau(x,0)=a. \tag{2.8}$$

Eliminating $q_\tau(x,x)$ from the diagonal relations and differentiating along $x=\tau$, one obtains a closed ODE for $p(x,x)$:

$$\frac{d^2}{dx^2}p(x,x)-a^2p(x,x)=0,\qquad p(0,0)=0,$$

whose solution is

$$p(x,x)=2a\sinh(ax).$$

Substituting this diagonal value into the kernel system gives the final Goursat problem

$$p_{xx} - p_{\tau\tau} = 2a^3 q - 2a q_{\tau\tau} - a^2 p, \tag{2.9}$$
$$p(x,x) = 2a\sinh(ax), \tag{2.10}$$
$$p(x,0) = 0, \quad p_\tau(x,0) = 2a^2, \tag{2.11}$$

and

$$q_{xx} - q_{\tau\tau} = -3a^2 q + 2ap, \tag{2.12}$$
$$q(x,x) = \sinh(ax), \tag{2.13}$$
$$q(x,0) = 0, \quad q_\tau(x,0) = a. \tag{2.14}$$

By standard results for linear hyperbolic systems posed on triangular domains, this problem admits a unique classical solution $(p,q) \in C^2(\triangle)$. A direct verification shows that the explicit solution is

$$p(x,\tau) = 2a\sinh(a\tau), \qquad q(x,\tau) = \sinh(a\tau).$$

These kernels satisfy all interior, diagonal, and boundary conditions, thereby completing the construction of the backstepping transformation.

It remains to examine the boundary condition at $x = 0$. Evaluating the transformation (2.1) at $x = 0$ and using the boundary equation in (1.1), we obtain

$$\kappa_{tt}(t,0) = \mathcal{Y}_x(t,0) - p(0,0)\mathcal{Y}(t,0) - q(0,0)\mathcal{Y}_t(t,0). \tag{2.15}$$

Therefore, in order to ensure that the transformed state satisfies the desired boundary relation

$$\kappa_{tt}(t,0) = \kappa_x(t,0),$$

it is sufficient to impose the compatibility conditions

$$p(0,0) = 0, \qquad q(0,0) = 0.$$

These conditions are consistent with the previously derived kernel boundary constraints.

Under the interior kernel equations and the boundary compatibility conditions established above, the transformed variable κ satisfies the target system

$$\begin{cases} \kappa_{tt}(t,x) = \kappa_{xx}(t,x), & \text{in } (0,\infty)\times(0,1), \\ \kappa_{tt}(t,0) = \kappa_x(t,0), & \text{in } (0,\infty), \\ \kappa(t,1) = \mathscr{W}(t), & \text{in } (0,\infty), \\ \kappa(0,x) =: \kappa_0(x), \quad \kappa_t(0,x) =: \kappa_1(x) & \text{in } (0,1), \end{cases} \tag{2.16}$$

where the new boundary input is given by

$$\mathscr{W}(t) = \cosh(a)\mathcal{U}(t) - \int_0^1 \sinh(a\tau)\big[2a\mathcal{Y}(t,\tau) + \mathcal{Y}_t(t,\tau)\big]\,d\tau + a\int_0^t \mathcal{Y}(r,0)\,dr. \tag{2.17}$$

System (2.16) is a standard one-dimensional wave equation with dynamic boundary condition at $x = 0$ and control input $\mathscr{W}(t)$ at $x = 1$. All distributed destabilizing terms have been eliminated by the backstepping transformation, and stabilization of the original system is therefore reduced to the design of $W(t)$ for the target model (2.16).

2.2. **The well-posedness of the closed-loop system (2.16).** In this subsection, we prove that the closed-loop system (2.16) is well-posed by employing a semigroup approach. More precisely, existence and uniqueness of solutions are obtained by recasting (2.16) as an abstract Cauchy problem associated with a linear operator defined on an appropriate Hilbert space.

We introduce the state space

$$\mathrm{X} := H_L^1 \times L^2 \times \mathbb{C}, \qquad H_L^1 = \{\zeta \in H^1(0,1) : \zeta(1) = 0\},$$

equipped with the inner product

$$\big\langle (\zeta, \nu, \rho_1), (\phi, \psi, \rho_2) \big\rangle_{\mathrm{X}} := \int_0^1 \left(\zeta_x \overline{\phi_x} + \nu \overline{\psi} \right) \, dx + \rho_1 \overline{\rho_2},$$

where $\overline{x}$ denotes the complex conjugate of $x \in \mathbb{C}$.

For $t \geq 0$, we define the state vector associated with the transformed variable as

$$\mathscr{X}(t) = \begin{pmatrix} \kappa(t,\cdot) - \mathscr{W}(t)\mathbf{1} \\ \kappa_t(t,\cdot) \\ \kappa_t(t,0) \end{pmatrix}, \qquad \mathbf{1}(x) \equiv 1,$$

and set

$$b = \begin{pmatrix} 0 \\ -\mathbf{1} \\ -1 \end{pmatrix}.$$

Next, we define the linear operator $\mathscr{A}$ by

$$\mathfrak{D}(\mathscr{A}) = \Big\{ (\zeta, \nu, \rho)^\top \in \mathrm{X} : \ \zeta \in H^2(0,1), \ \nu \in H_L^1, \ \nu(0) = \rho \Big\},$$

$$\mathscr{A} \begin{pmatrix} \zeta \\ \nu \\ \rho \end{pmatrix} = \begin{pmatrix} \nu \\ \zeta'' \\ \zeta'(0) \end{pmatrix}. \tag{2.18}$$

We also introduce the control operator

$$\mathscr{B}\mu = \mu \, \mathscr{A}_{-1} b, \qquad \mathscr{B} \in \mathcal{L}(\mathbb{R}; \mathrm{X}_{-1}),$$

where $\mathscr{A}_{-1}$ denotes the canonical extension of $\mathscr{A}$ to the extrapolation space X_{-1}, see [9].

With these definitions, the auxiliary system (2.16) can be written as the abstract evolution equation

$$\begin{cases} \dot{\mathscr{X}}(t) = \mathscr{A}\big(\mathscr{X}(t) + \dot{\mathscr{W}}(t) b\big) = \mathscr{A}_{-1}\mathscr{X}(t) + \mathscr{B}\dot{\mathscr{W}}(t), \quad t > 0, \\ \mathscr{X}(0) = \big(\kappa_0(\cdot) - \kappa_0(1)\mathbf{1}, \, \kappa_1(\cdot), \, \kappa_1(0)\big)^\top. \end{cases} \tag{2.19}$$

This formulation embeds the problem into the standard semigroup setting for linear boundary control systems. The following lemma ensures the well-posedness of the closed-loop system (2.16) and characterizes the structure of the adjoint of the associated control operator, see [4, 7, 8].

Lemma 2.1. (1) *The operator $\mathscr{A}$ is skew-adjoint on* X. *As a consequence, it generates a C_0-group of isometries $(e^{t\mathscr{A}})_{t\in\mathbb{R}}$ on* X.

(2) *The operator $\mathscr{B}$ is admissible for the group $(e^{t\mathscr{A}})_{t\in\mathbb{R}}$ on* X.

(3) *The adjoint operator $\mathscr{B}^*$ satisfies*

$$\mathscr{B}^* \in \mathcal{L}\big((\mathrm{X}_{-1})^*, \mathbb{R}\big), \quad \mathfrak{D}(\mathscr{A}) \subseteq \mathfrak{D}(\mathscr{B}^*) = (\mathrm{X}_{-1})^* \subseteq \mathrm{X},$$

and its restriction to $\mathfrak{D}(\mathscr{A})$ is given by

$$\mathscr{B}^*_{|\mathfrak{D}(\mathscr{A})} = -b^*\mathscr{A}.$$

Moreover, for any $\mathscr{X} = (\zeta, \nu, \rho)^\top \in \mathrm{X}$,

$$\mathscr{B}^*\mathscr{X} = -\zeta'(1).$$

In addition, the domain of $\mathscr{B}^*$ is characterized by

$$\mathfrak{D}(\mathscr{B}^*) = \big\{\mathscr{X} = (\zeta, \nu, \rho)^\top \in \mathrm{X} \;:\; \zeta \text{ is continuous at } x = 1\big\}.$$

(4) *The vector b belongs to $\mathfrak{D}(\mathscr{B}^*)$ and satisfies $\mathscr{B}^* b = 0$.*

3. A feedback stabilizer of system (2.16)

This section is devoted to the design of a boundary feedback law $\mathscr{W}$ ensuring exponential stability of the auxiliary system (2.16). The analysis relies on a Lyapunov approach tailored to capture both the distributed energy and the boundary coupling induced by the control.

We introduce the natural energy functional

$$\mathscr{V}(t) = \frac{1}{2}\|\mathscr{X}(t)\|^2 = \frac{1}{2}\left\{\int_0^1 \left(|\kappa_\tau(t,\tau)|^2 + |\kappa_t(t,\tau)|^2\right) d\tau + |\kappa_t(t,0)|^2\right\}, \tag{3.1}$$

where κ satisfies (2.16). A direct computation using the boundary conditions gives

$$\dot{\mathscr{V}}(t) = \frac{1}{2}\left(\kappa_x(t,1)\overline{\dot{\mathscr{W}}(t)} + \overline{\kappa_x(t,1)}\dot{\mathscr{W}}(t)\right). \tag{3.2}$$

We now select the feedback law

$$\dot{\mathscr{W}}(t) = -\kappa_x(t,1) = \mathscr{B}^*\mathscr{X}(t). \tag{3.3}$$

Substituting (3.3) into (3.2) yields

$$\dot{\mathscr{V}}(t) = -|\kappa_x(t,1)|^2 \le 0,$$

showing strict boundary dissipation. The next step is to strengthen this property to exponential decay.

Define $\mathscr{X}(t) = (\kappa - \mathscr{W}(t)\mathbf{1}; \kappa_t; \kappa_t(t,0))^\top$. Under (3.3), the closed-loop system becomes

$$\dot{\mathscr{X}}(t) = \mathscr{A}^o\mathscr{X}(t), \qquad \mathscr{X}(0) = \mathscr{X}_0, \tag{3.4}$$

where $\mathscr{A}^o := \mathscr{A}(I - b\mathscr{B}^*)$.

The operator A admits the representation

$$\mathfrak{D}(\mathscr{A}^o) = \Big\{(\zeta,\nu,\rho)^\top \in \mathrm{X} : \zeta \in H^2 \cap H^1_L,\ \nu \in H^2,\ \nu(0)=\rho,\ \nu(1)=-\zeta'(1)\Big\}, \tag{3.5}$$

$$\mathscr{A}^o \begin{pmatrix}\zeta\\ \nu\\ \rho\end{pmatrix} = \begin{pmatrix}\nu+\zeta'(1)\mathbf{1}\\ \zeta''\\ \zeta'(0)\end{pmatrix}. \tag{3.6}$$

The following theorem establishes the well-posedness of the Cauchy problem (3.4).

Theorem 3.1. *The operator $\mathscr{A}^o$ satisfies:*

(1) $\mathrm{Re}\langle \mathscr{A}^o\mathscr{X},\mathscr{X}\rangle = -|\mathscr{B}^*\mathscr{X}|^2$ *for all* $\mathscr{X}\in\mathfrak{D}(\mathscr{A}^o)$*;*
(2) $0\in\rho(\mathscr{A}^o)$*;*
(3) $\mathscr{A}^o$ *generates a C_0-semigroup of contractions* $(e^{\mathscr{A}^o t})_{t\geq 0}$ *on* X.

Proof. (1) Let $\mathscr{X}\in\mathfrak{D}(\mathscr{A}^o)$. By Lemma 2.1, we compute

$$\begin{aligned}\mathrm{Re}\,\langle \mathscr{A}^o\mathscr{X},\mathscr{X}\rangle &= \frac{1}{2}\Big(\langle \mathscr{A}^o\mathscr{X},\mathscr{X}\rangle + \langle \mathscr{X},\mathscr{A}^o\mathscr{X}\rangle\Big)\\ &= \frac{1}{2}\Big(\langle \mathscr{A}\mathscr{X},\mathscr{X}\rangle - \langle \mathscr{B}\mathscr{B}^*\mathscr{X},\mathscr{X}\rangle + \langle \mathscr{X},\mathscr{A}\mathscr{X}\rangle - \langle \mathscr{X},\mathscr{B}\mathscr{B}^*\mathscr{X}\rangle\Big).\end{aligned}$$

Since the operator $\mathscr{A}$ is skew-adjoint on its domain, the two terms involving $\mathscr{A}$ cancel each other. Hence,

$$\mathrm{Re}\,\langle \mathscr{A}^o\mathscr{X},\mathscr{X}\rangle = -\langle \mathscr{B}^*\mathscr{X},\mathscr{B}^*\mathscr{X}\rangle = -|\mathscr{B}^*\mathscr{X}|^2 \leq 0.$$

Therefore, $\mathscr{A}^o$ is dissipative.

(2) We now verify the maximality of $\mathscr{A}^o$. For this purpose, consider the equation

$$\mathscr{A}^o\mathscr{X} = F.$$

Using the definition of $\mathscr{A}^o$, this relation can be rewritten as

$$\mathscr{X} = \mathscr{A}^{-1}F + b\,\mathscr{B}^*\mathscr{X}. \tag{3.7}$$

Any solution of this equation must lie in $\mathfrak{D}(\mathscr{A}^o)$, which is contained in $\mathfrak{D}(\mathscr{B}^*)$. Applying the operator $\mathscr{B}^*$ to (3.7) yields

$$\mathscr{B}^*\mathscr{X} = \mathscr{B}^*\mathscr{A}^{-1}F + \mathscr{B}^* b\,\mathscr{B}^*\mathscr{X}.$$

Since $\mathscr{B}^*b=0$, and recalling from Lemma 2.1 that $\mathscr{B}^*_{|\mathfrak{D}(\mathscr{A})}$ is invertible, we obtain

$$\mathscr{B}^*\mathscr{X} = \mathscr{B}^*\mathscr{A}^{-1}F = -b^*\mathscr{A}\mathscr{A}^{-1}F = -b^*F.$$

Substituting this identity into (3.7) gives

$$\mathscr{X} = \mathscr{A}^{-1}F - b\,b^*F = (\mathscr{A}^{-1} - bb^*)F.$$

The obtained vector clearly belongs to $\mathfrak{D}(\mathscr{A}^o)$, which proves that the equation $\mathscr{A}^o\mathscr{X}=F$ admits a unique solution for every F. Consequently,

$$0\in\rho(\mathscr{A}^o) \quad \text{and} \quad (\mathscr{A}^o)^{-1} = \mathscr{A}^{-1} - bb^*.$$

(3) Since $\mathscr{A}^o$ is dissipative and maximal, the Lumer–Phillips theorem ensures that $\mathscr{A}^o$ generates a contraction semigroup. ☐

To establish exponential stability, we introduce the modified Lyapunov functional

$$\mathscr{V}_{\mathscr{R}}(t) = \langle \mathscr{R}\mathscr{X}(t), \mathscr{X}(t)\rangle,$$

where $\mathscr{R}$ is defined by

$$\mathscr{R}\begin{pmatrix}\zeta\\ \nu\\ \rho\end{pmatrix} = \begin{pmatrix}\zeta - \displaystyle\int_x^1 \theta(s)\nu(s)\,ds\\ \nu + \theta\zeta'\\ \rho\end{pmatrix}, \qquad \theta(x) = \frac{1}{4}(1-x). \tag{3.8}$$

Lemma 3.2. *The operator $\mathscr{R}$ satisfies:*

(1) *$\mathscr{R} = \mathscr{R}^*$;*
(2) *$\mathscr{R}$ is positive and invertible, and*

$$\frac{3}{4}\|\mathscr{X}\|^2 \leq \langle \mathscr{R}\mathscr{X}, \mathscr{X}\rangle \leq \frac{5}{4}\|\mathscr{X}\|^2; \tag{3.9}$$

(3) *For all $\mathscr{X} \in \mathfrak{D}(\mathscr{A}^o)$,*

$$\langle \mathscr{A}^o\mathscr{X}, \mathscr{R}\mathscr{X}\rangle + \langle \mathscr{R}\mathscr{X}, \mathscr{A}^o\mathscr{X}\rangle \leq -\frac{1}{4}\|\mathscr{X}\|^2. \tag{3.10}$$

Proof. (1) The self-adjointness of $\mathscr{R}$ follows by direct inspection of the definition.

(2) Let $\mathscr{X} = (\zeta, \nu, \rho)^\top \in \mathrm{X}$. Then

$$\langle \mathscr{R}\mathscr{X}, \mathscr{X}\rangle = \|\mathscr{X}\|^2 + 2\Re\int_0^1 \theta\,\zeta'\overline{\nu}\,dx.$$

Using $|\theta(x)| \leq \frac{1}{4}$ and Young's inequality,

$$\left|2\Re\int_0^1 \theta\,\zeta'\overline{\nu}\,dx\right| \leq \frac{1}{4}\int_0^1 (|\zeta'|^2 + |\nu|^2)\,dx \leq \frac{1}{4}\|\mathscr{X}\|^2.$$

This yields (3.9), hence $\mathscr{R}$ is positive and boundedly invertible.

(3) Let $\mathscr{X} \in \mathfrak{D}(\mathscr{A}^o)$. A straightforward integration by parts, together with the boundary conditions, gives

$$\begin{aligned}\langle \mathscr{A}^o\mathscr{X}, \mathscr{R}\mathscr{X}\rangle + \langle \mathscr{R}\mathscr{X}, \mathscr{A}^o\mathscr{X}\rangle = -\int_0^1 \theta'(|\zeta'|^2 + |\nu|^2)\,dx - \frac{1}{4}\left|\zeta'(0)\right|^2 - \frac{1}{4}\left|\nu(0)\right|^2\\ - 2\left|\zeta'(1)\right|^2.\end{aligned}$$

Since $\theta'(x) = -\frac{1}{4}$ and the boundary contributions are nonpositive, we infer

$$\langle \mathscr{A}^o\mathscr{X}, \mathscr{R}\mathscr{X}\rangle + \langle \mathscr{R}\mathscr{X}, \mathscr{A}^o\mathscr{X}\rangle \leq -\frac{1}{4}\|\mathscr{X}\|^2.$$

This proves (3.10). □

Let $\mathscr{X}(t) = e^{\mathscr{A}^o t}\mathscr{X}_0$. From the Lyapunov inequality (3.10), we get

$$\dot{V}_{\mathscr{R}}(t) \leq -\frac{1}{5}V_{\mathscr{R}}(t),$$

which yields

$$V_{\mathscr{R}}(t) \leq e^{-\frac{1}{5}t} V_{\mathscr{R}}(0).$$

Using norm equivalence (3.9),

$$\|e^{\mathscr{A}^o t}\mathscr{X}_0\| \leq \sqrt{\frac{5}{3}}\, e^{-\frac{1}{5}t}\|\mathscr{X}_0\|, \qquad t \geq 0.$$

We summarize these results in the following

Theorem 3.3. *The semigroup* $(e^{\mathscr{A}^o t})_{t\geq 0}$ *is exponentially stable. More precisely,*

$$\|e^{\mathscr{A}^o t}\| \leq \sqrt{\frac{5}{3}}\, e^{-\frac{1}{5}t}, \qquad t \geq 0.$$

Consequently, the boundary feedback law (3.3) ensures exponential stabilization of the auxiliary system (2.16), with decay rate $-\frac{1}{5}$.

4. Exponential stabilization of initial system (1.1)

We now establish exponential stability of the closed-loop system associated with (1.1). As in [6], the approach exploits the similarity between original system (1.1) and the exponentially stable target system (2.16).

To this end, we define the auxiliary operator

$$\mathfrak{R}\begin{pmatrix}\zeta\\ \nu\\ \rho\end{pmatrix} = \begin{pmatrix}\mathcal{B}\zeta - \mathcal{B}\zeta(1)\mathbb{1} := \mathfrak{P}\zeta\\ \mathcal{B}\nu\\ \mathcal{B}\rho\end{pmatrix}. \tag{4.1}$$

We also introduce

$$g := \mathcal{B}^{-1}\mathbb{1}, \quad \mathbb{1}_g := \frac{g}{g(1)}, \quad \text{if } g(1) \neq 0.$$

The following lemma can be established through straightforward computations.

Lemma 4.1. *If* $g(1) \neq 0$, *then:*

- *The operator* $\mathfrak{P}$ *admits a bounded inverse on* H^1_L, *given by*
$$\mathfrak{P}^{-1}\zeta = \mathcal{B}^{-1}\zeta - \mathcal{B}^{-1}\zeta(1)\mathbb{1}_g, \quad \zeta \in H^1_L.$$
- $\mathfrak{R}$ *is an isomorphism on* X *with inverse*
$$\mathfrak{R}^{-1} = (\mathfrak{P}^{-1}, \mathcal{B}^{-1}, \mathcal{B}^{-1}).$$

If $g(1) = 0$, then $\mathfrak{P}$ (and hence κ) is not invertible, since $\mathfrak{P}g = 0$ while $\mathbb{1} = \mathcal{B}g \neq 0$ would lead to a contradiction.

Let

$$\mathscr{X}(t) = (\kappa(t) - \mathscr{W}(t)\mathbb{1}, \kappa_t(t,x), \kappa_t(t,0))^\top$$

denote the solution of the auxiliary system (2.16)-(3.3). Substituting (3.3) into the boundary condition of (2.16) yields, for $t > 0$,

$$\mathcal{U}_t(t) = \frac{1}{\cosh(a)}\Big[-\kappa_x(t,1) + \int_0^1 \sinh(a\tau)\big(2a\mathcal{Y}_t(t,\tau) + u_{tt}(t,\tau)\big)d\tau + a\mathcal{Y}(t,0)\Big]. \tag{4.2}$$

Using the interior equation in (1.1), this simplifies to

$$\mathcal{U}_t(t) = \frac{1}{\cosh(a)}\Big[-\kappa_x(t,1) + \sinh(a)\mathcal{Y}_x(t,1) - a\cosh(a)\mathcal{U}(t)\Big]. \tag{4.3}$$

For a solution $\mathcal{Y}$ of (1.1), define

$$Y(t) = (\mathcal{Y} - \mathcal{U}(t)\mathbb{1}_g, \mathcal{Y}_t, \mathcal{Y}_t(t,0))^\top,$$

and consider the operator

$$\mathscr{A}_c^o(\zeta,\nu,\rho)^\top = \begin{pmatrix} \nu - \nu(1)\mathbb{1}_g \\ \zeta'' - 2a\nu - a^2\zeta \\ \zeta'(0) - a\nu(0) \end{pmatrix}, \tag{4.4}$$

with domain

$$\begin{aligned}\mathfrak{D}(\mathscr{A}_c^o) := \Big\{&(\zeta,\nu,\rho)^\top \in \mathrm{X} : \zeta \in (H^2 \cap H_L^1),\ \nu \in H^2 : \nu(0) = \rho,\\ &\nu(1) = -\frac{1}{\cosh(a)}\big[-(\mathcal{B}\zeta)_x(1) + \sinh(a)\zeta'(1) - a\cosh(a)\zeta(1)\big]\Big\}.\end{aligned} \tag{4.5}$$

Then, system (1.1) with control (4.2) can be expressed as

$$\dot{Y}(t) = \mathscr{A}_c^o Y(t), \quad Y(0) = Y_0. \tag{4.6}$$

Lemma 4.2. *Assume that $g(1) \neq 0$. Then the following statements hold:*

(i) $\mathscr{A}_c^o = \mathfrak{R}^{-1}\mathscr{A}^o\mathfrak{R}$;

(ii) *the operator $\mathscr{A}^o$ generates a C_0-semigroup $(e^{t\mathscr{A}^o})_{t\geq 0}$ satisfying*

$$e^{t\mathscr{A}_c^o} = \mathfrak{R}^{-1}e^{t\mathscr{A}^o}\mathfrak{R}. \tag{4.7}$$

Proof. (i) Let $\mathscr{X} = (\zeta,\nu,\rho)^\top \in \mathfrak{D}(\mathscr{A}^o)$. By definition of the transformation $\mathfrak{R}$, we have $\mathfrak{R}\mathscr{X} \in \mathfrak{D}(\mathscr{A}^o)$. Computing the action of $\mathscr{A}^o$ on $\mathfrak{R}\mathscr{X}$ gives

$$\mathscr{A}^o\mathfrak{R}\mathscr{X} = \begin{pmatrix} \mathcal{B}\nu - (\mathcal{B}\zeta)'(1)\mathbb{1} \\ (\mathcal{B}\zeta)'' \\ (\mathcal{B}\zeta)'(0) \end{pmatrix}. \tag{4.8}$$

On the other hand, applying $\mathfrak{R}$ to $\mathscr{A}_c^o\mathscr{X}$ yields

$$\mathfrak{R}\mathscr{A}_c^o\mathscr{X} = \begin{pmatrix} \mathfrak{P}\big[\nu - \nu(1)\mathbb{1}_g\big] \\ \mathcal{B}\big[\zeta'' + a\nu\big] \\ \mathcal{B}\big[\zeta'(0)\big] \end{pmatrix}. \tag{4.9}$$

Using the definition of the transformation operators, the first component becomes

$$\mathfrak{P}\big[\nu - \nu(1)\mathbb{1}_g\big] = \mathcal{B}\nu - (\mathcal{B}\zeta)'(1)\mathbb{1}.$$

Moreover, by construction of the kernel equations (2.9)–(2.14) (see Subsection 2), the transformation $\mathcal{B}$ satisfies the identity

$$(\mathcal{B}\zeta)'' = \mathcal{B}[\zeta'' + a\nu].$$

Combining these relations with (4.8)–(4.9) gives

$$\mathfrak{R}\mathscr{A}_c^o\mathscr{X} = \mathscr{A}^o\mathfrak{R}\mathscr{X}, \qquad \forall \mathscr{X} \in \mathfrak{D}(\mathscr{A}_c^o).$$

Consequently,

$$\mathscr{A}_c^o = \mathfrak{R}^{-1}\mathscr{A}^o\mathfrak{R},$$

which proves the similarity relation.

(ii) Since $g(1) \neq 0$, the operator $\mathfrak{R}$ is bounded and invertible on the state space. From part (i), the operators $\mathscr{A}^o$ and $\mathscr{A}_c^o$ are therefore similar. Because $\mathscr{A}^o$ generates a C_0-semigroup $(e^{t\mathscr{A}^o})_{t\geq 0}$, classical results from semigroup theory imply that $\mathscr{A}_c^o$ also generates a C_0-semigroup on the same space. Moreover, the two semigroups are related through the similarity transform

$$e^{t\mathscr{A}_c^o} = \mathfrak{R}^{-1}e^{t\mathscr{A}^o}\mathfrak{R}, \qquad t \geq 0,$$

which follows directly from the identity $\mathscr{A}_c^o = \mathfrak{R}^{-1}\mathscr{A}^o\mathfrak{R}$. The proof is complete. □

This shows that the solutions of the original closed-loop system (1.1), (4.2) are in one-to-one correspondence with those of the auxiliary system via the isomorphism κ:

$$\mathscr{X}(t) = \mathfrak{R}Y(t), \quad t \geq 0.$$

Consequently, the feedback (4.3) can be expressed in terms of Y as

$$\mathcal{U}_t(t) = -a\mathcal{U}(t) - \mathcal{Y}_x(t,1), \tag{4.10}$$

whose explicit solution with zero initial data is

$$\mathcal{U}(t) = -e^{-at}\int_0^t e^{as}\mathcal{Y}_x(s,1)ds. \tag{4.11}$$

Finally, using the isomorphism $\mathcal{B}$ and the exponential stability of the auxiliary system, we deduce exponential stabilization of the original system.

Theorem 4.3. *The boundary feedback*

$$\mathcal{U}(t) = -e^{-at}\int_0^t e^{as}\mathcal{Y}_x(s,1)ds$$

exponentially stabilizes system (1.1) *with a decay rate of* $1/5$*.*

Proof. The transformation $\mathcal{B}$ is an isomorphism on the state space. Since the target system generates an exponentially stable semigroup, the original system inherits this property under the inverse transformation. The decay estimate follows from the equivalence of norms. □

5. Numerical Results

In this section, we demonstrate the effectiveness of the boundary control law proposed in Theorem 4.3. We simulate the dynamical behavior of system (1.1) over the spatial domain $[0,1]$ and the time interval $[0,T]$.

The numerical scheme combines the implicit Euler method for time discretization with the central finite difference method (FDM) for spatial discretization.

The spatial grid composed of $N_x = 80$ points, while the time step is chosen as $\Delta t = 10^{-4}$. The initial condition is given by

$$u_0(x) = 10\sin(3\pi x).$$

The parameter $a = -0.1$ is chosen to introduce a slight destabilizing effect, allowing the effectiveness of the proposed boundary control law to be clearly illustrated. All simulations were performed using MATLAB R2024a.

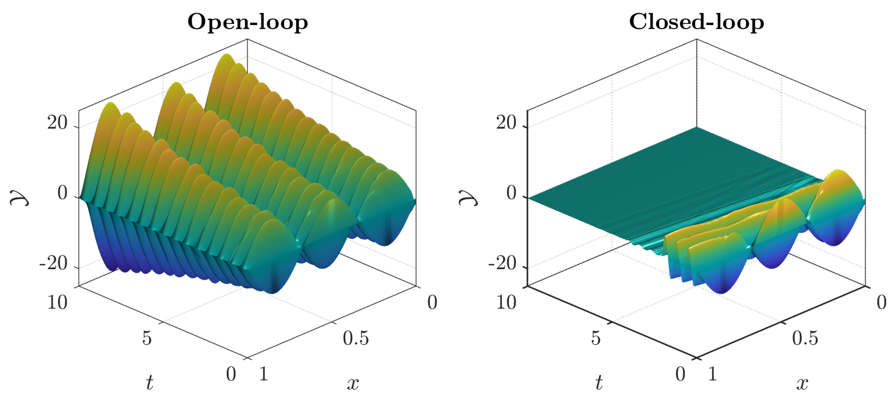

FIGURE 1. System displacement without control (left) and with the proposed boundary control (right).

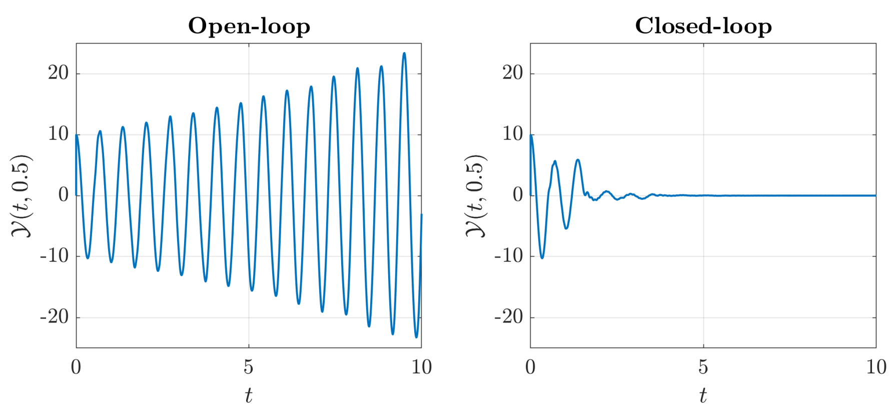

FIGURE 2. Cross-sectional displacement at $x = 0.5$ without control (left) and with the proposed boundary control (right).

Figure 1 illustrates the system behavior with and without control. In the uncontrolled case, the solution exhibits unstable behavior, indicating that the system energy grows over time. In contrast, when the proposed boundary controller is applied, the closed-loop system becomes stabilized and the displacement decays rapidly.

Figure 2 presents cross-sectional plots at $x = 0.5$, further illustrating the stabilizing effect of the proposed boundary feedback. The results show a rapid decay of the displacement in the controlled case.

These numerical simulations confirm that the proposed boundary feedback law achieves global exponential stabilization of the anti-damped wave equation, thereby validating the theoretical analysis.

6. Conclusion

This study examined the boundary stabilization of a class of one-dimensional wave equations affected by anti-damping. A constructive boundary control was developed to ensure exponential decay of the energy of the closed-loop system within a suitable Hilbert space setting.

The methodology relies on a Volterra integral transformation that converts the original unstable dynamics into an auxiliary system with improved stability behavior. The corresponding kernel functions were identified through a coupled system of hyperbolic equations posed on a triangular domain.

Once the well-posedness of the transformed system was established via semigroup arguments, a Lyapunov-based technique was employed to derive a stabilizing boundary feedback for the auxiliary model. The combination of this feedback with the inverse transformation yields an explicit controller for the initial system. In addition, a similarity property between the semigroups of the two systems was proved, which allows the exponential stability of the transformed dynamics to be inherited by the original closed-loop system.

Finally, numerical results were provided to demonstrate the relevance and efficiency of the proposed approach, as well as its consistency with the theoretical analysis.

Future work may focus on extending the present methodology to more general classes of distributed parameter systems, including nonlinear models, multidimensional PDEs, and systems with uncertainties or time delays. Another interesting direction concerns the development of observer-based boundary feedback laws when only partial state measurements are available.

Acknowledgement. The author sincerely thanks the referees for their thorough efforts and for their constructive remarks and recommendations.

References

1. Bastin, G. & Coron, J.-M. (2016). Stability and Boundary Stabilization of 1-D Hyperbolic Systems, Springer, 1, XIV–307. https://doi.org/10.1007/978-3-319-32062-5
2. Coron, J.-M. (2007). Control and Nonlinearity, American Mathematical Society, 136, . https://doi.org/10.1137/S0363012996302366
3. Curtain, R. F. & Zwart, H. (1995). An Introduction to Infinite-Dimensional Linear Systems Theory, Springer.
4. Elharfi, A. El hassani, M. and El Mourchid, S. (2023). Motion Planning Problem For a class of hyperbolic PDEs. Journal of Evolution Equation and Control Theory, 12(5), 1268-1286. https://doi.org/10.3934/eect.2023011
5. Elharfi, A. (2016). Exponential stabilization of a class of 1-D hyperbolic PDEs. J. Evol. Equ., 16, 665–679. https://link.springer.com/article/10.1007/s00028-015-0317-z
6. Elharfi, A. (2017). Exponential stabilization and motion planning of an overhead crane system. IMA Journal of Mathematical Control and Information, 34, 1299–1321. https://doi.org/10.1093/imamci/dnw026

7. El hassani, M. (2025). Output tracking problem for a class of 3-D hyperbolic PDEs, Gulf Journal of Mathematics, 19, 16-39. https://doi.org/10.56947/gjom.v19i1.2317
8. El hassani, M. (2026). Motion Planning Problem for Anti-Damped Wave Equation, Gulf Journal of Mathematics, 22(2), 1-15. https://doi.org/10.56947/gjom.v22i2.4230
9. Engel, K. and Nagel, R. (2000). One-parameter Semigroups for Linear Evolution Equations, Encyclopedia of Mathematics and its Applications, Springer–Verlag, 63(2), 278-280. https://doi.org/10.1007/s002330010042
10. Komornik, V. & Loreti, P. (2005). Fourier Series in Control Theory. Springer Monogr. Math., New York: Springer.
11. Krstic, M. & Smyshlyaev, A. (2008). Boundary Control of PDEs: A Course on Backstepping Designs, SIAM, 16.
12. Lions, J.-L. (1988). Exact Controllability, Stabilization and Perturbations for Distributed Systems, SIAM Review, 30(1), 1-68. https://doi.org/10.1137/1030001
13. Lasiecka, I. & Triggiani, R. (2000). Control Theory for Partial Differential Equations: Continuous and Approximation Theories, Cambridge University Press. https://doi.org/10.1017/CBO9780511574801
14. Russell, D. L. (1978). Controllability and stabilizability theory for linear partial differential equations: Recent progress and open questions. SIAM Rev., 20, 639–739. https://doi.org/10.1137/1020093
15. Smyshlyaev,A., Cerpa,E. & Krstic, M. (2010). Boundary stabilization of a 1-d wave equation with in-domain antidamping. SIAM J. Contr. Optim., 48, 4014-4031. https://doi.org/10.1137/080742646
16. Staffans, O. (2005). Well-Posed Linear Systems, Cambridge University Press, 103.
17. Tucsnak, M. (1993). Boundary stabilization for the stretched string equation. Diff. Integr. Eq., 6, 925–935.

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