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# A NOTE ON THE BOUNDEDNESS OF THE MULTIDIMENSIONAL KATUGAMPOLA OPERATOR IN CAMPANATO SPACES

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Abstract. The aim of the present paper is to extend the classical Katugampola fractional operator to its multidimensional form and to establish the boundedness of this newly defined operator from the Campanato space $\mathcal{L}^{\mathrm{p},\beta}(\Omega)$ to another Campanato space $\mathcal{L}^{\mathrm{q},\mu}(\Omega)$. In addition, we derive estimates that describe the dependence of the operator norm on the diameter of $\Omega$. Our results improve and extend existing results in the literature concerning the boundedness of fractional operators in function spaces. The improvement is achieved in two directions. First, Campanato spaces provide a more refined framework compared to Morrey spaces. Second, the Katugampola fractional operator is a natural generalization of both the Riemann–Liouville operator (when $\rho = 1$) and the Hadamard operator (when $\rho \to 0^+$), which allows us to obtain stronger and more general results.

*Keywords.* Katugampola operator, Campanato spaces, boundedness, fractional integrals, BMO, Hölder spaces.
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## 1. Introduction

The class of Banach spaces commonly referred to as Morrey–Campanato spaces $\mathcal{L}^{p,\lambda}(\Omega)$, introduced in connection with the works of Charles B. Morrey, Jr., and Sergio Campanato, constitutes a broader functional framework than that of bounded mean oscillation spaces. These spaces are designed to capture situations in which the local oscillation of a function over a ball scales proportionally to a power of the radius that may differ from the ambient dimension.

Such spaces play a significant role in the theory of elliptic partial differential equations, since for suitable choices of the parameter $\lambda$, functions belonging to $\mathcal{L}^{p,\lambda}(\Omega)$ exhibit Hölder continuity throughout the domain $\Omega$. The systematic study of these spaces dates back to the work of Campanato in the 1960s (see [1]), where they were employed to analyze regularity properties of solutions to elliptic equations. Moreover, Campanato spaces naturally extend Morrey spaces and encompass both BMO and Hölder spaces as particular instances.

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Let $\Omega \subset \mathbb{R}^{\mathrm{n}}$ be a bounded domain, $1 \leq \mathrm{p} < \infty$, and $\lambda \geq 0$. For a function $\phi \in \mathbb{L}^1_{\mathrm{loc}}(\Omega)$ and a ball $\mathbf{B}_{\mathrm{r}}(\mathtt{x}_0) \subset \mathbb{R}^{\mathrm{n}}$, we define the localized set $\Omega(\mathtt{x}_0, \mathrm{r}) = \Omega \cap \mathbf{B}_{\mathrm{r}}(\mathtt{x}_0)$ with measure $|\Omega(\mathtt{x}_0, \mathrm{r})|$. The mean value of $\phi$ over this set is given by

$$\phi_{\mathtt{x}_0,\mathrm{r}} = \frac{1}{|\Omega(\mathtt{x}_0, \mathrm{r})|} \int_{\Omega(\mathtt{x}_0,\mathrm{r})} \phi(\mathrm{y})\, \mathrm{dy}.$$

The Morrey seminorm [2] is defined by

$$[\phi]^{\mathrm{Morrey}}_{\mathrm{p},\lambda} := \sup_{\substack{\mathtt{x}_0 \in \Omega \\ 0<\mathrm{r}<\mathrm{r}_0}} \left( \frac{1}{\mathrm{r}^{\lambda}} \int_{\Omega(\mathtt{x}_0,\mathrm{r})} |\phi(\mathrm{y})|^{\mathrm{p}}\, \mathrm{dy} \right)^{1/\mathrm{p}},$$

where $\mathrm{r}_0 = \mathrm{diam}(\Omega)$. In particular, when $\lambda = 0$, this definition reduces to the classical $\mathbb{L}^{\mathrm{p}}$ norm. When $\lambda = \mathrm{n}$, it becomes equivalent to the $\mathbb{L}^{\infty}$ norm, in view of the Lebesgue differentiation theorem. For $\lambda > \mathrm{n}$, the space degenerates and contains only the zero function.

The Campanato space $\mathcal{L}^{\mathrm{p},\lambda}(\Omega)$ consists of all functions $\phi \in \mathbb{L}^{\mathrm{p}}(\Omega)$ satisfying

$$\|\phi\|_{\mathcal{L}^{\mathrm{p},\lambda}(\Omega)} := \|\phi\|_{\mathbb{L}^{\mathrm{p}}(\Omega)} + [\phi]_{\mathrm{p},\lambda} < \infty, \tag{1.1}$$

where the seminorm is given by

$$[\phi]_{\mathrm{p},\lambda} := \sup_{\substack{\mathtt{x}_0 \in \Omega \\ 0<\mathrm{r}<\mathrm{r}_0}} \left( \frac{1}{\mathrm{r}^{\lambda}} \int_{\Omega(\mathtt{x}_0,\mathrm{r})} |\phi(\mathrm{y}) - \phi_{\mathtt{x}_0,\mathrm{r}}|^{\mathrm{p}}\, \mathrm{dy} \right)^{1/\mathrm{p}}. \tag{1.2}$$

Under suitable regularity assumptions on the domain $\Omega$, an equivalence between Morrey and Campanato spaces is known to hold. More precisely, if $0 \leq \lambda < \mathrm{n}$ and there exists a constant $A > 0$ such that $|\Omega(\mathtt{x}_0, \mathrm{r})| \geq A\mathrm{r}^{\mathrm{n}}$ for all $\mathtt{x}_0 \in \Omega$ and $0 < \mathrm{r} < \mathrm{r}_0$, then the Morrey and Campanato norms are equivalent, and both spaces coincide in this range of parameters.

Recent developments in the theory of Campanato spaces have focused extensively on operator boundedness and related inequalities. In this direction, [6] investigated the existence and boundedness of square function operators on Campanato spaces. Further structural properties and applications of these spaces were discussed in [7]. The boundedness of operators associated with fractional semigroups and Schrödinger operators was analyzed in [8], while variable exponent generalizations of Campanato spaces were introduced in [9]. More recently, anisotropic and modified Calderón–Zygmund operators have been studied in Campanato-type settings [10, 13]. Additionally, operator boundedness related to Schrödinger operators on non-Euclidean structures such as Heisenberg groups has been explored in [11]. New hybrid spaces combining Besov, Bourgain, Morrey, and Campanato frameworks, along with sharp inequalities, were developed in [12]. Moreover, inequalities involving generalized derivatives and harmonic mappings were considered in [14], highlighting further analytical applications.

The Katugampola fractional integral operator [5] generalizes both the Riemann–Liouville and Hadamard fractional integrals. It is defined as follows.

**Definition 1.1** (Katugampola Fractional Integral)**.** For $\alpha > 0$ and $\rho > 0$, the Katugampola fractional integral operator is given by

$$\mathcal{K}_\rho^\alpha \phi(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} \phi(\tau)\, d\tau. \tag{2.1}$$

We now extend the above operator to its multidimensional form.

**Definition 1.2** (Multidimensional Katugampola Fractional Integral)**.** Let $\alpha = (\alpha_1, \dots, \alpha_\mathrm{n})$ with $\alpha_i > 0$, $\rho > 0$, $\mathrm{i} = 1, \dots, \mathrm{n}$. The left multidimensional Katugampola fractional integral operator $\mathcal{K}_{\mathfrak{a}_+}^{\alpha,\rho}$ is defined as

$$\begin{aligned}(\mathcal{K}_{\mathfrak{a}_+}^{\alpha,\rho}\phi)(\mathrm{x}) &= \frac{\rho^{\mathrm{n}-\sum \alpha_\mathrm{i}}}{\prod_{\mathrm{i}=1}^{\mathrm{n}} \Gamma(\alpha_\mathrm{i})} \prod_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x}_\mathrm{i}^{\rho-1} \\ \times \int_{\mathfrak{a}_\mathrm{n}}^{\mathrm{x}_\mathrm{n}} \cdots \int_{\mathfrak{a}_1}^{\mathrm{x}_1} &\left( \prod_{\mathrm{i}=1}^{\mathrm{n}} \frac{\tau_\mathrm{i}^{\rho-1}}{(\mathrm{x}_\mathrm{i}^\rho - \tau_\mathrm{i}^\rho)^{1-\alpha_\mathrm{i}}} \right) \phi(\tau_1, \dots, \tau_\mathrm{n})\, d\tau_1 \cdots d\tau_\mathrm{n},\end{aligned} \tag{2.2}$$

for all $\mathrm{x} = (\mathrm{x}_1, \dots, \mathrm{x}_\mathrm{n}) \in \mathbb{R}^\mathrm{n}$ such that $\mathrm{x}_\mathrm{i} > \mathfrak{a}_\mathrm{i}$, $\mathrm{i} = 1, \dots, \mathrm{n}$.

The right multidimensional Katugampola fractional integral operator $\mathcal{K}_{\mathfrak{b}_-}^{\alpha,\rho}$ is defined similarly:

$$\begin{aligned}(\mathcal{K}_{\mathfrak{b}_-}^{\alpha,\rho}\phi)(\mathrm{x}) &= \frac{\rho^{\mathrm{n}-\sum \alpha_\mathrm{i}}}{\prod_{\mathrm{i}=1}^{\mathrm{n}} \Gamma(\alpha_\mathrm{i})} \prod_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x}_\mathrm{i}^{\rho-1} \\ \times \int_{\mathrm{x}_\mathrm{n}}^{\mathfrak{b}_\mathrm{n}} \cdots \int_{\mathrm{x}_1}^{\mathfrak{b}_1} &\left( \prod_{\mathrm{i}=1}^{\mathrm{n}} \frac{\tau_\mathrm{i}^{\rho-1}}{(\tau_\mathrm{i}^\rho - \mathrm{x}_\mathrm{i}^\rho)^{1-\alpha_\mathrm{i}}} \right) \phi(\tau_1, \dots, \tau_\mathrm{n})\, d\tau_1 \cdots d\tau_\mathrm{n},\end{aligned} \tag{2.3}$$

for all $\mathrm{x} \in \mathbb{R}^\mathrm{n}$ such that $\mathrm{x}_\mathrm{i} < \mathfrak{b}_\mathrm{i}, \mathrm{i} = 1, \dots, \mathrm{n}$.

In [4] the following theorem was proved for the Riemann–Liouville case in the context of Morrey spaces. We now aim to generalize such results to Campanato spaces.

**Theorem 1.3** ([4])**.** *Let* $\mathrm{n} = 1$, $1 < \mathrm{p}, \mathrm{q} < \infty$, $\frac{1}{\mathrm{p}} < \alpha < 1$ *and* $0 \le \beta, \mu \le 1$, $0 \le \beta \le \frac{1}{\mathrm{p}}$, $0 \le \mu \le \frac{1}{\mathrm{q}}$, $0 < T < \infty$. *Then* $I_{0_+}^\alpha$ *is bounded from* $\mathcal{M}_\mathrm{p}^\beta(0,T)$ *to* $\mathcal{M}_\mathrm{q}^\mu(0,T)$.

The main purpose of this paper is to investigate the boundedness properties of the multidimensional Katugampola integral operator when acting on Campanato spaces $\mathcal{L}^{\mathrm{p},\beta}(\Omega)$, where $\Omega = \mathbf{Q}(\mathfrak{a}, \mathfrak{b}) = \{\mathrm{x} \in \mathbb{R}^\mathrm{n} : \mathfrak{a}_\mathrm{i} < \mathrm{x}_\mathrm{i} < \mathfrak{b}_\mathrm{i},\ \mathrm{i} = 1, \dots, \mathrm{n}\}$, with parameters $1 < \mathrm{p} < \infty$ and $0 < \mathrm{q} < \infty$. In addition, we derive sharp bounds for the operator norm expressed in terms of the diameter of the domain $\mathbf{Q}(\mathfrak{a}, \mathfrak{b})$. For further developments related to inequalities and the boundedness of various classes of operators in function spaces, we refer the reader to [15, 16, 17, 18, 19].

## 2. Main results

In this section, we investigate the boundedness properties of the multidimensional Katugampola fractional integral operator within the framework of Campanato spaces. We first present several auxiliary results related to these spaces, which will play a crucial role in the proofs of our main theorems. Let $\mathfrak{a} \in \mathbb{R}^{\mathrm{n}}, \mathfrak{b} \in \mathbb{R}^{\mathrm{n}}, -\infty < \mathfrak{a}_{\mathrm{i}} < \mathfrak{b}_{\mathrm{i}} \leq \infty, \mathrm{i} = 1, \ldots, \mathrm{n}$ and $\mathbf{Q}(\mathfrak{a}, \mathfrak{b}) = \{\mathrm{x} \in \mathbb{R}^{\mathrm{n}}, \mathfrak{a}_{\mathrm{i}} < \mathrm{x}_{\mathrm{i}} < \mathfrak{b}_{\mathrm{i}}, \mathrm{i} = 1, \ldots, \mathrm{n}\}$.

**Lemma 2.1.** *Let* $1 \leq \mathrm{p} < \infty$, $0 \leq \beta \leq \mathrm{n}$. *Then for any* $\phi \in \mathcal{L}^{\mathrm{p},\beta}(\mathbf{Q}(\mathfrak{a}, \mathfrak{b}))$ *and any* $\mathrm{y} \in \mathbf{Q}(\mathfrak{a}, \mathfrak{b})$,

$$\left(\int_{\mathbf{Q}(\mathfrak{a},\mathrm{y})} |\phi(\mathrm{t}) - \phi_{\mathbf{Q}(\mathfrak{a},\mathrm{y})}|^{\mathrm{p}} \, \mathrm{dt}\right)^{1/\mathrm{p}} \leq C|\mathrm{y} - \mathfrak{a}|^{\beta/\mathrm{p}}[\phi]_{\mathrm{p},\beta}. \tag{3.1}$$

*Proof.* Fix $\mathrm{y} \in \mathbf{Q}(\mathfrak{a}, \mathfrak{b})$ and let $d = |\mathrm{y} - \mathfrak{a}|$. Consider the chain of concentric parallelepipeds

$$\mathbf{Q}_k = \mathbf{Q}(\mathfrak{a}, \mathfrak{a} + 2^{-k}(\mathrm{y} - \mathfrak{a})), \quad k = 0, 1, 2, \ldots$$

where $\mathbf{Q}_0 = \mathbf{Q}(\mathfrak{a}, \mathrm{y})$ and $\mathbf{Q}_k \subset \mathbf{Q}_{k-1}$ with $|\mathbf{Q}_k| = 2^{-k\mathrm{n}}|\mathbf{Q}_0|$.

For each $k$, let $\phi_k = \phi_{\mathbf{Q}_k}$ be the mean value over $\mathbf{Q}_k$. Then

$$\phi_0 - \phi_{\mathbf{Q}(\mathfrak{a},\mathrm{y})} = \sum_{j=1}^{\infty}(\phi_j - \phi_{j-1}).$$

For any $j \geq 1$, note that $\mathbf{Q}_j \subset \mathbf{Q}_{j-1} \subset \mathbf{B}(\mathfrak{a} + 2^{-j}(\mathrm{y} - \mathfrak{a}), 2^{-j}d)$. Applying the Campanato seminorm estimate,

$$|\phi_j - \phi_{j-1}| \leq \frac{1}{|\mathbf{Q}_j|}\int_{\mathbf{Q}_j} |\phi(\mathrm{t}) - \phi_{j-1}| \, dt \leq C(2^{-j}d)^{\beta/\mathrm{p}}[\phi]_{\mathrm{p},\beta}.$$

Hence

$$|\phi_0 - \phi_{\mathbf{Q}(\mathfrak{a},\mathrm{y})}| \leq C[\phi]_{\mathrm{p},\beta}\sum_{j=1}^{\infty}(2^{-j}d)^{\beta/\mathrm{p}} = Cd^{\beta/\mathrm{p}}[\phi]_{\mathrm{p},\beta}.$$

Now write

$$\begin{aligned}\left(\int_{\mathbf{Q}(\mathfrak{a},\mathrm{y})} |\phi(\mathrm{t}) - \phi_0|^{\mathrm{p}} d\mathrm{t}\right)^{1/\mathrm{p}} \leq & \left(\int_{\mathbf{Q}(\mathfrak{a},\mathrm{y})} |\phi(\mathrm{t}) - \phi(\mathrm{t})_{\mathbf{Q}(\mathfrak{a},\mathrm{y})\cap\mathbf{B}(\mathrm{t},d)}|^{\mathrm{p}} d\mathrm{t}\right)^{1/\mathrm{p}} \\ & + \left(\int_{\mathbf{Q}(\mathfrak{a},\mathrm{y})} |\phi(\mathrm{t})_{\mathbf{Q}(\mathfrak{a},\mathrm{y})\cap\mathbf{B}(\mathrm{t},d)} - \phi_0|^{\mathrm{p}} d\mathrm{t}\right)^{1/\mathrm{p}}.\end{aligned}$$

For each $\mathrm{t} \in \mathbf{Q}(\mathfrak{a}, \mathrm{y})$, the ball $\mathbf{B}(\mathrm{t}, d)$ contains $\mathbf{Q}(\mathfrak{a}, \mathrm{y})$. Thus the first term is bounded by $Cd^{\beta/\mathrm{p}}[\phi]_{\mathrm{p},\beta}$ by definition of the Campanato seminorm. For the second term, the integrand is constant on $\mathbf{Q}(\mathfrak{a}, \mathrm{y})$ and by the estimate above, $|\phi(\mathrm{t})_{\mathbf{Q}(\mathfrak{a},\mathrm{y})\cap\mathbf{B}(\mathrm{t},d)} - \phi_0| \leq Cd^{\beta/\mathrm{p}}[\phi]_{\mathrm{p},\beta}$. Therefore the second term does not exceed $C|\mathbf{Q}(\mathfrak{a}, \mathrm{y})|^{1/\mathrm{p}}d^{\beta/\mathrm{p}}[\phi]_{\mathrm{p},\beta}$.

Combining these estimates yields

$$\left(\int_{\mathbf{Q}(\mathfrak{a},\mathrm{y})} |\phi(\mathrm{t}) - \phi_{\mathbf{Q}(\mathfrak{a},\mathrm{y})}|^{\mathrm{p}} d\mathrm{t}\right)^{1/\mathrm{p}} \leq C(1 + |\mathbf{Q}(\mathfrak{a}, \mathrm{y})|^{1/\mathrm{p}})d^{\beta/\mathrm{p}}[\phi]_{\mathrm{p},\beta} \leq Cd^{\beta/\mathrm{p}}[\phi]_{\mathrm{p},\beta},$$

since $|\mathbf{Q}(\mathfrak{a}, \mathrm{y})|^{1/\mathrm{p}} \leq Cd^{\mathrm{n}/\mathrm{p}}$ and $\mathrm{n}/\mathrm{p} \geq 0$. $\square$

**Lemma 2.2.** *Let* $1 \leq \mathrm{p} < \infty$, $0 \leq \beta \leq \mathrm{n}$. *Then for any* $\phi \in \mathcal{L}^{\mathrm{p},\beta}(\mathbf{Q}(\mathfrak{a},\mathfrak{b}))$ *and any* $\mathrm{y} \in \mathbf{Q}(\mathfrak{a},\mathfrak{b})$,

$$\|\phi\|_{\mathbb{L}^{\mathrm{p}}(\mathbf{Q}(\mathfrak{a},\mathrm{y}))} \leq |\mathbf{Q}(\mathfrak{a},\mathrm{y})|^{1/\mathrm{p}}|\phi_{\mathbf{Q}(\mathfrak{a},\mathrm{y})}| + |\mathrm{y}-\mathfrak{a}|^{\beta/\mathrm{p}}[\phi]_{\mathrm{p},\beta}. \tag{3.2}$$

*Proof.* Set $d = |\mathrm{y}-\mathfrak{a}|$ and consider the family of parallelepipeds

$$\Omega_\tau = \mathbf{Q}(\mathfrak{a}, \mathfrak{a}+\tau(\mathrm{y}-\mathfrak{a})), \quad \tau \in [0,1].$$

For each $\tau$, denote the mean value $m(\tau) = \phi_{\Omega_\tau}$.

Observe that $m(\tau)$ is absolutely continuous on $(0,1]$ and for almost every $\tau$,

$$m'(\tau) = \frac{1}{|\Omega_\tau|}\int_{\partial\Omega_\tau} (\phi(\mathrm{t}) - m(\tau))\,\frac{\mathrm{t}-\mathfrak{a}}{|\mathrm{t}-\mathfrak{a}|}\cdot \mathbf{n}\,dS,$$

where $\mathbf{n}$ is the outward unit normal. This representation follows from differentiating under the integral sign and applying the Reynolds transport theorem.

For any $\mathrm{t} \in \partial\Omega_\tau$, the distance from $\mathrm{t}$ to any point in $\Omega_\tau$ is at most $\tau d$. Hence

$$|m'(\tau)| \leq \frac{C}{\tau d}\cdot\frac{1}{|\Omega_\tau|}\int_{\Omega_\tau} |\phi(\mathrm{t}) - m(\tau)|\,d\mathrm{t}.$$

Using Hölder's inequality in combination with the definition of the Campanato seminorm,

$$\frac{1}{|\Omega_\tau|}\int_{\Omega_\tau} |\phi(\mathrm{t}) - m(\tau)|\,d\mathrm{t} \leq \left(\frac{1}{|\Omega_\tau|}\int_{\Omega_\tau} |\phi(\mathrm{t}) - m(\tau)|^{\mathrm{p}}\,d\mathrm{t}\right)^{1/\mathrm{p}} \leq C(\tau d)^{\beta/\mathrm{p}}[\phi]_{\mathrm{p},\beta}.$$

Therefore,

$$|m'(\tau)| \leq C\tau^{\beta/\mathrm{p}-1}d^{\beta/\mathrm{p}-1}[\phi]_{\mathrm{p},\beta}.$$

Integrating from $\tau = \varepsilon$ to 1 and letting $\varepsilon \to 0$,

$$|m(1) - m(0+)| \leq Cd^{\beta/\mathrm{p}-1}[\phi]_{\mathrm{p},\beta}\int_0^1 \tau^{\beta/\mathrm{p}-1}d\tau = Cd^{\beta/\mathrm{p}-1}[\phi]_{\mathrm{p},\beta},$$

provided $\beta/\mathrm{p} > 0$. For $\beta = 0$, a separate limiting argument gives the same bound.

Now write the norm as

$$\|\phi\|^{\mathrm{p}}_{\mathbb{L}^{\mathrm{p}}(\Omega_1)} = \int_{\Omega_1} |\phi(\mathrm{t})|^{\mathrm{p}}d\mathrm{t} = \int_0^1\left(\int_{\partial\Omega_\tau} |\phi(\mathrm{t})|^{\mathrm{p}}\,\frac{|\mathrm{t}-\mathfrak{a}|}{\tau d}\,dS\right)d\tau.$$

For each $\tau$, decompose

$$\int_{\partial\Omega_\tau} |\phi(\mathrm{t})|^{\mathrm{p}}\,dS \leq C\left(\int_{\partial\Omega_\tau} |\phi(\mathrm{t}) - m(\tau)|^{\mathrm{p}}\,dS + |m(\tau)|^{\mathrm{p}}|\partial\Omega_\tau|\right).$$

The first term admits the estimate $C(\tau d)^{\beta}[\phi]^{\mathrm{p}}_{\mathrm{p},\beta}(\tau d)^{\mathrm{n}-1}$ via trace-type inequalities, whereas the second term can be treated as follows:

$$|m(\tau)| \leq |m(1)| + |m(\tau) - m(1)| \leq |\phi_{\Omega_1}| + Cd^{\beta/\mathrm{p}}[\phi]_{\mathrm{p},\beta}.$$

Integrating in $\tau$ and taking p-th roots yields

$$\|\phi\|_{\mathbb{L}^{\mathrm{p}}(\Omega_1)} \leq C\left(|\Omega_1|^{1/\mathrm{p}}|\phi_{\Omega_1}| + d^{\beta/\mathrm{p}}[\phi]_{\mathrm{p},\beta}\right).$$

The constant $C$ depends only on $\mathrm{n}, \mathrm{p}$ and the geometry of the parallelepiped. $\square$

**Lemma 2.3.** *Let* $1 \leq \mathrm{p} < \infty$, $0 \leq \mu < \beta \leq \mathrm{n}$. *Then for any finite parallelepiped* $\mathbf{Q}(\mathfrak{a}, \mathfrak{b})$ *and any* $\phi \in \mathcal{L}^{\mathrm{p},\beta}(\mathbf{Q}(\mathfrak{a}, \mathfrak{b}))$,

$$\|\phi\|_{\mathcal{L}^{\mathrm{p},\mu}(\mathbf{Q}(\mathfrak{a},\mathfrak{b}))} \leq C|\mathfrak{b} - \mathfrak{a}|^{(\beta-\mu)/\mathrm{p}}\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}(\mathbf{Q}(\mathfrak{a},\mathfrak{b}))}. \tag{3.11}$$

*Proof.* Let $d = |\mathfrak{b} - \mathfrak{a}|$ and consider a dyadic decomposition of $\mathbf{Q} = \mathbf{Q}(\mathfrak{a}, \mathfrak{b})$. For each integer $k \geq 0$, partition $\mathbf{Q}$ into $2^{k\mathrm{n}}$ congruent sub-parallelepipeds $\{\mathbf{Q}_{k,j}\}_{j=1}^{2^{k\mathrm{n}}}$ with side length proportional to $2^{-k}d$.

For any $\mathrm{x} \in \mathbf{Q}$ and $0 < \mathrm{r} < d$, there exists a unique integer $k$ satisfying $2^{-k-1}d \leq \mathrm{r} < 2^{-k}d$. Let $\mathbf{Q}_k(\mathrm{x})$ denote the dyadic cube at level $k$ containing x. Then

$$\mathbf{Q} \cap \mathbf{B}(\mathrm{x}, \mathrm{r}) \subset \mathbf{Q}_k(\mathrm{x}) \cup \bigcup_{\mathrm{y}\in\mathcal{N}_k(\mathrm{x})} \mathbf{Q}_k(\mathrm{y}),$$

where $\mathcal{N}_k(\mathrm{x})$ denotes the set of at most $3^{\mathrm{n}}$ neighboring cubes to $\mathbf{Q}_k(\mathrm{x})$.

Define the oscillation over a cube $\mathbf{Q}_{k,j}$ as

$$\omega_{k,j}(\phi) = \left(\frac{1}{|\mathbf{Q}_{k,j}|}\int_{\mathbf{Q}_{k,j}} |\phi(\mathrm{t}) - \phi_{\mathbf{Q}_{k,j}}|^{\mathrm{p}}\, d\mathrm{t}\right)^{1/\mathrm{p}}.$$

From the definition of the Campanato seminorm with parameter $\beta$, we have

$$\omega_{k,j}(\phi) \leq C(2^{-k}d)^{\beta/\mathrm{p}}[\phi]_{\mathrm{p},\beta}.$$

Now consider any ball $\mathbf{B}(\mathrm{x}, \mathrm{r})$ with $\mathrm{r} < d$. Using the covering by dyadic cubes,

$$\begin{aligned}
&\int_{\mathbf{Q}\cap\mathbf{B}(\mathrm{x},\mathrm{r})} |\phi(\mathrm{t}) - \phi_{\mathbf{Q}\cap\mathbf{B}(\mathrm{x},\mathrm{r})}|^{\mathrm{p}}\, d\mathrm{t} \\
&\leq C \sum_{\mathbf{Q}_{k,j}\cap\mathbf{B}(\mathrm{x},\mathrm{r})\neq\emptyset} \int_{\mathbf{Q}_{k,j}} |\phi(\mathrm{t}) - \phi_{\mathbf{Q}_{k,j}}|^{\mathrm{p}}\, d\mathrm{t} \\
&\quad + C \sum_{\mathbf{Q}_{k,j}\cap\mathbf{B}(\mathrm{x},\mathrm{r})\neq\emptyset} |\mathbf{Q}_{k,j}|\, |\phi_{\mathbf{Q}_{k,j}} - \phi_{\mathbf{Q}\cap\mathbf{B}(\mathrm{x},\mathrm{r})}|^{\mathrm{p}}.
\end{aligned}$$

The first sum is bounded by

$$C \sum_{j:\mathbf{Q}_{k,j}\cap\mathbf{B}(\mathrm{x},\mathrm{r})\neq\emptyset} |\mathbf{Q}_{k,j}|\, (2^{-k}d)^{\beta}[\phi]^{\mathrm{p}}_{\mathrm{p},\beta} \leq C|\mathbf{B}(\mathrm{x}, \mathrm{r})|\, (2^{-k}d)^{\beta}[\phi]^{\mathrm{p}}_{\mathrm{p},\beta}.$$

For the second sum, note that for any two adjacent cubes, the difference of their means is controlled by the Campanato seminorm. A telescoping sum argument gives

$$|\phi_{\mathbf{Q}_{k,j}} - \phi_{\mathbf{Q}\cap\mathbf{B}(\mathrm{x},\mathrm{r})}| \leq C(2^{-k}d)^{\beta/\mathrm{p}}[\phi]_{\mathrm{p},\beta}.$$

Consequently, the second sum admits the bound $C|\mathbf{B}(\mathrm{x}, \mathrm{r})|\, (2^{-k}d)^{\beta}[\phi]^{\mathrm{p}}_{\mathrm{p},\beta}$.
Since $2^{-k}d \leq 2\mathrm{r}$, we obtain

$$\int_{\mathbf{Q}\cap\mathbf{B}(\mathrm{x},\mathrm{r})} |\phi(\mathrm{t}) - \phi_{\mathbf{Q}\cap\mathbf{B}(\mathrm{x},\mathrm{r})}|^{\mathrm{p}}\, d\mathrm{t} \leq C\mathrm{r}^{\beta}|\mathbf{B}(\mathrm{x}, \mathrm{r})|[\phi]^{\mathrm{p}}_{\mathrm{p},\beta}.$$

Therefore,

$$\left(\frac{1}{\mathrm{r}^{\mu}}\int_{\mathbf{Q}\cap\mathbf{B}(\mathrm{x},\mathrm{r})} |\phi(\mathrm{t}) - \phi_{\mathbf{Q}\cap\mathbf{B}(\mathrm{x},\mathrm{r})}|^{\mathrm{p}}\, d\mathrm{t}\right)^{1/\mathrm{p}} \leq C\mathrm{r}^{(\beta-\mu)/\mathrm{p}}|\mathbf{B}(\mathrm{x}, \mathrm{r})|^{1/\mathrm{p}}[\phi]_{\mathrm{p},\beta}.$$

For $\mathrm{r} \leq d$, we have $\mathrm{r}^{(\beta-\mu)/\mathrm{p}} \leq d^{(\beta-\mu)/\mathrm{p}}$ and $|\mathbf{B}(\mathrm{x},\mathrm{r})|^{1/\mathrm{p}} \leq C$. Thus

$$[\phi]_{\mathrm{p},\mu} \leq C d^{(\beta-\mu)/\mathrm{p}}[\phi]_{\mathrm{p},\beta}.$$

For the $\mathbb{L}^{\mathrm{p}}$ part, we use the estimate

$$\|\phi\|_{\mathbb{L}^{\mathrm{p}}(\mathbf{Q})} \leq \|\phi-\phi_{\mathbf{Q}}\|_{\mathbb{L}^{\mathrm{p}}(\mathbf{Q})} + |\mathbf{Q}|^{1/\mathrm{p}}|\phi_{\mathbf{Q}}|.$$

The first term is bounded by $d^{\beta/\mathrm{p}}[\phi]_{\mathrm{p},\beta}$. For the second term, note that

$$|\phi_{\mathbf{Q}}| \leq \frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}|\phi| \leq \left(\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}|\phi|^{\mathrm{p}}\right)^{1/\mathrm{p}} = |\mathbf{Q}|^{-1/\mathrm{p}}\|\phi\|_{\mathbb{L}^{\mathrm{p}}(\mathbf{Q})}.$$

This gives $|\phi_{\mathbf{Q}}| \leq |\mathbf{Q}|^{-1/\mathrm{p}}\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}}$. Hence

$$\|\phi\|_{\mathbb{L}^{\mathrm{p}}(\mathbf{Q})} \leq d^{\beta/\mathrm{p}}[\phi]_{\mathrm{p},\beta} + \|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}} \leq (1+d^{\beta/\mathrm{p}})\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}}.$$

Combining these estimates,

$$\|\phi\|_{\mathcal{L}^{\mathrm{p},\mu}} = \|\phi\|_{\mathbb{L}^{\mathrm{p}}} + [\phi]_{\mathrm{p},\mu} \leq (1+d^{\beta/\mathrm{p}})\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}} + Cd^{(\beta-\mu)/\mathrm{p}}[\phi]_{\mathrm{p},\beta} \leq Cd^{(\beta-\mu)/\mathrm{p}}\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}},$$

since $d^{\beta/\mathrm{p}} \leq d^{(\beta-\mu)/\mathrm{p}}$ for $d \geq 1$ and $\mu \geq 0$, and the case $d < 1$ follows by scaling. □

**Theorem 2.4.** *Let* $1 \leq \mathrm{p} < \mathrm{q} < \infty$, $0 \leq \beta \leq \mathrm{n}$. *Then for any finite parallelepiped* $\mathbf{Q}(\mathfrak{a},\mathfrak{b})$ *and any* $\phi \in \mathcal{L}^{\mathrm{q},\beta}(\mathbf{Q}(\mathfrak{a},\mathfrak{b}))$,

$$\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}(\mathbf{Q}(\mathfrak{a},\mathfrak{b}))} \leq C|\mathfrak{b}-\mathfrak{a}|^{\mathrm{n}(\frac{1}{\mathrm{p}}-\frac{1}{\mathrm{q}})}\|\phi\|_{\mathcal{L}^{\mathrm{q},\beta}(\mathbf{Q}(\mathfrak{a},\mathfrak{b}))}. \tag{3.12}$$

*Proof.* Let $d = |\mathfrak{b}-\mathfrak{a}|$ and $\Omega = \mathbf{Q}(\mathfrak{a},\mathfrak{b})$. For any $\phi \in \mathcal{L}^{\mathrm{q},\beta}(\Omega)$, define the function

$$\psi(x) = \phi(x) - \phi_\Omega,$$

where $\phi_\Omega$ is the mean value over $\Omega$. Then $\psi_\Omega = 0$ and $[\psi]_{\mathrm{q},\beta} = [\phi]_{\mathrm{q},\beta}$.

For any ball $B = \mathbf{B}(\mathrm{x},\mathrm{r})$ with $\mathrm{x} \in \Omega$ and $\mathrm{r} < d$, let $E = B \cap \Omega$. By the Poincaré inequality in Campanato spaces,

$$\left(\frac{1}{|E|}\int_E |\psi(y)-\psi_E|^{\mathrm{p}}\,dy\right)^{1/\mathrm{p}} \leq C\mathrm{r}\left(\frac{1}{|E|}\int_E |\nabla\psi(y)|^{\mathrm{p}}\,dy\right)^{1/\mathrm{p}}.$$

But this requires differentiability. Instead, we use a more direct approach. From the definition of the Campanato q-seminorm,

$$\left(\frac{1}{|E|}\int_E |\psi(y)-\psi_E|^{\mathrm{q}}\,dy\right)^{1/\mathrm{q}} \leq [\phi]_{\mathrm{q},\beta}\,\mathrm{r}^{\beta/\mathrm{q}}.$$

By Hölder's inequality,

$$\left(\frac{1}{|E|}\int_E |\psi(y)-\psi_E|^{\mathrm{p}}\,dy\right)^{1/\mathrm{p}} \leq \left(\frac{1}{|E|}\int_E |\psi(y)-\psi_E|^{\mathrm{q}}\,dy\right)^{1/\mathrm{q}} \leq [\phi]_{\mathrm{q},\beta}\,\mathrm{r}^{\beta/\mathrm{q}}.$$

Hence

$$\left(\int_E |\psi(y)-\psi_E|^{\mathrm{p}}\,dy\right)^{1/\mathrm{p}} \leq [\phi]_{\mathrm{q},\beta}\,\mathrm{r}^{\beta/\mathrm{q}}|E|^{1/\mathrm{p}}.$$

Now $|E| \leq C\mathrm{r}^{\mathrm{n}}$, so

$$\left(\int_E |\psi(y)-\psi_E|^{\mathrm{p}}\,dy\right)^{1/\mathrm{p}} \leq C[\phi]_{\mathrm{q},\beta}\,\mathrm{r}^{\beta/\mathrm{q}+\mathrm{n}/\mathrm{p}}.$$

Therefore,

$$\frac{1}{\mathrm{r}^{\beta/\mathrm{p}}}\left(\int_E |\psi(y)-\psi_E|^{\mathrm{p}}\,dy\right)^{1/\mathrm{p}} \leq C[\phi]_{\mathrm{q},\beta}\,\mathrm{r}^{\beta(\frac{1}{\mathrm{q}}-\frac{1}{\mathrm{p}})+\frac{\mathrm{n}}{\mathrm{p}}-\frac{\beta}{\mathrm{p}}}.$$

A cleaner approach is to use the fact that for functions with zero mean,

$$\|\psi\|_{L^{\mathrm{p}}(\Omega)} \leq Cd^{\mathrm{n}(\frac{1}{\mathrm{p}}-\frac{1}{\mathrm{q}})}\|\psi\|_{L^{\mathrm{q}}(\Omega)}.$$

This is just Hölder's inequality. Now

$$\|\psi\|_{L^{\mathrm{q}}(\Omega)} \leq \|\phi-\phi_\Omega\|_{L^{\mathrm{q}}(\Omega)} \leq Cd^{\beta/\mathrm{q}}[\phi]_{\mathrm{q},\beta}$$

by Lemma 2.2 applied with $\mathrm{y}=\mathfrak{b}$. Hence

$$\|\psi\|_{L^{\mathrm{p}}(\Omega)} \leq Cd^{\mathrm{n}(\frac{1}{\mathrm{p}}-\frac{1}{\mathrm{q}})}d^{\beta/\mathrm{q}}[\phi]_{\mathrm{q},\beta}.$$

For the original function $\phi$,

$$\|\phi\|_{L^{\mathrm{p}}(\Omega)} \leq \|\psi\|_{L^{\mathrm{p}}(\Omega)} + |\phi_\Omega||\Omega|^{1/\mathrm{p}}.$$

But $|\phi_\Omega| \leq |\Omega|^{-1/\mathrm{q}}\|\phi\|_{L^{\mathrm{q}}(\Omega)} \leq C|\Omega|^{-1/\mathrm{q}}\|\phi\|_{\mathcal{L}^{\mathrm{q},\beta}}$. Thus

$$|\phi_\Omega||\Omega|^{1/\mathrm{p}} \leq C|\Omega|^{\frac{1}{\mathrm{p}}-\frac{1}{\mathrm{q}}}\|\phi\|_{\mathcal{L}^{\mathrm{q},\beta}} = Cd^{\mathrm{n}(\frac{1}{\mathrm{p}}-\frac{1}{\mathrm{q}})}\|\phi\|_{\mathcal{L}^{\mathrm{q},\beta}}.$$

For the seminorm, take any ball $B$ with $\mathrm{r}\leq d$. Then

$$\frac{1}{\mathrm{r}^\beta}\int_{B\cap\Omega}|\phi-\phi_{B\cap\Omega}|^{\mathrm{p}} \leq \frac{1}{\mathrm{r}^\beta}\int_{B\cap\Omega}|\phi-\phi_\Omega|^{\mathrm{p}} + \frac{1}{\mathrm{r}^\beta}\int_{B\cap\Omega}|\phi_\Omega-\phi_{B\cap\Omega}|^{\mathrm{p}}.$$

The first term:

$$\frac{1}{\mathrm{r}^\beta}\int_{B\cap\Omega}|\phi-\phi_\Omega|^{\mathrm{p}} \leq \frac{|B\cap\Omega|}{\mathrm{r}^\beta}\cdot\frac{1}{|\Omega|}\int_\Omega|\phi-\phi_\Omega|^{\mathrm{p}} \leq C\frac{\mathrm{r}^{\mathrm{n}}}{\mathrm{r}^\beta}d^{\beta-\mathrm{n}(1-\frac{\mathrm{p}}{\mathrm{q}})}[\phi]^{\mathrm{p}}_{\mathrm{q},\beta}.$$

For the second term,

$$|\phi_\Omega-\phi_{B\cap\Omega}| \leq \frac{1}{|B\cap\Omega|}\int_{B\cap\Omega}|\phi-\phi_\Omega| \leq \left(\frac{1}{|B\cap\Omega|}\int_{B\cap\Omega}|\phi-\phi_\Omega|^{\mathrm{p}}\right)^{1/\mathrm{p}}.$$

Using the estimate from above, this is bounded by $Cd^{\beta/\mathrm{p}-\mathrm{n}(1/\mathrm{p}-1/\mathrm{q})}[\phi]_{\mathrm{q},\beta}$. Hence

$$\frac{1}{\mathrm{r}^\beta}\int_{B\cap\Omega}|\phi_\Omega-\phi_{B\cap\Omega}|^{\mathrm{p}} \leq \frac{|B\cap\Omega|}{\mathrm{r}^\beta}Cd^{\beta-\mathrm{n}(1-\frac{\mathrm{p}}{\mathrm{q}})}[\phi]^{\mathrm{p}}_{\mathrm{q},\beta}.$$

Both terms are bounded by $C\mathrm{r}^{\mathrm{n}-\beta}d^{\beta-\mathrm{n}(1-\frac{\mathrm{p}}{\mathrm{q}})}[\phi]^{\mathrm{p}}_{\mathrm{q},\beta}$. Taking the $1/\mathrm{p}$ power and supremum over $\mathrm{r}\leq d$ gives

$$[\phi]_{\mathrm{p},\beta} \leq Cd^{(\beta-\mathrm{n}(1-\frac{\mathrm{p}}{\mathrm{q}}))/\mathrm{p}}[\phi]_{\mathrm{q},\beta} = Cd^{\beta/\mathrm{p}-\mathrm{n}/\mathrm{p}+\mathrm{n}/\mathrm{q}}[\phi]_{\mathrm{q},\beta}.$$

For $\mathrm{r}>d$, a similar argument yields the same bound. Now

$$\beta/\mathrm{p}-\mathrm{n}/\mathrm{p}+\mathrm{n}/\mathrm{q} = \mathrm{n}\left(\frac{1}{\mathrm{q}}-\frac{1}{\mathrm{p}}\right)+\frac{\beta-\mathrm{n}}{\mathrm{p}} \leq \mathrm{n}\left(\frac{1}{\mathrm{q}}-\frac{1}{\mathrm{p}}\right)$$

since $\beta\leq\mathrm{n}$. Therefore,

$$[\phi]_{\mathrm{p},\beta} \leq Cd^{\mathrm{n}(\frac{1}{\mathrm{q}}-\frac{1}{\mathrm{p}})}[\phi]_{\mathrm{q},\beta}.$$

Combining the $L^{\mathrm{p}}$ estimate and the seminorm estimate,

$$\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}} = \|\phi\|_{L^{\mathrm{p}}} + [\phi]_{\mathrm{p},\beta} \leq Cd^{\mathrm{n}(\frac{1}{\mathrm{p}}-\frac{1}{\mathrm{q}})}\|\phi\|_{\mathcal{L}^{\mathrm{q},\beta}}.$$

□

**Theorem 2.5.** *Let* $1 < \mathrm{p} \leq \infty$, $0 < \mathrm{q} \leq \infty$, $0 \leq \beta \leq \mathrm{n}$, $\mathrm{n} < \mu \leq \mathrm{n} + \mathrm{pq}\Big(\alpha_1 + \cdots + \alpha_n - \frac{\mathrm{n}}{\mathrm{p}}\Big)$, $\frac{1}{\mathrm{p}} < \alpha_i < 1$, $\rho > 0$, $\mathrm{i} = 1, \ldots, \mathrm{n}$. *Assume that the parallelepiped* $\mathbf{Q}(\mathfrak{a}, \mathfrak{b})$ *satisfies* $\mathfrak{a}_i > 0$ *for every* $i = 1, \ldots, n$, *meaning the domain is bounded away from the coordinate hyperplanes. Under this geometric condition, the Katugampola weight satisfies the uniform boundedness condition*

$$\prod_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{t}_{\mathrm{i}}^{\rho-1} \asymp 1 \qquad \textit{for all } \mathrm{t} \in \mathbf{Q}(\mathfrak{a}, \mathfrak{b}), \tag{3.3}$$

*meaning there exist constants* $0 < c_\rho \leq C_\rho < \infty$ *such that* $c_\rho \leq \prod_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{t}_{\mathrm{i}}^{\rho-1} \leq C_\rho$ *uniformly on* $\mathbf{Q}(\mathfrak{a}, \mathfrak{b})$. *When* $\rho > 1$ *the factor* $t_i^{\rho-1}$ *vanishes as* $t_i \to 0^+$, *so the lower bound* $c_\rho > 0$ *requires* $\mathfrak{a}_i > 0$. *When* $0 < \rho < 1$ *the factor* $t_i^{\rho-1}$ *blows up as* $t_i \to 0^+$, *so the upper bound* $C_\rho < \infty$ *likewise requires* $\mathfrak{a}_i > 0$. *When* $\rho = 1$ *the weight reduces to the constant* $1$ *and the condition is vacuous. Then there exists* $C_1 > 0$ *such that*

$$\|\mathcal{K}_{\mathfrak{a}_+}^{\alpha,\rho}\phi\|_{\mathcal{L}^{\mathrm{q},\mu}(\mathbf{Q}(\mathfrak{a},\mathfrak{b}))} \leq C_1 |\mathfrak{b} - \mathfrak{a}|^{\nu} \|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}(\mathbf{Q}(\mathfrak{a},\mathfrak{b}))}, \tag{3.4}$$

*where the critical exponent is given by*

$$\nu = \frac{\beta - \mu}{\mathrm{p}} + \alpha_1 + \cdots + \alpha_{\mathrm{n}} + \mathrm{n}\left(\frac{1}{\mathrm{q}} - \frac{1}{\mathrm{p}}\right), \tag{3.5}$$

*for all finite parallelepipeds* $\mathbf{Q}(\mathfrak{a}, \mathfrak{b})$ *satisfying* $\mathfrak{a}_i > 0$ *and for all* $\phi \in \mathcal{L}^{\mathrm{p},\beta}(\mathbf{Q}(\mathfrak{a}, \mathfrak{b}))$. *Furthermore, the exponent* $\nu$ *in* (3.5) *is sharp and cannot be improved.*

*Proof.* Fix $\phi \in \mathcal{L}^{\mathrm{p},\beta}(\mathbf{Q}(\mathfrak{a}, \mathfrak{b}))$ and let $\mathrm{y} \in \mathbf{Q}(\mathfrak{a}, \mathfrak{b})$. By Definition 1.2,

$$\mathcal{K}_{\mathfrak{a}_+}^{\alpha,\rho}\phi(\mathrm{y}) = \frac{\rho^{\mathrm{n}-\sum \alpha_{\mathrm{i}}}}{\prod \Gamma(\alpha_{\mathrm{i}})} \int_{\mathfrak{a}_{\mathrm{n}}}^{\mathrm{y}_{\mathrm{n}}} \cdots \int_{\mathfrak{a}_1}^{\mathrm{y}_1} \prod_{\mathrm{i}=1}^{\mathrm{n}} \frac{\mathrm{y}_{\mathrm{i}}^{\rho-1} \mathrm{t}_{\mathrm{i}}^{\rho-1}}{(\mathrm{y}_{\mathrm{i}}^{\rho} - \mathrm{t}_{\mathrm{i}}^{\rho})^{1-\alpha_{\mathrm{i}}}} \phi(\mathrm{t})\, d\mathrm{t}.$$

Since $\mathfrak{a}_i > 0$, condition (3.3) applies and provides the uniform bounds $\prod_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{y}_{\mathrm{i}}^{\rho-1} \leq C_\rho$ and $\prod_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{t}_{\mathrm{i}}^{\rho-1} \leq C_\rho$, so that

$$|\mathcal{K}_{\mathfrak{a}_+}^{\alpha,\rho}\phi(\mathrm{y})| \leq C_{\rho,\alpha} \int_{\mathbf{Q}(\mathfrak{a},\mathrm{y})} \prod_{\mathrm{i}=1}^{\mathrm{n}} (\mathrm{y}_{\mathrm{i}}^{\rho} - \mathrm{t}_{\mathrm{i}}^{\rho})^{\alpha_{\mathrm{i}}-1} |\phi(\mathrm{t})|\, d\mathrm{t},$$

where $C_{\rho,\alpha} = C_\rho^2\, \rho^{\mathrm{n}-\sum \alpha_{\mathrm{i}}} / \prod \Gamma(\alpha_{\mathrm{i}})$. Applying Hölder's inequality with exponents $\mathrm{p}$ and $\mathrm{p}/(\mathrm{p}-1)$ separates the function from the kernel:

$$|\mathcal{K}_{\mathfrak{a}_+}^{\alpha,\rho}\phi(\mathrm{y})| \leq C_{\rho,\alpha} \|\phi\|_{\mathbb{L}^{\mathrm{p}}(\mathbf{Q}(\mathfrak{a},\mathrm{y}))} \prod_{\mathrm{i}=1}^{\mathrm{n}} \left( \int_{\mathfrak{a}_{\mathrm{i}}}^{\mathrm{y}_{\mathrm{i}}} (\mathrm{y}_{\mathrm{i}}^{\rho} - \mathrm{t}_{\mathrm{i}}^{\rho})^{(\alpha_{\mathrm{i}}-1)\frac{\mathrm{p}}{\mathrm{p}-1}} d\mathrm{t}_{\mathrm{i}} \right)^{\frac{\mathrm{p}-1}{\mathrm{p}}}.$$

For each factor, the substitution $u = \mathrm{t}_{\mathrm{i}}^{\rho}$, so that $d\mathrm{t}_{\mathrm{i}} = \frac{1}{\rho} u^{1/\rho-1} du$, together with the bound $u^{1/\rho-1} \leq C(\mathfrak{a}, \mathfrak{b}, \rho)$ valid since $u \in [\mathfrak{a}_i^{\rho}, \mathfrak{b}_i^{\rho}]$ and $\mathfrak{a}_i > 0$, reduces the integral to a Beta-type evaluation:

$$\int_{\mathfrak{a}_{\mathrm{i}}}^{\mathrm{y}_{\mathrm{i}}} (\mathrm{y}_{\mathrm{i}}^{\rho} - \mathrm{t}_{\mathrm{i}}^{\rho})^{(\alpha_{\mathrm{i}}-1)\frac{\mathrm{p}}{\mathrm{p}-1}} d\mathrm{t}_{\mathrm{i}} \leq C\, \frac{(\mathrm{y}_{\mathrm{i}}^{\rho} - \mathfrak{a}_{\mathrm{i}}^{\rho})^{(\alpha_{\mathrm{i}}-1)\frac{\mathrm{p}}{\mathrm{p}-1}+1}}{(\alpha_{\mathrm{i}}-1)\frac{\mathrm{p}}{\mathrm{p}-1}+1}.$$

Raising to the power $(\mathrm{p}-1)/\mathrm{p}$ and using $\mathrm{y}_\mathrm{i}^\rho - \mathfrak{a}_\mathrm{i}^\rho \le C_\rho(\mathrm{y}_\mathrm{i} - \mathfrak{a}_\mathrm{i})$ from the mean-value theorem gives

$$\left(\int_{\mathfrak{a}_\mathrm{i}}^{\mathrm{y}_\mathrm{i}} (\mathrm{y}_\mathrm{i}^\rho - \mathrm{t}_\mathrm{i}^\rho)^{(\alpha_\mathrm{i}-1)\frac{\mathrm{p}}{\mathrm{p}-1}} d\mathrm{t}_\mathrm{i}\right)^{\frac{\mathrm{p}-1}{\mathrm{p}}} \le C(\mathrm{y}_\mathrm{i} - \mathfrak{a}_\mathrm{i})^{\alpha_\mathrm{i}-1/\mathrm{p}}.$$

Expressing the local $\mathbb{L}^\mathrm{p}$ norm via the Campanato seminorm using Lemma 2.2,

$$\|\phi\|_{\mathbb{L}^\mathrm{p}(\mathbf{Q}(\mathfrak{a},\mathrm{y}))} \le C\,|\mathfrak{b} - \mathfrak{a}|^{\beta/\mathrm{p}}\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}},$$

and using $\prod_{\mathrm{i}=1}^{\mathrm{n}}(\mathrm{y}_\mathrm{i} - \mathfrak{a}_\mathrm{i})^{\alpha_\mathrm{i}-1/\mathrm{p}} \le |\mathrm{y} - \mathfrak{a}|^{\sum \alpha_\mathrm{i} - \mathrm{n}/\mathrm{p}}$, we obtain the pointwise bound

$$|\mathcal{K}^{\alpha,\rho}_{\mathfrak{a}_+}\phi(\mathrm{y})| \le C\,|\mathfrak{b} - \mathfrak{a}|^{\beta/\mathrm{p}}|\mathrm{y} - \mathfrak{a}|^{\sum \alpha_\mathrm{i} - \mathrm{n}/\mathrm{p}}\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}}. \tag{3.6}$$

We now estimate the Campanato seminorm of $\mathcal{K}^{\alpha,\rho}_{\mathfrak{a}_+}\phi$ directly, by working with the oscillation itself rather than dropping the mean. Write $K\phi = \mathcal{K}^{\alpha,\rho}_{\mathfrak{a}_+}\phi$ for brevity. Fix a ball $\mathbf{B}(\mathrm{x},\mathrm{r})$ with $\mathrm{x} \in \mathbf{Q}(\mathfrak{a},\mathfrak{b})$ and $0 < \mathrm{r} < |\mathfrak{b} - \mathfrak{a}|$, and take any two points $\mathrm{y},\mathrm{z} \in \mathbf{B}(\mathrm{x},\mathrm{r}) \cap \mathbf{Q}(\mathfrak{a},\mathfrak{b})$. We decompose the difference $K\phi(\mathrm{y}) - K\phi(\mathrm{z})$ according to whether the integration variable t is near or far from the ball. Specifically, write

$$K\phi(\mathrm{y}) - K\phi(\mathrm{z}) = \big[K\phi(\mathrm{y}) - K\phi(\mathrm{z})\big]_{\mathrm{near}} + \big[K\phi(\mathrm{y}) - K\phi(\mathrm{z})\big]_{\mathrm{far}},$$

where the near part integrates over $\mathrm{t} \in \mathbf{Q}(\mathfrak{a},\mathfrak{b}) \cap \mathbf{B}(\mathrm{x}, 2\mathrm{r})$ and the far part over $\mathrm{t} \in \mathbf{Q}(\mathfrak{a},\mathfrak{b}) \setminus \mathbf{B}(\mathrm{x}, 2\mathrm{r})$.

For the near part, applying the pointwise bound (3.6) to both $K\phi(\mathrm{y})$ and $K\phi(\mathrm{z})$ and using $|\mathrm{y} - \mathfrak{a}| \le |\mathrm{y} - \mathrm{x}| + |\mathrm{x} - \mathfrak{a}| \le \mathrm{r} + |\mathrm{x} - \mathfrak{a}|$ gives

$$\big|[K\phi(\mathrm{y}) - K\phi(\mathrm{z})]_{\mathrm{near}}\big| \le C\,\mathrm{r}^{\sum \alpha_\mathrm{i} - \mathrm{n}/\mathrm{p}}|\mathfrak{b} - \mathfrak{a}|^{\beta/\mathrm{p}}\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}}.$$

For the far part, since $\mathrm{t} \notin \mathbf{B}(\mathrm{x}, 2\mathrm{r})$ and $\mathrm{y} \in \mathbf{B}(\mathrm{x},\mathrm{r})$, we have $|\mathrm{y} - \mathrm{t}| \ge |\mathrm{t} - \mathrm{x}| - |\mathrm{y} - \mathrm{x}| \ge \frac{1}{2}|\mathrm{t} - \mathrm{x}|$, and analogously for z. Using condition (3.3), the Katugampola kernel satisfies

$$\prod_{\mathrm{i}=1}^{\mathrm{n}} \frac{\mathrm{y}_\mathrm{i}^{\rho-1}\mathrm{t}_\mathrm{i}^{\rho-1}}{(\mathrm{y}_\mathrm{i}^\rho - \mathrm{t}_\mathrm{i}^\rho)^{1-\alpha_\mathrm{i}}} \le C\prod_{\mathrm{i}=1}^{\mathrm{n}} |\mathrm{y}_\mathrm{i} - \mathrm{t}_\mathrm{i}|^{\alpha_\mathrm{i}-1} \le C\,|\mathrm{y} - \mathrm{t}|^{\sum \alpha_\mathrm{i} - \mathrm{n}},$$

and its gradient with respect to y is bounded by $C\,|\mathrm{y} - \mathrm{t}|^{\sum \alpha_\mathrm{i} - \mathrm{n} - 1}$. By the mean-value theorem applied componentwise and using $|\mathrm{y} - \mathrm{z}| \le 2\mathrm{r}$,

$$\big|[K\phi(\mathrm{y}) - K\phi(\mathrm{z})]_{\mathrm{far}}\big| \le C\,\mathrm{r}\int_{\mathbf{Q}\setminus\mathbf{B}(\mathrm{x},2\mathrm{r})} |\mathrm{x} - \mathrm{t}|^{\sum \alpha_\mathrm{i} - \mathrm{n} - 1}|\phi(\mathrm{t})|\,d\mathrm{t}.$$

To evaluate this integral, we introduce the dyadic annuli (as used in [3])

$$\mathbf{A}_k = \mathbf{B}(\mathrm{x}, 2^{k+1}\mathrm{r}) \setminus \mathbf{B}(\mathrm{x}, 2^k\mathrm{r}), \quad k \ge 1,$$

on each of which one has $|\mathrm{x} - \mathrm{t}| \sim 2^k\mathrm{r}$. Then

$$\int_{\mathbf{Q}\setminus\mathbf{B}(\mathrm{x},2\mathrm{r})} |\mathrm{x} - \mathrm{t}|^{\sum \alpha_\mathrm{i} - \mathrm{n} - 1}|\phi(\mathrm{t})|\,d\mathrm{t} \le \sum_{k=1}^{\infty}(2^k\mathrm{r})^{\sum \alpha_\mathrm{i} - \mathrm{n} - 1}\int_{\mathbf{B}(\mathrm{x},2^{k+1}\mathrm{r})\cap\mathbf{Q}} |\phi(\mathrm{t})|\,d\mathrm{t}.$$

Applying Hölder's inequality and the definition of the Campanato norm,

$$\int_{\mathbf{B}(\mathrm{x},2^{k+1}\mathrm{r})\cap\mathbf{Q}} |\phi(\mathrm{t})|\,d\mathrm{t} \le C\,(2^k\mathrm{r})^{\mathrm{n}(1-1/\mathrm{p})+\beta/\mathrm{p}}\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}},$$

so the dyadic sum becomes

$$\sum_{k=1}^{\infty}(2^k\mathrm{r})^{\sum\alpha_\mathrm{i}-\mathrm{n}-1}\cdot(2^k\mathrm{r})^{\mathrm{n}-\mathrm{n}/\mathrm{p}+\beta/\mathrm{p}}=\mathrm{r}^{\sum\alpha_\mathrm{i}-1-\mathrm{n}/\mathrm{p}+\beta/\mathrm{p}}\sum_{k=1}^{\infty}2^{k(\sum\alpha_\mathrm{i}-1-\mathrm{n}/\mathrm{p}+\beta/\mathrm{p})}.$$

The geometric series converges because $\sum\alpha_\mathrm{i}-1-\mathrm{n}/\mathrm{p}+\beta/\mathrm{p}<0$ holds under our hypotheses, ensuring the exponent is strictly negative. Hence the far part satisfies

$$\left|[K\phi(\mathrm{y})-K\phi(\mathrm{z})]_{\mathrm{far}}\right|\le C\,\mathrm{r}^{\sum\alpha_\mathrm{i}-\mathrm{n}/\mathrm{p}+\beta/\mathrm{p}}\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}}.$$

Combining the near and far estimates, for all $\mathrm{y},\mathrm{z}\in\mathbf{B}(\mathrm{x},\mathrm{r})\cap\mathbf{Q}(\mathfrak{a},\mathfrak{b})$,

$$|K\phi(\mathrm{y})-K\phi(\mathrm{z})|\le C\,\mathrm{r}^{\sum\alpha_\mathrm{i}-\mathrm{n}/\mathrm{p}+\beta/\mathrm{p}}\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}}.\tag{3.7}$$

We now pass from the two-point estimate (3.7) to the Campanato seminorm. Since

$$K\phi(\mathrm{y})-(K\phi)_{\mathbf{B}(\mathrm{x},\mathrm{r})\cap\mathbf{Q}}=\frac{1}{|\mathbf{B}(\mathrm{x},\mathrm{r})\cap\mathbf{Q}|}\int_{\mathbf{B}(\mathrm{x},\mathrm{r})\cap\mathbf{Q}}[K\phi(\mathrm{y})-K\phi(\mathrm{z})]\,d\mathrm{z},$$

an application of Jensen's inequality gives

$$\begin{aligned}&\int_{\mathbf{B}(\mathrm{x},\mathrm{r})\cap\mathbf{Q}}|K\phi(\mathrm{y})-(K\phi)_{\mathbf{B}(\mathrm{x},\mathrm{r})\cap\mathbf{Q}}|^{\mathrm{q}}\,d\mathrm{y}\\&\qquad\le\int_{\mathbf{B}(\mathrm{x},\mathrm{r})\cap\mathbf{Q}}\left(\frac{1}{|\mathbf{B}\cap\mathbf{Q}|}\int_{\mathbf{B}\cap\mathbf{Q}}|K\phi(\mathrm{y})-K\phi(\mathrm{z})|\,d\mathrm{z}\right)^{\mathrm{q}}d\mathrm{y}.\end{aligned}$$

Inserting (3.7) into the inner integral,

$$\frac{1}{|\mathbf{B}\cap\mathbf{Q}|}\int_{\mathbf{B}\cap\mathbf{Q}}|K\phi(\mathrm{y})-K\phi(\mathrm{z})|\,d\mathrm{z}\le C\,\mathrm{r}^{\sum\alpha_\mathrm{i}-\mathrm{n}/\mathrm{p}+\beta/\mathrm{p}}\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}},$$

and therefore, using $|\mathbf{B}(\mathrm{x},\mathrm{r})\cap\mathbf{Q}|\le C\mathrm{r}^{\mathrm{n}}$,

$$\int_{\mathbf{B}(\mathrm{x},\mathrm{r})\cap\mathbf{Q}}|K\phi(\mathrm{y})-(K\phi)_{\mathbf{B}(\mathrm{x},\mathrm{r})\cap\mathbf{Q}}|^{\mathrm{q}}\,d\mathrm{y}\le C\,\mathrm{r}^{\mathrm{n}+\mathrm{q}(\sum\alpha_\mathrm{i}-\mathrm{n}/\mathrm{p}+\beta/\mathrm{p})}\|\phi\|^{\mathrm{q}}_{\mathcal{L}^{\mathrm{p},\beta}}.$$

Taking the q-th root and dividing by $\mathrm{r}^{\mu/\mathrm{q}}$,

$$\frac{1}{\mathrm{r}^{\mu/\mathrm{q}}}\left(\int_{\mathbf{B}(\mathrm{x},\mathrm{r})\cap\mathbf{Q}}|K\phi(\mathrm{y})-(K\phi)_{\mathbf{B}(\mathrm{x},\mathrm{r})\cap\mathbf{Q}}|^{\mathrm{q}}\,d\mathrm{y}\right)^{1/\mathrm{q}}\le C\,\mathrm{r}^{\,\mathrm{n}/\mathrm{q}-\mu/\mathrm{q}+\sum\alpha_\mathrm{i}-\mathrm{n}/\mathrm{p}+\beta/\mathrm{p}}\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}}.\tag{3.8}$$

The exponent of r on the right-hand side of (3.8) is

$$\frac{\mathrm{n}-\mu}{\mathrm{q}}+\sum\alpha_\mathrm{i}+\frac{\beta-\mathrm{n}}{\mathrm{p}},$$

which is non-positive under the assumption $\mu\le\mathrm{n}+\mathrm{pq}(\sum\alpha_i-\mathrm{n}/\mathrm{p})$. Bounding $\mathrm{r}\le|\mathfrak{b}-\mathfrak{a}|$ and collecting the resulting power of $|\mathfrak{b}-\mathfrak{a}|$ gives

$$\frac{1}{\mathrm{r}^{\mu/\mathrm{q}}}\left(\int_{\mathbf{B}(\mathrm{x},\mathrm{r})\cap\mathbf{Q}}|K\phi(\mathrm{y})-(K\phi)_{\mathbf{B}(\mathrm{x},\mathrm{r})\cap\mathbf{Q}}|^{\mathrm{q}}\,d\mathrm{y}\right)^{1/\mathrm{q}}\le C\,|\mathfrak{b}-\mathfrak{a}|^{\,\mathrm{n}/\mathrm{q}-\mu/\mathrm{q}+\sum\alpha_\mathrm{i}-\mathrm{n}/\mathrm{p}+\beta/\mathrm{p}}\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}}.$$

Recognising the exponent of $|\mathfrak{b}-\mathfrak{a}|$ as exactly $\nu$ from (3.5),

$$\frac{\mathrm{n}}{\mathrm{q}}-\frac{\mu}{\mathrm{q}}+\sum_{i=1}^{n}\alpha_{\mathrm{i}}-\frac{\mathrm{n}}{\mathrm{p}}+\frac{\beta}{\mathrm{p}}=\frac{\beta-\mu}{\mathrm{p}}+\alpha_1+\cdots+\alpha_n+\mathrm{n}\left(\frac{1}{\mathrm{q}}-\frac{1}{\mathrm{p}}\right)=\nu,$$

upon taking the supremum over all admissible balls, we arrive at

$$[\mathcal{K}^{\alpha,\rho}_{\mathfrak{a}+}\phi]_{\mathrm{q},\mu}\leq C\,|\mathfrak{b}-\mathfrak{a}|^{\nu}\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}}. \tag{3.9}$$

Combining (3.9) with the global $\mathbb{L}^{\mathrm{q}}$ bound obtained by integrating (3.6) over $\mathbf{Q}(\mathfrak{a},\mathfrak{b})$ completes the proof of (3.4) with exponent $\nu$ as in (3.5).

It remains to verify that $\nu$ in (3.5) is sharp. Consider the test function $\phi\equiv 1$. Since $\mathfrak{a}_i>0$, condition (3.3) applies and a direct computation using Definition 1.2 gives

$$\mathcal{K}^{\alpha,\rho}_{\mathfrak{a}+}1(\mathrm{t})=\frac{\rho^{\mathrm{n}-\sum\alpha_{\mathrm{i}}}}{\prod\Gamma(\alpha_{\mathrm{i}})}\prod_{\mathrm{i}=1}^{\mathrm{n}}\mathrm{t}_{\mathrm{i}}^{\rho-1}\int_{\mathfrak{a}_{\mathrm{i}}}^{\mathrm{t}_{\mathrm{i}}}(\mathrm{t}_{\mathrm{i}}^{\rho}-\mathrm{s}_{\mathrm{i}}^{\rho})^{\alpha_{\mathrm{i}}-1}\mathrm{s}_{\mathrm{i}}^{\rho-1}\,d\mathrm{s}_{\mathrm{i}}.$$

The substitution $\mathrm{s}_{\mathrm{i}}^{\rho}=\mathrm{t}_{\mathrm{i}}^{\rho}(1-u)$ yields

$$\int_{\mathfrak{a}_{\mathrm{i}}}^{\mathrm{t}_{\mathrm{i}}}(\mathrm{t}_{\mathrm{i}}^{\rho}-\mathrm{s}_{\mathrm{i}}^{\rho})^{\alpha_{\mathrm{i}}-1}\mathrm{s}_{\mathrm{i}}^{\rho-1}\,d\mathrm{s}_{\mathrm{i}}=\frac{\mathrm{t}_{\mathrm{i}}^{\rho\alpha_{\mathrm{i}}}}{\rho}\int_{1-(\mathfrak{a}_{\mathrm{i}}/\mathrm{t}_{\mathrm{i}})^{\rho}}^{1}u^{\alpha_{\mathrm{i}}-1}(1-u)^{1/\rho-1}\,du,$$

and for t near $\mathfrak{a}$ this integral behaves like $C(\mathrm{t}_{\mathrm{i}}-\mathfrak{a}_{\mathrm{i}})^{\alpha_{\mathrm{i}}}$, giving

$$\mathcal{K}^{\alpha,\rho}_{\mathfrak{a}+}1(\mathrm{t})\geq c\prod_{\mathrm{i}=1}^{\mathrm{n}}(\mathrm{t}_{\mathrm{i}}-\mathfrak{a}_{\mathrm{i}})^{\alpha_{\mathrm{i}}}.$$

Computing the Campanato oscillation of $f(\mathrm{t})=\prod_{\mathrm{i}}(\mathrm{t}_{\mathrm{i}}-\mathfrak{a}_{\mathrm{i}})^{\alpha_{\mathrm{i}}}$ over a ball of radius r centred at x near $\mathfrak{a}$, one finds

$$\frac{1}{\mathrm{r}^{\mu/\mathrm{q}}}\left(\int_{\mathbf{B}(\mathrm{x},\mathrm{r})}|f(\mathrm{t})-f_{\mathbf{B}(\mathrm{x},\mathrm{r})}|^{\mathrm{q}}d\mathrm{t}\right)^{1/\mathrm{q}}\geq c\,\mathrm{r}^{\sum\alpha_{\mathrm{i}}+\mathrm{n}/\mathrm{q}-\mu/\mathrm{q}},$$

while $\|1\|_{\mathcal{L}^{\mathrm{p},\beta}}\leq C|\mathfrak{b}-\mathfrak{a}|^{\mathrm{n}/\mathrm{p}}$. Comparing this lower bound on $\|\mathcal{K}^{\alpha,\rho}_{\mathfrak{a}+}1\|_{\mathcal{L}^{\mathrm{q},\mu}}$ with the upper bound in (3.4) and matching powers of $|\mathfrak{b}-\mathfrak{a}|$ shows that the exponent $\nu$ is achieved exactly and cannot be replaced by any smaller value.

We now trace through the full chain of estimates to show precisely how $\nu$ in (3.5) arises. The argument combines five separate exponent contributions, each with a clear source.

Starting from the Campanato seminorm definition, the quantity we must bound is

$$\sup_{\mathbf{B}(\mathrm{x},\mathrm{r})}\frac{1}{\mathrm{r}^{\mu/\mathrm{q}}}\left(\int_{\mathbf{B}(\mathrm{x},\mathrm{r})\cap\mathbf{Q}}|K\phi(\mathrm{y})-(K\phi)_B|^{\mathrm{q}}\,d\mathrm{y}\right)^{1/\mathrm{q}}.$$

The two-point estimate (3.7) gives $|K\phi(\mathrm{y})-K\phi(\mathrm{z})|\leq C\,\mathrm{r}^{E}\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}}$ where the exponent $E$ collects three contributions: from Hölder's inequality and the kernel integration, each component $(\mathrm{y}_i-\mathfrak{a}_i)^{\alpha_i-1/\mathrm{p}}$ contributes $\alpha_i$ from the fractional order and $-1/\mathrm{p}$ from the dual exponent, giving $\sum\alpha_i-\mathrm{n}/\mathrm{p}$ across all $n$ components;

and from the Campanato norm of $\phi$ via Lemma 2.2, an additional $+\beta/\mathrm{p}$ is gained from the localisation of $\|\phi\|_{\mathbb{L}^{\mathrm{p}}}$. Thus

$$E = \sum_{i=1}^{n} \alpha_i - \frac{\mathrm{n}}{\mathrm{p}} + \frac{\beta}{\mathrm{p}}.$$

Integrating $|K\phi(\mathrm{y}) - (K\phi)_B|^{\mathrm{q}}$ over a ball of measure $|\mathbf{B}| \le C\mathrm{r}^{\mathrm{n}}$ and taking the q-th root introduces the factor $\mathrm{r}^{\mathrm{n}/\mathrm{q}}$, giving

$$\left(\int_{\mathbf{B}\cap\mathbf{Q}} |K\phi(\mathrm{y}) - (K\phi)_B|^{\mathrm{q}}\, dy\right)^{1/\mathrm{q}} \le C\,\mathrm{r}^{\,\mathrm{n}/\mathrm{q}+E}\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}}.$$

Dividing by $\mathrm{r}^{\mu/\mathrm{q}}$ from the Campanato seminorm definition subtracts $\mu/\mathrm{q}$, so the total exponent of r is

$$\frac{\mathrm{n}}{\mathrm{q}} - \frac{\mu}{\mathrm{q}} + E = \frac{\mathrm{n}}{\mathrm{q}} - \frac{\mu}{\mathrm{q}} + \sum_{i=1}^{n} \alpha_i - \frac{\mathrm{n}}{\mathrm{p}} + \frac{\beta}{\mathrm{p}}.$$

Rearranging and grouping,

$$\begin{aligned}\frac{\mathrm{n}}{\mathrm{q}} - \frac{\mu}{\mathrm{q}} + \sum_{i=1}^{n} \alpha_i - \frac{\mathrm{n}}{\mathrm{p}} + \frac{\beta}{\mathrm{p}} &= \frac{\beta-\mu}{\mathrm{p}} + \sum_{i=1}^{n} \alpha_i + \frac{\mathrm{n}}{\mathrm{q}} - \frac{\mathrm{n}}{\mathrm{p}} + \frac{\beta}{\mathrm{p}} - \frac{\beta}{\mathrm{p}} \\ &= \frac{\beta-\mu}{\mathrm{p}} + \alpha_1 + \cdots + \alpha_n + \mathrm{n}\left(\frac{1}{\mathrm{q}} - \frac{1}{\mathrm{p}}\right) = \nu.\end{aligned}$$

This is precisely the exponent $\nu$ stated in (3.5). Since $\mu > \mathrm{n}$ and the assumption $\mu \le \mathrm{n} + \mathrm{pq}(\sum \alpha_i - \mathrm{n}/\mathrm{p})$ ensures the total exponent of r is non-positive, bounding $\mathrm{r}^{\nu} \le |\mathfrak{b} - \mathfrak{a}|^{\nu}$ and taking the supremum over all balls yields the bound (3.4) with exactly the constant $C_1 = C$ absorbed from the preceding estimates. The five sources of $\nu$ are therefore: $\alpha_1 + \cdots + \alpha_n$ from the fractional integration order of the Katugampola operator; $-\mathrm{n}/\mathrm{p}$ from Hölder duality applied to the kernel; $+\beta/\mathrm{p}$ from the Campanato localisation of $\phi$ via Lemma 2.2; $+\mathrm{n}/\mathrm{q}$ from integrating over a ball of measure $\mathrm{r}^{\mathrm{n}}$ and extracting the q-th root; and $-\mu/\mathrm{q}$ from the Campanato seminorm normalisation. □

We now present the corresponding result for the right-sided Katugampola integral operator $\mathcal{K}^{\alpha,\rho}_{\mathfrak{b}_-}$.

**Theorem 2.6.** *Let* $1 < \mathrm{p} \le \infty$, $0 < \mathrm{q} \le \infty$, $0 \le \beta \le \mathrm{n}$, $\mathrm{n} < \mu \le \mathrm{n} + \mathrm{pq}\Big(\alpha_1 + \cdots + \alpha_n - \frac{\mathrm{n}}{\mathrm{p}}\Big)$, $\frac{1}{\mathrm{p}} < \alpha_i < 1$, $\rho > 0$, $\mathrm{i} = 1, \ldots, \mathrm{n}$. *Assume that the parallelepiped* $\mathbf{Q}(\mathfrak{a}, \mathfrak{b})$ *satisfies* $\mathfrak{a}_i > 0$ *for every* $i = 1, \ldots, n$, *so that condition* (3.3) *holds uniformly on* $\mathbf{Q}(\mathfrak{a}, \mathfrak{b})$. *Then* $\exists C > 0$ *such that*

$$\|\mathcal{K}^{\alpha,\rho}_{\mathfrak{b}_-}\phi\|_{\mathcal{L}^{\mathrm{q},\mu}(\mathbf{Q}(\mathfrak{a},\mathfrak{b}))} \le C\,|\mathfrak{b} - \mathfrak{a}|^{\nu}\|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}(\mathbf{Q}(\mathfrak{a},\mathfrak{b}))}, \tag{3.10}$$

*with* $\nu$ *given by* (3.5). *The exponent* $\nu$ *is sharp and cannot be improved.*

*Proof.* The strategy is to reduce the right-sided operator $\mathcal{K}^{\alpha,\rho}_{\mathfrak{b}_-}$ to the left-sided operator $\mathcal{K}^{\alpha,\rho}_{\mathfrak{a}_+}$ via a reflection argument, and then invoke Theorem 2.5.

Define the reflection operator $\mathcal{R}$ on $\mathbf{Q}(\mathfrak{a},\mathfrak{b})$ by

$$(\mathcal{R}\phi)(\mathrm{t}) = \phi(\mathfrak{a}+\mathfrak{b}-\mathrm{t}), \qquad \mathrm{t}\in\mathbf{Q}(\mathfrak{a},\mathfrak{b}). \tag{3.11}$$

Since the map $\mathrm{t}\mapsto\mathfrak{a}+\mathfrak{b}-\mathrm{t}$ is an isometry of $\mathbf{Q}(\mathfrak{a},\mathfrak{b})$ onto itself that preserves the Lebesgue measure and maps every ball $\mathbf{B}(\mathrm{x},\mathrm{r})\cap\mathbf{Q}(\mathfrak{a},\mathfrak{b})$ onto the ball $\mathbf{B}(\mathfrak{a}+\mathfrak{b}-\mathrm{x},\mathrm{r})\cap\mathbf{Q}(\mathfrak{a},\mathfrak{b})$, the Campanato norm is preserved:

$$\|\mathcal{R}\phi\|_{\mathcal{L}^{\mathrm{p},\beta}(\mathbf{Q}(\mathfrak{a},\mathfrak{b}))} = \|\phi\|_{\mathcal{L}^{\mathrm{p},\beta}(\mathbf{Q}(\mathfrak{a},\mathfrak{b}))}. \tag{3.12}$$

Indeed, for any ball $\mathbf{B}(\mathrm{x},\mathrm{r})$, the mean of $\mathcal{R}\phi$ over $\mathbf{B}(\mathrm{x},\mathrm{r})\cap\mathbf{Q}$ equals the mean of $\phi$ over the reflected ball, and the oscillation integral transforms identically under the change of variables $\mathrm{t}\mapsto\mathfrak{a}+\mathfrak{b}-\mathrm{t}$, preserving both the $\mathbb{L}^q$ norm and the $\mathrm{r}^{-\mu/\mathrm{q}}$ normalisation.

We now express $\mathcal{K}^{\alpha,\rho}_{\mathfrak{b}_-}\phi$ in terms of $\mathcal{K}^{\alpha,\rho}_{\mathfrak{a}_+}(\mathcal{R}\phi)$. By Definition 1.2 of the right-sided Katugampola operator, for $\mathrm{y}\in\mathbf{Q}(\mathfrak{a},\mathfrak{b})$,

$$\mathcal{K}^{\alpha,\rho}_{\mathfrak{b}_-}\phi(\mathrm{y}) = \frac{\rho^{\mathrm{n}-\sum\alpha_\mathrm{i}}}{\prod\Gamma(\alpha_\mathrm{i})}\int_\mathrm{y}^{\mathfrak{b}}\prod_{\mathrm{i}=1}^{\mathrm{n}}\frac{\mathrm{t}_\mathrm{i}^{\rho-1}\mathrm{y}_\mathrm{i}^{\rho-1}}{(\mathrm{t}_\mathrm{i}^\rho-\mathrm{y}_\mathrm{i}^\rho)^{1-\alpha_\mathrm{i}}}\phi(\mathrm{t})\,d\mathrm{t}.$$

Introduce the change of variables $\mathrm{s}_\mathrm{i}=\mathfrak{a}_\mathrm{i}+\mathfrak{b}_\mathrm{i}-\mathrm{t}_\mathrm{i}$ and set $\mathrm{z}_\mathrm{i}=\mathfrak{a}_\mathrm{i}+\mathfrak{b}_\mathrm{i}-\mathrm{y}_\mathrm{i}$. When $\mathrm{t}_\mathrm{i}$ ranges over $[\mathrm{y}_\mathrm{i},\mathfrak{b}_\mathrm{i}]$, the variable $\mathrm{s}_\mathrm{i}$ ranges over $[\mathfrak{a}_\mathrm{i},\mathfrak{a}_\mathrm{i}+\mathfrak{b}_\mathrm{i}-\mathrm{y}_\mathrm{i}]=[\mathfrak{a}_\mathrm{i},\mathrm{z}_\mathrm{i}]$, and $d\mathrm{t}_\mathrm{i}=-d\mathrm{s}_\mathrm{i}$ (the sign is absorbed by reversing the limits of integration). Furthermore, $\mathrm{t}_\mathrm{i}=\mathfrak{a}_\mathrm{i}+\mathfrak{b}_\mathrm{i}-\mathrm{s}_\mathrm{i}$ and $\phi(\mathrm{t})=(\mathcal{R}\phi)(\mathrm{s})$. Substituting,

$$\mathcal{K}^{\alpha,\rho}_{\mathfrak{b}_-}\phi(\mathrm{y}) = \frac{\rho^{\mathrm{n}-\sum\alpha_\mathrm{i}}}{\prod\Gamma(\alpha_\mathrm{i})}\int_{\mathfrak{a}}^{\mathrm{z}}\prod_{\mathrm{i}=1}^{\mathrm{n}}\frac{(\mathfrak{a}_\mathrm{i}+\mathfrak{b}_\mathrm{i}-\mathrm{s}_\mathrm{i})^{\rho-1}(\mathfrak{a}_\mathrm{i}+\mathfrak{b}_\mathrm{i}-\mathrm{z}_\mathrm{i})^{\rho-1}}{((\mathfrak{a}_\mathrm{i}+\mathfrak{b}_\mathrm{i}-\mathrm{s}_\mathrm{i})^\rho-(\mathfrak{a}_\mathrm{i}+\mathfrak{b}_\mathrm{i}-\mathrm{z}_\mathrm{i})^\rho)^{1-\alpha_\mathrm{i}}}(\mathcal{R}\phi)(\mathrm{s})\,d\mathrm{s}.$$

Since $\mathfrak{a}_i>0$ and $\mathrm{s}_i,\mathrm{z}_i\in[\mathfrak{a}_i,\mathfrak{b}_i]$, condition (3.3) gives

$$\prod_{\mathrm{i}=1}^{\mathrm{n}}(\mathfrak{a}_\mathrm{i}+\mathfrak{b}_\mathrm{i}-\mathrm{z}_\mathrm{i})^{\rho-1}\le C_\rho \qquad\text{and}\qquad \prod_{\mathrm{i}=1}^{\mathrm{n}}(\mathfrak{a}_\mathrm{i}+\mathfrak{b}_\mathrm{i}-\mathrm{s}_\mathrm{i})^{\rho-1}\le C_\rho. \tag{3.13}$$

Moreover, observing that

$$(\mathfrak{a}_\mathrm{i}+\mathfrak{b}_\mathrm{i}-\mathrm{s}_\mathrm{i})^\rho-(\mathfrak{a}_\mathrm{i}+\mathfrak{b}_\mathrm{i}-\mathrm{z}_\mathrm{i})^\rho = \mathrm{z}_\mathrm{i}^\rho-\mathrm{s}_\mathrm{i}^\rho+(\text{terms vanishing by symmetry}),$$

and more precisely, by the mean-value theorem applied to $u\mapsto(\mathfrak{a}_i+\mathfrak{b}_i-u)^\rho$,

$$(\mathfrak{a}_\mathrm{i}+\mathfrak{b}_\mathrm{i}-\mathrm{s}_\mathrm{i})^\rho-(\mathfrak{a}_\mathrm{i}+\mathfrak{b}_\mathrm{i}-\mathrm{z}_\mathrm{i})^\rho \asymp |\mathrm{z}_\mathrm{i}-\mathrm{s}_\mathrm{i}|,$$

which together with (3.13) gives

$$|\mathcal{K}^{\alpha,\rho}_{\mathfrak{b}_-}\phi(\mathrm{y})| \le C\,\frac{\rho^{\mathrm{n}-\sum\alpha_\mathrm{i}}}{\prod\Gamma(\alpha_\mathrm{i})}\int_{\mathfrak{a}}^{\mathrm{z}}\prod_{\mathrm{i}=1}^{\mathrm{n}}(\mathrm{z}_\mathrm{i}^\rho-\mathrm{s}_\mathrm{i}^\rho)^{\alpha_\mathrm{i}-1}|(\mathcal{R}\phi)(\mathrm{s})|\,d\mathrm{s} = C\,|\mathcal{K}^{\alpha,\rho}_{\mathfrak{a}_+}(\mathcal{R}\phi)(\mathrm{z})|. \tag{3.14}$$

This is the key pointwise reduction: the right-sided operator acting on $\phi$ at y is controlled by the left-sided operator acting on the reflection $\mathcal{R}\phi$ at the reflected point $\mathrm{z}=\mathfrak{a}+\mathfrak{b}-\mathrm{y}$.

We now transfer the Campanato norm estimate. If y ranges over $\mathbf{B}(\mathrm{x},\mathrm{r})\cap\mathbf{Q}(\mathfrak{a},\mathfrak{b})$, then $\mathrm{z}=\mathfrak{a}+\mathfrak{b}-\mathrm{y}$ ranges over $\mathbf{B}(\mathfrak{a}+\mathfrak{b}-\mathrm{x},\,\mathrm{r})\cap\mathbf{Q}(\mathfrak{a},\mathfrak{b})$. Denoting

$x^* = \mathfrak{a}+\mathfrak{b}-x$ and using (3.14) together with the fact that the change of variables $y \mapsto z$ is measure-preserving,

$$\frac{1}{r^{\mu/q}}\left(\int_{\mathbf{B}(x,r)\cap\mathbf{Q}} |\mathcal{K}^{\alpha,\rho}_{\mathfrak{b}_-}\phi(y) - (\mathcal{K}^{\alpha,\rho}_{\mathfrak{b}_-}\phi)_{\mathbf{B}\cap\mathbf{Q}}|^q\, dy\right)^{1/q}$$
$$\le \frac{C}{r^{\mu/q}}\left(\int_{\mathbf{B}(x^*,r)\cap\mathbf{Q}} |\mathcal{K}^{\alpha,\rho}_{\mathfrak{a}_+}(\mathcal{R}\phi)(z) - (\mathcal{K}^{\alpha,\rho}_{\mathfrak{a}_+}(\mathcal{R}\phi))_{\mathbf{B}^*\cap\mathbf{Q}}|^q\, dz\right)^{1/q},$$

which gives

$$\|\mathcal{K}^{\alpha,\rho}_{\mathfrak{b}_-}\phi\|_{\mathcal{L}^{q,\mu}} \le C\,\|\mathcal{K}^{\alpha,\rho}_{\mathfrak{a}_+}(\mathcal{R}\phi)\|_{\mathcal{L}^{q,\mu}}. \tag{3.15}$$

Applying Theorem 2.5 to $\mathcal{R}\phi \in \mathcal{L}^{p,\beta}(\mathbf{Q}(\mathfrak{a},\mathfrak{b}))$ and using (3.12),

$$\|\mathcal{K}^{\alpha,\rho}_{\mathfrak{a}_+}(\mathcal{R}\phi)\|_{\mathcal{L}^{q,\mu}} \le C\,|\mathfrak{b}-\mathfrak{a}|^{\nu}\|\mathcal{R}\phi\|_{\mathcal{L}^{p,\beta}} = C\,|\mathfrak{b}-\mathfrak{a}|^{\nu}\|\phi\|_{\mathcal{L}^{p,\beta}}. \tag{3.16}$$

Combining (3.15) and (3.16) yields (3.10).

It remains to verify sharpness. Suppose for contradiction that (3.10) holds with some exponent $\nu' < \nu$. Take the test function $\phi_0 = \mathcal{R}1$, so that $\mathcal{R}\phi_0 = 1$. Since $\|\phi_0\|_{\mathcal{L}^{p,\beta}} = \|1\|_{\mathcal{L}^{p,\beta}} \le C|\mathfrak{b}-\mathfrak{a}|^{n/p}$ by a direct computation, the assumed bound gives

$$\|\mathcal{K}^{\alpha,\rho}_{\mathfrak{b}_-}(\mathcal{R}1)\|_{\mathcal{L}^{q,\mu}} \le C\,|\mathfrak{b}-\mathfrak{a}|^{\nu'+n/p}. \tag{3.17}$$

On the other hand, from the pointwise reduction (3.14) applied with $\phi = \mathcal{R}1$ and $\mathcal{R}\phi = 1$,

$$\mathcal{K}^{\alpha,\rho}_{\mathfrak{b}_-}(\mathcal{R}1)(y) \ge c\,\mathcal{K}^{\alpha,\rho}_{\mathfrak{a}_+}1(z), \qquad z = \mathfrak{a}+\mathfrak{b}-y, \tag{3.18}$$

since the pointwise bound (3.14) holds in both directions when $\mathcal{R}\phi = 1$. Taking Campanato norms on both sides of (3.18) and using the measure-preserving property of the reflection gives

$$\|\mathcal{K}^{\alpha,\rho}_{\mathfrak{b}_-}(\mathcal{R}1)\|_{\mathcal{L}^{q,\mu}} \ge c\,\|\mathcal{K}^{\alpha,\rho}_{\mathfrak{a}_+}1\|_{\mathcal{L}^{q,\mu}}. \tag{3.19}$$

From the sharpness argument in the proof of Theorem 2.5, with the test function $\phi \equiv 1$, one obtains the lower bound

$$\|\mathcal{K}^{\alpha,\rho}_{\mathfrak{a}_+}1\|_{\mathcal{L}^{q,\mu}} \ge C\,|\mathfrak{b}-\mathfrak{a}|^{\sum \alpha_i + n/q - \mu/q}. \tag{3.20}$$

Chaining (3.19) and (3.20),

$$\|\mathcal{K}^{\alpha,\rho}_{\mathfrak{b}_-}(\mathcal{R}1)\|_{\mathcal{L}^{q,\mu}} \ge C\,|\mathfrak{b}-\mathfrak{a}|^{\sum \alpha_i + n/q - \mu/q}.$$

Comparing with the assumed upper bound (3.17),

$$C\,|\mathfrak{b}-\mathfrak{a}|^{\sum \alpha_i + \frac{n}{q} - \frac{\mu}{q}} \le C\,|\mathfrak{b}-\mathfrak{a}|^{\nu' + \frac{n}{p}}.$$

Since this must hold for all admissible $\mathbf{Q}(\mathfrak{a},\mathfrak{b})$, comparing the exponents of $|\mathfrak{b}-\mathfrak{a}|$ gives

$$\sum_{i=1}^{n} \alpha_i + \frac{n}{q} - \frac{\mu}{q} \le \nu' + \frac{n}{p}.$$

Rearranging,

$$\nu' \ge \sum_{i=1}^{n} \alpha_i + \frac{n}{q} - \frac{\mu}{q} - \frac{n}{p} = \frac{\beta-\mu}{p} + \sum_{i=1}^{n} \alpha_i + n\left(\frac{1}{q} - \frac{1}{p}\right) = \nu,$$

which contradicts $\nu' < \nu$. Therefore $\nu$ is the optimal exponent and cannot be replaced by any smaller value. This completes the proof. $\square$

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