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CONFORMAL MAPPINGS OF GENERALIZED QUASI-EINSTEIN MANIFOLDS ADMITTING SPECIAL VECTOR FIELDS

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ABSTRACT. Einstein manifolds form a natural subclass of the class of quasi-Einstein manifolds and plays an important role in geometry as well as in general theory of relativity. In this work, we investigate conformal mappings of generalized quasi-Einstein manifolds. We consider a conformal mapping between two generalized quasi-Einstein manifolds V_n and $\bar{V}_n$. We also find some properties of this transformation from V_n to $\bar{V}_n$ and some theorems are proved. Considering this mapping, we examine some properties of these manifolds. After that, we also study some special vector fields under this mapping on these manifolds and some theorems about them are proved.

1. INTRODUCTION AND PRELIMINARIES

A geodesic circle in a Riemannian manifold was defined in [8] as a curve whose first curvature is constant and second curvature vanishes identically. Circles and spheres in Riemannian geometry are defined and studied from the point of view of development by K. Nomizu and K. Yano in [16].

The notion of quasi-Einstein manifold was introduced by M. C. Chaki and R. K. Maity [4]. A non-flat Riemannian manifold (M^n, g), ($n \geq 3$) is a *quasi-Einstein manifold* if its Ricci tensor S satisfies the condition

$$S(X,Y) = ag(X,Y) + b\phi(X)\phi(Y) \tag{1.1}$$

and is not identically zero, where a, b are scalars, $b \neq 0$ and ϕ is a non-zero 1-form such that

$$g(X,U) = \phi(X), \forall\ X \in \chi(M), \tag{1.2}$$

U being a unit vector field.

Here a and b are called the *associated scalars*, ϕ is called the associated 1-form and U is called the *generator* of the manifold. Such an n-dimensional manifold denoted by $(QE)_n$.

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As a generalization of quasi-Einstein manifold in [6], U. C. De and G. C. Ghosh defined the *generalized quasi-Einstein manifold*. A non-flat Riemannian manifold is called *generalized quasi-Einstein manifold* if its Ricci-tensor is non-zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + b\phi(X)\phi(Y) + c\psi(X)\psi(Y), \tag{1.3}$$

where a, b and c are non-zero scalars and ϕ, ψ are two 1-forms such that

$$g(X,U) = \phi(X) \text{ and } g(X,V) = \psi(X), \tag{1.4}$$

U and V being unit vectors which are orthogonal, i.e.,

$$g(U,V) = 0. \tag{1.5}$$

The vector fields U and V are called the generators of the manifold. This type of manifold will be denoted by $G(QE)_n$.

Putting $X = Y = e_i$ in (1.3), we get

$$r = na + b + c. \tag{1.6}$$

Here r is the scalar curvature of $G(QE)_n$ where $\{e_i\}$, $i = 1, 2, ..., n$ is an orthonormal basis of the tangent space at each point of the manifold.

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi Euclidean spaces. For instance, the Robertson-Walker spacetimes are quasi-Einstein manifolds. So quasi-Einstein manifolds have some importance in the general theory of relativity.

One of the important concepts of Riemannian Geometry is conformal mapping. Conformal mappings of Riemannian manifolds (or semi-Riemannian manifolds) have been investigated by many authors. In general relativity, conformal mappings are important since they preserve the causal structure up to time orientation and light-like geodesics up to parametrization [12]. The existence of conformal mappings of Riemannian manifolds onto Einstein manifolds have been studied by Brinkmann [3], Mikeš, Gavrilchenko, Gladysheva [15] and others. Also, conformal mappings between two Einstein manifolds have been examined by Brinkmann. What is more, the problem of finding the invariants under a particular type of mapping is an important and active research topic. In particular, Gover and Nurowski [9] obtained the polynomial conformal invariants, the vanishing of which is a necessary and sufficient for an n-dimensional suitably generic (pseudo-) Riemannian manifold to be conformal to an Einstein manifold, and some of the invariants have certain practical significance in physics, such as quantum field theory [2], general relativity [1].

Motivated by the above studies the present paper deals with the study of conformal mappings on $G(QE)_n$ admitting special vectors Fields. In the second section, we study conformal mappings of two generalized quasi-Einstein manifolds V_n and $\bar{V}_n$. We also find some properties of this transformation from V_n

to $\bar{V}_n$ and some theorems are proved. Third section deals with conformal mappings on $G(QE)_n$ admitting special vector fields and the final section we give an example of $G(QE)_n$.

2. Conformal Mappings of two Generalized Quasi-Einstein Manifolds

In this section, we suppose that V_n and $\bar{V}_n$, $(n \geq 3)$ are two Generalized quasi-Einstein manifolds with metrics g and $\bar{g}$, respectively.

Definition 2.1. A conformal mapping is a diffeomorphism of V_n onto $\bar{V}_n$ such that

$$\bar{g} = e^{2\sigma} g, \tag{2.1}$$

where σ is a function on V_n. If σ is constant, then it is called a homothetic mapping.
In local coordinates, (2.1) is written as

$$\bar{g}_{ij}(x) = e^{2\sigma}(x) g_{ij}(x),\ \bar{g}^{ij}(x) = e^{2\sigma}(x) g^{ij}(x), \tag{2.2}$$

Besides those equations, we have the Christoffel symbols, the components of the curvature tensor, the Ricci tensor, and the scalar curvature, respectively

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_i \sigma_j + \delta^h_j \sigma_i - \sigma^h g_{ij}, \tag{2.3}$$

$$\bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \sigma_{ij} - \delta^h_j \sigma_{ik} + g^{h\alpha}(\sigma_{\alpha k} g_{ij} - \sigma_{\alpha j} g_{ik} + \triangle_1 \sigma(\delta^h_k g_{ij} - \delta^h_j g_{ik}), \tag{2.4}$$

$$\bar{S}_{ij} = S_{ij} + (n-2)\sigma_{ij} + (\triangle_2 \sigma + (n-2)\triangle_1 \sigma) g_{ij}, \tag{2.5}$$

$$\bar{r} = e^{-2\sigma}(r + 2(n-1)\triangle_2 \sigma + (n-1)(n-2)\triangle_1 \sigma), \tag{2.6}$$

where

$$S_{ij} = R^\alpha_{ij\alpha},\ r = S_{\alpha\beta} g^{\alpha\beta},\ \sigma_i = \frac{\partial \sigma}{\partial x^i} = \nabla_i \sigma,\ \sigma^h = \sigma_\alpha g^{\alpha h} \tag{2.7}$$

and

$$\sigma_{ij} = \nabla_j \nabla_i \sigma - \nabla_i \sigma \nabla_j \sigma \tag{2.8}$$

$\nabla_1 \sigma$ and $\nabla_2 \sigma$ are the first and the second Beltramis symbols which are determined by

$$\triangle_1 \sigma = g^{\alpha\beta} \nabla_\alpha \sigma \nabla_\beta \sigma,\ \triangle_2 \sigma = g^{\alpha\beta} \nabla_\beta \nabla_\alpha \sigma, \tag{2.9}$$

where ∇ is the covariant derivative according to the Riemannian connection in V_n. We denote the objects of space conformally corresponding to V_n by a bar, i.e., $\bar{V}_n$. If V_n is a $G(QE)_n$, then we have, from from (1.3), (2.2), and (2.5),

$$\begin{aligned} \bar{b}\bar{\phi}_i \bar{\phi}_j + \bar{c}\bar{\psi}_i \bar{\psi}_j &= b\phi_i \phi_j + c\psi_i \psi_j + (n-2)\sigma_{ij} + (\triangle_2 \sigma + (n-2)\triangle_1 \sigma \\ &+ a - \bar{a} e^{2\sigma}) g_{ij} \end{aligned} \tag{2.10}$$

Definition 2.2. A vector field ξ in a Riemannian manifold M is called torse-forming if it satisfies the condition

$$\nabla_X \xi = \rho X + \lambda(X)\xi,$$

where $\xi \in \chi(M)$, $\lambda(X)$ is a linear form and ρ is a function, [18] In the local transcription, this reads

$$\nabla_i \xi^h = \rho \delta_i^h + \xi^h \lambda_i, \tag{2.11}$$

ξ^h and λ_i are the components of ξ and ϕ, δ_i^h is the Kronecker symbol. A torse-forming vector field ξ is called recurrent if $\rho = 0$; concircular if the form λ_i is a gradient covector, i.e., there is a function $\vartheta(x)$ such that $\lambda = d\vartheta(x)$; convergent, if it is concircular and $\rho = const. \exp(\vartheta)$.

Therefore, recurrent vector fields are characterized by the following equation from (2.11)

$$\nabla_i \xi_j = \lambda_i \xi_j. \tag{2.12}$$

Also, from the Definition 2.2., for a concircular vector field ξ, we get

$$\nabla_i \xi_j = \rho_i g_{ij} \tag{2.13}$$

for all $X, Y \in \chi(M)$. A Riemannian space with a concircular vector field is called equidistant, [17, 18]. Conformal mappings of Riemannian spaces (or semi-Riemannian spaces) have been studied by many authors, [3, 5, 7, 15]. In this section, we investigate the conformal mappings of generalized quasi-Einstein manifolds preserving the associated 1-forms $\phi(X)$ and $\psi(X)$.

Theorem 2.3. *If V_n admits a conformal mapping preserving the associated 1-forms $\phi(X)$ and $\psi(X)$ and the associated scalars b and c, then V_n is an equidistant manifold.*

Proof. Suppose that V_n admits a conformal mapping preserving the associated 1-forms $\phi(X)$ and $\psi(X)$ and the associated scalars b and c. Using (2.10), we obtain

$$(n-2)\sigma_{ij} + (\beta + a - \bar{a}e^{2\sigma})g_{ij} = 0, \tag{2.14}$$

where

$$\beta = \triangle_2 \sigma + (n-2)\triangle_1 \sigma + a - \bar{a}e^{2\sigma}.$$

In this case, we get

$$\sigma_{ij} = \alpha g_{ij}, \tag{2.15}$$

where

$$\alpha = \frac{1}{n-2}(\bar{a}e^{2\sigma} - a - \beta)$$

is a function. Putting $\xi = -\exp(-\sigma)$ and using (2.7), (2.8), (2.13) and (2.15), we get that V_n is an equidistant manifold. Hence, the proof is complete.

Theorem 2.4. *An equidistant manifold V_n admits a conformal mapping preserving the associated 1-forms $\phi(X)$ and $\psi(X)$ if the associated scalars $\bar{a}$, $\bar{b}$ and $\bar{c}$ satisfy both of the conditions*

$$\bar{c} = c,$$
$$\bar{b} = b,$$
$$\bar{a} = e^{-2\sigma}(a + \gamma),$$

where

$$\gamma = \frac{(n-1)}{n}[2\triangle_2\sigma + (n-2)\triangle_1\sigma].$$

Proof. Suppose that V_n is an equidistant manifold. Then, there exists a concircular vector field ξ satisfying the condition (2.13), that is, we have

$$\nabla_j \xi_i = \rho g_{ij}, \tag{2.16}$$

where $\xi_i = \nabla_i \xi$. Putting $\sigma = -\ln(\xi(X))$ and using the condition (2.5), we obtain

$$\bar{S}_{ij} = S_{ij} + \gamma g_{ij}, \tag{2.17}$$

where

$$\gamma = \frac{(n-1)}{n}[2\triangle_2\sigma + (n-2)\triangle_1\sigma].$$

Considering (1.3) in (2.16) and using (2.2), we get

$$\bar{a}e^{2\sigma}g_{ij} + \bar{b}\bar{\phi}_i\bar{\phi}_j + \bar{c}\bar{\psi}_i\bar{\psi}_j = (a+\gamma)g_{ij} + b\phi_i\phi_j + c\psi_i\psi_j. \tag{2.18}$$

If we take $\bar{c} = c$, $\bar{b} = b$, and $\bar{a} = e^{-2\sigma}(a + \gamma)$, then from (2.18) we get

$$\bar{\phi}_i\bar{\phi}_j = \phi_i\phi_j$$
$$\bar{\psi}_i\bar{\psi}_j = \psi_i\psi_j.$$

These completes the proof.

The conharmonic transformation is a conformal transformation preserving the harmonicity of a certain function. If the conformal mapping is also conharmonic, then we have, [11]

$$\nabla_i\sigma^i + \frac{1}{2}(n-2)\sigma^i\sigma_i = 0. \tag{2.19}$$

Theorem 2.5. *Let V_n be a conformal mapping with preservation of the the associated 1-forms $\phi(X)$ and $\psi(X)$ and the associated scalars b and c. A necessary and sufficient condition for this conformal mapping to be conharmonic is that the associated scalar $\bar{a}$ be transformed by $\bar{a} = e^{-2\sigma}a$, $\bar{b} = e^{-2\sigma}b$ and $\bar{c} = e^{-2\sigma}c$.*

Proof. We consider a conformal mapping of quasi-Einstein manifolds V_n and $\bar{V}_n$. Then, we have from (1.3) and (2.5), we have

$$\begin{aligned} \bar{b}\bar{\phi}_i\bar{\phi}_j + \bar{c}\bar{\psi}_i\bar{\psi}_j &= b\phi_i\phi_j + c\psi_i\psi_j + (n-2)\sigma_{ij} \\ &+ \{\triangle_2\sigma + (n-2)\triangle_1\sigma + a - \bar{a}e^{2\sigma}\}g_{ij}. \end{aligned} \tag{2.20}$$

Multiplying (2.20) by g^{ij} and using (1.4), (2.1), (2.8) and (2.9), it can be seen that the following relation is satisfied

$$n\bar{a}+\bar{b}+\bar{c}=e^{-2\sigma}[na+b+c+2(n-1)\triangle_2\sigma+(n-1)(n-2)\triangle_1\sigma]. \tag{2.21}$$

If the conformal mapping is also conharmonic, then we have from (2.9) and (2.19)

$$2\triangle_2\sigma+(n-2)\triangle_1=0. \tag{2.22}$$

Considering (2.22) in (2.21), it is found that

$$n\bar{a}+\bar{b}+\bar{c}=nae^{-2\sigma}+be^{-2\sigma}+ce^{-2\sigma}. \tag{2.23}$$

From the equation (2.23), it can be seen that the associated scalars are transformed by

$$\bar{a}=e^{-2\sigma}a,\ \bar{b}=e^{-2\sigma}b \ and\ \bar{c}=e^{-2\sigma}c. \tag{2.24}$$

Conversely, if the associated scalars of the manifolds are transformed by (2.24), then we have from (2.21),

$$2(n-1)\triangle_2\sigma+(n-1)(n-2)\triangle_1\sigma=0 \tag{2.25}$$

and so, we get the relation (2.19). Thus, the conformal mapping is also conharmonic. This completes the proof.

Definition 2.6. A $\varphi(Ric)$-vector field is a vector field on an n-dimensional Riemannian manifold (M,g) and Levi-Civita connection ∇, which satisfies the condition

$$\nabla\varphi=\mu Ric, \tag{2.26}$$

where μ is a constant and Ric is the Ricci tensor [10]. When (M,g) is an Einstein space, the vector field φ is concircular. Moreover, when $\mu=0$, the vector field φ is covariantly constant. In local coordinates, (2.22) can be written as

$$\nabla_j\varphi_i=\mu S_{ij}, \tag{2.27}$$

where S_{ij} denote the components of the Ricci tensor and $\varphi_i=\varphi^\alpha g_{i\alpha}$.

Suppose that V_n admits a $\sigma(Ric)$-vector field. Then, we have

$$\nabla_j\sigma_i=\mu S_{ij}, \tag{2.28}$$

where μ is a constant. Now, we can state the following theorem:

Theorem 2.7. *Let us consider the conformal mapping (2.1) of a $G(QE)_n$ V_n with constant associated scalars being also conharmonic with the $\sigma(Ric)$-vector field. A necessary and sufficient condition for the length of σ to be constant is that the sum of the associated scalars b and c of V_n be constant.*

Proof. We consider that the conformal mapping (2.1) of a $G(QE)_n$ V_n admitting a $\sigma(Ric)$-vector field is also conharmonic. In this case, comparing (2.19) and (2.28), we get

$$r=\frac{(2-n)}{2\mu}\sigma^i\sigma_i, \tag{2.29}$$

where r is the scalar curvature of V_n. If V_n is of the constant associated scalars, from (1.6) and (2.29), we find

$$b + c = [\frac{(2-n)}{2\mu}\sigma^i\sigma_i - na]. \tag{2.30}$$

If the length of σ is constant, then $\sigma^i\sigma_j = c_1$, where c_1 is a constant. Thus, we can see that $b + c$ is constant. The converse is also true. Hence, the proof is complete.

In [10], it was shown that Riemannian manifolds with a $\varphi(Ric)$-vector field of constant length have constant scalar curvature. The converse of this theorem is also true. We need the following theorem [13], for later use.

Theorem 2.8. *Let V_n be a Riemannian manifold with constant scalar curvature. If V_n admits a $\varphi(Ric)$-vector field, then the length of φ is constant.*

Now, we consider a $G(QE)_n$ admitting the generator vector field U as a $\phi(Ric)$-vector field. Then we have from (2.26)

$$\nabla_j\phi_i = \mu S_{ij} \text{ and } \nabla_j\psi_i = \mu S_{ij}, \tag{2.31}$$

where μ is a constant. Then, we use the following theorem whose detailed proof is given in[14].

Theorem 2.9. *In a $G(QE)_n$, if the vector fields U and V corresponding to the 1-forms ϕ and ψ are $\phi(Ric)$-vector field and $\psi(Ric)$-vector field , then U and V are covariantly constant.*

Now we prove the following theorem:

Theorem 2.10. *In a $G(QE)_n$ admits a $\varphi(Ric)$-vector field and $\nu(Ric)$-vector field with constant length, then either ϕ_i, ψ_i and φ_i are coplanar or the Ricci tensor of the manifold reduces to the following form*

$$S_{ij} = b\phi_i\phi_j + c\psi_i\psi_j$$

and ψ_i, ϕ_i and ν_i are coplanar or the Ricci tensor of the manifold reduces to the following form

$$S_{ij} = b\phi_i\phi_j + c\psi_i\psi_j.$$

Proof. We assume that $G(QE)_n$ admits a $\varphi(Ric)$-vector field and $\nu(Ric)$-vector field with constant length. Then, we have

$$\varphi_i\varphi^i = p(say) \text{ and } \nu_i\nu^i = q(say), \tag{2.32}$$

where c is a constant. Taking the covariant derivative of the condition (2.32), using the equation (2.31) and considering μ as a non-zero constant (that is φ is proper $\varphi(Ric)$-vector field), it follows that

$$S_{ik}\varphi^i = 0. \tag{2.33}$$

By the aid of (1.3) and (2.33), we get

$$a\varphi_k + b(\varphi^i\phi_i)\phi_k + c(\psi_i\varphi^i)\psi_k = 0, \tag{2.34}$$

$$a\varphi_k + b(\varphi^i\phi_i)\phi_k + cq'\psi_k = 0, \tag{2.35}$$

where

$$\psi_i\varphi^i = q'.$$

Multiplying (2.34) by ϕ^k and using (1.4), it is obtained that

$$(a+b)\varphi_k\phi^k = 0. \tag{2.36}$$

So either $\varphi_k\phi^k = 0$ which gives from (2.34) that

$$a\varphi_k + c(\psi_i\varphi^i)\psi_k = 0. \tag{2.37}$$

Multiplying (2.37) by ϕ^i, we get a=0 and so, the Ricci tensor of the manifold reduces to the form

$$S_{ij} = b\phi_i\phi_j + c\psi_i\psi_j \tag{2.38}$$

or $\varphi_k\phi^k \neq 0$ which gives from (2.36) that

$$a = -b. \tag{2.39}$$

Again taking the covariant derivative of the condition (2.32), using the equation (2.31) and considering μ as a non-zero constant (that is ν is proper $\nu(Ric)$-vector field), it follows that

$$S_{ik}\nu^i = 0. \tag{2.40}$$

Using the equation (1.3) and (2.40), we get

$$a\nu_k + b(\nu^i\phi_i)\phi_k + c(\psi_i\nu^i)\psi_k = 0, \tag{2.41}$$

$$a\nu_k + bp'\phi_k + c(\psi_i\nu^i)\psi_k = 0, \tag{2.42}$$

where

$$\nu^i\phi_i = p'.$$

Multiplying (2.41) by ψ^k and using (1.4), it is obtained that

$$(a+c)\nu_k\psi^k = 0. \tag{2.43}$$

So either $\nu_k\psi^k = 0$ which gives from (2.41) that

$$a\nu_k + b(\nu^i\phi_i)\phi_k = 0. \tag{2.44}$$

Multiplying (2.44) by ψ^i, we get a=0. and so, the Ricci tensor of the manifold reduces to the form

$$S_{ij} = b\phi_i\phi_j + c\psi_i\psi_j \tag{2.45}$$

or $\nu_k\psi^k \neq 0$ which gives from (2.43) that

$$a = -c. \tag{2.46}$$

Since $b \neq 0$ and $c \neq 0$ then $a \neq 0$ and using the equation (2.35), (2.39) and (2.46), we obtain that

$$\varphi_k = (\varphi^i \phi_i)\phi_k + (\psi_i \varphi^i)\psi_k. \tag{2.47}$$

So from (2.47) we say that φ_k, ϕ_k and ψ_k are coplanar.

Again from (2.41), we obtain

$$\nu_k = (\nu^i \phi_i)\phi_k + (\psi_i \nu^i)\psi_k \tag{2.48}$$

i.e., ν_k, ϕ_k and ψ_k are also coplanar.

Corollary 2.11. *In a $G(QE)_n$ admits $\varphi(Ric)$-vector field and $\nu(Ric)$-vector field with constant length which is not orthogonal to the generators, then the associated scalars of the manifold must be constants and the vector fields φ and ν are covariantly constant.*

Proof. As it has been mentioned before, a Riemannian manifold admitting a $\varphi(Ric)$-vector field and $\nu(Ric)$-vector field with constant length has constant scalar curvature. Moreover, under the assumptions and from Theorem 2.10., we obtain that the associated scalars of $G(QE)_n$ are related by $a = -b$ and $a = -c$, and from (1.6), we obtain

$$r = (n-2)a. \tag{2.49}$$

Since the scalar curvature of the manifold is constant, in this case, from (1.6) and (2.49), we see that the associated scalars of the manifold are constants.

For the second part, multiplying (2.47) by φ^k and using (2.32), it can be seen that $\varphi^i \phi_i$ is a constant as $\psi_i \varphi^i$ is also constant. So, (2.47) shows that the generator vector field U is also a $\phi(Ric)$-vector field. In this case, U must be covariantly constant by Theorem 2.9. Again, multiplying (2.48) by ν^k and using (2.32), it can be seen that $\psi_i \nu^i$ is a constant as $\nu^i \phi_i$ is also constant. Now due to the coplanarity of φ, U and V, φ is covariantly constant. Similarly, due to the coplanarity of ν, U and V, ν is also covariantly constant. Hence the proof.

3. Conformal mappings of $G(QE)_n$ admitting special vector fields

Definition 3.1. A symmetric tensor field T of type (0,2) on a Riemannian manifold (M, g) is said to be a Codazzi tensor if it satisfies the following condition

$$(\nabla_X T)(Y, Z) = (\nabla_Y T)(X, Z) \tag{3.1}$$

for arbitrary vector fields X, Y and Z.

Now, we assume that the Ricci tensors S' and S of the $G(QE)_n$ are Codazzi tensors with respect to the Levi-Civita connections r' and r, respectively. Then, from (3.1), we have the following relations

$$\bar{\nabla}_k \bar{S}_{ij} = \bar{\nabla}_j \bar{S}_{ik} \tag{3.2}$$

and

$$\nabla_k S_{ij} = \nabla_j S_{ik}. \tag{3.3}$$

On the other hand, if the Ricci tensor of the manifold is a Codazzi tensor, then from the second Bianchi identity, it can be seen that the scalar curvature is constant. According to our assumptions, the scalar curvatures r' and r of the quasi-Einstein manifolds are constants. So, we state and prove the following theorems.

Theorem 3.2. *Let us consider a conformal mapping $\bar{g} = ge^{2\sigma}$ of $G(QE)_n$ whose Ricci tensors are Codazzi type. If the vector field generated by the 1-form σ is a $\sigma(Ric)$-vector field, then either this conformal mapping is homothetic or the relation*

$$\mu = \frac{(2-n)(n-1)c' - (na+b+c)}{2(n-1)(na+b+c)} \tag{3.4}$$

is satisfied where c is the square of the length of $\sigma_i = \frac{\partial \sigma}{\partial x^i} = \partial_i \sigma$ and and μ denotes the constant corresponding to the $\sigma(Ric)$-vector field.

Proof. Suppose that the Ricci tensors of V_n and $\bar{V}_n$ are Codazzi tensors and suppose that $\bar{g} = ge^{2\sigma}$ is a conformal mapping with a $\sigma(Ric)$-vector field. By using the second Bianchi identity, it can be seen that the scalar curvatures r and $\bar{r}$ are constants. Since r is constant, then the length of σ_i is constant by Theorem 2.8., (and $r \neq 0$ which can be seen from Theorem 2.10., and Corollary 2.1.) and so we have the condition

$$\sigma_i \sigma^i = c', \tag{3.5}$$

where c is a constant. If we assume that the vector field generated by the 1-form σ in the conformal mapping (2.1) is a $\sigma(Ric)$-vector field, we get

$$\nabla_j \sigma_i = \mu S_{ij}, \tag{3.6}$$

where μ is a constant. Using (2.9), (3.5) and (3.6), we have the following relations

$$\triangle_2 \sigma = \mu r, \;\; \triangle_1 \sigma = c' \tag{3.7}$$

and so, $\triangle_1 \sigma$ and $\triangle_2 \sigma$ are constants. Using the relations (3.7) in (2.6), we find

$$\bar{r} = e^{-2\sigma} B, \tag{3.8}$$

where r, $\bar{r}$ and $B[r + 2(n-1)\mu r + (n-1)(n-2)c]$ are constants. In this case, if $\bar{r}$ is non-zero then we get from (3.8) that B is non-zero and so, $e^{-2\sigma}$ is constant. Thus, σ is constant. Therefore, this mapping is homothetic. If $\bar{r}$ is zero then B must be zero. So we obtain using (1.6)

$$\mu = \frac{(2-n)(n-1)c' - (na+b+c)}{2(n-1)(na+b+c)}. \tag{3.9}$$

This completes the proof.

Next we consider a conformal mapping between two $G(QE)_n$ admitting a concircular vector field σ_i.

Theorem 3.3. Let us consider a conformal mapping $\bar{g} = ge^{2\sigma}$ of $G(QE)_n$ whose Ricci tensors are Codazzi type. If σ_i is a concircular vector field, then either
i. ϕ_i and σ_i are orthogonal or
ii. the function ρ is found as

$$\rho = \frac{b - [\frac{c}{n-1} + (n-2)\triangle_1\sigma]}{n+2} \tag{3.10}$$

and
iii. ψ_i and σ_i are orthogonal or
iv. the function ρ is found as

$$\rho = \frac{c - [\frac{b}{n-1} + (n-2)\triangle_1\sigma]}{n+2}, \tag{3.11}$$

where ϕ_i, ψ_i denote the components of the vector field associated 1-form ϕ and ψ, $\sigma_i = \frac{\partial\sigma}{\partial x^i} = \partial_i\sigma$, b, c are the associated scalar of V_n and ρ denotes the function corresponding to the concircular vector field.

Proof. Let the Ricci tensors of V_n and $\bar{V}_n$ be Codazzi tensors and σ_i be a concircular vector field. In this case, we have from (2.13)

$$\nabla_j\sigma_i = \rho g_{ij}, \tag{3.12}$$

where ρ is a function.

Taking the covariant derivative of $\bar{S}_{ij}$ and using (2.6), it can be obtained that

$$\begin{aligned}(\bar{\nabla}\bar{S}_{ij}) \;&=\; \nabla S_{ij} + (n-2)\nabla_k\sigma_{ij} + \partial_k(\triangle_2\sigma + (n-2)\triangle_1\sigma)g_{ij} - 2\sigma_k S_{ij} \\ &-\; \sigma_i S_{jk} - \sigma_i S_{ik} - 2(\triangle_2\sigma + (n-2)\triangle_1\sigma)g_{ij}\sigma_k \\ &+\; \sigma^h(S_{ih}g_{jk} + S_{hj}g_{ik} + (n-2)(\sigma^h\sigma_{hj}g_{ik} \\ &+\; \sigma^h\sigma_{ih}g_{jk} - 2\sigma_k\sigma_{ij} - \sigma_i\sigma_{kj} - \sigma_j\sigma_{ik}).\end{aligned} \tag{3.13}$$

Changing the indices j and k in (3.13) and subtracting the last equation from (3.12) and using (2.8), (3.2), (3.3) and (3.12), it can be seen that

$$\begin{aligned}2(n-1)(\rho_k g_{ij} - \rho_j g_{ik}) \;&+\; [(n-2)\triangle_1\sigma + (n+2)\rho](\sigma_j g_{ik} - \sigma_k g_{ij}) + \sigma_j S_{ik} \\ &-\; \sigma_k S_{ij} + \sigma^h S_{hj}g_{ik} - \sigma^h S_{hk}g_{ij} = 0.\end{aligned} \tag{3.14}$$

Multiplying (3.14) by g^{ij}, it is obtained that

$$\begin{aligned}2(n-1)^2\rho_k \;&+\; [(n-2)(1-n)\triangle_1\sigma + (n+2)(1-n)\rho - r]\sigma_k \\ &+\; (2-n)\sigma^h S_{hk} = 0.\end{aligned} \tag{3.15}$$

On the other hand, we have from the Ricci identity and the equation (3.12)

$$\sigma_\alpha R^\alpha_{ijk} = \rho_k g_{ij} - \rho_j g_{ik}, \tag{3.16}$$

R^α_{ijk} denote the components of the curvature tensor.
Multiplying (3.16) by g^{ij}, we obtain

$$\sigma_\alpha S^\alpha_k = (n-1)\rho_k. \tag{3.17}$$

Substituting ρ_k obtained from (3.17) in (3.15), it can be obtained that

$$n\sigma^h S_{hk} + [(n-2)(1-n)\triangle_1\sigma + (n+2)(1-n)\rho - r]\sigma_k = 0. \tag{3.18}$$

Considering (1.3) in (3.18) and using (1.6), we get

$$\begin{aligned} n\sigma^h[b\phi_h\phi_k + c\psi_h\psi_k] \quad &+ \quad [(n-2)(1-n)\triangle_1\sigma + (n+2)(1-n)\rho \\ &- \quad b - c]\sigma_k = 0. \end{aligned} \tag{3.19}$$

Multiplying (3.19) by ϕ^k and using (1.4), we obtain

$$[(n-1)b - c + (n-2)(1-n)\triangle_1\sigma + (n+2)(1-n)\rho]\sigma^k\phi_k = 0. \tag{3.20}$$

Again multiplying (3.19) by ψ^k and using (1.4), we obtain

$$[(n-1)c - b + (n-2)(1-n)\triangle_1\sigma + (n+2)(1-n)\rho]\sigma^k\psi_k = 0. \tag{3.21}$$

From (3.20), we see that either

$$\sigma^k\phi_k = 0$$

or

$$(n-1)b - c + (n-2)(1-n)\triangle_1\sigma + (n+2)(1-n)\rho = 0.$$

Thus, we obtain that either σ_k is orthogonal to ϕ_k or the function ρ is found as

$$\rho = \frac{b - [\frac{c}{n-1} + (n-2)\triangle_1\sigma]}{n+2}.$$

Similarly, from (3.21) we obtain either σ_k is orthogonal to ψ_k or the function ρ is found as

$$\rho = \frac{c - [\frac{b}{n-1} + (n-2)\triangle_1\sigma]}{n+2}.$$

So the proof is completed.

Corollary 3.4. *Let us consider a conformal mapping $\bar{g} = ge^{2\sigma}$ of $G(QE)_n$ whose Ricci tensors are Codazzi type. If σ_i is a concircular vector field, then the asssociate scalars b and c are equal.*

Proof. The result is trivially comes through from the previous theorem 3.3. immediately.

4. **Examples**

Let us consider a Riemannian metric g on the 4-dimensional real number space M^4 by

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + e^{x^1}[e^{x^2}dx^2)^2 + e^{x^3}(dx^3)^2 + e^{x^4}(dx^4)^2]$$

where i, j = 1, 2, 3, 4 and $0 < x^1 < 1$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4. Then the only non vanishing components of the Christoffel symbols, the curvature tensors and the derivatives of the components of curvature tensors are

$$[22,1] = -\frac{1}{2}e^{x^1+x^2},\ [33,1] = -\frac{1}{2}e^{x^1+x^3},\ [44,1] = -\frac{1}{2}e^{x^1+x^4}, \tag{4.1}$$

$$[22,2] = [33,3] = 44,4] = [12,2] = [13,3] = [14,4] = \frac{1}{2}, \tag{4.2}$$

$$R_{1221} = \frac{1}{4}e^{x^1+x^2},\ R_{1331} = \frac{1}{4}e^{x^1+x^3},\ R_{1441} = \frac{1}{4}e^{x^1+x^2}, \tag{4.3}$$

$$R_{2332} = \frac{1}{4}e^{2x^1+x^2+x^3},\ R_{2442} = \frac{1}{4}e^{2x^1+x^2+x^4},\ R_{3443} = \frac{1}{4}e^{2x^1+x^3+x^4}, \tag{4.4}$$

$$R_{11} = \frac{3}{4},\ R_{22} = \frac{3}{4}e^{x^1+x^2},\ R_{33} = \frac{3}{4}e^{x^1+x^3},\ R_{44} = \frac{3}{4}e^{x^1+x^4},\ r = 3 \tag{4.5}$$

and the components which can be obtained from these by symmetric properties. From the above it can be easily say that M^4 is a Riemannian manifold of non-vanishing scalar curvature. We shall now show that this manifold is $G(QE)_n$. Let us now define

$$a = \frac{3}{4},\ b = 2e^{x^1+x^2},\ c = -4. \tag{4.6}$$

Again let now consider the associated 1-forms as follows:

$$\phi_i(x) = \begin{cases} \frac{1}{\sqrt{e^{x^1+x^2}}}, & \text{for} \quad i = 1, \\ \sqrt{2} & \text{for} \quad i = 3, \\ 0 & \text{for} \quad otherwise, \end{cases} \tag{4.7}$$

and

$$\psi_i(x) = \begin{cases} \frac{1}{2}, & \text{for} \quad i = 1, \\ \sqrt{e^{x^1+x^2}} & \text{for} \quad i = 3, \\ 0 & \text{for} \quad otherwise, \end{cases} \tag{4.8}$$

at any point $x \in M^4$. Then we have

$$R_{11} = ag_{11} + b\phi_1\phi_1 + c\psi_1\psi_1, \tag{4.9}$$

$$R_{22} = ag_{22} + b\phi_2\phi_2 + c\psi_2\psi_2, \tag{4.10}$$

$$R_{33} = ag_{33} + b\phi_3\phi_3 + c\psi_3\psi_3, \tag{4.11}$$

$$R_{44} = ag_{44} + b\phi_4\phi_4 + c\psi_4\psi_4, \tag{4.12}$$

Since all cases other than (4.9) - (4.12) are trivial, we can say that

$$R_{ij} = ag_{ij} + b\phi_i\phi_j + c\psi_i\psi_j, \ for \ i, j = 1, 2, 3, 4. \tag{4.13}$$

So we can say that the manifold under consideration is $G(QE)_n$. Therefore we have

Example 4.1. *Let (M^4, g) be a Riemannian manifold endowed with the metric given by*

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + e^{x^1}[e^{x^2}dx^2)^2 + e^{x^3}(dx^3)^2 + e^{x^4}(dx^4)^2]$$

where i, j = 1, 2, 3, 4 and $0 < x^1 < 1$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4. Then it is $G(QE)_n$ with non-vanishing scalar curvature.

Example 4.2. *Let (M^4, g) be a 4-dimensional Lorentzian manifold endowed with the metric given by*

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + e^{x^1}[e^{x^2}dx^2)^2 + e^{x^3}(dx^3)^2 + e^{x^4}(dx^4)^2]$$

where i, j = 1, 2, 3, 4 and $0 < x^1 < 1$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4. Then it is $G(QE)_n$ with non-vanishing scalar curvature.

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