Gulf Journal of Mathematics
Vol 23, Issue 1 (2026) 1-15

ISSN: 2309-4966
https://doi.org/10.56947/xj8zd415

EXISTENCE AND STABILITY RESULTS FOR IMPLICIT CAPUTO-TEMPERED FRACTIONAL DELAY INTEGRO-DIFFERENTIAL EQUATIONS

MOUSSA HAOUES

Abstract. We aim to study a class of implicit Caputo-tempered fractional integro-differential equations with retarded and advanced delays in Banach spaces with nonlocal conditions. By applying Sadovskii's fixed point theorem and measure of noncompactness under suitable assumptions, we establish sufficient conditions for the existence of solutions to the proposed fractional integro-differential equations in Banach spaces under nonlocal conditions. Furthermore, we prove that the solution is Hyers-Ulam stable. These results also provide a significant expansion of the existing literature in this field. Finally, an example is provided to illustrate the main results.

Keywords. Implicit fractional differential equations, retarded and advanced delay, Caputo tempered fractional derivative, fixed point theorems, measure of noncompactness.
2020 Mathematics Subject Classification. Primary 34A12, 34K37; Secondary 26A33.

1. Introduction

Fractional differential equations, both with and without delay, arise naturally in a wide range of scientific and engineering applications. These include, but are not limited to, applied sciences, mechanical systems, engineering methods, economics, control theory, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic and nonlinear oscillators, conservative systems, the stability and instability of geodesics on Riemannian manifolds, and the dynamics of Hamiltonian systems. In particular, the qualitative analysis of linear and nonlinear fractional differential equations, with or without delay, has attracted considerable research interest, as evidenced by the works cited in [1]-[18],[20, 24], and the references therein.

The so-called tempered fractional calculus is a particularly interesting kind of fractional calculus that expands on the traditional idea of fractional derivatives by taking into account functions with exponentially decaying tails. This concept has been rediscovered at least twice under different names[5, 7, 12]. It

Date: Received: Mar 22, 2026; Revised: Apr 19, 2026; Accepted: May 6, 2026.

is interesting because it is the only point where fractional calculus with analytic kernels and weighted fractional calculus overlap. It has been used in Lévy processes such as Brownian motion and to understand turbulence in geophysical flows. In addition, it is particularly relevant in applications where memory effects are essential, such as in viscoelastic materials, nonlocal models in physics, and fractional-order control systems. The Caputo-tempered derivative is a modification of the classical Caputo space-fractional derivative, in which the underlying convolution kernel is exponentially truncated. The tempered fractional derivative allows for a more accurate description of the underlying dynamics, capturing both long-range memory and fast-decaying behaviours. Further elaboration on this topic can be found in [4]-[7] and [13]-[14]. A subfield of mathematics called Ulam-Hyers stability examines the stability of specific functional equations. The behaviour of solutions to functional equations that are disturbed by slight variations in the input parameters is specifically examined by the Ulam-Hyers stability theory. The Ulam-Hyers stability theory generally addresses the following query: If an equation holds roughly, how near to the exact solution may the solution get? The theory offers a framework for examining the stability of a wide range of functional equations, including those from different branches of mathematics, including number theory, differential equations, and functional analysis. In the 1940s, S. M. Ulam and D. H. Hyers separately devised the Ulam-Hyers stability hypothesis, which bears their names. Since then, numerous studies have investigated the stability of various differential and integral equations. Readers seeking further information can consult [13, 23] and the references therein

In interesting contributions, in [14], the authors discussed the existence and uniqueness results for a class of problems for nonlinear implicit fractional differential equations subject to boundary conditions and delay:

$$\begin{cases} \left({}_0^C\mathfrak{D}_\theta^{\zeta,\varepsilon}\chi\right)(\theta) = \Psi\left(\theta, \chi_{\rho(\theta,\chi_\theta)}, \mathfrak{D}_0^\zeta\chi(\theta)\right); \theta\in\Theta \\ \chi(\theta) = \wp(\theta), \quad \theta\in[-\kappa,0] \\ \imath\chi(0) + \jmath\chi(\varpi) = \varrho \end{cases}$$

where $0<\zeta<1, \varepsilon\geq 0, {}_0^C\mathfrak{D}_\theta^{\zeta,\varepsilon}$ is the Caputo tempered fractional derivative, $\Psi:\Theta\times C([-\kappa,0],\mathbb{R})\times\mathbb{R}$ is a continuous function, $\wp\in C([-\kappa,\varpi],\mathbb{R}), 0<\varpi<+\infty, \iota, \jmath, \varrho$ are real constants, and $\kappa>0$ is the time delay.

Rahou in [19] discussed the existence of solutions for the nonlinear implicit Caputo-Tempered fractional differential equations with retarded and advanced arrguments in Banach Spaces with nonlocal conditions

$$\begin{cases} {}_0^C D_\theta^{\alpha,\varrho}\xi(\theta) = \Psi\left(\theta, \xi^\theta, {}_0^C D_\theta^{\alpha,o}\xi(\theta)\right), \quad \theta\in\Theta := [0,\varkappa], \\ \xi(\theta) = \Lambda_1(\theta), \quad \theta\in[-\varpi,0], \\ \xi(\theta) = \Lambda_2(\theta), \quad \theta\in[\varkappa,\varkappa+\delta], \end{cases}$$

where ${}_0^C D_\theta^{\alpha,\varrho}$ represents the Caputo tempered fractional derivative of order $\alpha\in(0,1), \varrho\geq 0,\ \varpi,\delta>0, \Psi:\Theta\times C([-\varpi,\delta],\Xi)\times\Xi\to\Xi$ is a given function, $\Lambda_1\in C([-\varpi,0],\Xi)$, and $\Lambda_2\in C([\varkappa,\varkappa+\delta],\Xi)$. The authors derived existence and stability conclusions by applying the fixed point theory for condensing maps.

Inspired by the aforementioned research, we examine the following nonlinear implicit Caputo-Tempered fractional integro-differential equations with advanced and retarded arguments in Banach spaces with nonlocal conditions in this paper:

$$\begin{cases} {}_0^C D_\theta^{\varepsilon,\rho}\xi(\theta) = \mathfrak{F}\left(\theta, \xi^\theta, {}_0^C D_\theta^{\varepsilon,\rho}\xi(\theta), \int_0^\theta \mathfrak{g}\left(\theta, s, \xi(s)\right) ds\right), \quad \theta \in \mathfrak{J} := [0,\kappa], \\ \xi(\theta) + \left(F_1\xi\right)(\theta) = \Lambda_1(\theta), \quad \theta \in [-\mathfrak{r}, 0], \\ \xi(\theta) + \left(F_2\xi\right)(\theta) = \Lambda_2(\theta), \quad \theta \in [\kappa, \kappa + \mathfrak{h}], \end{cases} \tag{1.1}$$

where ${}_0^C D_\theta^{\varepsilon,\rho}$ represents the Caputo tempered fractional derivative of order $\varepsilon \in (0,1)$, $\rho \geq 0$, $\mathfrak{r}, \mathfrak{h} > 0$, and $\mathfrak{F} : \mathfrak{J} \times C([-\mathfrak{r}, \mathfrak{h}], \Xi) \times \Xi \times \Xi \to \Xi$ is a given function, $\mathfrak{g} : \mathfrak{J} \times \mathfrak{J} \times \Xi \to \Xi$, $F_1 : C([-\mathfrak{r}, \kappa + \mathfrak{h}], \Xi) \to C([-\mathfrak{r}, 0], \Xi)$ and $F_2 : C\left([-\mathfrak{r}, \kappa + \mathfrak{h}], \Xi\right) \to C\left([\kappa, \kappa + \mathfrak{h}], \Xi\right)$ Moreover, $\Lambda_1 \in C\left([-\mathfrak{r}, 0], \Xi\right)$ and $\Lambda_2 \in C\left([\kappa, \kappa + \mathfrak{h}], \Xi\right)$.

For each function ξ defined on $[-\mathfrak{r}, \kappa + \mathfrak{h}]$ and for any $\theta \in \mathfrak{J}$, we denote by ξ_θ the element of $C\left([-\mathfrak{r}, \mathfrak{h}], \Xi\right)$ defined by

$$\xi^\theta(t) = \xi(\theta + t), \ t \in [-\mathfrak{r}, \mathfrak{h}]$$

Outline of the paper. The work is divided as follows. In Section 2, we give some notation and materials needed for our work. In Section 3, we transform (1.1) into an integral equation and then use Sadovskii's fixed point theorem to show the existence results. Section 4 is devoted to obtaining Ulam-Hyers and generalised Ulam-Hyers stability results of (1.1). Finally, we provide an example to illustrate the obtained results.

2. Preliminaries

The terminology, definitions, and background information utilised in this study are introduced in this section.

The Banach space of all continuous functions with the norm

$$\|\xi\|_\infty = \sup\left\{\|\xi(\theta)\| : 0 \leq \theta \leq \kappa\right\}$$

is represented by $C\left([0,\kappa], \Xi\right)$.

Let $C([-\mathfrak{r}, 0], \Xi)$ be the Banach space with the norm

$$\|\xi\|_{[-\mathfrak{r},0]} = \sup\{\|\xi(\theta)\| : \theta \in [-\mathfrak{r}, 0]\}.$$

Consider $C([\kappa, \kappa + \mathfrak{h}], \Xi)$ the Banach space with the norm

$$\|\xi\|_{[\kappa,\kappa+\mathfrak{h}]} = \sup\{\|\xi(\theta)\| : \theta \in [\kappa, \kappa + \mathfrak{h}]\},$$

and $C([-\mathfrak{r}, \mathfrak{h}], \Xi)$ the Banach space with the norm

$$\|\xi\|_{[-\mathfrak{r},\mathfrak{h}]} = \sup\{\|\xi(\theta)\| : \theta \in [-\mathfrak{r}, \mathfrak{h}]\}.$$

Let

$$\Omega_\Xi = \left\{\xi : [-\mathfrak{r}, \kappa + \mathfrak{h}] \to \Xi \mid \xi|_{[0,\kappa]} \in C(\mathfrak{J}, \Xi), \ \xi|_{[-\mathfrak{r},0]} \in C([-\mathfrak{r}, 0], \Xi), \ \text{and } \xi|_{[\kappa,\kappa+\mathfrak{h}]} \in C([\kappa, \kappa + \mathfrak{h}], \Xi)\right\}.$$

We note that Ω_Ξ is a Banach space with the norm

$$\|\xi\|_{\Omega_\Xi} = \sup_{\theta \in [-\mathfrak{r},\kappa+\mathfrak{h}]} \|\xi(\theta)\|.$$

Definition 2.1. Suppose that the function $\mathfrak{F} \in C(\mathfrak{J}, \Xi)$ and $\rho \geq 0$. Then, the Riemann-Liouville tempered fractional integral of order ε is defined by

$$ {}_0I_\theta^{\varepsilon,\rho}\mathfrak{F}(\theta) = e^{-\rho\theta}\, {}_0I_\theta^{\varepsilon}\left(e^{\rho\theta}\mathfrak{F}(\theta)\right) = \frac{1}{\Gamma(\varepsilon)}\int_0^\theta e^{-\rho(\theta-\sigma)}\mathfrak{F}(\sigma)(\theta-\sigma)^{\varepsilon-1}d\sigma, \tag{2.1} $$

where ${}_0I_\theta^{\varepsilon}$ denotes the Riemann-Liouville fractional integral [13], defined by

$$ {}_0I_\theta^{\varepsilon}\mathfrak{F}(\theta) = \frac{1}{\Gamma(\varepsilon)}\int_0^\theta \mathfrak{F}(\sigma)(\theta-\sigma)^{\varepsilon-1}d\sigma. \tag{2.2} $$

The Riemann-Liouville fractional integral (2.2) is the obvious reduction of the tempered fractional integral (2.1) if $\rho = 0$.

Definition 2.2. [21]. For $j-1 < \varepsilon < j$, $j \in \mathbb{N}$, and $\rho \geq 0$, the Riemann-Liouville tempered fractional derivative is defined by

$$ {}_0D_\theta^{\varepsilon,\rho}\mathfrak{F}(\theta) = e^{-\rho\theta}\, {}_0D_\theta^{\varepsilon}\left(e^{\rho\theta}\mathfrak{F}(\theta)\right) = e^{-\rho\theta}\frac{1}{\Gamma(j-\varepsilon)}\frac{d^j}{d\theta^j}\int_0^\theta e^{\rho\sigma}\mathfrak{F}(\sigma)(\theta-\sigma)^{\varepsilon-j+1}d\sigma, $$

where ${}_0D_\theta^{\varepsilon}(e^{\rho\theta}\mathfrak{F}(\theta))$ denotes the Riemann-Liouville fractional derivative [13], is defined as

$$ {}_0D_\theta^{\varepsilon}(e^{\rho\theta}\mathfrak{F}(\theta)) = \frac{1}{\Gamma(j-\varepsilon)}\frac{d^j}{d\theta^j}\int_0^\theta (e^{\rho\sigma}\mathfrak{F}(\sigma))(\theta-\sigma)^{\varepsilon-j+1}d\sigma. $$

Definition 2.3. [21]. For $j-1 < \varepsilon < j$, $j \in \mathbb{N}$, and $\rho \geq 0$, the definition of the Caputo tempered fractional derivative is

$$ \begin{aligned} {}_0^CD_\theta^{\varepsilon,\rho}\mathfrak{F}(\theta) &= e^{-\rho\theta}\, {}_0^CD_\theta^{\varepsilon}\left(e^{\rho\theta}\mathfrak{F}(\theta)\right) \\ &= e^{-\rho\theta}\frac{1}{\Gamma(j-\varepsilon)}\int_0^\theta \frac{1}{(\theta-\sigma)^{\varepsilon-j+1}}\frac{d^j}{d\sigma^j}(e^{\rho\sigma}\mathfrak{F}(\sigma))d\sigma, \end{aligned} $$

where

$$ {}_0^CD_\theta^{\varepsilon}(e^{\rho\theta}\mathfrak{F}(\theta)) = \frac{1}{\Gamma(j-\varepsilon)}\int_0^\theta \frac{1}{(\theta-\sigma)^{\varepsilon-j+1}}\frac{d^j}{d\sigma^j}(e^{\rho\sigma}\mathfrak{F}(\sigma))d\sigma. $$

Lemma 2.4. [21]. *For a constant C,*

$$ \begin{aligned} {}_0D_\theta^{\varepsilon,\rho}C &= Ce^{-\rho\theta}{}_0D_\theta^{\varepsilon}e^{\rho\theta}, \\ {}_0^CD_\theta^{\varepsilon,\rho}C &= Ce^{-\rho\theta}{}_0^CD_\theta^{\varepsilon}e^{\rho\theta}. \end{aligned} $$

Obviously,

$$ {}_0D_\theta^{\varepsilon,\rho}(C) \neq {}_0^CD_\theta^{\varepsilon,\rho}(C). $$

Moreover,

$$ {}_0^CD_\theta^{\varepsilon,\rho}(C) \text{ is no longer equal to zero, differing from } {}_0^CD_\theta^{\varepsilon}(C) \equiv 0. $$

Lemma 2.5. [21]. *Let $\mathfrak{F} \in C^j(\Theta, \Xi)$ and $j-1 < \varepsilon < j$, $j \in \mathbb{N}$. Then, the Caputo tempered fractional derivative and the Riemann-Liouville tempered fractional integral satisfy the composition properties*

$$ {}_0I_\theta^{\varepsilon,\rho}\left({}_0^CD_\theta^{\varepsilon,\rho}\mathfrak{F}(\theta)\right) = \mathfrak{F}(\theta) - \sum_{k=0}^{j-1} e^{-\rho\theta}\frac{\theta^k}{k!}\left[\frac{d^k}{d\theta^k}\left(e^{\rho\theta}\mathfrak{F}(\theta)\right)\Big|_{\theta=0}\right], $$

and

$$ {}_0^C D_\theta^{\varepsilon,\rho} \left({}_0 I_\theta^{\varepsilon,\rho} \mathfrak{F}(\theta) \right) = \mathfrak{F}(\theta), \quad \text{for } \varepsilon \in (0,1). $$

Definition 2.6. [3]. Let $\mathbb{X}$ be a Banach space and let $\Delta_{\mathbb{X}}$ be the family of bounded subsets of $\mathbb{X}$. The *Kuratowski measure of noncompactness* is the map $\varphi : \Delta_{\mathbb{X}} \to [0, \infty)$ defined by

$$ \varphi(\Phi) = \inf \left\{ \alpha > 0 : \Phi \subset \bigcup_{j=1}^{m} \Phi_j, \ diam(\Phi_j) \leq \alpha \right\}, $$

where $\Phi \in \Delta_{\mathbb{X}}$.

Definition 2.7. [3]. The map φ satisfies the following properties:

- $\varphi(\Phi) = 0 \iff \Phi$ is compact (i.e., Φ is relatively compact);
- $\varphi(\Phi) = \varphi(\overline{\Phi})$;
- If $\Phi_1 \subset \Phi_2$, then $\varphi(\Phi_1) \leq \varphi(\Phi_2)$;
- $\varphi(\Phi_1 + \Phi_2) \leq \varphi(\Phi_1) + \varphi(\Phi_2)$;
- $\varphi(c\Phi) = |c|\varphi(\Phi)$, for $c \in \mathbb{R}$;
- $\varphi(conv\Phi) = \varphi(\Phi)$.

Lemma 2.8. [9]. *Let $B \subset \Omega_\Xi$ be a bounded and equicontinuous set. Then:*

(1) *The function $\theta \mapsto \varphi(B(\theta))$ is continuous, and*

$$ \varphi_{\Omega_\Xi}(B) = \sup_{\theta \in [-\mathfrak{r}, \kappa + \mathfrak{h}]} \varphi(B(\theta)). $$

(2)

$$ \varphi \left(\left\{ \int_0^\kappa \xi(\sigma) d\sigma : \xi \in B \right\} \right) \leq \int_0^\kappa \varphi(B(\sigma)) d\sigma, $$

where $B(\theta) = \{\xi(\theta) : \xi \in B\}, \quad \theta \in \mathfrak{J}$.

Definition 2.9. [3] Let $\mathbb{X}$ be a Banach space and $H : \mathbb{X} \to \mathbb{X}$ be a continuous mapping. H is said to be a *condensing mapping* if for each bounded set B with $\varphi(B) \neq 0$, we have

$$ \varphi(H(B)) < \varphi(B). $$

Theorem 2.10. [8] *Let D be a non-empty, closed, bounded, and convex subset of a Banach space $\mathbb{X}$, and let $H : D \to D$ be a condensing mapping. Then H has a fixed point in D.*

3. Existence Results

To demonstrate our existence results, let's state a few presumptions:

(H1) The function $\mathfrak{F} : \mathfrak{J} \times C([-\mathfrak{r}, \mathfrak{h}], \Xi) \times \Xi \times \Xi \to \Xi$ is continuous.

(H2) There exist constants $K_1 > 0$ and $0 < K_2 < 1$ such that

$$ \|\mathfrak{F}(\theta, \gamma, \wp, v) - \mathfrak{F}(\theta, \bar{\gamma}, \bar{\wp}, \bar{v})\| \leq K_1 \|\gamma - \bar{\gamma}\|_{[-\mathfrak{r}, \mathfrak{h}]} + K_2 \|\wp - \bar{\wp}\| + K_3 \|v - \bar{v}\|, $$

for any $\gamma, \bar{\gamma} \in C([-\mathfrak{r}, \mathfrak{h}], \Xi)$, $\wp, \bar{\wp}, v, \bar{v} \in \Xi$, and $\theta \in \mathfrak{J}$.

(H3) There exist constants $0 < K_4 < 1$ and $K_5 > 0$ such that
$$\|\digamma_1\xi_1 - \digamma_1\xi_2\|_{[-\mathfrak{r},0]} \leq K_4\|\xi_1 - \xi_2\|_{[-\mathfrak{r},\kappa+\mathfrak{h}]}.$$
and
$$\|\mathfrak{g}\,(\theta, s, u_1) - \mathfrak{g}\,(\theta, s, u_2)\,\| \leq K_5\|u_1 - u_2\|.$$

(H4) There exist constants $M_{\digamma_1} \geq K_4 > 0$ and $M_{\digamma_2} > 0$ such that
$$\|\digamma_1\xi\|_{[-\mathfrak{r},0]} \leq M_{\digamma_1} \text{ and } \|\digamma_2\xi\|_{[\kappa,\kappa+\mathfrak{h}]} \leq M_{\digamma_2}.$$

(H5) For each $\theta \in \mathfrak{J}$ and bounded sets $B_1 \subseteq C([-\mathfrak{r}, \mathfrak{h}], \Xi)$, $B_2, B_3 \subseteq \Xi$, we have
$$\zeta(\mathfrak{F}(\theta, B_1, B_2, B_3)) \leq l_1 \sup_{\sigma\in[-\mathfrak{r},\mathfrak{h}]} \zeta(B_1(\sigma)) + l_2\zeta(B_2) + l_3\zeta(B_3).$$

(H6) There exist a function $\eta \in L^1(\mathfrak{J}, \Xi)$ such that
$$\int_0^\theta |\mathfrak{g}\,(\theta, s, u)|\, ds \leq \eta\,(\theta)\,(1 + |u|)\,,$$

Lemma 3.1. *Let ϑ be a continuous function. Then the linear problem*
$$\begin{cases} {}_0^C D_\theta^{\varepsilon,\rho}\xi(\theta) = \vartheta\,(\theta)\,, & \theta \in \mathfrak{J} := [0, \kappa]\,, \\ \xi(\theta) + (\digamma_1\xi)\,(\theta) = \Lambda_1(\theta), & \theta \in [-\mathfrak{r}, 0]\,, \\ \xi(\theta) + (\digamma_2\xi)\,(\theta) = \Lambda_2(\theta), & \theta \in [\kappa, \kappa + \mathfrak{h}]\,, \end{cases} \tag{3.1}$$
has a unique solution which is given by
$$\xi(\theta) = \begin{cases} \Lambda_1(\theta) - (\digamma_1\xi)\,(\theta)\,, & \theta \in [-\mathfrak{r}, 0]\,, \\ (\Lambda_1\,(0) - (\digamma_1\xi)\,(0))\, e^{-\rho\theta} + \frac{1}{\Gamma(\varepsilon)}\int_0^\theta e^{-\rho(\theta-\sigma)}\vartheta\,(\sigma)\,(\theta - \sigma)^{\varepsilon-1}\, d\sigma, & \theta \in \mathfrak{J} := [0, \kappa]\,, \\ \Lambda_2(\theta) - (\digamma_2\xi)\,(\theta)\,, & \theta \in [\kappa, \kappa + \mathfrak{h}]\,, \end{cases} \tag{3.2}$$

Proof. Suppose that ξ satisfies (3.1). By applying the tempered fractional integral of Riemann-Liouville of order ε and by Lemma 2.5 , we get
$${}_0I_\theta^{\varepsilon,\rho}\,{}_0^C D_\theta^{\varepsilon,\rho}\xi(\theta) = {}_0I_\theta^{\varepsilon,\rho}\vartheta\,(\theta)\,.$$
Implies that
$$\xi(\theta) - \xi(0)e^{-\rho\theta} = \frac{1}{\Gamma(\varepsilon)}\int_0^\theta e^{-\rho(\theta-\sigma)}\vartheta\,(\sigma)\,(\theta - \sigma)^{\varepsilon-1}\, d\sigma.$$
Then,
$$\xi(\theta) = \xi(0)e^{-\rho\theta} + \frac{1}{\Gamma(\varepsilon)}\int_0^\theta e^{-\rho(\theta-\sigma)}\vartheta\,(\sigma)\,(\theta - \sigma)^{\varepsilon-1}\, d\sigma.$$
Finally,
$$\xi(\theta) = (\Lambda_1(0) - (\digamma_1\xi)\,(0))\, e^{-\rho\theta} + \frac{1}{\Gamma(\varepsilon)}\int_0^\theta e^{-\rho(\theta-\sigma)}\vartheta\,(\sigma)\,(\theta - \sigma)^{\varepsilon-1}\, d\sigma.$$
On the other hand, it can be demonstrated, using Definition 2.1 , Lemma 2.4 and Lemma 2.5 that if ξ verifies (3.2), then the problem (3.1) is also satisfied. $\square$

The next result is based on Sadovskii's fixed point theorem.

Theorem 3.2. *Assume that* (**H1**) – (**H6**) *hold. If*

$$\lambda + \frac{(K_1 + K_3 K_5 \kappa)\, \kappa^{\epsilon}}{(1 - K_2)\, \Gamma(\epsilon + 1)} < 1$$

where $\lambda = \sup_{\theta \in \mathfrak{J}} K_4 e^{-\rho\theta}$, *then there exists at least one solution of* (1.1)

Proof. Consider the operator $T : \Omega_{\Xi} \to \Omega_{\Xi}$ defined by

$$T\xi(\theta) = \begin{cases} \Lambda_1(\theta) - (\digamma_1 \xi)(\theta), & \theta \in [-\mathfrak{r}, 0], \\ (\Lambda_1(0) - (\digamma_1 \xi)(0))\, e^{-\rho\theta} + \frac{1}{\Gamma(\varepsilon)} \int_0^{\theta} e^{-\rho(\theta - \sigma)} \mathcal{N}(\sigma)\, (\theta - \sigma)^{\varepsilon - 1}\, d\sigma, & \theta \in \mathfrak{J} := [0, \kappa], \\ \Lambda_2(\theta) - (\digamma_2 \xi)(\theta), & \theta \in [\kappa, \kappa + \mathfrak{h}], \end{cases}$$

where $\mathcal{N} \in C(\mathfrak{J}, \Xi)$ is such that

$$\mathcal{N}(\theta) = \mathfrak{F}\left(\theta, \xi^{\theta}, \mathcal{N}(\theta), \int_0^{\theta} \mathfrak{g}(\theta, s, \xi(s))\, ds\right)$$

Clearly, the fixed points of operator T are solution of problem (1.1). The proof is divided into three steps.

Step 1. We prove that T is continuous.

Let u_n be a sequence such that $u_n \to u$ in Ω_{Ξ}. If $\theta \in [-\mathfrak{r}, 0]$ or $\theta \in [\kappa, \kappa + \mathfrak{h}]$ then

$$|(Tu_n)(\theta) - (Tu)(\theta)| = 0$$

For $\theta \in \mathfrak{J}$ we have

$$|(Tu_n)(\theta) - (Tu)(\theta)| \le |(\digamma_1 u_n)(0) - (\digamma_1 u)(0)|\, e^{-\rho\theta} + \frac{1}{\Gamma(\varepsilon)} \int_0^{\theta} e^{-\rho(\theta - \sigma)} |\mathcal{N}_n(\sigma) - \mathcal{N}(\sigma)|\, (\theta - \sigma)^{\varepsilon - 1}\, d\sigma$$

where$\mathcal{N}_n, \mathcal{N} \in C(\mathfrak{J}, \Xi)$ are such that

$$\mathcal{N}_n(\theta) = \mathfrak{F}\left(\theta, u_n^{\theta}, \mathcal{N}_n(\theta), \int_0^{\theta} \mathfrak{g}(\theta, s, u_n(s))\, ds\right)$$

and

$$\mathcal{N}(\theta) = \mathfrak{F}\left(\theta, u^{\theta}, \mathcal{N}(\theta), \int_0^{\theta} \mathfrak{g}(\theta, s, u(s))\, ds\right)$$

By (H2) and (H3) we have

$$\begin{aligned} |\mathcal{N}_n(\theta) - \mathcal{N}(\theta)| &= \left| \mathfrak{F}\left(\theta, u_n^{\theta}, \mathcal{N}_n(\theta), \int_0^{\theta} \mathfrak{g}(\theta, s, u_n(s))\, ds\right) - \mathfrak{F}\left(\theta, u^{\theta}, \mathcal{N}(\theta), \int_0^{\theta} \mathfrak{g}(\theta, s, u(s))\, ds\right) \right| \\ &\le K_1 \|u_n^{\theta} - u^{\theta}\|_{[-\mathfrak{r}, \mathfrak{h}]} + K_2\, |\mathcal{N}_n(\theta) - \mathcal{N}(\theta)| + K_3 \int_0^{\theta} |\mathfrak{g}(\theta, s, u_n(s)) - \mathfrak{g}(\theta, s, u(s))|\, ds \\ &\le K_1 \|u_n^{\theta} - u^{\theta}\|_{[-\mathfrak{r}, \mathfrak{h}]} + K_2\, |\mathcal{N}_n(\theta) - \mathcal{N}(\theta)| + K_3 \int_0^{\theta} K_5 \|u_n - u\| ds, \\ &\le K_1 \|u_n^{\theta} - u^{\theta}\|_{[-\mathfrak{r}, \mathfrak{h}]} + K_2\, |\mathcal{N}_n(\theta) - \mathcal{N}(\theta)| + K_3 K_5 \left(\int_0^{\theta} ds\right) \|u_n - u\|, \end{aligned}$$

Thus,

$$|\mathcal{N}_n(\theta) - \mathcal{N}(\theta)| \le \frac{K_1 - K_3 K_5 \theta}{1 - K_2} \|u_n - u\|_{[-\mathfrak{r}, \mathfrak{h}]}.$$

And by (H3) we have

$$|(\digamma_1 u_n)(0) - (\digamma_1 u)(0)| \leq K_4 \|u_n^\theta - u^\theta\|_{[-\mathfrak{r},\kappa+\mathfrak{h}]}.$$

So, for each $\theta \in \mathfrak{J}$, we have

$$|(Tu_n)(\theta)-(Tu)(\theta)| \leq K_4 e^{-\rho\theta}\|u_n^\theta-u^\theta\|_{[-\mathfrak{r},\kappa+\mathfrak{h}]}+\frac{K_1-K_3K_5\theta}{1-K_2}\int_0^\theta e^{-\rho(\theta-\sigma)}(\theta-\sigma)^{\varepsilon-1}\|u_n^\theta-u^\theta\|_{[-\mathfrak{r},\mathfrak{h}]}d\sigma$$

Then the Lebesgue dominated convergence theorem imply that

$$|(Tu_n)(\theta) - (Tu)(\theta)| \to 0 \text{ as } n \to \infty$$

Hence

$$\|Tu_n - Tu\|_{\Omega_\Xi} \to 0 \text{ as } n \to \infty$$

Consequently, T is continuous.

Choose

$$R \geq \max\left\{\frac{\|\Lambda_1(0)\| + M_{\digamma_1} + \frac{\left(\mathfrak{F}_1^* + K_3\|\eta\|_{L_1}\right)\kappa^\epsilon}{(1-K_2)\Gamma(\varepsilon+1)}}{1 - \frac{\kappa^\epsilon\left(K_1 + K_3\|\eta\|_{L_1}\right)}{(1-K_2)\Gamma(\epsilon+1)}}, M_{\digamma_1} + \|\Lambda_1\|_{[-\mathfrak{r},0]}, M_{\digamma_2} + \|\Lambda_2\|_{[\kappa,\kappa+\mathfrak{h}]}\right\},$$

and define the set

$$D_R = \{\xi \in \Omega_\Xi : \|\xi\|_{\Omega_\Xi} \leq R\}.$$

It is clear D_R that is a bounded, closed and convex subset of Ω_Ξ .

Step 2. We prove that $T(D_R) \subset D_R$.

If $\theta \in [-\mathfrak{r}, 0]$, then

$$\begin{aligned}\|T\xi(\theta)\| &\leq \|\digamma_1\xi\|_{[-\mathfrak{r},0]} + \|\Lambda_1\|_{[-\mathfrak{r},0]}\\ &\leq M_{\digamma_1} + \|\Lambda_1\|_{[-\mathfrak{r},0]}\\ &\leq R.\end{aligned}$$

If $\theta \in [\kappa, \kappa+\mathfrak{h}]$, then

$$\begin{aligned}\|T\xi(\theta)\| &\leq \|\digamma_2\xi\|_{[\kappa,\kappa+\mathfrak{h}]} + \|\Lambda_2\|_{[\kappa,\kappa+\mathfrak{h}]}\\ &\leq M_{\digamma_2} + \|\Lambda_2\|_{[\kappa,\kappa+\mathfrak{h}]}\\ &\leq R.\end{aligned}$$

For each $\theta \in \mathfrak{J} := [0, \kappa]$, then

$$|T\xi(\theta)| \leq (|\Lambda_1(0)| + |(\digamma_1\xi)(0)|)\, e^{-\rho\theta} + \frac{1}{\Gamma(\varepsilon)}\int_0^\theta e^{-\rho(\theta-\sigma)}|\mathcal{N}(\sigma)|(\theta-\sigma)^{\varepsilon-1}d\sigma,$$

From hypothesis (H2), we have

$$\begin{aligned}|\mathcal{N}(\theta)| &= \left|\mathfrak{F}\left(\theta, \xi^\theta, \mathcal{N}(\theta), \int_0^\theta \mathfrak{g}(\theta, s, u(s))\,ds\right)\right|\\ &\leq \mathfrak{F}_1(\theta) + K_1\|\xi^\theta\|_{[-\mathfrak{r},\mathfrak{h}]} + K_2|\mathcal{N}(\theta)| + K_3\int_0^\theta |\mathfrak{g}(\theta, s, \xi(s))|\,ds\\ &\leq \mathfrak{F}_1^* + K_1\|\xi\|_{\Omega_\Xi} + K_2|\mathcal{N}(\theta)| + K_3\eta(\theta)(1 + \|\xi\|_{\Omega_\Xi})\\ &\leq \mathfrak{F}_1^* + K_1R + K_2|\mathcal{N}(\theta)| + K_3\|\eta\|_\infty(1+R)\end{aligned}$$

where $\mathfrak{F}_1^* = \sup_{\theta\in\mathfrak{J}}\mathfrak{F}_1(\theta) = \sup_{\theta\in\mathfrak{J}}\mathfrak{F}(\theta,0,0)$ and $\|\eta\|_\infty = \sup_{\theta\in\mathfrak{J}}\eta(\theta)$

Then,

$$|\mathcal{N}(\theta)| \leq \frac{\mathfrak{F}_1^* + K_1 R + K_3 \|\eta\|_\infty (1+R)}{1-K_2}$$

Therefore,

$$\begin{aligned}
|T\xi(\theta)| &\leq (|\Lambda_1(0)| + |(F_1\xi)(0)|)\, e^{-\rho\theta} + \frac{\mathfrak{F}_1^* + K_1 R + K_3 \|\eta\|_\infty (1+R)}{(1-K_2)\,\Gamma(\varepsilon)} \int_0^\theta e^{-\rho(\theta-\sigma)} (\theta-\sigma)^{\varepsilon-1}\, d\sigma, \\
|T\xi(\theta)| &\leq (|\Lambda_1(0)| + |(F_1\xi)(0)|)\, e^{-\rho\theta} + \frac{(\mathfrak{F}_1^* + K_1 R + K_3 \|\eta\|_\infty (1+R))\, \kappa^\epsilon}{(1-K_2)\,\Gamma(\varepsilon+1)} \\
|T\xi(\theta)| &\leq \|\Lambda_1(0)\| + M_{F_1} + \frac{(\mathfrak{F}_1^* + K_1 R + K_3 \|\eta\|_\infty (1+R))\, \kappa^\epsilon}{(1-K_2)\,\Gamma(\varepsilon+1)} \\
&\leq R
\end{aligned}$$

Consequently,

$$T(D_R) \subset D_R.$$

Step 3. T is condensing.
Let B be an equicontinuous subset of D_R.
If $\theta \in [-\mathfrak{r}, 0]$, then

$$\begin{aligned}
\zeta(T(B(\theta))) &= \zeta\{T\xi(\theta), \quad \xi \in B\} \\
&= \zeta\left\{\Lambda_1(\theta) - F_1\xi(\theta), \quad \xi \in B\right\} \\
&\leq M_{F_1}
\end{aligned}$$

and if $\theta \in [\kappa, \kappa+\mathfrak{h}]$, then

$$\begin{aligned}
\zeta(T(B(\theta))) &= \zeta\{T\xi(\theta), \quad \xi \in B\} \\
&= \zeta\left\{\Lambda_2(\theta) - F_2\xi(\theta), \quad \xi \in B\right\} \\
&\leq M_{F_2}
\end{aligned}$$

For each $\theta \in \mathfrak{J}$, we have

$$\begin{aligned}
\zeta(T(B(\theta))) &= \zeta\{T\xi(\theta), \quad \xi \in B\} \\
&= \zeta\left\{(\Lambda_1(0) - (F_1\xi)(0))\, e^{-\rho\theta} + \frac{1}{\Gamma(\epsilon)} \int_0^\theta e^{-\varrho(\theta-\sigma)} (\theta-\sigma)^{\epsilon-1} \mathcal{N}(\sigma) d\sigma, \quad \xi \in B\right\} \\
&\leq \zeta\left\{(\Lambda_1(0) - (F_1\xi)(0))\, e^{-\rho\theta}, \quad \xi \in B\right\} \\
&\quad + \zeta\left\{\frac{1}{\Gamma(\epsilon)} \int_0^\theta e^{-\varrho(\theta-\sigma)} (\theta-\sigma)^{\epsilon-1} \mathcal{N}(\sigma) d\sigma, \quad \xi \in B\right\} \\
&\leq K_4 e^{-\rho\theta} \zeta_\xi(B) + \frac{1}{\Gamma(\epsilon)} \int_0^\theta (\theta-\sigma)^{\epsilon-1} \{\zeta(\mathcal{N}(\sigma)) d\sigma, \quad \xi \in B\}
\end{aligned}$$

We have by condition (H5)

$$\begin{aligned}\zeta(\mathcal{N}(\theta)) &= \zeta\left(\mathfrak{F}\left(\theta,\xi^{\theta},\mathcal{N}(\theta),\int_0^{\theta}\mathfrak{g}\left(\theta,s,\xi(s)\right)ds\right)\right)\\ &\leq K_1\sup_{\theta\in[-\mathfrak{r},\mathfrak{h}]}\zeta\left(\xi^{\theta}\right)+K_2\zeta(\mathcal{N}(\theta))+K_3\zeta\left(\int_0^{\theta}\mathfrak{g}\left(\theta,s,\xi(s)\right)ds\right)\\ &\leq K_1\sup_{\theta\in[-\mathfrak{r},\kappa+\mathfrak{h}]}\zeta(\xi(\theta))+K_2\zeta(\mathcal{N}(\theta))+K_3K_5\left(\int_0^{\theta}\zeta\left(\xi(s)\right)ds\right)\end{aligned}$$

Thus,

$$\zeta(\mathcal{N}(\theta))\leq\frac{K_1+K_3K_5\kappa}{1-K_2}\sup_{\theta\in[-\mathfrak{r},\kappa+\mathfrak{h}]}\zeta(\xi(\theta))$$

Then,

$$\begin{aligned}\zeta(T(B(\theta)) &\leq K_4e^{-\rho\theta}\zeta_{\xi}(B)+\frac{K_1+K_3K_5\kappa}{(1-K_2)\,\Gamma(\epsilon)}\int_0^{\theta}(\theta-\sigma)^{\epsilon-1}\{\sup_{\theta\in[-\mathfrak{r},\kappa+\mathfrak{h}]}\zeta(\xi(\theta))d\sigma,\quad \xi\in B\}\\ &\leq\left(\lambda+\frac{(K_1+K_3K_5\kappa)\,\kappa^{\epsilon}}{(1-K_2)\,\Gamma(\epsilon+1)}\right)\zeta_{\xi}(B)\end{aligned}$$

Therefore,

$$\zeta_{\Omega_{\Xi}}(T(B))\leq\left(\lambda+\frac{(K_1+K_3K_5\kappa)\,\kappa^{\epsilon}}{(1-K_2)\,\Gamma(\epsilon+1)}\right)\zeta_{\Omega_{\Xi}}(B)<\zeta_{\Omega_{\Xi}}(B).$$

Then, we have $\zeta_{\Omega_{\Xi}}(T(B))\leq\zeta_{\Omega_{\Xi}}(B)$, which implies that T is a condensing operator.

As a result of Sadovskii's fixed point theorem, the operator T has at least one fixed point that solves the problem (1.1). □

4. Ulam-Hyers stability

We adopt the definitions of Ulam-Hyers stability and generalized Ulam-Hyers stability given in [13, 23].

Definition 4.1. We say that (1.1) has Ulam-Hyers stability if there exists a real number $K_{\mathfrak{F}}>0$ such that for each $\epsilon>0$, if $\xi\in\Omega_{\Xi}$ satisfies

$$\left\|{}_0^CD_{\theta}^{\varepsilon,\rho}\xi(\theta)-\mathfrak{F}\left(\theta,\xi^{\theta},{}_0^CD_{\theta}^{\varepsilon,\rho}\xi(\theta),\int_0^{\theta}\mathfrak{g}\left(\theta,s,\xi(s)\right)ds\right)\right\|\leq\epsilon,\quad\theta\in\mathfrak{J}:=[0,\kappa],$$

then there exists a solution $\bar{\xi}\in\Omega_{\Xi}$ of (1.1) such that

$$\left\|\xi(\theta)-\bar{\xi}(\theta)\right\|\leq K_{\mathfrak{F}}\epsilon,\quad\theta\in\mathfrak{J}:=[0,\kappa].$$

Definition 4.2. We say that (1.1) has generalized Ulam–Hyers stability if there exists a function $\varphi_{\mathfrak{F}}\in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\varphi_{\mathfrak{F}}(0)=0$ such that for each $\epsilon>0$, if $\xi\in\Omega_{\Xi}$ satisfies

$$\left\|{}_0^CD_{\theta}^{\varepsilon,\rho}\xi(\theta)-\mathfrak{F}\left(\theta,\xi^{\theta},{}_0^CD_{\theta}^{\varepsilon,\rho}\xi(\theta),\int_0^{\theta}\mathfrak{g}\left(\theta,s,\xi(s)\right)ds\right)\right\|\leq\epsilon,\quad\theta\in\mathfrak{J}:=[0,\kappa],$$

then there exists a solution $\bar{\xi} \in \Omega_{\Xi}$ of (1.1) such that

$$\left\|\xi(\theta)-\bar{\xi}(\theta)\right\| \leq \varphi_{\mathfrak{F}}(\epsilon), \quad \theta \in \mathfrak{J} := [0,\kappa].$$

Theorem 4.3. *Assume that the conditions (H1)-(H2) hold and that the condition*

$$\left\|{}_0^C D_\theta^{\varepsilon,\rho}\xi(\theta) - \mathfrak{F}\left(\theta, \xi^\theta, {}_0^C D_\theta^{\varepsilon,\rho}\xi(\theta), \int_0^\theta \mathfrak{g}\left(\theta, s, \xi(s)\right) ds\right)\right\| \leq \epsilon, \quad \theta \in \mathfrak{J} := [0,\kappa],$$

is satisfied for every $\epsilon > 0$.

Then the problem (1.1) is Ulam-Hyers stable.

Proof. Let $\xi \in \Omega_{\Xi}$ be a solution of the inequality

$$\left\|{}_0^C D_\theta^{\varepsilon,\rho}\xi(\theta) - \mathfrak{F}\left(\theta, \xi^\theta, {}_0^C D_\theta^{\varepsilon,\rho}\xi(\theta), \int_0^\theta \mathfrak{g}\left(\theta, s, \xi(s)\right) ds\right)\right\| \leq \epsilon, \quad \theta \in \mathfrak{J} := [0,\kappa],$$

and $\bar{\xi} \in \Omega_{\Xi}$ the solution of the problem (1.1). Then there exists a function $y_\xi \in C(\mathfrak{J}, \Xi)$ (depending on ξ) such that

$$\|y_\xi(\theta)\| \leq \epsilon, \quad \theta \in \mathfrak{J} := [0,\kappa],$$

and

$${}_0^C D_\theta^{\varepsilon,\rho}\xi(\theta) = \mathfrak{F}\left(\theta, \xi^\theta, {}_0^C D_\theta^{\varepsilon,\rho}\xi(\theta), \int_0^\theta \mathfrak{g}\left(\theta, s, \xi(s)\right) ds\right) + y_\xi(\theta), \quad \theta \in \mathfrak{J} := [0,\kappa],$$

In the light of Lemma 3.1, ξ satisfies the fractional integral equation

$$\xi(\theta) = \begin{cases} \Lambda_1(\theta) - \left(\mathcal{F}_1\xi\right)(\theta), \quad \theta \in [-\mathfrak{r}, 0], \\ \left(\Lambda_1(0) - \left(\mathcal{F}_1\xi\right)(0)\right) e^{-\rho\theta} + \frac{1}{\Gamma(\varepsilon)}\int_0^\theta e^{-\rho(\theta-\sigma)}\mathcal{N}(\sigma)(\theta-\sigma)^{\varepsilon-1} d\sigma \\ +\frac{1}{\Gamma(\varepsilon)}\int_0^\theta e^{-\rho(\theta-\sigma)} y_\xi(\sigma)(\theta-\sigma)^{\varepsilon-1} d\sigma, \quad \theta \in \mathfrak{J} := [0,\kappa], \\ \Lambda_2(\theta) - \left(\mathcal{F}_2\xi\right)(\theta), \quad \theta \in [\kappa, \kappa+\mathfrak{h}], \end{cases}$$

Then we have

$$\begin{aligned} \left\|\xi(\theta)-\bar{\xi}(\theta)\right\| = \Big\| & \left(\left(\mathcal{F}_1\xi\right)(0) - \left(\mathcal{F}_1\bar{\xi}\right)(0)\right) e^{-\rho\theta} + \tfrac{1}{\Gamma(\varepsilon)}\int_0^\theta e^{-\rho(\theta-\sigma)}\left(\mathcal{N}(\sigma) - \mathcal{M}(\sigma)\right)(\theta-\sigma)^{\varepsilon-1} d\sigma \\ & - \tfrac{1}{\Gamma(\varepsilon)}\int_0^\theta e^{-\rho(\theta-\sigma)} y_\xi(\sigma)(\theta-\sigma)^{\varepsilon-1} d\sigma \Big\| \end{aligned}$$

which implies that

$$\begin{aligned} \left\|\xi(t)-\bar{\xi}(t)\right\| \leq\ & \left\|\left(\mathcal{F}_1\xi\right)(0) - \left(\mathcal{F}_1\bar{\xi}\right)(0)\right\| e^{-\rho\theta} + \frac{1}{\Gamma(\varepsilon)}\int_0^\theta e^{-\rho(\theta-\sigma)} \left\|\mathcal{N}(\sigma) - \mathcal{M}(\sigma)\right\|(\theta-\sigma)^{\varepsilon-1} d\sigma \\ & + \frac{1}{\Gamma(\varepsilon)}\int_0^\theta e^{-\rho(\theta-\sigma)} \left\|y_\xi(\sigma)\right\|(\theta-\sigma)^{\varepsilon-1} d\sigma \\ \leq\ & \left\|\left(\mathcal{F}_1\xi\right)(0) - \left(\mathcal{F}_1\bar{\xi}\right)(0)\right\| e^{-\rho\theta} + \frac{1}{\Gamma(\varepsilon)}\int_0^\theta e^{-\rho(\theta-\sigma)} \left\|\mathcal{N}(\sigma) - \mathcal{M}(\sigma)\right\|(\theta-\sigma)^{\varepsilon-1} d\sigma \\ & + \frac{\kappa^\varepsilon \epsilon}{\Gamma(\varepsilon+1)}, \end{aligned}$$

where the function $\mathcal{N}$ and $\mathcal{M}$ are defined such that they each meet distinct sets of functional equations.

$$\mathcal{N}(\theta)=\mathfrak{F}\left(\theta,\xi^{\theta},\mathcal{N}(\theta),\int_0^{\theta}\mathfrak{g}_1(\theta,s,\xi(s))\,ds\right)$$

and

$$\mathcal{M}(\theta)=\mathfrak{F}\left(\theta,\bar{\xi}^{\theta},\mathcal{M}(\theta),\int_0^{\theta}\mathfrak{g}_2\left(\theta,s,\bar{\xi}(s)\right)ds\right)$$

From hypothesis (H3), we have

$$\|\left(F_1\xi\right)(0)-\left(F_1\bar{\xi}\right)(0)\|\leq K_4\|\xi^{\theta}-\bar{\xi}^{\theta}\|_{[-\mathfrak{r},\kappa+\mathfrak{h}]}.$$

And by hypothesis (H2), we have

$$\begin{aligned}
\|\mathcal{N}(\sigma)-\mathcal{M}(\sigma)\| &= \|\mathfrak{F}\left(\theta,\xi^{\theta},\mathcal{N}(\theta),\int_0^{\theta}\mathfrak{g}_1(\theta,s,\xi(s))\,ds\right)-\mathfrak{F}\left(\theta,\bar{\xi}^{\theta},\mathcal{M}(\theta),\int_0^{\theta}\mathfrak{g}_2\left(\theta,s,\bar{\xi}(s)\right)ds\right)\| \\
&\leq K_1\left\|\xi^{\theta}-\bar{\xi}^{\theta}\right\|_{[-\mathfrak{r},\mathfrak{h}]}+K_2\|\mathcal{N}(\sigma)-\mathcal{M}(\sigma)\| \\
&\quad +K_3\int_0^{\theta}\left|\mathfrak{g}_1(\theta,s,\xi(s))-\mathfrak{g}_2\left(\theta,s,\bar{\xi}(s)\right)\right|ds \\
&\leq K_1\left\|\xi^{\theta}-\bar{\xi}^{\theta}\right\|_{[-\mathfrak{r},\mathfrak{h}]}+K_2\|\mathcal{N}(\sigma)-\mathcal{M}(\sigma)\|+K_3K_5\int_0^{\theta}\|\xi-\bar{\xi}\|ds \\
&\leq K_1\left\|\xi^{\theta}-\bar{\xi}^{\theta}\right\|_{[-\mathfrak{r},\mathfrak{h}]}+K_2\|\mathcal{N}(\sigma)-\mathcal{M}(\sigma)\|+K_3K_5\left(\int_0^{\theta}ds\right)\|\xi-\bar{\xi}\|,
\end{aligned}$$

which implies that,

$$\|\mathcal{N}(\sigma)-\mathcal{M}(\sigma)\|\leq\frac{K_1+K_3K_5\theta}{1-K_2}\|\xi-\bar{\xi}\|_{[-\mathfrak{r},\mathfrak{h}]}.$$

Then

$$\begin{aligned}
\left\|\xi(t)-\bar{\xi}(t)\right\| &\leq e^{-\rho\theta}K_4\|\xi^{\theta}-\bar{\xi}^{\theta}\|_{[-\mathfrak{r},\kappa+\mathfrak{h}]}+\frac{K_1+K_3K_5\theta}{(1-K_2)\Gamma(\varepsilon)}\left(\int_0^{\theta}e^{-\rho(\theta-\sigma)}(\theta-\sigma)^{\varepsilon-1}d\sigma\right)\|\xi-\bar{\xi}\|_{[-\mathfrak{r},\mathfrak{h}]} \\
&\quad +\frac{\kappa^{\varepsilon}\epsilon}{\Gamma(\varepsilon+1)}, \\
&\leq \left(e^{-\rho\theta}K_4+\frac{K_1+K_3K_5\theta\kappa^{\varepsilon}}{(1-K_2)\Gamma(\varepsilon+1)}\right)\|\xi-\bar{\xi}\|_{\Omega_{\Xi}}+\frac{\kappa^{\varepsilon}\epsilon}{\Gamma(\varepsilon+1)}.
\end{aligned}$$

Thus

$$\|\xi-\bar{\xi}\|_{\Omega_{\Xi}}\leq\frac{\frac{\kappa^{\varepsilon}\epsilon}{\Gamma(\varepsilon+1)}}{1-\left(\lambda+\frac{K_1+K_3K_5\theta\kappa^{\varepsilon}}{(1-K_2)\Gamma(\varepsilon+1)}\right)}=K_{\mathfrak{F}}\epsilon$$

Therefore, the problem (1.1) is Ulam-Hyers stable. If we set $\varphi_{\mathfrak{F}}(\epsilon)=K_{\mathfrak{F}}\epsilon$, then $\varphi_{\mathfrak{F}}(0)=0$, which proves that (1.1) is generalized Ulam-Hyers stable with the initial conditions. □

5. Example

Examine the implicit fractional integro-differential equation that follows

$$\begin{cases} {}_0^C D_\theta^{\frac{3}{5},2}\xi(\theta) = \frac{1}{4e^\theta} + \frac{1}{5e^\theta}\left(\left|\xi^\theta\right| + \left|{}_0^C D_\theta^{\frac{3}{5},2}\xi(\theta)\right|\right), \text{ for each } \theta \in [0,e], \\ \xi(\theta) + \frac{\cos(\xi(\theta))e^\theta}{e^{-\theta}+8} = \Lambda_1(\theta), \theta \in [-1,0], \\ \xi(\theta) + \frac{6}{(\theta+9)^3}\frac{|\xi(\theta)|}{1+|\xi(\theta)|} = \Lambda_2(\theta), \theta \in [e, e+2], \end{cases} \tag{5.1}$$

where $\Lambda_1 \in C([-1,0],\Xi)$ and $\Lambda_2 \in C([e,e+2],\Xi)$. Set

$$\Xi = \ell^1 = \left\{\xi = (\xi_1, \xi_2, \ldots, \xi_j, \ldots) \mid \sum_{j=1}^{\infty} |\xi_j| < \infty\right\}$$

where Ξ is a Banach space. And for $\theta \in [0,e], u \in C([-1,2],\Xi)$, $v \in \Xi$, we set

$$\mathfrak{F}(\theta, \xi^\theta, {}_0^C D_\theta^{\frac{3}{5},2}\xi(\theta), v) = \frac{5e^2 + \cos\left(\|\xi^\theta\|_{[-1,2]}\right) + \cos\left(\|{}_0^C D_\theta^{\frac{3}{5},2}\xi(\theta)\|\right) + \frac{1}{4}\|v\|}{50e^{\theta+5}\left(1 + \|\xi^\theta\|_{[-1,2]} + \|{}_0^C D_\theta^{\frac{3}{5},2}\xi(\theta)\| + \|v\|\right)},$$

$$\mathfrak{g}\left(\theta, s, \xi(s)\right) = \frac{\sin\left(\theta s\right)}{10}\xi(s)$$

Clearly, the functions $\mathfrak{F}$ and $\mathfrak{g}$ are continuous. For any $\gamma, \bar{\gamma} \in\in C([-1,2];\Xi)$, $\wp, \bar{\wp}, v, \bar{v} \in \Xi$, and $\theta \in [0,e]$, we have:

$$\|\mathfrak{F}(\theta,\gamma,\wp,v) - \mathfrak{F}(\theta,\bar{\gamma},\bar{\wp},\bar{v})\| \le \frac{1}{50e^5}\|\gamma - \bar{\gamma}\|_{[-1,2]} + \frac{1}{50e^5}\|\wp - \bar{\wp}\| + \frac{1}{200e^5}\left\|v - \bar{v}\right\|.$$

$$\|\mathfrak{g}\left(\theta, s, u_1\right) - \mathfrak{g}\left(\theta, s, u_2\right)\| \le \frac{1}{10}\|u_1 - u_2\|.$$

And set

$$\left(\digamma_1\xi\right)(\theta) = \frac{\cos(\|\xi(\theta)\|)e^\theta}{e^{-\theta}+8}, \theta \in [-1,0]$$

$$\left(\digamma_2\xi\right)(\theta) = \frac{6}{(\theta+9)^3}\frac{\|\xi(\theta)\|}{1+\|\xi(\theta)\|}, \theta \in [e,e+2]$$

for any $\xi \in C([-1,e+2];\Xi)$. For $\xi_1, \xi_2 \in C([-1,e+2];\Xi)$ we have, if $\theta \in [-1,0]$

$$\|\left(\digamma_1\xi_1\right)(t) - \left(\digamma_1\xi_2\right)(t)\|_{[-1,0]} \le \frac{1}{9}\|\xi_1 - \xi_2\|_{[-1,e+2]},$$

and if $\theta \in [e,e+2]$

$$\|\left(\digamma_2\xi_1\right)(t) - \left(\digamma_2\xi_2\right)(t)\|_{[e,e+2]} \le \frac{1}{270}\|\xi_1 - \xi_2\|_{[-1,e+2]},$$

So, we have $\epsilon = \frac{3}{5}, \kappa = e$, $K_1 = \frac{1}{50e^5}$, $\quad K_2 = \frac{1}{50e^5}$, $\quad K_3 = \frac{1}{200e^5}$, $K_5 = \frac{1}{10}, \lambda = \sup_{\theta\in\mathfrak{J}}\frac{e^{-2\theta}}{9} = \frac{1}{9}$ the contraction condition is

$$\lambda + \frac{K_1 + K_3\ K_5 \theta \kappa^{\epsilon}}{(1 - K_2)\, \Gamma(\epsilon + 1)} \approx 0.1114 < 1$$

All assumptions in Theorem 3.2 are satisfied, then the problem (5.1) has at least one solution. Moreover, it verifies the Ulam-Hyers stability.

References

1. Abdeljawad, T., Shah, K., Abdo, M. S., & Jarad, F. (2023). An analytical study of fractional delay impulsive implicit systems with Mittag-Leffler law. https://doi.org/10.30546/1683-6154.22.1.2023.31.
2. Abbas, S., Benchohra, M., Graef, J. R., & Henderson, J. (2018). Implicit fractional differential and integral equations: existence and stability (Vol. 26). Walter de Gruyter GmbH & Co KG. https://doi.org/10.1515/9783110553819
3. Banaś, J. (1980). On measures of noncompactness in Banach spaces. Commentationes Mathematicae Universitatis Carolinae, 21(1), 131-143.
4. Benchohra, M., Bouriah, S., & Henderson, J. (2018). Nonlinear implicit Hadamard's fractional differential equations with retarded and advanced arguments. Azerbaijan J. Math, 8(2), 72-85.
5. Cao, J., Li, C., & Chen, Y. (2014, September). On tempered and substantial fractional calculus. In 2014 IEEE/ASME 10th International Conference on Mechatronic and Embedded Systems and Applications (MESA) (pp. 1-6). IEEE. https://doi.org/10.1109/MESA.2014.6935561
6. Fazli, H., Nieto, J. J., & Bahrami, F. (2018). On the existence and uniqueness results for nonlinear sequential fractional differential equations. Appl. Comput. Math, 17(1), 36-47.
7. Friedrich, R., Jenko, F., Baule, A., & Eule, S. (2006). Anomalous diffusion of inertial, weakly damped particles. Physical review letters, 96(23), 230601. https://doi.org/10.1103/PhysRevLett.96.230601
8. Goebel, K. (2002). Concise course on fixed point theorems.
9. Guo, D., Lakshmikantham, V., & Liu, X. (2013). Nonlinear integral equations in abstract spaces (Vol. 373). Springer Science & Business Media.
10. Haoues, M., Ardjouni, A., & Djoudi, A. (2020). Existence and uniqueness of solutions for the nonlinear retarded and advanced implicit Hadamard fractional differential equations with nonlocal conditions. Nonlinear studies, 27(2), 433-445. https://www.nonlinearstudies.com/index.php/nonlinear/article/view/1978
11. Haoues, M., Ardjouni, A., & Djoudi, A. (2020). Existence and stability for nonlinear Caputo-Hadamard fractional delay differential equations. Acta Mathematica Universitatis Comenianae, 89(2), 225-242. http://www.iam.fmph.uniba.sk/amuc/ojs/index.php/amuc/article/view/1097
12. Jarad, F., Abdeljawad, T., & Alzabut, J. (2017). Generalized fractional derivatives generated by a class of local proportional derivatives. The European Physical Journal Special Topics, 226(16), 3457-3471.https://doi.org/10.1140/epjst/e2018-00021-7
13. Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). Theory and applications of fractional differential equations (Vol. 204). elsevier. https://doi.org/10.1016/S03040208(06)800010
14. Krim, S., Salim, A., & Benchohra, M. (2023). On implicit Caputo tempered fractional boundary value problems with delay. https://doi.org/10.5281/zenodo.7682064
15. Li, C., Deng, W., & Zhao, L. (2015). Well-posedness and numerical algorithm for the tempered fractional ordinary differential equations. arXiv preprint arXiv:1501.00376.
16. Luo, D., Luo, Z., & Qiu, H. (2020). Existence and Hyers–Ulam Stability of Solutions for a Mixed Fractional-Order Nonlinear Delay Difference Equation with Parameters. Mathematical Problems in Engineering, 2020(1), 9372406. https://doi.org/10.1155/2020/9372406

17. Malek, A., Derguine, M., & Saadallah, A. (2025). Asymptotic behavior of solutions for a non-isothermal coupled problem with nonlinear boundary conditions in a thin domain. Gulf Journal of Mathematics, 21(1), 398-411. https://doi.org/10.56947/gjom.v21i1.3422
18. Podlubny, I. (1998). Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications (Vol. 198). elsevier. https://doi.org/10.1016/S00765392(99)X80015
19. Rahou, W., Salim, A., Lazreg, J. E., & Benchohra, M. (2025). Implicit Caputo tempered fractional differential equations with retarded and advanced arguments in Banach spaces. Palestine Journal of Mathematics, 14(1).
20. Roufaida, K., Abdelmoumene Djabi, & Ilyas Boukaroura. (2025). A fractional model for electro-mechanical viscoelastic contact with normal compliance condition. Gulf Journal of Mathematics, 21(1), 534-545. https://doi.org/10.56947/gjom.v21i1.3530
21. Sabzikar, F., Meerschaert, M. M., & Chen, J. (2015). Tempered fractional calculus. Journal of Computational Physics, 293, 14-28. https://doi.org/10.1016/j.jcp.2014.04.024
22. Shiri, B., Wu, G. C., & Baleanu, D. (2020). Collocation methods for terminal value problems of tempered fractional differential equations. Applied Numerical Mathematics, 156, 385-395. https://doi.org/10.1016/j.apnum.2020.05.007
23. Wang, J., & Zhang, Y. (2014). Ulam–Hyers–Mittag-Leffler stability of fractional-order delay differential equations. Optimization, 63(8), 1181-1190. https://doi.org/10.1080/02331934.2014.906597
24. Zhou, Y. (2023). Basic theory of fractional differential equations. World scientific. https://doi.org/10.1142/9069

[1] Department of Mechanics, Blida 1 University, Algeria
Email address: Moussa.haoues@yahoo.com (haoues_moussa@univ-blida.dz)