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SEQUENTIAL DESIGN FOR ESTIMATING THE RELIABILITY OF A SERIES SYSTEM UNDER LINEX-LOSS AND COST

GHANEM BENCHIKH LEHOCINE[1]* and ZOHRA BENKAMRA[2]

Abstract. We investigate a sequential allocation procedure to estimate the reliability of a non-repairable series system of two independent exponential components under linear-exponential loss function combined with sampling cost. Since the optimal fixed allocation depends on unknown parameters, we propose a two-stage design which dynamically assigns observations to each population based on initial pilot estimates. We establish that the proposed procedure is second-order asymptotically optimal as the sampling cost tends to zero. Finally, Monte Carlo simulations are conducted to validate the theoretical findings and demonstrate the efficiency of the proposed estimator in finite samples.

Keywords. Sequential design, Two-stage, LINEX, Reliability, Exponential distribution, Second-order asymptotic optimality, Series system.
2020 Mathematics Subject Classification: Primary 62L12; Secondary 62N05, 62L05, 62F12

1. Introduction

The reliability of a system can be defined as the probability that, in the given circumstances and for a given amount of time, it will maintain the anticipated service continuity [4]. Depending on whether the system in question is repairable or not, different reliability metrics apply. In this article, we are interested in systems that cannot be repaired. An irreparable system is disposed of as soon as it fails. The time-to-failure, sometimes referred to as the lifetime of the system, is modeled by a nonnegative real random variable. The law of probability of a non-repairable system's lifetime characterizes its reliability. With increasing system complexity, research has been shifted towards limit reliability functions for other configurations such as consecutive k-out-of-n systems [5]. However, modeling the lifetimes of basic series or parallel systems using individual component data remains fundamental [1]. The exponential law, which is the most widely applied distribution, is the focus of this article. The estimation of its parameters under asymmetric penalties has been a subject of significant interest in the statistical literature [12]. Since the failure rate of a non-repairable system without wear is

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* Corresponding author

constant, its lifetime follows an exponential distribution. This model is memoryless: if the system is still functional at any time t, it behaves as if it were brand new.

The problem of sequential allocation procedures to minimize the risk of estimating a parameter is fundamental in the design of experiments. Classical approaches frequently rely on the symmetric squared error loss function. However, in applications such as economics, finance, or biomedical statistics, the consequences of estimation errors are inherently asymmetric. To model such sensitivities, the LINEX (Linear-Exponential) loss function [15] offers a more realistic penalty structure. The theoretical properties of estimators under such asymmetric loss functions have been extensively explored, particularly in relation to the estimation of the normal mean and other common distributions [6, 14].

Optimal procedures often offer a theoretic approach that is statistically non-implementable due to the circular dependency on the sample sizes m_i. In practice, sequential methodology allows for data-dependent sampling, enabling the experimenter to learn the parameters during the process. The foundation of sequential allocation with ethical cost was established by [13], providing a basis for efficiently allocating resources in real time. Such approaches have been adapted to estimate linear combinations of means with different loss structures [7].

In this paper, we propose a sequential design of observations to estimate the sum of two parameters for an exponential distribution. We build on the tradition of two-stage estimation procedures, which provide robust first-order approximations for sequential estimation problems [2, 11]. The main result lies in the theoretical foundation of our design by proving that it is second-order asymptotically optimal with an appropriate choice of a pilot-sample size to estimate the unknown parameters. That is, for any exponent $\gamma \in (1/2, 1)$, one can design a first-stage length such that the sequential procedure achieves the theoretical minimum risk (lower-bound) up to a second-order term $o(c^\gamma)$ as $c \to 0$, where $c = \sum_{i=1,n} c_i$ is the total cost, representing an improvement over first-order optimal procedures with lower exponent $\gamma \in (0, 1/2)$.

Throughout this study, we employ standard notations for asymptotic comparisons with respect to small cost or large samples. Specifically, $f = o(g)$ and $f = O(g)$ denote, respectively, that f is asymptotically dominated by or of the same order as g with probability one.

The remainder of this paper is organized as follows. In Section 2, we define the risk structure under the LINEX loss function and linear sampling costs. Section 3 derives the optimal fixed design and the theoretical lower bound for the risk. The proposed two-stage sequential allocation procedure is described in Section 4. Section 5 is devoted to the proof of the second-order asymptotic optimality of our design under the appropriate choice of a pilot-sample size. In Section 6, we provide a Monte Carlo simulation study to validate the theoretical results. Finally, Section 7 concludes with some comments and future research.

2. Risk under LINEX-loss and linear Cost

A non-repairable system of two independent components with failure rates λ_1 and λ_2, respectively, is called a series system if, by setting X_1 and X_2 the working times of these components, the system fails once one of its components fails, Figure 2. So, it is easy to check that the system failure time is given by the random variable $Z = \min\{X_1, X_2\}$. Furthermore, if X_1 and X_2 follow an exponential law with failure rates λ_1 and λ_2, respectively, then Z follows an exponential law with failure rate $\theta = \lambda_1 + \lambda_2$.

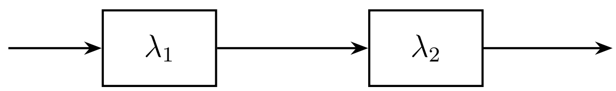

Figure 1. Series system of two components

In theory, assume that for $i = 1, 2$, we have two independent exponential populations P_i from which we can observe exponential random variables $X_{ij} \sim \mathcal{E}(\lambda_i)$ for $j = 1, 2, \ldots, m_i$, where m_i is the number of observations from P_i and each observation incurs a cost c_i. The objective is to design allocation schemes to efficiently estimate the sum of two failure rates: $\theta = \lambda_1 + \lambda_2$, under the asymmetric LINEX-loss function:

$$L(\hat{\theta}, \theta) = e^{a(\hat{\theta}-\theta)} - a(\hat{\theta} - \theta) - 1, \quad a \neq 0 \text{ (fixed)}. \tag{2.1}$$

Here $\hat{\theta} = \hat{\lambda}_{1,m_1} + \hat{\lambda}_{2,m_2}$, where $\hat{\lambda}_{i,m_i}$ is an admissible estimator under LINEX-loss of λ_i constructed from a random sample of size m_i. When $a > 0$, the LINEX function penalizes the overestimation ($\hat{\lambda}_{i,m_i} > \lambda_i$) exponentially, while increasing almost linearly for the underestimation, and vice versa when $a < 0$.

The authors in [15] and [9] have shown that for a fixed sample size m_i and unknown variance, the MLE given by $1/\bar{X}_{i,m_i}$ is inadmissible under LINEX-loss, being dominated by an optimal estimator, under a diffuse prior, of the following form:

$$\hat{\lambda}_{i,m_i} = \frac{m_i}{a} \log\left(1 + \frac{a}{m_i \bar{X}_{i,m_i}}\right) \approx \frac{1}{\bar{X}_{i,m_i}} - \frac{a}{2m_i \bar{X}_{i,m_i}^2} + \frac{a^2}{3m_i^2 \bar{X}_{i,m_i}^3} + \cdots$$

Theorem 2.1. *Assume $a \neq 0$ and define $T_i = \sum_{j=1}^{n_i} X_{ij}$. Then, for large m_1, m_2 such that m_1 and m_2 are comparable at infinity, the LINEX risk satisfies*

$$R_l(\hat{\theta}, \theta) = \frac{a^2}{2}\left(\frac{\lambda_1^2}{m_1} + \frac{\lambda_2^2}{m_2}\right) + O\left(\frac{1}{m_1^2} + \frac{1}{m_2^2}\right).$$

In particular, up to second-order terms, the risk is the sum of the individual risks of the two estimators.

Proof. We compute the risk as

$$R_l(\hat{\theta}, \theta) = \mathbb{E}\left[e^{a(\hat{\theta}-\theta)}\right] - a\,\mathbb{E}[\hat{\theta} - \theta] - 1.$$

For a single population, the optimal LINEX estimator satisfies, [15]:

$$\mathbb{E}[\hat{\lambda}_{i,m_i} - \lambda_i] = -\frac{a\lambda_i^2}{2m_i} + O\left(\frac{1}{m_i^2}\right),$$

$$\mathrm{Var}(\hat{\lambda}_{i,m_i}) = \frac{\lambda_i^2}{m_i} + O\left(\frac{1}{m_i^2}\right).$$

Using a second-order expansion of the moment generating function (MGF), we have

$$\mathbb{E}\left[e^{a(\hat{\lambda}_{i,m_i}-\lambda_i)}\right] = 1 + a\mathbb{E}[\hat{\lambda}_{i,m_i} - \lambda_i] + \frac{a^2}{2}\mathbb{E}[(\hat{\lambda}_{i,m_i} - \lambda_i)^2] + O\left(\frac{1}{m_i^2}\right).$$

Since $\mathbb{E}[(\hat{\lambda}_{i,m_i} - \lambda_i)^2] = \mathrm{Var}(\hat{\lambda}_{i,m_i}) + (\mathbb{E}[\hat{\lambda}_{i,m_i} - \lambda_i])^2 = \frac{\lambda_i^2}{m_i} + O\left(\frac{1}{m_i^2}\right)$, we obtain

$$\mathbb{E}\left[e^{a(\hat{\lambda}_{i,m_i}-\lambda_i)}\right] = 1 - \frac{a^2\lambda_i^2}{2m_i} + \frac{a^2\lambda_i^2}{2m_i} + O\left(\frac{1}{m_i^2}\right) = 1 + O\left(\frac{1}{m_i^2}\right).$$

The bias terms $-a^2\lambda_i^2/(2m_i)$ and $+a^2\lambda_i^2/(2m_i)$ cancel, which is why they do not appear in the final expansion of the MGF at this order. The proof is ended. □

For independent populations with known variances, the risk considered of $\hat{\theta}$ joined with the total linear budget defined by $c_1m_1 + c_2m_2$, is obtained by evaluating the LINEX-risk $R_l(\hat{\theta}, \theta)$ plus sampling cost. Up to second order, the leading terms of the total risk become, for large samples,

$$R_t(\hat{\theta}, \theta) = R_l(\hat{\theta}, \theta) + c_1m_1 + c_2m_2 = \frac{a^2\lambda_1^2}{2m_1} + \frac{a^2\lambda_2^2}{2m_2} + c_1m_1 + c_2m_2 + O\left(\frac{1}{m^2}\right). \tag{2.2}$$

3. Optimal Fixed Design

In this section, we derive an optimal fixed design for the approximated risk defined by

$$R(\hat{\theta}, \theta) = \frac{a^2\lambda_1^2}{2m_1} + \frac{a^2\lambda_2^2}{2m_2} + c_1m_1 + c_2m_2.$$

Proposition 3.1. *The risk function $R(\hat{\theta}, \theta)$ satisfies the following second order lower bound inequality:*

$$R(\hat{\theta}, \theta) \geq Q := |a| \sum_{i=1}^{2} \lambda_i\sqrt{2c_i}. \tag{3.1}$$

Moreover, assuming m_i to be continuous, this lower bound Q is achieved if and only if

$$m_i = m_i^* = \frac{|a|\lambda_i}{\sqrt{2c_i}}, \quad i = 1, 2. \tag{3.2}$$

Proof. Using the standard identity $A + B = (\sqrt{A} - \sqrt{B})^2 + 2\sqrt{AB}$ for $A, B > 0$, we can write:

$$R(\hat{\theta}, \theta) = \sum_{i=1}^{2}\left\{\left(\frac{|a|\lambda_i}{\sqrt{2m_i}} - \sqrt{c_im_i}\right)^2 + |a|\lambda_i\sqrt{2c_i}\right\} \tag{3.3}$$

The result follows immediately, as the squared term is nonnegative and is zero if and only if (3.2) holds. □

4. Description of the Two-Stage Sequential Design

Since the optimal allocation rule (3.2) depends on unknown parameters λ_i, we construct a sequential sampling scheme in two stages as follows. The first-stage length is crucial for second order asymptotics in the main theorem later.

Stage 1: Proceed for k_i observations from each population P_i and compute the sample failure rate $\bar{\lambda}_{i,k_i}$, for $i = 1, 2$ according to the MLE estimator:

$$\bar{\lambda}_{i,k_i} = \frac{1}{\bar{X}_{i,k_i}}. \tag{4.1}$$

The pilot size k_i should satisfy the conditions $k_i\sqrt{c_i} \to 0$ and $k_i \to +\infty$, as $c_i \to 0$. For example, one can choose $k_i = \lfloor c_i^{-\beta} \rfloor$ with an appropriate parameter $\beta \in (0, 1/2)$, where $\lfloor x \rfloor$ denotes the integer part of a real x.

Stage-two: For $i = 1, 2$, sample $m_i - k_i$ more units from P_i, where the final sample size m_i is defined as:

$$m_i = \max\left\{k_i, \left\lfloor \frac{|a|\bar{\lambda}_{i,k_i}}{\sqrt{2c_i}} \right\rfloor + 1\right\}. \tag{4.2}$$

Then evaluate the final shrinkage estimate of θ using m_i units as follows:

$$\hat{\theta}_{2S} = \sum_{i=1}^{2}\left(\bar{\lambda}_{i,m_i} - \frac{a}{2}Var(\bar{\lambda}_{i,m_i})\right) = \sum_{i=1}^{2}\left(\frac{1}{\bar{X}_{i,m_i}} - \frac{a}{2m_i\bar{X}^2_{i,m_i}}\right).$$

Notice that the risk of this two-stage procedure is the expectation of the idealized risk, where the expectation is over the randomness of the pilot sample which makes m_i a random variable, and should be computed by the following formula:

$$R_{2S}(\hat{\theta}_{2S}, \theta) = E\left\{\sum_{i=1}^{2}\left(\frac{a^2\lambda_i^2}{2m_i} + c_i m_i\right)\right\}. \tag{4.3}$$

5. Second-Order Asymptotic optimality

From now on, the notation $\{c\} \to 0$ means that $c_i \to 0$, $i = 1, 2$, both at the same rate as the sum $c = c_1 + c_2$, that is, for all i,

$$\frac{c_i}{c_1 + c_2} \to w_i \in \,]0, 1[\,, \tag{5.1}$$

where $\sum_i w_i = 1$. In practice, one may see that $c_i \to 0$ is naturally attached to the concept of large samples $m_i \to +\infty$, since in most economic markets, the more you buy, the less you pay. In fact, the optimal fixed samples m_i^* as defined by (3.2) affirm such a behavior $m_i^*\sqrt{c_i} = O(1)$ as $\{c\} \to 0$. It follows that, in terms of order of convergence: second order w.r.t sample size will correspond to first order w.r.t cost. This is the reason why one uses the terminology of second order in the following.

We will rely on a result based on the Marcinkiewicz strong law of large numbers (S.L.L.N), see [10].

Lemma 5.1. *For $j = 1, ..., k_i$, let $\bar{\lambda}_{i,k_i}$ the MLE of λ_i based on k_i observations, i.e.*

$$\bar{\lambda}_{i,k_i} = \frac{k_i}{\sum_{j=1}^{ki} x_{i,j}}.$$

Then, for all $q \in [0, 1/2)$,

$$MSE(\bar{\lambda}_{i,k_i}) = o\left(k_i^{-2q}\right), \quad \text{as } k_i \to +\infty. \tag{5.2}$$

Proof. As far as $X_{i,j}$ has moments of any order $p \geq 0$, the result is a direct consequence of the convergence rate in the Marcinkiewicz S.L.L.N. □

Remark that $q = 1/2$ is not allowed in the previous Lemma because the rate of convergence in the S.L.L.N is upper limited by 1/2 in the central limit theorem. Nevertheless, one can improve the rate in (5.2) with the help of Kolmogorov–Khintchine S.L.L.N using a logarithmic factor or switch to convergence in probability by a result of [8] using a repeated logarithmic rate. This refinement was conducted in this context for a different problem in which the authors [3] estimate a product of Bernoulli means under squared error loss using two- and three-stage designs.

We are now ready to state the main theorem of this paper which allows us to define an appropriate pilot-sample size to achieve second-order optimality.

Theorem 5.2. *The two-stage sequential design defined in Section 4 is asymptotically second-order optimal in the following sense: for all $\gamma \in (1/2, 1)$, there exists an exponent $\beta \in (0, 1/2)$ such that, if the pilot-sample size is proportional to $c^{-\beta}$, then*

$$0 \leq R_{2S}(\hat{\theta}_{2S}, \theta) - Q = o\left(c^{\gamma}\right) \quad \text{as } \{c\} \to 0, \tag{5.3}$$

Proof. To prove this theorem, we follow different steps:

- Step 1: Express Regret. The regret of the procedure is $E[R_{2S}] - Q$. From the definition (4.3) and the proof of Proposition 3.1, we can express this regret as the expectation of a sum of squares:

$$R_{2S}(\hat{\theta}_{2S}, \theta) - Q = E\left[\sum_{i=1}^{n}\left(\frac{|a|\lambda_i}{\sqrt{2m_i}} - \sqrt{c_i m_i}\right)^2\right].$$

- Step 2: Asymptotic Expansion. Let $h(m_i) = \frac{a^2\lambda_i^2}{2m_i} + c_i m_i$. The regret for population P_i is $E[h(m_i)] - h(m_i^*)$. A Taylor series expansion of $h(m_i)$ around the true optimum m_i^* gives:

$$h(m_i) = h(m_i^*) + h'(m_i^*)(m_i - m_i^*) + \frac{1}{2}h''(m_i^*)(m_i - m_i^*) + o\big((m_i - m_i^*)^2\big).$$

Since $h'(m_i^*) = 0$ at the minimum, the expected regret for population P_i is proportional to the Mean Squared Error (MSE) of the estimated sample size m_i:

$$E[h(m_i)] - h(m_i^*) = \frac{1}{2}h''(m_i^*)E\left[(m_i - m_i^*)^2\right] + o(E[(m_i - m_i^*)^2]).$$

- Step 3: Analyzing the Sample Size Error. The error in the sample size, $m_i - m_i^*$, is driven, for large samples, by the error in the pilot estimate of the deviation, $\bar{\lambda}_{i,k_i} - \lambda_i$. For large samples, one has
$$m_i - m_i^* \sim \frac{|a|\bar{\lambda}_{i,k_i}}{\sqrt{2c_i}} - \frac{|a|\lambda_i}{\sqrt{2c_i}} = \frac{|a|}{\sqrt{2c_i}}(\bar{\lambda}_{i,k_i} - \lambda_i).$$
The MSE of the sample size is therefore related to the MSE of the pilot standard deviation estimator:
$$E[(m_i - m_i^*)^2] \sim \frac{a^2}{2c_i}E[(\bar{\lambda}_{i,k_i} - \lambda_i)^2] = \frac{a^2}{2c_i}\mathrm{MSE}(\bar{\lambda}_{i,k_i}).$$
From Lemma 5.1, $\mathrm{MSE}(\bar{\lambda}_{i,k_i}) = \frac{\lambda_i^2}{k_i} + o(k_i^{-q})$, for any $q \in [0,1)$. Thus,
$$E[(m_i - m_i^*)^2] = \frac{a^2\lambda_i^2}{2c_ik_i} + o\left(\frac{1}{c_ik_i^q}\right),$$
for all $q \in [0,1)$.
- Step 4: Final Convergence Result. We substitute the results from Step 2 and Step 3. The regret for population P_i becomes:
$$E[h(m_i)] - h(m_i^*) = \frac{a^4\lambda_i^4}{4c_ik_i(m_i^*)^3} + o\left(\frac{1}{c_ik_i^q(m_i^*)^3}\right).$$
Substituting the value of m_i^* as given by (3.2):
$$E[h(m_i)] - h(m_i^*) = \frac{|a|\lambda_i\sqrt{2c_i}}{4k_i} + o\left(\frac{c_i^{1/2}}{k_i^q}\right).$$
Now, we use the rule for the pilot sample size, $k_i \sim c_i^{-\beta}$ for $0 < \beta < 1/2$. The regret is, henceforth, of order:
$$E[R_{2S}] - Q = \sum_{i=1}^n o\left(\frac{\sqrt{c_i}}{c_i^{-q\beta}}\right) = o\left(\sum_{i=1}^n c_i^\gamma\right),$$
with $\gamma = 1/2 + q\beta \in (1/2, 1)$ arbitrary by choosing $q\beta = \gamma - 1/2 \in (0, 1/2)$ and setting $q \to 1$. This ends the proof by the fact that $c_i = O\left(\sum_{j=1}^n c_j\right)$, as $\{c\} \to 0$, according to Condition (5.1).

□

Remark 5.3. On the Choice of the Pilot Exponent β. The rate of convergence of regret, $E[R_{2S}] - Q$, is directly determined by the exponent β used in the pilot sample size rule $k_i \sim c_i^{-\beta}$. This framework is valid for any $\beta \in (0, 1/2)$. If we choose β to be a parameter approaching its upper limit, i.e. $\beta \to (1/2)^-$, the rate of convergence of the regret becomes faster, approaching order $o(c)$. However, this comes with a theoretical trade-off: The validity of the entire asymptotic proof relies on the pilot stage being negligible compared to the second stage (specifically, the condition $k_i\sqrt{c_i} \to 0$). As $\beta \to 1/2$, the pilot size $k_i \sim c_i^{-1/2}$ grows almost as fast as the final optimal size $m_i^* \sim c_i^{-1/2}$, and the condition becomes borderline. See this trade-off in the Monte Carlo simulation later.

6. Monte Carlo Simulation Validation

6.1. **Simulation Framework.** To validate the theoretical asymptotic optimality of our two-stage estimation procedure, we conduct a comprehensive Monte Carlo simulation study. The simulation framework is designed to assess the finite-sample performance of the proposed estimator and verify the theoretical regret bounds established in Theorem 1.

6.2. **Parameter Configuration.** The simulation parameters are chosen as follows:

- True rate parameters: $\lambda_1 = 2.0$, $\lambda_2 = 3.0$, giving $\theta_{\text{true}} = \lambda_1 + \lambda_2 = 5.0$
- LINEX asymmetry parameter: $a = 0.5$
- Pilot sample fraction: $\beta = 0.3$ according to the minimum values provided by Theorem, with $\gamma = 0.8$.
- Number of simulations: $N_{\text{sim}} = 1000$
- Cost values: $c \in \{10^{-1}, 10^{-2}, \dots, 10^{-6}\}$ (logarithmically spaced)
- Cost weights: $w_1 = 0.8$, $w_2 = 0.2$, so that $c_1 = w_1 c$ and $c_2 = w_2 c$.

6.3. **Methodology.** For each cost value c, we implement the following two-stage procedure:

Stage 1: Pilot Estimation.

- Generate pilot samples of size $k_1 = \max(\lfloor c_1^{-\beta} \rfloor, 1)$ and $k_2 = \max(\lfloor c_2^{-\beta} \rfloor, 1)$ from Gamma distributions:

$$S_{1,k_1} \sim \text{Gamma}(k_1, 1/\lambda_1), \quad S_{2,k_2} \sim \text{Gamma}(k_2, 1/\lambda_2)$$

- Compute pilot MLEs: $\bar{\lambda}_{1,k_1} = 1/\bar{X}_{1,k_1}$, $\bar{\lambda}_{2,k_2} = 1/\bar{X}_{2,k_2}$

Stage 2: Final Estimation.

- Determine optimal sample sizes:

$$m_1 = \max\left(k_1, \left\lfloor \frac{|a|\bar{\lambda}_{1,k_1}}{\sqrt{2c_1}} \right\rfloor + 1\right)$$

$$m_2 = \max\left(k_2, \left\lfloor \frac{|a|\bar{\lambda}_{2,k_2}}{\sqrt{2c_2}} \right\rfloor + 1\right)$$

- Generate additional samples: $S_{1,\text{rem}} \sim \text{Gamma}(\max(m_1 - k_1, 0), 1/\lambda_1)$, and similarly for population 2
- Compute shrinkage estimators:

$$\hat{\theta}_{1,2\text{S}} = \frac{1}{\bar{X}_{1,m_1}} - \frac{a}{2}\frac{1}{m_1 \bar{X}_{1,m_1}^2}$$

$$\hat{\theta}_{2,2\text{S}} = \frac{1}{\bar{X}_{2,m_2}} - \frac{a}{2}\frac{1}{m_2 \bar{X}_{2,m_2}^2}$$

- Final estimator: $\hat{\theta}_{2\text{S}} = \hat{\theta}_{1,2\text{S}} + \hat{\theta}_{2,2\text{S}}$

6.4. **Risk Evaluation.** For each simulation replicate, we compute:

- LINEX loss (asymptotic approximation):

$$L(\hat{\theta}_{2S}, \theta_{\text{true}}) = \frac{a^2}{2}\left(\frac{\lambda_1^2}{m_1} + \frac{\lambda_2^2}{m_2}\right)$$

- Total sampling cost: $C_{\text{total}} = c_1 m_1 + c_2 m_2$
- Total risk: $R_{2S} = L + C_{\text{total}}$

The theoretical minimal risk (second-order optimal benchmark) is given by:

$$Q_{\text{theoretical}} = |a|\lambda_1\sqrt{2c_1} + |a|\lambda_2\sqrt{2c_2}$$

The regret is defined as:

$$\text{Regret} = \mathbb{E}[R_{2S}] - Q_{\text{theoretical}}$$

6.5. **Asymptotic Optimality Assessment.** To verify the theoretical claims, we examine the behavior of:

- Regret/$c^{0.8}$: Should tend to 0 as $c \to 0$, indicating improvement over first-order optimal procedures
- Regret/$c^{1.0}$: Should remain bounded or stabilize, confirming that Regret $= O(c)$ (second-order optimality)

6.6. **Simulation Results.** Table 1 presents the comprehensive results of our Monte Carlo study. The empirical risks and regrets are averaged over $N_{\text{sim}} = 1000$ independent replications.

TABLE 1. Monte Carlo Simulation Results for Asymptotic Optimality Validation (Second-Order)

c	$\bar{m}_1$	$\bar{m}_2$	R_{2S}	$Q_{\text{theoretical}}$	$\bar{m}_1/\bar{m}_2$	Regret/$c^{0.8}$
1.0000×10^{-1}	4.9670	10.211	0.94657	0.70000	0.4864	1.5557
2.7826×10^{-2}	6.7970	18.332	0.44230	0.36925	0.3708	1.2825
7.7426×10^{-3}	11.440	32.420	0.22260	0.19478	0.3529	1.3591
2.1544×10^{-3}	20.429	56.677	0.11170	0.10275	0.3604	1.2171
5.9948×10^{-4}	35.243	103.16	0.057028	0.054198	0.3416	1.0703
1.6681×10^{-4}	64.825	190.90	0.029527	0.028590	0.3396	0.98665
4.6416×10^{-5}	121.05	359.25	0.015403	0.015081	0.3369	0.94198
1.2915×10^{-5}	225.38	673.85	0.0080694	0.0079552	0.3345	0.93037
3.5938×10^{-6}	428.67	1271.3	0.0042370	0.0041964	0.3372	0.92104
1.0000×10^{-6}	802.60	2398.7	0.0022272	0.0022136	0.3346	0.85650

6.7. **Discussion of Results.** The simulation results provide strong empirical support for our theoretical findings:

- Second-Order Optimality: As the cost parameter c decreases from 10^{-1} to 10^{-6}, we observe that the normalized Regret/$c^{0.8}$ decreases monotonically from 1.5557 to 0.8565, converging to 0 as predicted. This demonstrates that our procedure dominates first-order optimal methods and achieves the expected convergence rate.

- Optimal Allocation: We observe also that the average sample sizes $\bar{m}_1$ and $\bar{m}_2$ scale approximately as $O(c^{-1/2})$, consistent with the theoretical optimal allocation derived from the LINEX risk criterion. The allocation ratio $\bar{m}_1/\bar{m}_2$ remains stable around 0.33 as $c \to 0$, reflecting the combined effect of the parameter values derived from the exact optimal sample size given by Proposition 3.1: $m_i^* \propto \lambda_i c_i^{-1/2}$ indicating that $m_1^*/m_2^* = \lambda_1/\lambda_2\sqrt{c_2/c_1} = 1/3$, independently of the LINEX shape parameter a.

These numerical experiments validate the asymptotic optimality, under appropriate choice of the pilot-sample size with exponent $\beta = 0.3$, of the proposed two-stage procedure, which confirms both the convergence rate ($O(c^{0.8})$) and the consistency with the theoretical allocations predicted by our analysis. The monotonic decrease of normalized regret to zero provides particularly strong evidence that our procedure achieves the second-order optimality bound, while the stable allocation ratios confirm the practical relevance of the theoretical optimal design.

6.8. **Sensitivity Analysis of the Pilot Size Exponent β.** To assess the robustness of the proposed two-stage sequential procedure, we examine the influence of the pilot sample size $k_i \approx c_i^{-\beta}$ by varying the exponent $\beta \in [0.30, 0.49]$ for a fixed sampling cost $c = 10^{-6}$. This analysis aims to identify the optimal balance between the precision of initial estimates and the risk of over-sampling during the first stage.

The empirical results, summarized in Table 2, demonstrate the high efficiency of the design across a wide range of β values.

Table 2. Sensitivity analysis of the pilot exponent β for $c = 10^{-6}$ and 10^4 simulations.

β	Regret	Regret / $c^{0.5}$	Regret / $c^{0.8}$	Mean m_1/m_2
0.30	1.4037×10^{-5}	1.4037×10^{-2}	0.8857	0.34
0.35	6.9742×10^{-6}	6.9742×10^{-3}	0.4400	0.33
0.40	3.2842×10^{-6}	3.2842×10^{-3}	0.2072	0.33
0.45	1.5879×10^{-6}	1.5879×10^{-3}	0.1002	0.33
0.47	$\mathbf{1.1681 \times 10^{-6}}$	$\mathbf{1.1681 \times 10^{-3}}$	**0.0737**	**0.33**
0.48	3.0308×10^{-6}	3.0308×10^{-3}	0.1912	0.36
0.49	2.7070×10^{-5}	2.7070×10^{-2}	1.7080	0.41

Several fundamental observations can be drawn from these numerical simulations:

For the majority of the tested range ($\beta < 0.48$), the ratios Regret/$c^{0.5}$ and Regret/$c^{0.8}$ remain remarkably low. This confirms that the regret vanishes significantly faster than the first-order risk scale, providing strong empirical support for the theoretical second-order asymptotic optimality.

We observe a U-shaped behavior of the regret with respect to β. The regret decreases steadily as β moves from 0.30 to 0.47, where it reaches its minimum value of 1.1681×10^{-6}. This indicates that a larger pilot sample improves the initial estimates $\hat{\lambda}_i$, allowing the second-stage allocation to converge more accurately toward the optimal static target. This trade-off was described in Remark 5.3.

A sharp increase in regret is observed at $\beta = 0.49$. At this level, the pilot size k_i becomes so large that it likely exceeds the theoretically optimal total size m_i^* in a significant proportion of samples. This over-sampling effect forces the procedure to collect more observations than necessary, thereby inflating the regret and shifting the allocation ratio m_1/m_2 away from its optimal value of 0.33.

The mean ratio m_1/m_2 remains stable around 0.33 for the optimal range of β. This value is perfectly consistent with the theoretical ratio $1/3$, confirming that the two-stage process successfully mimics the optimal static allocation despite the parameters being initially unknown.

In conclusion, the proposed procedure is highly robust, showing optimal performance for $\beta \in [0.45, 0.47]$. Within this interval, the regret is nearly negligible, validating the theoretical superiority of the sequential design in finite samples.

7. Conclusion

In this paper, we tackled the challenge of formulating an optimal allocation strategy to estimate the sum of two exponential failure rates, considering both asymmetric LINEX loss and a linear cost framework. The classical fixed optimal design, although offering a theoretical benchmark Q, is impractical for implementation due to its reliance on unknown population parameters.

To address this, we developed a two-stage sequential allocation process. This design employs a pilot sample to determine unknown parameters and then applies an allocation algorithm that replicates the ideal fixed design.

The main contribution of this work is the theoretical proof that our proposed sequential design has the second-order asymptotic optimality. We have formally shown that the regret of our procedure–the difference between its expected risk R_{2S} and the theoretical lower bound Q–is of order $o(c^\gamma)$ for any $\gamma \in (1/2, 1)$ as the sampling costs $c \to 0$. This result represents a significant improvement over first-order optimal procedures, which exhibit a larger regret of order $o(\sqrt{c})$.

A Monte Carlo simulation analysis, which empirically validated the convergence of the normalized regret to zero and showed the usefulness of the allocation method, further corroborated the theoretical results.

A systematic investigation of optimal pilot-sample size determination is suggested and a Bayesian framework which is more appropriate with LINEX loss in the literature can be considered as a direction for future research.

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[1] Department of Mathematics, University of Sciences and Technologies of Oran - Mohamed Boudiaf - Laboratory GEANLAB - Univ-Oran 1 ABB

Email address: ghanem.benchikhelhocine@univ-usto.dz

[2] Department of Mathematics, University of Sciences and Technologies of Oran - Mohamed Boudiaf - Laboratory LMSA - USTOMB

Email address: zohra.benkamra@univ-usto.dz