

## ON GEOMETRY OF WARPED PRODUCT PSEUDO-SLANT SUBMANIFOLDS ON GENERALIZED SASAKIAN SPACE FORMS

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**ABSTRACT.** In the present paper, we consider warped product pseudo-slant submanifolds of generalized Sasakian space forms. We obtain an interesting inequality in terms of the second fundamental forms and the scalar curvature using Gauss equation. Next we compute to characterize the relation between the warping function and the squared norm of mean curvature of warped product pseudo-slant submanifolds of generalized Sasakian space forms under some certain conditions. Finally, we derive warped product pseudo-slant submanifolds in generalized Sasakian space forms to become Riemannian product with various mathematical and physical expressions.

### 1. INTRODUCTION

The geometry of submanifolds plays very important role in differential geometry. In [9], B. Y. Chen study warped product slant submanifolds as a generalization of both holomorphic and totally real submanifolds of a Kähler manifold and also obtained an inequality for the second fundamental forms in terms of warping functions. Semi-slant submanifolds of a Kähler manifold were proposed by N. Papaghiuc [16]. A. Carriazo established the notion of bi-slant submanifolds of an almost Hermitian manifold in [8], as a generalization of semi-slant submanifolds. The notion of quasi-slant submanifolds as another important generalized class of slant submanifold was introduced by Etayo [11]. One of the classes of bi-slant submanifolds in almost Hermitian manifolds were studied by A. Carriazo but B. Sahin [17] changed the name of these submanifolds, called pseudo-slant submanifolds. Many authors studied pseudo-slant submanifolds in different almost contact manifolds. V. A. Khan and M. A. Khan studied the contact version of pseudo-slant submanifolds in [13]. Uddin, et al.[20] obtained some inequalities of warped product submanifolds in different structures. Sahin renamed pseudo-slant submanifolds as hemi-slant submanifolds in [18]. Many authors like [12], [19] etc studied hemi-slant submanifolds in different contact metric manifolds.

The concept of warped product takes an important role in differential geometry as

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well as in physics, particularly in general theory of relativity [15] which provides the best mathematical model of the universe. The warped product model plays a crucial role to build basic cosmological models such as the Robertson-Walker spacetime, the Friedmann cosmological model, the standard static spacetime etc in [9].

Let  $(B, g_B)$  and  $(F, g_F)$  be two Riemannian manifolds and  $f > 0$  is a differential function on  $B$ . Consider the product manifold  $B \times F$  with its projections  $\pi : B \times F \rightarrow B$  and  $\sigma : B \times F \rightarrow F$ . The warped product  $M = B \times_f F$  is the manifold  $B \times F$  with the Riemannian structure such that  $\|X\|^2 = \|\pi^*(X)\|^2 + f^2(\pi(p))\|\sigma^*(X)\|^2$ , for any vector field  $X$  on  $M$ . Thus we have  $g_M = g_B + f^2g_F$  holds on  $M$ . Here  $B$  is called the base of  $M$  and  $F$  the fiber. The function  $f$  is called the warping function of the warped product [15]. The concept of warped products was first introduced by Bishop and O’Neil [4] to construct examples of Riemannian manifold with negative curvature. Now we have the following lemma from [15].

**Lemma 1.1.** *Let  $M = B \times_f F$  be a warped product,  $\nabla, \nabla^B, \nabla^F$  be the Levi-Civita connection on  $M, B$  and  $F$  respectively. If  $X, Y \in \chi(B), U, W \in \chi(F)$ , then*

- (1)  $\nabla_X Y = \nabla_X^B Y,$
- (2)  $\nabla_X U = \nabla_U X = (X \ln f)U$
- (3)  $\nabla_U W = -\frac{g(U,W)}{f} \text{grad}_B f + \nabla_U^F W,$

for any  $X, Y \in \Gamma(TB)$  and  $U, W \in \Gamma(TF)$  where  $\nabla$  and  $\nabla^F$  denote the Levi-Civita connections on  $M$  and  $F$ , respectively, and  $\text{grad } f$  is the gradient of  $f$ .

Many authors study the warped product submanifolds and give many fruitful results for various submanifolds in different known spaces in [3], [21], etc.

In this paper, first we give basic formulas in preliminaries and define pseudo slant submanifolds. Next we establish some basic lemmas which are helpful to prove our main results and give an example of warped product pseudo slant submanifolds. Next we obtain a geometric inequality for the warped product pseudo slant submanifolds in generalized Sasakian space forms and derive the relation between the second fundamental forms and the scalar curvature of the submanifold with the warping functions. Later we discuss about the geometric inequalities for the squared norm of the mean curvature and the warping functions of warped product pseudo slant submanifolds in generalized Sasakian space forms. Then we find an application of the Hessian of warping function. Later we discuss about warped product pseudo slant submanifolds to be a simply Riemannian product in generalized Sasakian space forms in terms of kinetic energy function and Hamiltonian under some certain conditions.

## 2. PRELIMINARIES

Let  $\tilde{M}$  be a  $(2p + 1)$ -dimensional smooth manifold with an almost contact structure  $(\phi, \xi, \eta, g)$  satisfying[5]

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi) \quad (2.1)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \quad (2.2)$$

P. Alegre, D. E. Blair and A. Carriazo introduced the concept of generalized Sasakian space forms in [2]. An almost contact metric manifold  $\tilde{M}$  with an almost contact metric structure  $(\phi, \xi, \eta, g)$  is called a generalized Sasakian space forms if there exist three functions  $f_1, f_2, f_3$  on  $\tilde{M}$  such that the curvature tensor  $R$  is given by

$$\begin{aligned} R(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y \\ &\quad - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} + f_3\{g(X, Z)\eta(Y)\xi \\ &\quad - g(Y, Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\}. \end{aligned} \quad (2.3)$$

for all vector fields  $X, Y, Z$  on  $\tilde{M}$ .

If  $f_1 = \frac{c+3}{4}$ ,  $f_2 = f_3 = \frac{c-1}{4}$  then  $\tilde{M}$  is a Sasakian space forms.

If  $f_1 = \frac{c-3}{4}$ ,  $f_2 = f_3 = \frac{c+1}{4}$  then  $\tilde{M}$  is a Kenmotsu space forms.

If  $f_1 = f_2 = f_3 = \frac{c}{4}$  then  $\tilde{M}$  is a cosymplectic space forms.

In generalized Sasakian space forms  $\tilde{M}^{(2p+1)}(f_1, f_2, f_3)$ , we have the following relations [2]

$$\nabla_X \xi = (f_3 - f_1)\phi X, \quad (2.4)$$

$$(\nabla_X \phi)(Y) = (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X], \quad (2.5)$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y) = (f_3 - f_1)g(\phi X, Y). \quad (2.6)$$

**Example 2.1.** In [2] P. Alegre, D. Blair and A. Carriazo showed that  $\mathbb{R} \times \mathbb{C}^m$  is a generalized Sasakian space forms with

$$f_1 = -\frac{(f')^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f},$$

where  $f = f(t)$ ,  $t \in \mathbb{R}$  and  $f'$  denotes derivative of  $f$  with respect to  $t$ .

Let  $M$  be a submanifold of an almost contact metric manifold  $\tilde{M}$  with induced metric  $g$  and let  $\nabla$  and  $\nabla^\perp$  be the induced connections on the tangent bundle  $TM$  and normal bundle  $T^\perp M$  of  $M$  respectively. Let  $\mathcal{F}$  denote the algebra of

smooth function on  $M$  and  $\Gamma(TM)$  denotes the  $\mathcal{F}$ -module of smooth sections of  $TM$  over  $M$ . Then the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (2.7)$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad (2.8)$$

for each  $X, Y \in \Gamma(TM)$  and  $N \in \Gamma(T^\perp M)$ , where  $h$  and  $A_N$  are the second fundamental forms and the shape operator (corresponding to the normal vector field  $N$ ), respectively, for the immersion of  $M$  into  $\tilde{M}$ . They are related as

$$g(h(X, Y), N) = g(A_N X, Y), \quad (2.9)$$

where  $g$  denotes the Riemannian metric on  $\tilde{M}$  as well as the one induced on  $M$ . The mean curvature vector  $H$  of  $M$  is given by  $H = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i)$ , where  $n$  is the dimension of  $M$  and  $\{e_1, e_2, \dots, e_n\}$  is a local orthonormal frame of vector fields on  $M$ .

Moreover, for a submanifold  $M$ , the Gauss equation is defined as

$$\begin{aligned} \tilde{R}(X, Y, Z, W) &= R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) \\ &\quad - g(h(X, W), h(Y, Z)), \end{aligned} \quad (2.10)$$

for any  $X, Y, Z, W \in \Gamma(TM)$ .

The scalar curvature  $\tau$  for a submanifold  $M$  of an almost contact metric manifold  $\tilde{M}$  is given by

$$\tau(TM) = \sum_{1 \leq r \neq s \leq n} K(e_r \wedge e_s), \quad (2.11)$$

where  $K(e_r \wedge e_s)$  denotes the sectional curvature of the  $n$ -plane section spanned by  $e_r$  and  $e_s$ .

**Definition 2.2.** A submanifold  $M$  of an almost contact metric manifold  $\tilde{M}$  is said to be totally umbilical if the second fundamental forms satisfies  $h(X, Y) = g(X, Y)H$ , for all  $X, Y \in \Gamma(TM)$ .

**Definition 2.3.** A submanifold  $M$  is said to be totally geodesic if  $h(X, Y) = 0$ , for all  $X, Y \in \Gamma(TM)$  and minimal if  $H = 0$ .

Let  $\{e_1, e_2, \dots, e_n\}$  be a local orthonormal basis of tangent space  $TM$  and  $e_s$  belong to the orthonormal basis  $\{e_{n+1}, e_{n+2}, \dots, e_{2p+1}\}$  of the normal bundle  $T^\perp M$ , we put

$$h_{ij}^s = g(h(e_i, e_j), e_s) \text{ and } \|h\|^2 = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)). \quad (2.12)$$

For any  $X \in \Gamma(TM)$  and  $N \in \Gamma(T^\perp M)$

$$\phi X = PX + FX, \quad \text{and } \phi N = tN + fN \quad (2.13)$$

where  $PX$  ( $tN$ ) is the tangential component and  $FX$  ( $fN$ ) is the normal component of  $\phi X$ .

A submanifold  $M$  of an almost contact metric manifold  $\tilde{M}$  is said to be invariant if  $F$  is identically zero, that is  $\phi X \in \Gamma(TM)$  and anti-invariant if  $P$  is identically zero, that is  $\phi X \in \Gamma(T^\perp M)$ , for any  $X \in \Gamma(TM)$ .

Also, we have

$$\begin{aligned} (1)g(PX, Y) &= -g(X, PY) \\ (2)\|P\|^2 &= \sum_{i,j=1}^n g^2(\phi e_i, e_j). \end{aligned} \quad (2.14)$$

for each vector fields  $X, Y$ , where  $g$  is Riemannian metric and  $P$  is an  $(1, 1)$ -tensor field.

Now, let  $f$  be a differential function defined on  $M^n$ . Hence the gradient  $\nabla f$  is given by

$$\begin{aligned} (1)g(\nabla f, X) &= Xf \\ (2)\|\nabla f\|^2 &= \sum_{i=1}^n (e_i(f))^2. \end{aligned} \quad (2.15)$$

From the above equations, the Hamiltonian in a local orthonormal frame is defined by

$$H(df, x) = \frac{1}{2} \sum_{j=1}^n df(e_j)^2 = \frac{1}{2} \sum_{i=1}^n (e_i(f))^2 = \frac{1}{2} \|f\|^2. \quad (2.16)$$

Furthermore, the Laplacian  $\Delta f$  of  $f$  is given by

$$\Delta f = \sum_{i=1}^n \{(\nabla_{e_i} e_i) f - e_i(e_i(f))\} = - \sum_{i=1}^n g(\nabla_{e_i} \text{grad} f, e_i). \quad (2.17)$$

The Hessian of the function  $f$ , denoted by  $H^f$ , is defined by

$$\Delta f = -\text{trace} H^f = - \sum_{i=1}^n H^f(e_i, e_i). \quad (2.18)$$

Now, we consider the compact manifold  $M^n$  without boundary. Thus The following lemma is given by:

**Lemma 2.4.** [7] *Let  $M^n$  be a connected, compact Riemannian manifold and  $f$  a smooth function on  $M^n$  such that  $\Delta f \geq 0$  ( $\Delta f \leq 0$ ). Then  $f$  is a constant function on  $M^n$ .*

The above lemma is known as Hopf’s lemma.

Moreover, for a compact orientable Riemannian manifold  $M^n$  without boundary, then we have the following formula

$$\int_M \Delta f dV = 0 \tag{2.19}$$

such that  $dV$  denotes the volume of  $M^n$  [22].

If we consider  $M^n$  to be a manifold with boundary, the Hopf lemma becomes the uniqueness theorem for the Dirichlet problem. Thus we find the following result

**Theorem 2.5.** [7] *Let  $M^n$  be a connected, compact manifold and  $f$  a positive differentiable function on  $M^n$  such that  $\Delta f = 0$ , on  $M$  and  $f/\partial M = 0$ . Then  $f = 0$ , where  $\partial M$  is the boundary of  $M^n$ .*

on  $M$  and  $f/\partial M = 0$ . Then  $f = 0$ , where  $\partial M$  is the boundary of  $M^n$ .

Let us assume  $M^n$  to be a compact Riemannian manifold and  $f$  to be a positive differentiable function on  $M^n$ . Thus the kinetic energy function is defined by [10] :

$$E(f) = \frac{1}{2} \int_M \|f\|^2 dV, \tag{2.20}$$

where  $dV$  denotes the volume of  $M^n$ .

**Theorem 2.6.** [7] *The Euler-Lagrange equation for the Lagrangian function  $L = \frac{1}{2}\|f\|^2$  is  $\Delta f = 0$ .*

There is another class of submanifolds that is called the slant submanifold. For each non-zero vector  $X$  tangent to  $M$  at  $x$ , such that  $X$  is not proportional to  $\xi_x$ , we denote by  $0 \leq \theta(X) \leq \frac{\pi}{2}$ , the angle between  $\phi X$  and  $T_x M$  is called the Wirtinger angle. If the angle  $\theta(X)$  is constant for all nonzero  $X \in T_x M - \langle \xi_x \rangle$  and  $x \in M$ , then  $M$  is said to be a slant submanifold [6] and the angle  $\theta$  is the slant angle of  $M$ . Obviously if  $\theta = 0$ ,  $M$  is invariant and if  $\theta = \frac{\pi}{2}$ ,  $M$  is an anti-invariant submanifold. A slant submanifold is said to be proper slant if it is neither invariant nor anti-invariant.

We recall the following result which was obtained by Cabreizo et al. [6] for a slant submanifold of an almost contact metric manifold.

**Theorem 2.7.** *Let  $M$  be a submanifold of an almost contact metric manifold  $\tilde{M}$ , such that  $\xi \in TM$ . Then,  $M$  is slant iff  $\exists$  a constant  $\lambda \in [0, 1]$  such that*

$$P^2 = \lambda(-I + \eta \otimes \xi). \tag{2.21}$$

Again, if  $\theta$  is slant angle of  $M$ , then  $\lambda = \cos^2 \theta$ .

The following relations are straightforward consequences of (2.21):

$$g(PX, PY) = \cos^2 \theta [g(X, Y) - \eta(X)\eta(Y)], \quad (2.22)$$

$$g(FX, FY) = \sin^2 \theta [g(X, Y) - \eta(X)\eta(Y)], \quad (2.23)$$

for any  $X, Y \in \Gamma(TM)$ .

Now, we give the brief introduction of pseudo-slant submanifold of an almost contact metric manifold  $\tilde{M}$ .

**Definition 2.8.** [17] A submanifold  $M$  of an almost contact metric manifold  $\tilde{M}$  is said to be a pseudo-slant submanifold if there exists a pair of orthogonal distributions  $\mathcal{D}^\perp$  and  $\mathcal{D}^\theta$  on  $M$  such that:

- (1)  $TM$  admits the orthogonal direct decomposition  $TM = \mathcal{D}^\perp \oplus \mathcal{D}^\theta \oplus \langle \xi \rangle$ .
- (2) The distribution  $\mathcal{D}^\perp$  is anti-invariant, i.e.,  $\phi(\mathcal{D}^\perp) \subset T^\perp M$ .
- (3) The distribution  $\mathcal{D}^\theta$  is slant with angle  $\theta \neq \frac{\pi}{2}$ .

If  $\mu$  is invariant subspace under  $\phi$  of the normal bundle  $T^\perp M$ , then in the case of pseudo-slant submanifold, the normal bundle  $T^\perp M$  can be decomposed as

$$T^\perp M = \mu \oplus \phi\mathcal{D}^\perp \oplus F\mathcal{D}^\theta.$$

From the definition it is clear that if  $\theta = 0$ , then pseudo-slant submanifold becomes semi-invariant submanifold.

A pseudo-slant submanifold is said to be mixed totally geodesic if  $h(X, Z) = 0$ ,  $\forall X \in \Gamma(\mathcal{D}^\perp)$  and  $Z \in \Gamma(\mathcal{D}^\theta)$ .

Let  $\varrho : M^n = N_\perp^{n_1} \times_f N_\theta^{n_2} \rightarrow \tilde{M}^{2p+1}$  be an isometric immersion of warped product pseudo-slant submanifolds into generalized Sasakian space forms  $\tilde{M}$ . Then for unit vector fields  $X, Z$  tangent to  $N_\perp$  and  $N_\theta$  respectively,

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{\{(\nabla_X X)f - X^2 f\}}{f}. \quad (2.24)$$

We consider a local orthonormal frame  $\{e_1, e_2, \dots, e_n\}$  such that  $e_1, e_2, \dots, e_{n_1}$  tangent to  $N_\perp$  and  $e_{n_1+1}, \dots, e_n$  are tangent to  $N_\theta$ . Then

$$\sum_{i=1}^{n_1} \sum_{j=n_1+1}^n K(e_i \wedge e_j) = \frac{n_2 \Delta f}{f}. \quad (2.25)$$

**Lemma 2.9.** [9] *Let  $x_1, x_2, \dots, x_{n+1}$  be  $(n + 1)$ ,  $(n \geq 2)$  real numbers such that*

$$\left(\sum_{i=1}^n x_i\right)^2 = (n - 1)\left(\sum_{i=1}^n x_i^2 + x_{n+1}\right).$$

*Then  $2x_1x_2 \geq x_3$  with the equality holds if and only if  $x_1 + x_2 = x_3 = \dots = x_k$ .*

### 3. WARPED PRODUCT PSEUDO-SLANT SUBMANIFOLDS

Now, we discuss the warped product pseudo-slant submanifolds of the type  $N_{\perp} \times_f N_{\theta}$  of generalized Sasakian space forms  $\tilde{M}$ . Now We establish the following lemmas for later use

**Lemma 3.1.** *Let  $M = N_{\perp} \times_f N_{\theta}$  be a warped product pseudo-slant submanifold of a generalized Sasakian space forms  $\tilde{M}$  such that the structure vector field  $\xi$  tangent to the base manifold  $N_{\perp}$  then:*

- (1)  $\xi \ln f = 0$ ,
- (2)  $g(h(X, Z), \phi Y) = 0 = 2g(h(\phi X, Z), \phi Y)$ ,  
 (ii)  $g(h(X, Z), \phi Y) = 0 = 2g(h(\phi X, Z), \phi Y)$ ,
- (3)  $g(h(X, Z), FZ) = -\{(\phi X \ln f)g(Z, W) + (f_1 - f_3)\eta(X)\}\|Z\|^2$ ,
- (4)  $g(h(X, Z), FPZ) = -g(h(X, PZ), FW) = \cos^2 \theta(X \ln f)\|Z\|^2$ ,

for any  $X, Y \in \Gamma(TN_{\perp})$  and  $Z, W \in \Gamma(TN_{\theta})$ .

*Proof.* Using lemma 1.1(ii) and taking  $g(h(Z, Z), \phi X) = g(\tilde{\nabla}_Z Z, \phi X) = -g(Z, \tilde{\nabla}_Z X)$  and  $g(Z, PZ) = 0$  we get the above lemma whose proof is similar to [20]. □

**Lemma 3.2.** *Let  $\varrho : M = N_{\perp} \times_f N_{\theta} \rightarrow \tilde{M}$  be a warped product immersion into generalized Sasakian space forms  $\tilde{M}$  such that  $\xi$  tangent to the base manifold  $N_{\perp}$ . Then we have*

i)  $\|h\|^2 \geq 2(n_2\|f\|^2 - n_2\Delta(\ln f) + \tau(TM) - \tau(TN_{\perp}) - \tau(TN_{\theta}))$ , where  $\tau(TM) = \sum_{1 \leq r < s \leq n} K(e_r \wedge e_s)$  denotes the scalar curvature of the  $n$ -plane section and  $n_2$  is the dimension of  $N_{\theta}$ .

ii) *The equality holds if and only if  $N_{\perp}$  is totally geodesic and  $N_{\theta}$  is totally umbilical submanifolds in  $\tilde{M}$ .*

*Proof.* The lemma can be computed easily with the help of [14]. □

**Example 3.3.** Let  $M$  be a submanifold of  $R^5$  with coordinates  $(x_1, y_1, x_2, y_2, z)$  Let us consider an isometric immersion  $x$  into  $R^5$  as follows:

$$x(u, v, \alpha, z) = (u, -v\sqrt{3}, \sin \alpha, \cos \alpha, z).$$

We can easily to see that the tangent bundle  $TM$  is spanned by the tangent vectors  $e_1 = \frac{\partial}{\partial x_1}$ ,  $e_2 = \sin \alpha \frac{\partial}{\partial x_2} + \cos \alpha \frac{\partial}{\partial y_2} - \sqrt{3} \frac{\partial}{\partial y_1}$ ,  $e_3 = \frac{\partial}{\partial z} = \xi$ .



For any vector field  $X = \gamma_i \frac{\partial}{\partial x_i} + \delta_j \frac{\partial}{\partial y_j} + v \frac{\partial}{\partial z} \in \Gamma(TR^5)$ , then we have  $g(X, X) = \gamma_i^2 + \delta_j^2 + v^2$ ,  $g(\phi X, \phi X) = \gamma_i^2 + \delta_j^2$  and  $\phi(X) = -\gamma_i \frac{\partial}{\partial x_i} - \delta_j \frac{\partial}{\partial y_j} = -X + \eta(X)\xi$ , for any  $i, j = 1, 2$ . It is clear that  $g(\phi X, \phi X) = g(X, X) - \eta(X)\eta(X)$ . Thus  $(\phi, \xi, \eta, g)$  is an almost contact metric structure on  $R^5$ .

We define the almost contact structure  $\phi$  of  $R^5$ , by

$$\phi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \phi\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}, \frac{\partial}{\partial z} = 0, \quad i, j \in \{1, 2\}.$$

Thus we have  $\phi(e_1) = \frac{\partial}{\partial y_1}$ ,  $\phi(e_2) = \sin \alpha \frac{\partial}{\partial y_2} - \cos \alpha \frac{\partial}{\partial x_2} + \sqrt{3} \frac{\partial}{\partial x_1}$ ,  $e_3 = \frac{\partial}{\partial z} = \xi$ .

By direct calculations, we can infer  $D_\theta = \text{span}\{e_1, e_2\}$  is a slant distribution with slant angle  $\cos \theta = \frac{g(e_1, \phi e_2)}{\|e_1\| \|\phi e_2\|} = \frac{\sqrt{3}}{2}$ . Hence  $\theta = \frac{\pi}{6}$ . Since  $g(\phi(e_3), e_i) = 0$ , for  $i = 1, 2$ .  $\phi(e_3)$  is orthogonal to  $M$ ,  $D_\perp = \text{span}\{e_3\}$  is an anti-invariant distribution. Thus  $M$  is a 3-dimensional proper pseudo-slant submanifold of  $R^5$  with its usual almost contact metric structure. We can write that  $M^3 = N_\perp \times_f N_\theta$  is non-trivial warped product pseudo-slant submanifold of  $M^5$  with warping function  $f = 2$ .

#### 4. APPLICATION OF WARPED PRODUCT PSEUDO SLANT SUBMANIFOLD OF GENERALIZED SASAKIAN SPACE FORMS

In this section we will obtain some results on warped product pseudo slant submanifold which is isometrically immersed into generalized Sasakian space forms.

**Theorem 4.1.** *Let  $\rho : M = N_\perp \times_f N_\theta \rightarrow \tilde{M}$  be an isometric immersion of a warped product pseudo-slant submanifold into generalized Sasakian space forms  $\tilde{M}$  such that  $\xi$  tangent to the base manifold  $N_\perp$ . Then we have*

$$(1) \|h\|^2 \geq 2n_2 \|\nabla \ln f\|^2 - 2n_2 \Delta(\ln f) + 2f_1 n_1 n_2 - 2f_3 n_2 + 3f_2 (n_1 - \cos^2 \theta), \quad (4.1)$$

where  $\tau(TM) = \sum_{1 \leq r < s \leq n} K(e_r \wedge e_s)$  denotes the scalar curvature of the  $n$ -plane section and  $n_1$  and  $n_2$  are the dimensions of  $N_\perp$  and  $N_\theta$  respectively.

(2) The equality holds if and only if  $N_\perp$  is totally geodesic and  $N_\theta$  is totally umbilical submanifolds in  $\tilde{M}$ . Furthermore,  $M$  is a minimal submanifold of  $\tilde{M}$ .

*Proof.* From the equation (2.3) we obtain

$$\begin{aligned} \tilde{R}(X, Y, Z, W) &= f_1 \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\ &+ f_2 \{g(X, \phi Z)g(\phi Y, W) - g(Y, \phi Z)g(\phi X, W) \\ &+ 2g(X, \phi Y)g(\phi Z, W)\} + f_3 \{g(X, Z)\eta(Y)\eta(W) \\ &- g(Y, Z)\eta(X)\eta(W) + \eta(X)\eta(Z)g(Y, W) \\ &- \eta(Y)\eta(Z)g(X, W)\}. \end{aligned} \quad (4.2)$$

Putting  $X = W = e_i$ ,  $Y = Z = e_j$ , in the above equation, we have

$$\begin{aligned}\tilde{R}(e_i, e_j, e_j, e_i) &= f_1\{g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\} \\ &+ f_2\{g(e_i, \phi e_j)g(\phi e_j, e_i) - g(e_j, \phi e_j)g(\phi e_i, e_i) \\ &+ 2g(e_i, \phi e_j)g(\phi e_j, e_i)\} + f_3\{g(e_i, e_j)\eta(e_j)\eta(e_i) \\ &- g(e_j, e_j)\eta(e_i)\eta(e_i) + \eta(e_i)\eta(e_j)g(e_j, e_i) \\ &- \eta(e_j)\eta(e_j)g(e_i, e_i)\}.\end{aligned}\quad (4.3)$$

Taking summing up in (4.3) over the orthonormal vector fields of  $TN_\perp$ , we derive

$$2\tau(TN_\perp) = f_1(n_1^2 - n_1) + 3f_2\|P\|^2 - 2f_3(n_1 - 1).\quad (4.4)$$

For  $n_1$ -dimensional anti-invariant submanifold  $TN_\perp$ , we have  $\|P\| = 0$  and hence we get

$$2\tau(TN_\perp) = f_1(n_1^2 - n_1) - 2f_3(n_1 - 1).\quad (4.5)$$

Taking summing up in (4.3) over the orthonormal vector fields of  $TN_\theta$ , we obtain

$$2\tau(TN_\theta) = f_1(n_2^2 - n_2) + 3f_2\|P\|^2.\quad (4.6)$$

For a slant submanifold, by using (2.14)(2) and (2.23) we have  $\|P\|^2 = n_2 \cos^2 \theta$ . Hence we derive

$$2\tau(TN_\theta) = f_1(n_2^2 - n_2) + 3f_2 n_2 \cos^2 \theta\quad (4.7)$$

Again, summing up in (4.3) over basis vectors of  $TM$ , we obtain

$$2\tau(TM) = f_1(n^2 - n) + 3f_2 \sum_{i,j=1}^n g^2(\phi e_i, e_j) - 2f_3(n - 1).\quad (4.8)$$

Let  $M^n = N_\perp^{n_1} \times_f N_\theta^{n_2}$  be a proper pseudo-slant submanifold of generalized Sasakian space forms  $\tilde{M}^{(2p+1)}$  with  $\dim(N_\perp^{n_1}) = n_1 = 2\alpha + 1$  and  $\dim(N_\theta^{n_2}) = n_2 = 2\beta$ . Now, we define the following frame  $e_1, e_2 = \phi e_1, \dots, e_{2\alpha} = \phi e_{2\alpha-1}, e_{2\alpha+1} = \phi e_{2\alpha}, e_{2\alpha+2}, e_{2\alpha+3} = \sec \theta P e_{2\alpha+1}, \dots, e_{2\alpha+2\beta} = \sec \theta P e_{2\alpha+2\beta-2}, e_{2\alpha+2\beta+1} = \xi$ .

We can obtain

$$\begin{aligned}g^2(\phi e_i, e_{i+1}) &= 1, \text{ for } i \in \{1, \dots, 2\alpha + 1\} \\ &= \cos^2 \theta, \text{ for } i \in \{2\alpha + 2, \dots, 2\alpha + 2\beta - 1\}.\end{aligned}\quad (4.9)$$

Therefore, we can derive

$$2\tau(TM) = f_1(n^2 - n) + 3f_2((2\alpha + 1) + (2\beta - 1) \cos^2 \theta) - 2f_3(n - 1).\quad (4.10)$$

Using the equations (4.5), (4.7) and (4.10) in Lemma 3.2 we establish the required result i.e.

$$\|h\|^2 \geq 2n_2\|\nabla \ln f\|^2 - 2n_2\Delta(\ln f) + 2f_1 n_1 n_2 - 2f_3 n_2 + 3f_2(n_1 - \cos^2 \theta),\quad (4.11)$$

. Also the equality holds if and only if  $N_\perp$  is totally geodesic and  $N_\theta$  is totally umbilical submanifolds in  $\tilde{M}$ . Hence the theorem is proved.  $\square$

Next theorem shows that the inequality for warped product pseudo slant submanifold of generalized sasakian space forms such that  $\xi$  is tangent to the base of warped product and also gives the relation between the warping fuction and the squared norm of mean curvature.

**Theorem 4.2.** *Let  $\varrho : M = N_{\perp} \times_f N_{\theta} \rightarrow \tilde{M}$  be an isometric immersion of a warped product pseudo-slant submanifold into generalized Sasakian space forms  $\tilde{M}$  such that  $\xi$  tangent to the base manifold  $N_{\perp}$ . Then we have*

$$(1) \frac{n_2 \Delta f}{f} \leq f_1 n_1 n_2 - f_3 n_2 + \frac{3}{2} f_2 (n_1 - \cos^2 \theta) + \frac{n^2}{4} \|H\|^2. \quad (4.12)$$

(2) *The equality holds if and only if  $n_1 H_{\perp} = n_2 H_{\theta}$ , where  $H_{\perp}$  and  $H_{\theta}$  are partially mean curvature vectors on  $N_{\perp}$  and  $N_{\theta}$  respectively. Moreover  $\varrho$  is a mixed totally geodesic immersion.*

*Proof.* Let  $M^n = N_{\perp}^{n_1} \times_f N_{\theta}^{n_2}$  be a proper warped product pseudo-slant submanifold of generalized Sasakian space forms  $\tilde{M}^{(2p+1)}$ . Then from the equations (2.10) and (4.2) and then using the frame of the Theorem 4.1 we obtain

$$\begin{aligned} 2\tau(TM) &= f_1(n^2 - n) + 3f_2((2\alpha + 1) + (2\beta - 1)\cos^2 \theta) \\ &\quad - 2f_3(n - 1) + n^2\|H\|^2 - \|h\|^2. \end{aligned} \quad (4.13)$$

Now we consider

$$\begin{aligned} \varpi &= 2\tau(TM) - f_1(n^2 - n) - 3f_2((2\alpha + 1) + (2\beta - 1)\cos^2 \theta) \\ &\quad + 2f_3(n - 1) - \frac{n^2}{2}\|H\|^2. \end{aligned} \quad (4.14)$$

Then, from the equation (4.13) and (4.14) we get

$$n^2\|H\|^2 = 2(\varpi - \|h\|^2). \quad (4.15)$$

Using the equations (2.11) and (2.25) we derive

$$\frac{n_2 \Delta f}{f} = \tau(TM) - \sum_{1 \leq i \neq j \leq n_1} K(e_i \wedge e_j) - \sum_{n_1+1 \leq q \neq r \leq n} K(e_q \wedge e_r) \quad (4.16)$$

$$\begin{aligned} \frac{n_2 \Delta f}{f} &= \tau(TM) - \frac{f_1}{2}(n_1^2 - n_1) + f_3(n_1 - 1) \\ &\quad - \sum_{s=1}^{2p+1} \sum_{1 \leq i \neq j \leq n_1} (h_{ii}^s h_{jj}^s - (h_{ij}^s)^2) - \frac{f_1}{2}(n_2^2 - n_2) \\ &\quad - \sum_{s=1}^{2p+1} \sum_{n_1+1 \leq q \neq r \leq n} (h_{qq}^s h_{rr}^s - (h_{qr}^s)^2) \\ &\quad - \frac{3f_2}{2} \sum_{1 \leq i \neq j \leq n_1} g^2(\phi e_i, e_j). \end{aligned} \quad (4.17)$$

Using the orthonormal frame  $\{e_1, e_2, \dots, e_n\}$ , the equation (4.15) gives

$$\begin{aligned} \left( \sum_{s=n+1}^{2p+1} \sum_{i=1}^n h_{ii}^s \right)^2 &= 2(\varpi + \left( \sum_{s=n+1}^{2p+1} \sum_{i=1}^n h_{ii}^s \right)^2 + \left( \sum_{s=n+1}^{2p+1} \sum_{i<j=1}^n h_{ij}^s \right)^2 \\ &+ \left( \sum_{s=n+1}^{2p+1} \sum_{i,j=1}^n h_{ij}^s \right)^2). \end{aligned} \quad (4.18)$$

After some calculations and using Lemma 2.9 we can derive

$$\begin{aligned} \varpi + \sum_{i<j=1}^n (h_{ij}^{n+1})^2 + \left( \sum_{s=n+1}^{2p+1} \sum_{i,j=1}^n h_{ij}^s \right)^2 &\leq 2 \sum_{2 \leq j \neq k \leq n_1} h_{jj}^{n+1} h_{kk}^{n+1} \\ &+ 2 \sum_{n_1+1 \leq q \neq r \leq n} h_{qq}^{n+1} h_{rr}^{n+1}. \end{aligned} \quad (4.19)$$

The above equality holds if and only if

$$\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{q=n_1+1}^n h_{qq}^{n+1} \quad (4.20)$$

Using the equation (4.17) and (4.18), we get

$$\frac{n_2 \Delta f}{f} \leq \tau(TM) - \frac{f_1}{2}(n_1^2 - n_1) + f_3(n_1 - 1) - \frac{3}{2}f_2 n_2 \cos^2 \theta - \frac{\varpi}{2}. \quad (4.21)$$

Taking the equation (4.14), we obtain

$$\frac{n_2 \Delta f}{f} \leq \frac{f_1}{2} n_1 n_2 - f_3 n_2 + \frac{3}{2} f_2 (n_1 - \cos^2 \theta) + \frac{n^2}{4} \|H\|^2. \quad (4.22)$$

The equality holds if and only if from the equations (4.18) and (4.19) we have  $\sum_{s=n+2}^{2p+1} \sum_{j=1}^{n_1} h_{jj}^s = \sum_{s=n+2}^{2p+1} \sum_{i=n_1+1}^n h_{ii}^s$  and  $n_1 H_{\perp} = n_2 H_{\theta}$ .

From (4.18), we find that  $h_{ij}^s = 0$ , for each  $1 \leq i \leq n_1, n_1 + 1 \leq j \leq n, 1 \leq s \leq 2p + 1$ . Hence  $\varrho$  is a mixed totally geodesic immersion. Therefore the proof of the theorem is completed.  $\square$

Next theorem is considered that  $M$  is a compact Riemannian manifold without boundary and then we find under some certain conditions  $M$  becomes a simply Riemannian product.

**Theorem 4.3.** *Let  $M = N_{\perp}^{n_1} \times_f N_{\theta}^{n_2}$  be a compact orientable proper warped product pseudo-slant submanifold of generalized Sasakian space forms  $\tilde{M}$ . Then  $M$  is a Riemannian product if and only if*

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_{\mu}(e_i, e_j)\|^2 = f_1 n_1 n_2 - f_3 n_2 + \frac{3}{2} f_2 (n_1 - \cos^2 \theta), \quad (4.23)$$

where  $h_{\mu}$  is a component of  $h$  in  $\Gamma(\mu)$ .

*Proof.* We consider the equality holds in the equation (4.1) Then we obtain

$$\begin{aligned} \|h(\mathcal{D}^\perp, \mathcal{D}^\perp)\|^2 &+ \|h(\mathcal{D}^\theta, \mathcal{D}^\theta)\|^2 + 2\|h(\mathcal{D}^\perp, \mathcal{D}^\theta)\|^2 = 2n_2\|\nabla \ln f\|^2 \\ &+ 2f_1n_1n_2 + 3f_2(n_1 - \cos^2 \theta) - 2f_3n_2, \end{aligned} \quad (4.24)$$

As,  $N_\perp$  is totally geodesic and  $N_\theta$  is totally umbilical submanifolds in  $\tilde{M}$  and also  $M$  is a minimal submanifold of  $\tilde{M}$   $\|h(\mathcal{D}^\perp, \mathcal{D}^\perp)\|^2 = 0$  and  $\|h(\mathcal{D}^\theta, \mathcal{D}^\theta)\|^2 = 0$ . Therefore, the equation (4.24) becomes

$$\begin{aligned} 2\|h(\mathcal{D}^\perp, \mathcal{D}^\theta)\|^2 &= (2n_2\|\nabla \ln f\|^2 - 2n_2\Delta(\ln f) + 2f_1n_1n_2 - 2f_3n_2 \\ &+ 3f_2(n_1 - \cos^2 \theta)). \end{aligned} \quad (4.25)$$

Let  $M = N_\perp^{n_1} \times_f N_\theta^{n_2}$  be an  $n$ -dimensional warped product pseudo-slant submanifold of a  $(2p+1)$ -dimensional generalized Sasakian space forms  $\tilde{M}$  such that  $N_\perp$  is a  $n_1 = 2\alpha + 1$ -dimensional anti-invariant submanifold and  $N_\theta$  is a  $n_2 = 2\beta$ -dimensional slant submanifold where  $N_\perp$  and  $N_\theta$  are the integral manifolds of  $\mathcal{D}^\perp$  and  $\mathcal{D}^\theta$  respectively. Then the orthonormal frame fields of  $N_\perp$  and  $N_\theta$ , respectively, are  $\{e_1, e_2, \dots, e_\alpha, e_{\alpha+1} = \phi e_1, e_{\alpha+2} = \phi e_2, \dots, e_{2\alpha} = \phi e_\alpha, e_{2\alpha+1} = \xi\}$  and  $\{e_{2\alpha+2} = e_1^*, \dots, e_{2\alpha+\beta+1} = e_\beta^*, e_{2\alpha+\beta+2} = e_{\beta+1}^* = \sec \theta P e_1^*, \dots, e_{2\alpha+2\beta+1} = e_{2\beta}^* = \sec \theta P e_\beta^*\}$ . The orthonormal frames of the normal subbundles  $\phi\mathcal{D}^\perp$ ,  $\mathcal{F}\mathcal{D}^\theta$  and  $\mu$  respectively, are  $\{e_{n+1} = \tilde{e}_1 = \phi e_1, \dots, e_{n+2\alpha} = \tilde{e}_{2\alpha} = \phi e_{2\alpha}\}$ ,  $\{e_{n+2\alpha+1} = \tilde{e}_1 = \csc \theta F e_1^*, \dots, e_{n+2\alpha+\beta} = \tilde{e}_\beta = \csc \theta F e_\beta^*, e_{n+2\alpha+\beta+1} = \tilde{e}_{\beta+1} \csc \theta \sec \theta F P e_1^*, \dots, e_{n+2\alpha+2\beta} = \tilde{e}_{2\beta} = \csc \theta \sec \theta F P e_\beta^*\}$  and  $\{e_{n+2\alpha+2\beta+1}, \dots, e_{2p+1}\}$ . Taking summation over the vector fields  $N_\perp^{n_1}$  and  $N_\theta^{n_2}$  and then using adapted frame for the pseudo-slant submanifold we derive

$$\begin{aligned} &\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} g(h(e_i, e_j), h(e_i, e_j)) = \sum_{i=1}^\alpha \sum_{j=1}^\beta \sum_{k=1}^{2\alpha} g(h(e_i, e_j^*), \phi e_k)^2 \\ &+ \sec^2 \theta \sum_{i=1}^\alpha \sum_{j=1}^\beta \sum_{k=1}^{2\alpha} g(h(e_i, P e_j^*), \phi e_k)^2 \\ &+ \csc^2 \theta \sum_{i=1}^\alpha \sum_{j,k=1}^\beta g(h(e_i, e_j^*), F e_k^*)^2 \\ &+ \csc^2 \theta \sec^2 \theta \sum_{i=1}^\alpha \sum_{j,k=1}^\beta g(h(e_i, P e_j^*), F e_k^*)^2 \\ &+ \csc^2 \theta \sec^2 \theta \sum_{i=1}^\alpha \sum_{j,k=1}^\beta g(h(e_i, e_j^*), F P e_k^*)^2 \\ &+ \csc^2 \theta \sec^2 \theta \sum_{i=1}^\alpha \sum_{j,k=1}^\beta g(h(\phi e_i, e_j^*), F P e_k^*)^2 \\ &+ \csc^2 \theta \sec^4 \theta \sum_{i=1}^\alpha \sum_{j,k=1}^\beta g(h(\phi e_i, P e_j^*), F P e_k^*)^2 \\ &+ \csc^2 \theta \sec^2 \theta \sum_{i=1}^\alpha \sum_{j,k=1}^\beta g(h(\phi e_i, P e_j^*), F e_k^*)^2 \\ &+ \csc^2 \theta \sum_{i=1}^\alpha \sum_{j,k=1}^\beta g(h(\phi e_i, e_j^*), F e_k^*)^2 \\ &+ \csc^2 \theta \sec^4 \theta \sum_{i=1}^\alpha \sum_{j,k=1}^\beta g(h(e_i, P e_j^*), F P e_k^*)^2 \\ &+ \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{r=n+n_2+1}^{2p+1} g(h(e_i, e_j), e_r)^2. \end{aligned} \quad (4.26)$$

Using the Lemma 3.1 we get

$$\|h(\mathcal{D}^\perp, \mathcal{D}^\theta)\|^2 = n_2(2 \csc^2 \theta + 2 \cot^2 \theta)\|\nabla \ln f\|^2 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2. \quad (4.27)$$

From the equations (4.25) and (4.27) one can derive

$$\begin{aligned} n_2(2 \csc^2 \theta + 2 \cot^2 \theta - 1) \|\nabla \ln f\|^2 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2 \\ = f_1 n_1 n_2 - f_3 n_2 + \frac{3}{2} f_2 (n_1 - \cos^2 \theta) \\ - n_2 \Delta(\ln f). \end{aligned} \tag{4.28}$$

Now, taking the integration volume element  $dV$  of compact orientable warped product pseudo-slant submanifold  $M$  and also using the equation (4.28) we can obtain

$$\begin{aligned} n_2 \int_M [(2 \csc^2 \theta + 2 \cot^2 \theta - 1) \|\nabla \ln f\|^2] dV + \int_M [\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2] dV \\ = \int_M [f_1 n_1 n_2 - f_3 n_2 \\ + \frac{3}{2} f_2 (n_1 - \cos^2 \theta)] dV. \end{aligned} \tag{4.29}$$

If we consider the equation (4.23), then from the equation (4.29) we establish that  $f$  is a constant function on the proper warped product pseudo-slant submanifold  $M$  i.e.  $M$  becomes simply Riemannian product.

If we consider  $M$  is a simply Riemannian product then  $\nabla \ln f = 0$ . Hence from (4.29) we get the equality (4.23). The proof of the theorem is completed.  $\square$

**Corollary 4.4.** *Let  $\varrho : M = N_{\perp}^{n_1} \times_f N_{\theta}^{n_2}$  be an isometric immersion of a compact orientable proper warped product pseudo-slant submanifold into generalized Sasakian space forms  $\tilde{M}$ . Then  $M$  is a Riemannian product if and only if*

$$\|h\|^2 \geq 2f_1 n_1 n_2 - 2f_3 n_2 + 3f_2 (n_1 - \cos^2 \theta)$$

holds.

Now from (4.1) we have

$$2n_2 \Delta(\ln f) \geq -\|h\|^2 + 2n_2 \|\nabla \ln f\|^2 + 2f_1 n_1 n_2 - 2f_3 n_2 + 3f_2 (n_1 - \cos^2 \theta).$$

Again, taking the integration volume element  $dV$  of compact orientable warped product pseudo-slant submanifold  $M$  and also using the equation (2.20) we can get

$$0 \geq \int_M [-\|h\|^2 + 2n_2 \|\nabla \ln f\|^2 + 2f_1 n_1 n_2 - 2f_3 n_2 + 3f_2 (n_1 - \cos^2 \theta)] dV.$$

If the inequality

$$\|h\|^2 \geq 2f_1 n_1 n_2 - 2f_3 n_2 + 3f_2 (n_1 - \cos^2 \theta)$$

holds, we get

$$\int_M \|\nabla \ln f\|^2 \leq 0. \tag{4.30}$$

Hence from the equation (4.30) we get  $\|\nabla \ln f\|^2 \leq 0$ , but  $\|\nabla \ln f\|^2 \geq 0$ . Therefore,  $\nabla \ln f = 0$ , i.e.  $f$  is a constant function on  $M^n = N_\perp^{n_1} \times_f N_\theta^{n_2}$ .

Also, if  $f$  is a constant function on  $M^n = N_\perp^{n_1} \times_f N_\theta^{n_2}$  then the inequality  $\|h\|^2 \geq 2f_1n_1n_2 - 2f_3n_2 + 3f_2(n_1 - \cos^2\theta)$  holds. Hence we have the following corollary.

**Proposition 4.5.** *Let  $\varrho : M = N_\perp^{n_1} \times_f N_\theta^{n_2}$  be an isometric immersion of warped product pseudo-slant submanifold into generalized Sasakian space forms  $\tilde{M}$ . Then*

$$\begin{aligned} \int_{N_\perp^{n_1}} \|h\|^2 dV_\perp &\geq \int_{N_\perp^{n_1}} (2f_1n_1n_2 - 2f_3n_2 + 3f_2(n_1 - \cos^2\theta)) dV_\perp \\ &+ 2n_2\lambda_\perp \int_{N_\perp^{n_1}} (\ln f)^2 dV_\perp \end{aligned} \quad (4.31)$$

where  $\lambda_\perp$  is a non-zero eigenvalue of the Laplacian on the compact anti-invariant submanifold  $N_\perp^{n_1}$  and  $dV_\perp$  is the volume element of  $N_\perp^{n_1}$ .

*Proof.* Using the minimum principal property we get

$$\int_{N_\perp} \|\nabla f\|^2 dV_\perp \geq \lambda_\perp \int_{N_\perp} (\ln f)^2 dV_\perp \quad (4.32)$$

and then using the equation (4.1) we obtain the result.  $\square$

**Corollary 4.6.** *Let  $\varrho : M = N_\perp^{n_1} \times_f N_\theta^{n_2}$  be a warped product pseudo-slant submanifold into generalized Sasakian space forms  $\tilde{M}$ . Then*

$$\begin{aligned} \int_{N_\perp^{n_1}} \left[ \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2 \right] dV_\perp &\geq \int_{N_\perp^{n_1}} [f_1n_1n_2 - f_3n_2n_2 \\ &+ \frac{3}{2}f_2(n_1 - \cos^2\theta)] dV_\perp \\ &- \lambda_\perp n_2 \int_{N_\perp^{n_1}} [(2 \csc^2\theta \\ &+ 2 \cot^2\theta - 1) \|\nabla \ln f\|^2] dV_\perp, \end{aligned} \quad (4.33)$$

where  $\lambda_\perp$  is a non-zero eigenvalue of the Laplacian on the compact anti-invariant submanifold  $N_\perp^{n_1}$  and  $dV_\perp$  is the volume element of  $N_\perp^{n_1}$ .

*Proof.* Using the equations (4.23) and (4.31) we get the required result.  $\square$

## 5. APPLICATIONS OF HOPF'S LEMMA WARPED PRODUCT PSEUDO SLANT SUBMANIFOLD OF GENERALIZED SASAKIAN SPACE FORMS

In this section we apply  $M$  to be connected, compact Riemannian manifolds with boundary. Next theorems show the necessary and sufficient conditions of  $M$  to be Riemannian product in terms of some physical applications i.e. kinetic

energy functions, Hamiltonian under some certain conditions.

**Theorem 5.1.** *Let  $\varrho : M = N_{\perp}^{n_1} \times_f N_{\theta}^{n_2}$  be an isometric immersion from a warped product pseudo-slant submanifold into generalized Sasakian space forms  $\tilde{M}$ . Then  $M$  is a Riemannian product if and only if*

$$\begin{aligned}
 E(\ln f) &= -\frac{1}{n_2(4 \csc^2 \theta + 4 \cot^2 \theta - 2)} n_2 \int_M (16 \csc^3 \theta \cot \theta \frac{d\theta}{dV} E(\ln f) dV \\
 &- \frac{1}{n_2(4 \csc^2 \theta + 4 \cot^2 \theta - 2)} \int_M [\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_{\mu}(e_i, e_j)\|^2] dV \\
 &+ \frac{1}{n_2(4 \csc^2 \theta + 4 \cot^2 \theta - 2)} \int_M [f_1 n_1 n_2 - f_3 n_2 \\
 &+ \frac{3}{2} f_2 (n_1 - \cos^2 \theta)] dV. \tag{5.1}
 \end{aligned}$$

*Proof.* From the equation we have

$$\begin{aligned}
 n_2(2 \csc^2 \theta + 2 \cot^2 \theta - 1) \|\nabla \ln f\|^2 &+ \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_{\mu}(e_i, e_j)\|^2 \\
 &= f_1 n_1 n_2 - f_3 n_2 + \frac{3}{2} f_2 (n_1 - \cos^2 \theta) \\
 &- n_2 \Delta(\ln f). \tag{5.2}
 \end{aligned}$$

Taking the integration volume element  $dV$  of connected, compact warped product pseudo-slant submanifold  $M$  with boundary in the equation (5.2) we find

$$\begin{aligned}
 n_2 \int_M [(2 \csc^2 \theta + 2 \cot^2 \theta - 1) \|\nabla \ln f\|^2] dV &+ \int_M [\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_{\mu}(e_i, e_j)\|^2] dV \\
 &= \int_M [f_1 n_1 n_2 - f_3 n_2 + \frac{3}{2} f_2 (n_1 - \cos^2 \theta)] dV \\
 &- n_2 \int_M [\Delta(\ln f)] dV, \tag{5.3}
 \end{aligned}$$

which implies that

$$\begin{aligned}
 n_2 [(2 \csc^2 \theta + 2 \cot^2 \theta - 1) \int_M \|\nabla \ln f\|^2] dV &+ \int_M (8 \csc^3 \theta \cot \theta \frac{d\theta}{dV} \{ \int_M \|\nabla \ln f\|^2 \} dV) \\
 &+ \int_M [\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_{\mu}(e_i, e_j)\|^2] dV = \int_M [f_1 n_1 n_2 - f_3 n_2 \\
 &+ \frac{3}{2} f_2 (n_1 - \cos^2 \theta)] dV - n_2 \int_M [\Delta(\ln f)] dV. \tag{5.4}
 \end{aligned}$$



Using the equation (2.21), the above equation becomes

$$\begin{aligned}
& n_2(4 \csc^2 \theta + 4 \cot^2 \theta - 2)E(\ln f) + n_2 \int_M (16 \csc^3 \theta \cot \theta \frac{d\theta}{dV} E(\ln f) dV \\
& + \int_M [\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2] dV = \int_M [f_1 n_1 n_2 - f_3 n_2 \\
& + \frac{3}{2} f_2 (n_1 - \cos^2 \theta)] dV - n_2 \int_M [\Delta(\ln f)] dV. \quad (5.5)
\end{aligned}$$

Now taking the equality (5.1) in the equation (5.5), we can write

$$\int_M [\Delta(\ln f)] dV = 0, \quad (5.6)$$

i.e.,

$$\Delta(\ln f) = 0. \quad (5.7)$$

From the theorem and the equation, we can establish that  $f$  is constant on  $M$ . Thus the proof is completed.  $\square$

We compute to characterize in terms of Hamiltonian in the following corollary.

**Corollary 5.2.** *Let  $\varrho : M = N_\perp^{n_1} \times_f N_\theta^{n_2}$  be an isometric immersion from a warped product pseudo-slant submanifold into generalized Sasakian space forms  $\tilde{M}$ . Then  $M$  is a Riemannian product if and only if*

$$\begin{aligned}
H(d(\ln f), x) &= \frac{1}{n_2(4 \csc^2 \theta + 4 \cot^2 \theta - 2)} [f_1 n_1 n_2 - f_3 n_2 \\
&- \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2] + \frac{3}{2} f_2 (n_1 - \cos^2 \theta).
\end{aligned}$$

*Proof.* The equations (2.16), (4.28) and (5.8) imply

$$\Delta(\ln f) = 0, \quad (5.8)$$

i.e.  $f$  is constant on  $M^n$ . Hence completes the proof.  $\square$

## 6. APPLICATIONS OF HESSIAN OF WARPING FUNCTION ON PSEUDO SLANT SUBMANIFOLD OF GENERALIZED SASAKIAN SPACE FORMS

In this section we characterize to find a warped product pseudo slant submanifold of generalized Sasakian space forms to be a Riemannian product under some certain conditions.

To prove our main results we consider the following lemma.

**Lemma 6.1.** *Let  $\varrho : M = N_{\perp}^{n_1} \times_f N_{\theta}^{n_2}$  be an isometric immersion from a warped product pseudo-slant submanifold into generalized Sasakian space forms  $\tilde{M}$ . Then*

$$\Delta(\ln f) = - \sum_{i=1}^{\alpha} (H^{\ln f}(e_i, e_i) + H^{\ln f}(\phi e_i, \phi e_i) - 2n_2 \|\ln f\|^2), \quad (6.1)$$

where  $H^{\ln f}$  is Hessian of warping function  $\ln f$ ,  $n_1 = 2\alpha + 1$  is the dimension of  $N_{\perp}$  and  $n_2 = 2\beta$  is the dimension of  $N_{\theta}$ .

*Proof.* We skip the proof as the proof is similar to [3]. □

**Proposition 6.2.** *Let  $\varrho : M = N_{\perp}^{n_1} \times_f N_{\theta}^{n_2}$  be an isometric immersion from a warped product pseudo-slant submanifold into generalized Sasakian space forms  $\tilde{M}$ . Then  $M$  is a Riemannian product if*

$$\begin{aligned} \|h\|^2 &\geq 2f_1 n_1 n_2 - 2f_3 n_2 + 3f_2 (n_1 - \cos^2 \theta) \\ &+ H^{\ln f}(\phi e_i, \phi e_i) + 2n_2 \left( \sum_{i=1}^{\alpha} (H^{\ln f}(e_i, e_i)) \right) \end{aligned}$$

holds, where  $H^{\ln f}$  is Hessian of warping function  $\ln f$ .

*Proof.* Using the Lemma 6.1 and the equation (4.1) we derive

$$\begin{aligned} \|h\|^2 &\geq 2f_1 n_1 n_2 - 2f_3 n_2 + 3f_2 (n_1 - \cos^2 \theta) + (2n_2 + 4n_2^2) \|\nabla \ln f\|^2 \\ &+ 2n_2 \left( \sum_{i=1}^{\alpha} (H^{\ln f}(e_i, e_i) + H^{\ln f}(\phi e_i, \phi e_i)) \right). \end{aligned} \quad (6.2)$$

Considering the equation (6.2), the above equation reduces to  $\|\nabla \ln f\|^2 \leq 0$ . This contradicts to the definition of the norm. Therefore,  $\nabla \ln f = 0$ , i.e.  $f$  is a constant function on  $M^n = N_{\perp}^{n_1} \times_f N_{\theta}^{n_2}$ . Therefore, the proof is completed. □

**Proposition 6.3.** *Let  $\varrho : M = N_{\perp}^{n_1} \times_f N_{\theta}^{n_2}$  be an isometric immersion from a warped product pseudo-slant submanifold into generalized Sasakian space forms  $\tilde{M}$  with the slant function  $\theta \neq \text{arccot} \sqrt{n_2}$ . Then  $M$  is a Riemannian product if and only if*

$$\begin{aligned} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_{\mu}(e_i, e_j)\|^2 &= f_1 n_1 n_2 - f_3 n_2 + \frac{3}{2} f_2 (n_1 - \cos^2 \theta) \\ &+ n_2 \left( \sum_{i=1}^{\alpha} (H^{\ln f}(e_i, e_i) + H^{\ln f}(\phi e_i, \phi e_i)) \right). \end{aligned} \quad (6.3)$$

*Proof.* Using the Lemma 6.1 and the equation (4.28) it follows that

$$\begin{aligned} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \|h_\mu(e_i, e_j)\|^2 &= f_1 n_1 n_2 - f_3 n_2 + \frac{3}{2} f_2 (n_1 - \cos^2 \theta) \\ &+ n_2 \left( \sum_{i=1}^{\alpha} (H^{lnf}(e_i, e_i) + H^{lnf}(\phi e_i, \phi e_i)) \right) \\ &+ [n_2^2 - n_2(2 \csc^2 \theta + 2 \cot^2 \theta - 1)] \|\nabla lnf\|^2. \end{aligned} \tag{6.4}$$

Applying the equation (6.3), we derive  $\|\nabla lnf\|^2 \leq 0$ . But we know  $\|\nabla lnf\|^2 \geq 0$ . Hence  $\nabla lnf = 0$ , i.e.  $M$  becomes Riemmanian Product. Thus completes the required result.  $\square$

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