

## SEMI-SLANT RIEMANNIAN MAPS FROM COSYMPLECTIC MANIFOLDS INTO RIEMANNIAN MANIFOLDS

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**ABSTRACT.** In this paper we study semi-slant Riemannian maps from Cosymplectic manifolds into Riemannian manifolds. Several fundamental results on integrability of distributions and geometry of foliations are proved for such maps. Also we find the conditions for Riemannian maps to be totally geodesic and investigate some decomposition theorems. Finally, we give some examples of semi-slant Riemannian maps such that the characteristic vector field  $\xi$  is either vertical or horizontal.

### 1. INTRODUCTION

Let  $(M, g_M)$  and  $(N, g_N)$  be Riemannian manifolds of dimensional  $m$  and  $n$  respectively. A differentiable map  $f : (M, g_M) \rightarrow (N, g_N)$  is called a Riemannian submersion if (i)  $f$  has maximal rank (ii) The differential  $f_*$  preserves the lengths of horizontal vectors. Riemannian submersions between Riemannian manifolds were studied by O'Neill [10] and Gray [7], while Riemannian submersions between Riemannian manifolds equipped with an additional structure of almost complex type was initially studied by Watson in [15]. The concept of Riemannian maps between Riemannian manifolds was introduced by A. Fischer [6] in 1992. It generalizes and unifies the notion of an isometric immersion, a Riemannian submersion and an isometry. Let  $(M, g_M)$  and  $(N, g_N)$  be two Riemannian manifolds such that  $\dim M = m$  and  $\dim N = n$ . A smooth map  $f : (M, g_M) \rightarrow (N, g_N)$   $0 < \text{rank } f_* < \min(m, n)$  is called a Riemannian map if it satisfies the equation

$$g_N(f_*X, f_*Y) = g_M(X, Y), \quad (1.1)$$

for all  $X, Y \in (\ker f_*)$ . Let  $f : (M, g_M) \rightarrow (N, g_N)$  be a Riemannian map such that  $0 < \text{rank } f_* < \min(m, n)$ . Then we denote the kernel space of  $f_*$  by  $\ker f_*$  and consider the orthogonal complementary space  $(\ker f_*)^\perp$  of  $\ker f_*$  in  $TM$ . Then the tangent bundle of  $M$  has the following decomposition:

$$TM = \ker f_* \oplus (\ker f_*)^\perp. \quad (1.2)$$

In [2], isometric immersions and Riemannian submersions are particular cases of Riemannian maps with  $\ker f_* = \{0\}$  and  $(\text{range } f_*)^\perp = \{0\}$  respectively.

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In, 2010 anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds were introduced by B. Sahin [14]. Erken and Murathan introduced the notion anti-invariant Riemannian submersions from Sasakian manifolds onto Riemannian manifolds [5] and anti-invariant Riemannian submersions from Cosymplectic manifolds onto Riemannian manifolds [9]. Further anti-invariant Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds were studied by Beri et al. [2]. In 2013, B. Sahin introduced and studied the notion of slant Riemannian maps from almost Hermitian manifolds into Riemannian manifolds [13]. Further, in 2017, Prasad and Pandey introduced the concept of slant Riemannian maps from almost contact metric manifolds into Riemannian manifolds [16].

The paper is arranged as follows: In Section 2, we recall basic formulae and definitions for Cosymplectic manifolds and semi-slant submersions, which are needed in present paper. In section 3, we give the definition of a semi-slant Riemannian map from an almost contact metric manifold into a Riemannian manifold admitting vertical structure vector field and find some results. In section 4, we give the definition of a semi-slant Riemannian map from almost contact metric manifold to Riemannian manifolds admitting horizontal structure vector field and establish some results. Finally in section 5, we give examples of semi-slant Riemannian maps in which the characteristic vector field  $\xi$  is either vertical or horizontal.

## 2. PRELIMINARIES

An odd dimensional smooth manifold  $M$  is said to have an almost contact structure if there exist on  $M$ , a tensor field  $\phi$  of type  $(1, 1)$ , a vector field  $\xi$  and 1-form  $\eta$  such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad (2.1)$$

$$\eta(\xi) = 1. \quad (2.2)$$

If there exists a Riemannian metric  $g$  on an almost contact manifold  $M$  satisfying the conditions:

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.3)$$

$$g(X, \phi Y) = -g(\phi X, Y),$$

$$g(X, \xi) = \eta(X), \quad (2.4)$$

where  $X, Y$  are any vector fields on  $M$ . Then  $M$  is called almost contact metric manifold with almost contact structure  $(\phi, \xi, \eta, g)$ . It is denoted by  $(M, \phi, \xi, \eta, g)$ .

An almost contact structure  $(\phi, \xi, \eta)$  is said to be normal if the almost complex structure  $J$  on the product manifold  $M \times R$  given by

$$J(X, F \frac{d}{dt}) = (\phi X - F\xi, \eta(X) \frac{d}{dt}) \quad (2.5)$$

where  $J^2 = -I$  and  $F$  is the differentiable function on  $M \times R$  has no torsion i.e.,  $J$  is integrable. The form for normality in terms of  $\phi$ ,  $\xi$  and  $\eta$  is  $[\phi, \phi] + 2d\eta \otimes \xi = 0$  on  $M$ , where  $[\phi, \phi]$  is the Nijenhuis tensor of  $\phi$ . Lastly, the fundamental 2-form  $\Phi$  is defined by  $\Phi(X, Y) = g(X, \phi Y)$ .

An almost contact metric manifold  $M$  with almost contact structure  $(\phi, \xi, \eta, g)$  is said to be Cosymplectic manifold [8], if

$$(\nabla_X \phi)Y = 0, \quad (2.6)$$

for any vector fields  $X, Y$  on  $M$ , where  $\nabla$  denotes the Riemannian connection of the metric  $g$  on  $M$ . For a Cosymplectic manifold  $\nabla_X \xi = 0$ , for any vector field  $X$  on  $M$

**Example 2.1.** ([17]) We consider  $R^{2k+1}$  with Cartesian coordinates  $(x_i, y_i, z)$  ( $i = 1, 2, \dots, k$ ) and its usual contact form  $\eta = dz$ .

The characteristic vector field  $\xi$  is given by  $\frac{\partial}{\partial z}$  and its Riemannian metric  $g$  and tensor field  $\phi$  are given by

$$g_{R^{2k+1}} = \sum_{i=1}^k ((dx_i)^2 + (dy_i)^2) + (dz)^2, \quad \phi = \begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad i, j = 1, \dots, k.$$

This gives a Cosymplectic structure  $R^{2k+1}$ . The vector fields  $E_i = \frac{\partial}{\partial y_i}$ ,  $E_{k+i} = \frac{\partial}{\partial x_i}$ ,  $\xi = \frac{\partial}{\partial z}$  form a  $\phi$ -basis for the Cosymplectic manifold. On the other hand, it can be shown that  $(R^{2k+1}, \phi, \xi, \eta, g)$  is a Cosymplectic manifold.

Let  $f : (M, g_M, J) \rightarrow (N, g_N)$  be a differentiable map, where  $(M, g_M, J)$  be a Hermitian manifold and  $(N, g_N)$  a Riemannian manifold. A Riemannian map  $f : (M, g_M, J) \rightarrow (N, g_N)$  is called a semi-slant Riemannian map if there is a distribution  $D_1 \subseteq (\ker f_*)$  such that

$$\ker f_* = D_1 \oplus D_2, \quad J(D_1) = D_1,$$

and the angle  $\theta = \theta(X)$  between  $JX$  and the space  $(D_2)_q$  is constant for non-zero vector field  $X \in (D_2)_q$ ,  $q \in M$ , where  $D_2$  is the orthogonal complement of  $D_1$  in  $\ker f_*$  [12]. We call the angle  $\theta$  a semi-slant angle.

Let  $f : (M, g_M) \rightarrow (N, g_N)$  be a differentiable map. The second fundamental form of  $f$  is given by

$$(\nabla f_*)(X, Y) = \nabla_X^f f_* Y - f_*(\nabla_X Y), \quad \text{for all } X, Y \in \Gamma(TM). \quad (2.7)$$

where  $\nabla^f$  is the pullback connection and we denote conveniently by  $\nabla$  the Riemannian connections of the metrics  $g_M$  and  $g_N$ . Memories that  $f$  is said to be harmonic if we have the tensor field  $\tau(f) = \text{trace}(\nabla f_*) = 0$  and we call the map  $f$  a totally geodesic map if  $(\nabla f_*)(X, Y) = 0$ , for  $X, Y \in \Gamma(TM)$  [1]. Denote the range of  $f_{*q}$  by  $\text{range } f_{*q}$  at  $q \in M$  and consider the orthogonal complement  $(\text{range } f_{*q})^\perp$  of  $\text{range } f_{*q}$  in the tangent space  $T_f(q)N$  of  $N$ . Since for proper Riemannian maps  $0 < \text{rank} < f_* < \min(m, n)$ , we always have  $(\text{range } f_{*q})^\perp \neq 0$ . Thus the tangent space  $T_f(q)N$  of  $N$  has the following decomposition

$$f^{-1}TN = \text{range } f_* \oplus (\text{range } f_*)^\perp.$$

In 1966 [10], the fundamental tensors of a submersion were introduced by O'Neill. They have a similar role to that of the second fundamental form of an immersion. More exactly, O'Neill's tensors  $\mathcal{T}$  and  $\mathcal{A}$  defined by

$$\mathcal{A}_E F = \mathcal{H} \nabla_{\mathcal{H}E} \mathcal{V} F + \mathcal{V} \nabla_{\mathcal{H}E} \mathcal{H} F, \quad (2.8)$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \quad (2.9)$$

for vector fields  $E, F \in \Gamma(TM)$ , where  $\mathcal{V}$  and  $\mathcal{H}$  are the vertical and horizontal projections,  $\nabla$  is the Riemannian connection of  $M$ . Let tensors  $\mathcal{T}$  and  $\mathcal{A}$  for Riemannian map  $f$  are also defined by equations (2.8) and (2.9). On the other hand, from equations (2.8) and (2.9), we have

$$\nabla_U V = \mathcal{T}_U V + \widehat{\nabla}_U V, \quad (2.10)$$

$$\nabla_U X = \mathcal{H}\nabla_U X + \mathcal{T}_U X, \quad (2.11)$$

$$\nabla_X U = \mathcal{A}_X U + \mathcal{V}\nabla_X U, \quad (2.12)$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \quad (2.13)$$

for  $U, V \in \Gamma(\ker f_*)$  and  $X, Y \in \Gamma((\ker f_*)^\perp)$ . Now, we discuss the harmonicity of a map  $F$ . Let  $(M, g_M)$  and  $(N, g_N)$  be Riemannian manifolds with Riemannian metrics  $g_M$  and  $g_N$  respectively and let  $f : (M, g_M) \rightarrow (N, g_N)$  be a differentiable map between them. The energy density of  $f$  is the differentiable function  $e(f) : M \rightarrow [0, \infty]$  given by

$$e(f)(x) = \frac{1}{2} |e(f_*)(x)|^2, \quad x \in M,$$

where  $|e(f_*)(x)|$  denotes the Hilbert-Schmidt norm of  $(f_*)(x)$  defined by

$$|e(f_*)(x)|^2 = \sum_{i=1}^k g(f_*e_i, f_*e_i),$$

where  $e_i$  is an orthonormal basis for  $(T_x M)$  [1]. We call  $e(f)$  the energy density of  $f$  and let  $L$  be a compact domain of  $M$ . The energy integral of  $f$  over  $L$  is the integral of its energy density

$$E(f, K) = \int_K e(f) v_{g_M} = \frac{1}{2} \int_K |e(f_*)|^2 v_{g_M},$$

where  $\omega_{g_M}$  is the volume resume on  $M$  defined by the metric  $g_M$ . A differentiable map  $f : (M, g_M) \rightarrow (N, g_N)$  is said to be harmonic if it is a critical point of the energy functional  $E(f, K) : C^\infty(M, N) \rightarrow R$  for any compact domain  $L \subset M$  where  $C^\infty(M, N)$  denote the space of all differentiable maps from  $M$  to  $N$ . These results provided by J. Eells and J. Sampson [4], implies that the map  $f$  is harmonic if and only if the tension field  $\tau(f) = \text{trace}(\nabla f_*) = 0$ .

Next, we discuss for decomposition theorems by using the existence of semi-slant Riemannian map. We have memories the following results from [17]. Consider a distribution  $D$  on Riemannian manifold  $(M, g_M)$ . The distribution  $D$  is called autoparallel (a totally geodesic foliation) if  $\nabla_X Y \in \Gamma(D)$ , for every  $X, Y \in \Gamma(D)$ . If  $D$  is autoparallel, then it is integrable and its leaves are totally geodesic in  $M$ . The distribution  $D$  is said to be parallel if for  $V \in \Gamma(D)$  and  $W \in \Gamma(TM)$  implies  $\nabla_W V \in \Gamma(D)$ . If  $D$  is parallel then its orthogonal complement distribution  $D^\perp$  is also parallel. In this case  $M$  is locally Riemannian product manifold of the leaves of  $D$  and  $D^\perp$ . It is also simple to prove that if the distributions  $D$  and  $D^\perp$  are simultaneously autoparallel, then they are also parallel

**Lemma 2.2.** *Let  $f : (M, g_M) \rightarrow (N, g_N)$ , a Riemannian map between two Riemannian manifolds [6]. We have*

$$(\nabla f_*)(X, Y) \in \Gamma((\text{range } f_*)^\perp), \quad \text{for all } X, Y \in \Gamma((\ker f_*)^\perp).$$

**Lemma 2.3.** *For any  $X, Y$  vertical and  $V, W$  horizontal vector fields, the tensor fields  $\mathcal{T}$  and  $\mathcal{A}$  satisfy [12]:*

$$\begin{aligned} \mathcal{T}_X Y &= \mathcal{T}_Y X, \\ \mathcal{A}_V W &= -\mathcal{A}_W V = \frac{1}{2} \mathcal{V}[V, W]. \end{aligned}$$

**Lemma 2.4.** *For a Riemannian map  $f : (M, g_M) \rightarrow (N, g_N)$  between two Riemannian manifolds, we have*

$$2e(f) = \|f_*\|^2 = \text{rank } f.$$

Here  $\|f_*\|^2$  is a continuous function on  $M$  and  $\text{rank } f$  is integer-valued so that  $\text{rank } f$  is locally constant. Hence, if  $M$  is connected, then  $\text{rank } f$  is a constant function.

### 3. SEMI-SLANT RIEMANNIAN MAPS ADMITTING VERTICAL STRUCTURE VECTOR FIELD

In this section we define semi-slant Riemannian maps from almost contact metric manifolds to Riemannian manifolds. We have investigated integrability of distributions and harmonicity conditions for semi-slant Riemannian map. Further, we have found the conditions for semi-slant Riemannian map to be totally geodesic and proved decomposition theorems. Throughout this section we have taken semi-slant Riemannian maps admitting vertical structure vector field

**Definition 3.1.** Let  $(M, \phi, \xi, \eta, g_M)$  be a almost contact metric manifold and  $(N, g_N)$  be a Riemannian manifold. A Riemannian map  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  is called a semi-slant Riemannian map if there is a distribution  $D_1 \subset (\ker f_*)$  such that

$$\ker f_* = D_1 \oplus D_2 \oplus \langle \xi \rangle, \phi(D_1) = D_1,$$

and the angle  $\theta = \theta(U)$  between  $\phi U$  and the space  $(D_2)_x$  is constant for non-zero vector fields  $U \in (D_2)_x$  and  $x \in M$ , where  $D_1, D_2$  and  $\xi$  are mutually orthogonal distributions in  $\ker f_*$ . We call the angle  $\theta$  a semi-slant angle of the semi-slant Riemannian map.

Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map. Then there is a distribution  $D_1 \subset (\ker f_*)$  such that

$$\ker f_* = D_1 \oplus D_2 \oplus \langle \xi \rangle, \phi(D_1) = D_1,$$

and the angle  $\theta = \theta(U)$  between  $\phi U$  and the space  $(D_2)_x$  is constant for non zero vector fields  $(D_2)_x$  and  $x \in M$  where  $D_1, D_2$  and  $\xi$  are mutually orthogonal distributions in  $\ker f_*$ . For  $U \in \Gamma(\ker f_*)$ , we get

$$U = PU + QU + \eta(U)\xi, \quad (3.1)$$

where  $PU \in \Gamma(D_1)$  and  $QU \in \Gamma(D_2)$ .

Also, for  $U \in \Gamma(\ker f_*)$ , we get

$$\phi U = \psi U + \omega U, \quad (3.2)$$

where  $\psi U \in \Gamma(\ker f_*)$  and  $\omega U \in \Gamma((\ker f_*)^\perp)$ .

For  $X \in \Gamma((\ker f_*)^\perp)$ , we write

$$\phi X = BX + CX, \quad (3.3)$$

where  $BX \in \Gamma(\ker f_*)$  and  $CX \in \Gamma((\ker f_*)^\perp)$ .

For  $W \in \Gamma(TM)$ , we obtain

$$W = \mathcal{V}W + \mathcal{H}W, \quad (3.4)$$

where  $\mathcal{V}W \in \Gamma(\ker f_*)$  and  $\mathcal{H}W \in \Gamma((\ker f_*)^\perp)$ .

For  $Z \in \Gamma(f^{-1}TN)$ , we have

$$Z = \bar{P}Z + \bar{Q}Z, \quad (3.5)$$

where  $\bar{P}Z \in \Gamma(\text{range } f_*)$  and  $\bar{Q}Z \in \Gamma(\text{range } f_*)^\perp$ .

Then  $(\ker f_*)^\perp = \omega D_2 \oplus \mu$ , where  $\mu$  is the orthogonal complement of  $\omega D_2$  in  $(\ker f_*)^\perp$  and is invariant under  $\phi$ .

Further more,

$$\psi D_1 = D_1, \quad \omega D_1 = 0.$$

For  $U, V \in \Gamma(\ker f_*)$ , define

$$\widehat{\nabla}_U V = \mathcal{V}\nabla_U V,$$

$$(\nabla_U \psi)V = \widehat{\nabla}_U \psi V - \psi \widehat{\nabla}_U V, \quad (3.6)$$

$$(\nabla_U \omega)V = \mathcal{H}\nabla_U \omega V - \omega \widehat{\nabla}_U V. \quad (3.7)$$

**Lemma 3.2.** *Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian maps, where  $(M, \phi, \xi, \eta, g_M)$  is a Cosymplectic manifold and  $(N, g_N)$  be a Riemannian manifold. Then we have following results*

(1)

$$(\nabla_U \psi)V = B\mathcal{T}_U V - \mathcal{T}_U \omega V, \quad (3.8)$$

$$(\nabla_U \omega)V = C\mathcal{T}_U V - \mathcal{T}_U \psi V, \quad (3.9)$$

for all  $U, V \in \Gamma(\ker f_*)$ .

(2)

$$\mathcal{V}\nabla_X BY + \mathcal{A}_X CY = \psi \mathcal{A}_X Y + B\mathcal{H}\nabla_X Y, \quad (3.10)$$

$$\mathcal{A}_X BY + \mathcal{H}\nabla_X CY = \omega \mathcal{A}_X Y + C\mathcal{H}\nabla_X Y, \quad (3.11)$$

for all  $X, Y \in \Gamma((\ker f_*)^\perp)$ .

$$(3) \quad \widehat{\nabla}_U BX + \mathcal{T}_U CX = \psi \mathcal{T}_U X + B \mathcal{H} \nabla_U X, \quad (3.12)$$

$$\mathcal{T}_U BX + \mathcal{H} \nabla_U CX = C \mathcal{H} \nabla_U X + \omega \mathcal{T}_U X, \quad (3.13)$$

$$\mathcal{V} \nabla_X \psi U + \mathcal{A}_X \omega U = \psi \mathcal{V} \nabla_X U + B \mathcal{A}_X U, \quad (3.14)$$

$$\mathcal{A}_X \psi U + \mathcal{H} \nabla_X \omega U = \omega \mathcal{V} \nabla_X U + C \mathcal{A}_X U, \quad (3.15)$$

for all  $U \in \Gamma(\ker f_*)$  and  $X \in \Gamma((\ker f_*)^\perp)$ .

Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map with the semi-slant angles  $\theta$ . Then given non-zero vector field  $U \in \Gamma(D_2)$ , we have

$$\cos \theta = \frac{|\psi U|}{|\phi U|}, \quad (3.16)$$

and

$$\cos \theta = \frac{g_M(\phi U, \psi U)}{|\phi U| |\psi U|}.$$

By using equation (3.2), we have

$$\begin{aligned} \cos \theta &= \frac{g_M(\psi U, \psi U)}{|\phi U| |\psi U|}, \\ \cos \theta &= -\frac{g_M(U, \psi^2 U)}{|\phi U| |\psi U|}. \end{aligned} \quad (3.17)$$

From equations (2.1), (3.16) and (3.17), we get

$$\psi^2 U = -\cos^2 \theta \cdot U, \quad \text{for all } U \in \Gamma(D_2).$$

**Theorem 3.3.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map with the semi-slant angles  $\theta$ . Then*

$$\psi^2 U = -\cos^2 \theta \cdot U, \quad \text{for all } U \in \Gamma(D_2).$$

The following corollary can be easily obtained by above theorem.

**Corollary 3.4.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map with the semi-slant angles  $\theta$ . Then*

$$\begin{aligned} g_M(\psi U, \psi V) &= \cos^2 \theta g_M(U, V), \\ g_M(\omega U, \omega V) &= \sin^2 \theta g_M(U, V), \end{aligned}$$

for all  $U, V \in \Gamma(D_2)$ , where  $\theta \in (0, \frac{\pi}{2})$

We can locally choose an orthonormal frame  $\{e_1, \psi e_1, \dots, e_k, \psi e_k, l_1, \sec \theta \psi l_1, \csc \theta \omega l_1, \dots, l_s, \sec \theta \psi l_s, \csc \theta \omega l_s, \xi, g_1, \psi g_1, \dots, g_t, \psi g_t\}$  of  $TM$  such that  $\{e_1, \psi e_1, \dots, e_k, \psi e_k\}$  is an orthonormal frame of  $D_1$ ,  $\{l_1, \sec \theta \psi l_1, \dots, l_s, \sec \theta \psi l_s\}$  an orthonormal frame of  $D_2$ ,  $\langle \xi \rangle$  an orthogonal  $D_1$  and  $D_2$ ,  $\{\csc \theta \omega l_1, \dots, \csc \theta \omega l_s\}$  an orthonormal frame of  $\omega D_2$ , and  $\{g_1, \psi g_1, \dots, g_t, \psi g_t\}$  an orthonormal frame of  $\mu$ .

**Lemma 3.5.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map with the semi-slant angles  $\theta$ . If tensor  $\omega$  is parallel, then*

$$\mathcal{T}_{\psi U}\psi U = -\cos^2\theta.\mathcal{T}_U U, \quad \text{for all } U \in \Gamma(D_2). \quad (3.18)$$

*Proof.* If the tensor  $\omega$  is parallel such that

$$(\nabla_U\omega)V = 0. \quad (3.19)$$

From equation (3.9), we have

$$\mathcal{T}_V\psi U = \mathcal{T}_U\psi V.$$

Replace  $V \rightarrow \psi U$ , we have

$$\mathcal{T}_{\psi U}\psi U = -\cos^2\theta.\mathcal{T}_U U, \quad \text{for all } U \in \Gamma(D_2).$$

□

**Theorem 3.6.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map. Then the slant distribution  $D_1$  is integrable if and only if,*

$$\omega(\widehat{\nabla}_U V - \widehat{\nabla}_V U) = 0, \quad \text{for all } U, V \in \Gamma(D_1).$$

*Proof.* It is known that for all  $X, Y \in \Gamma(\ker f_*) \Rightarrow [X, Y] \in \Gamma((\ker f_*)$ . For all  $U, V \in \Gamma(D_1)$  and  $Z \in (\ker f_*)^\perp \Rightarrow [U, V] \in \Gamma((\ker f_*)$ . We know that

$$\begin{aligned} g_M(V, \xi) &= 0, \\ \nabla_U(g_M(V, \xi)) &= 0, \\ g_M(\nabla_U V, \xi) &= -g_M(V, \nabla_U \xi), \\ g_M(\nabla_U V, \xi) &= 0. \end{aligned}$$

Since  $\nabla_U \xi = 0$ , for Cosymplectic manifolds. Now, consider

$$\begin{aligned} g_M([U, V], \xi) &= g_M(\nabla_U V, \xi) - g_M(\nabla_V U, \xi), \\ g_M([U, V], \xi) &= 0. \end{aligned}$$

Consider

$$g_M(\phi[U, V], Z) = g_M(\phi(\nabla_U V - \nabla_V U), Z),$$

Using equation (3.2), (3.3) and (2.10), we get

$$\begin{aligned} g_M(\phi([U, V]), Z) &= g_M(B\mathcal{T}_U V + C\mathcal{T}_U V + \psi\widehat{\nabla}_U V + \omega\widehat{\nabla}_U V \\ &\quad - B\mathcal{T}_V U - C\mathcal{T}_V U - \psi\widehat{\nabla}_V U - \omega\widehat{\nabla}_V U, Z). \end{aligned}$$

Hence,  $D_1$  is integrable  $\Leftrightarrow P(\psi(\widehat{\nabla}_U V - \widehat{\nabla}_V U)) = 0$ , for all  $U, V \in \Gamma(D_1)$ . □

**Theorem 3.7.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map. Then the slant distribution  $D_2$  is integrable if and only if,*

$$P(\psi(\widehat{\nabla}_U V - \widehat{\nabla}_V U)) = 0, \quad \text{for all } U, V \in \Gamma(D_2).$$



*Proof.* For all  $U, V \in \Gamma(D_2)$  and  $X \in \Gamma(D_1)$ , since  $[U, V] \in \Gamma((\ker f_*)$ . Similar way as in theorem (3.6) we have

$$g_M([U, V], \xi) = 0.$$

Now, consider

$$g_M(\phi[U, V], X) = g_M(\phi(\nabla_U V - \nabla_V U), X),$$

Using equation (3.2), (3.3) and (2.10), we get

$$\begin{aligned} g_M(\phi([U, V]), X) &= g_M(B\mathcal{T}_U V + C\mathcal{T}_U V + \psi\widehat{\nabla}_U V + \omega\widehat{\nabla}_U V \\ &\quad - B\mathcal{T}_V U - C\mathcal{T}_V U - \psi\widehat{\nabla}_V U - \omega\widehat{\nabla}_V U, X). \end{aligned}$$

Hence,  $D_2$  is integrable  $\Leftrightarrow P(\psi(\widehat{\nabla}_U V - \widehat{\nabla}_V U)) = 0$ , for all  $U, V \in \Gamma(D_2)$ .  $\square$

**Theorem 3.8.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map such that  $D_1$  is integrable. Then  $f$  is harmonic if and only if  $\text{trace}(\nabla f_*) = 0$  on  $D_2$  and  $\overline{H}$ , where  $\overline{H}$  denotes the mean curvature vector field of range  $f_*$ .*

*Proof.* Using Lemma (2.2), we get  $\text{trace}(\nabla f_*)|_{\ker f_*} \in (\text{range } f_*)$  and  $\text{trace}(\nabla f_*)|_{(\ker f_*)^\perp} \in (\text{range } f_*)^\perp$  so that

$$\text{trace}(\nabla f_*) = 0 \Leftrightarrow \text{trace}(\nabla f_*)|_{\ker f_*} = 0,$$

and

$$\text{trace}(\nabla f_*)|_{(\ker f_*)^\perp} = 0.$$

Since  $D_1$  is invariant under  $\phi$ , we can choose locally orthonormal frame  $\{e_1, \phi e_1, \dots, \dots, e_k, \phi e_k\}$  of  $D_1$ . Using the integrability of the  $D_1$

$$\begin{aligned} (\nabla f_*)(\phi e_i, \phi e_i) &= -f_*(\nabla_{\phi e_i} \phi e_i), \\ &= f_*(\nabla_{e_i} e_i), \\ &= -(\nabla f_*)(e_i, e_i), \quad \text{for } 1 \leq i \leq k. \end{aligned}$$

Thus,

$$\text{trace}(\nabla f_*)|_{(\ker f_*)} = 0 \Leftrightarrow \text{trace}(\nabla f_*)|_{D_2} = 0, \text{ and } \text{trace}(\nabla f_*)(\xi, \xi) = 0.$$

Moreover, it is easy to get that

$$\text{trace}(\nabla f_*)|_{(\ker f_*)^\perp} = l\overline{H}, \quad \text{for } l = \dim(\ker f_*)^\perp,$$

so that

$$\text{trace}(\nabla f_*)|_{(\ker f_*)^\perp} = 0 \Leftrightarrow \overline{H} = 0.$$

$\square$

**Theorem 3.9.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map. Then  $f$  is a totally geodesic map if and only if*

$$\begin{aligned}\omega(\widehat{\nabla}_U\psi V + \mathcal{T}_U\omega V) + C(\mathcal{T}_U\psi V + \mathcal{H}\nabla_U\omega V) &= 0, \\ \omega(\widehat{\nabla}_UBX + \mathcal{T}_UCX) + C(\mathcal{T}_UBX + \mathcal{H}\nabla_UCX) &= 0, \\ \overline{Q}(\nabla_Y^f f_*Z) &= 0,\end{aligned}$$

for all  $U, V \in \Gamma(\ker f_*)$  and  $X, Y, Z \in \Gamma((\ker f_*)^\perp)$ .

*Proof.* If  $X, Y, Z \in \Gamma(\ker f_*)^\perp$ , from Lemma (2.2), we get

$$(\nabla f_*)(Y, Z) = 0 \Leftrightarrow \overline{Q}((\nabla f_*)(Y, Z)) = \overline{Q}(\nabla_Y^f f_*Z) = 0.$$

For  $U, V \in \Gamma(\ker f_*)$ , using equations (2.1), (3.2), (3.3), (2.10) and (2.11), we have

$$\begin{aligned}(\nabla f_*)(U, V) &= -f_*(\nabla_U V), \\ &= f_*(\phi(\widehat{\nabla}_U\psi V + \mathcal{T}_U\psi V + \mathcal{T}_U\omega V + \mathcal{H}\nabla_U\omega V), \\ &= f_*(\psi\widehat{\nabla}_U\psi V + \omega\widehat{\nabla}_U\psi V + B\mathcal{T}_U\psi V + C\mathcal{T}_U\psi V + \psi\mathcal{T}_U\omega V \\ &\quad + \omega\mathcal{T}_U\omega V + B\mathcal{H}\nabla_U\omega V + C\mathcal{H}\nabla_U\omega V).\end{aligned}$$

Hence

$$(\nabla f_*)(U, V) = 0 \Leftrightarrow \omega(\widehat{\nabla}_U\psi V + \mathcal{T}_U\omega V) + C(\mathcal{T}_U\omega V + \mathcal{H}\nabla_U\omega V) = 0.$$

If  $U \in \Gamma(\ker f_*)$  and  $X \in \Gamma(\ker f_*)^\perp$ , since the tensor  $(\nabla f_*)$  is symmetric. Using equations (2.1), (3.2), (3.3), (2.10) and (2.11), we get

$$\begin{aligned}(\nabla f_*)(U, X) &= -f_*(\nabla_U X), \\ &= f_*(\phi(\widehat{\nabla}_UBX + \mathcal{T}_UBX + \mathcal{T}_UCX + \mathcal{H}\nabla_UCX)), \\ &= f_*(\psi\widehat{\nabla}_UBX + \omega\widehat{\nabla}_UBX + B\mathcal{T}_UBX + C\mathcal{T}_UBX + \psi\mathcal{T}_UCX \\ &\quad + \omega\mathcal{T}_UCX + B\mathcal{H}\nabla_UCX + C\mathcal{H}\nabla_UCX).\end{aligned}$$

Thus,

$$(\nabla f_*)(U, X) = 0 \Leftrightarrow \omega(\widehat{\nabla}_UBX + \mathcal{T}_UCX) + C(\mathcal{T}_UBX + \mathcal{H}\nabla_UCX) = 0.$$

□

Let  $f : (M, g_M) \rightarrow (N, g_N)$  be a Riemannian map. The map  $f$  is called a Riemannian map with totally umbilical fibres if

$$\mathcal{T}_UV = g_M(U, V)H, \quad \text{for all } U, V \in \Gamma(\ker f_*),$$

where  $H$  is mean curvature vector field of the fibre.

**Lemma 3.10.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map. Then*

$$H \in \Gamma(\omega D_2)$$

*Proof.* For all  $U, V \in \Gamma(D_2)$  and  $Z \in \Gamma(\mu)$ , from equations (3.2), (3.3) and (2.10) we get

$$\begin{aligned}\phi\nabla_U V &= \phi\widehat{\nabla}_U V + \phi\mathcal{T}_U V, \\ \nabla_U \phi V &= \psi\widehat{\nabla}_U V + \omega\widehat{\nabla}_U V + \psi\mathcal{T}_U V + \omega\mathcal{T}_U V.\end{aligned}$$

We have

$$g_M(\mathcal{T}_U \phi V, Z) = g_M(C\mathcal{T}_U V, Z).$$

By using result

$$\begin{aligned}\mathcal{T}_U \phi V &= g_M(U, V)H, \\ \phi\mathcal{T}_U \phi V &= g_M(U, \phi V)H, \\ g_M(U, \phi V)g_M(H, Z) &= -g_M(\mathcal{T}_U V, \phi Z). \\ g_M(U, \phi V)g_M(H, Z) &= -g_M(U, V)g_M(H, \phi Z).\end{aligned}\tag{3.20}$$

Interchanging the role of  $U$  and  $V$ , we get

$$g_M(V, \phi U)g_M(H, Z) = -g_M(V, U)g_M(H, \phi Z).\tag{3.21}$$

So that comparing the equations (3.20) and (3.21), we have

$$g_M(V, U)g_M(H, \phi Z) = 0.$$

which means  $H \in \Gamma(\omega D_2)$  □

**Theorem 3.11.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map. Then  $(M, \phi, \xi, \eta, g_M)$  is locally a Riemannian product manifold of the leaves of  $\Gamma(\ker f_*)$  and  $\Gamma(\ker f_*)^\perp$  if and only if*

$$\begin{aligned}\omega(\widehat{\nabla}_U \psi V + \mathcal{T}_U \omega V) + C(\mathcal{T}_U \psi V + \mathcal{H}\nabla_U \omega V) &= 0, \\ \psi(\mathcal{V}\nabla_X BZ + \mathcal{A}_X CZ) + B(\mathcal{A}_X BZ + \mathcal{H}\nabla_X CZ) &= 0,\end{aligned}$$

for all  $U, V \in \Gamma(\ker f_*)$  and  $X, Z \in \Gamma((\ker f_*)^\perp)$ .

*Proof.* For  $U, V \in \Gamma(\ker f_*)$ , using equation (2.1), (2.6), (3.2), (3.3), (2.10) and (2.11), we get

$$\begin{aligned}\nabla_U V &= -(-\nabla_U V), \\ &= -\phi(\widehat{\nabla}_U \psi V + \mathcal{T}_U \psi V + \mathcal{T}_U \omega V + \mathcal{H}\nabla_U \omega V) \\ &= -(\psi\widehat{\nabla}_U \psi V + \omega\widehat{\nabla}_U \psi V + B\mathcal{T}_U \psi V + C\mathcal{T}_U \psi V + \psi\mathcal{T}_U \omega V + \omega\mathcal{T}_U \omega V \\ &\quad + B\mathcal{H}\nabla_U \omega V + C\mathcal{H}\nabla_U \omega V).\end{aligned}$$

Thus,

$$\nabla_U V \in \Gamma(\ker f_*) \Leftrightarrow \omega(\widehat{\nabla}_U \psi V + \mathcal{T}_U \omega V) + C(\mathcal{T}_U \omega V + \mathcal{H}\nabla_U \omega V) = 0.$$

For all  $X, Z \in \Gamma((\ker f_*)^\perp)$ , using equations (2.1), (2.6), (3.2), (3.3), (2.12) and (2.13), we have

$$\begin{aligned}\nabla_X Z &= -(-\nabla_X Z), \\ &= -\phi(\mathcal{V}\nabla_X BZ + \mathcal{A}_X BZ + \mathcal{A}_X CZ + \mathcal{H}\nabla_X CZ), \\ &= -(\psi\mathcal{V}\nabla_X BZ + \omega\mathcal{V}\nabla_X BZ + B\mathcal{A}_X BZ + C\mathcal{A}_X BZ \\ &\quad + \psi\mathcal{A}_X CZ + \omega\mathcal{A}_X CZ + B\mathcal{H}\nabla_X CZ + C\mathcal{H}\nabla_X CZ).\end{aligned}$$

Hence  $\nabla_X Z \in \Gamma((\ker f_*)^\perp) \Leftrightarrow \psi(\mathcal{V}\nabla_X BZ + \mathcal{A}_X CZ) + B(\mathcal{A}_X BZ + \mathcal{H}\nabla_X CZ)$ .  $\square$

**Theorem 3.12.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map. Then the fibers of  $f$  are locally Riemannian product manifolds of the leaves of  $D_1$  and  $D_2$  if and only if*

$$Q(\psi\widehat{\nabla}_U \psi V + B\mathcal{T}_U \psi V) = 0, \omega\widehat{\nabla}_U \psi V + C\mathcal{T}_U \psi V = 0,$$

$$P(\psi(\widehat{\nabla}_X \psi Y + \mathcal{T}_X \omega Y) + B(\mathcal{T}_X \psi Y + \mathcal{H}\nabla_X \omega Y)) = 0,$$

$$\omega(\widehat{\nabla}_X \psi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \psi Y + \mathcal{H}\nabla_X \omega Y) = 0,$$

for all  $U, V \in \Gamma(D_1)$ , and  $X, Y \in \Gamma(D_2)$ .

*Proof.* For  $U, V \in \Gamma(D_1)$ , using equations (2.1), (2.10), (3.2), (3.3) and (2.11), we have

$$\begin{aligned}\nabla_U V &= -\phi(\widehat{\nabla}_U \psi V + \mathcal{T}_U \psi V) \\ &= -(\psi\widehat{\nabla}_U \psi V + \omega\widehat{\nabla}_U \psi V + B\mathcal{T}_U \psi V + C\mathcal{T}_U \psi V).\end{aligned}$$

Hence

$$\nabla_U V \in \Gamma(D_1) \Leftrightarrow Q(\psi\widehat{\nabla}_U \psi V + B\mathcal{T}_U \psi V) = 0,$$

and

$$\omega\widehat{\nabla}_U \psi V + C\mathcal{T}_U \psi V = 0.$$

For  $X, Y \in \Gamma(D_2)$ , using equations (2.1), (3.2), (3.3), (2.11) and (2.12), we have

$$\begin{aligned}\nabla_X Y &= -(\psi\widehat{\nabla}_X \psi Y + \omega\widehat{\nabla}_X \psi Y + B\mathcal{T}_X \psi Y + C\mathcal{T}_X \psi Y + \psi\mathcal{T}_X \omega Y \\ &\quad + \omega\mathcal{T}_X \omega Y + B\mathcal{H}\nabla_X \omega Y + C\mathcal{H}\nabla_X \omega Y).\end{aligned}$$

Thes  $\nabla_X Y \in \Gamma(D_2) \Leftrightarrow P(\psi\widehat{\nabla}_X \psi Y + \psi\mathcal{T}_X \omega Y + B\mathcal{T}_X \psi Y + B\mathcal{H}\nabla_X \omega Y) = 0$  and  $\omega\widehat{\nabla}_X \psi Y + \omega\mathcal{T}_X \omega Y + C\mathcal{T}_X \psi Y + C\mathcal{H}\nabla_X \omega Y = 0$ .

Therefore, we get the result.  $\square$

#### 4. SEMI-SLANT RIEMANNIAN MAPS ADMITTING HORIZONTAL STRUCTURE VECTOR FIELD

In this section, we define semi-slant Riemannian maps from almost contact metric manifold into Riemannian manifolds admitting horizontal structure vector field. Integrability and harmonic conditions for such maps are found. We have investigated the conditions for semi-slant Riemannian to be totally geodesic and proved decomposition theorems

Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map. Then there is a distribution  $D_1 \subset (\ker f_*)$  such that

$$\ker f_* = D_1 \oplus D_2, \phi(D_1) = D_1,$$

$$(\ker f_*)^\perp = \omega(D_2) \oplus \mu \oplus \langle \xi \rangle, \phi(D_1) = D_1,$$

and the angle  $\theta = \theta(U)$  between  $\phi U$  and the space  $(D_2)_x$  is constant for non-zero vector fields  $U \in \Gamma(D_2)_x$  and  $x \in M$ , where  $D_1, D_2$  and  $\xi$  are mutually orthogonal distributions in  $\ker f_*$ .

We call the angle  $\theta$  a semi-slant angle of the semi-slant Riemannian map.

For  $U \in \Gamma(\ker f_*)$ , we get

$$U = PU + QU, \tag{4.1}$$

where  $PU \in \Gamma(D_1)$  and  $QU \in \Gamma(D_2)$ .

Also for  $U \in \Gamma(\ker f_*)$ , we get

$$\phi U = \psi U + \omega U, \tag{4.2}$$

where  $\psi U \in \Gamma(\ker f_*)$  and  $\omega U \in \Gamma((\ker f_*)^\perp)$ .

For  $X \in \Gamma((\ker f_*)^\perp)$ , we write

$$\phi X = BX + CX, \tag{4.3}$$

where  $BX \in \Gamma(\ker f_*)$  and  $CX \in \Gamma((\ker f_*)^\perp)$ . Then  $(\ker f_*)^\perp = \omega(D_2) \oplus \mu \oplus \langle \xi \rangle$ , where  $\mu$  is the orthogonal complement of  $\omega(D_2)$  in  $(\ker f_*)^\perp$  and is invariant under  $\phi$ .

*Remark 4.1.* Lemma (3.2) we found same results in this section.

Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map with the semi-slant angles  $\theta$ . Then given non-zero vector fields  $U \in \Gamma(D_2)$ , we have

$$\cos \theta = \frac{|\psi U|}{|\phi U|}, \tag{4.4}$$

and

$$\cos \theta = \frac{g_M(\phi U, \psi U)}{|\phi U| |\psi U|}.$$

By using equation (4.2), we have

$$\cos \theta = \frac{g_M(\psi U, \psi U)}{|\phi U| |\psi U|},$$

$$\cos \theta = -\frac{g_M(U, \psi^2 U)}{|\phi U| |\psi U|}. \quad (4.5)$$

From equations (2.1), (4.4) and (4.5), we get

$$\psi^2 U = -\cos^2 \theta \cdot U, \quad \text{for } U \in \Gamma(D_2).$$

**Theorem 4.2.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map with the semi-slant angles  $\theta$ . Then*

$$\psi^2 U = -\cos^2 \theta \cdot U, \quad \text{for all } U \in \Gamma(D_2).$$

**Corollary 4.3.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map with the semi-slant angles  $\theta$ . Then*

$$g_M(\psi U, \psi V) = \cos^2 \theta g_M(U, V),$$

$$g_M(\omega U, \omega V) = \sin^2 \theta g_M(U, V),$$

for all  $U, V \in \Gamma(D_2)$ , where  $\theta \in (0, \frac{\pi}{2})$ .

We can locally choose an orthonormal frame  $\{e_1, \psi e_1, \dots, e_k, \psi e_k, l_1, \sec \theta \psi l_1, \csc \theta \omega l_1, \dots, l_s, \sec \theta \psi l_s, \csc \theta \omega l_s, g_1, \psi g_1, \dots, g_t, \psi g_t, \xi\}$  of  $TM$  such that  $\{e_1, \psi e_1, \dots, e_k, \psi e_k\}$  is an orthonormal frame of  $D_1$ ,  $\{l_1, \sec \theta \psi l_1, \dots, l_s, \sec \theta \psi l_s\}$  an orthonormal frame of  $D_2$ ,  $\{\csc \theta \omega l_1, \dots, \csc \theta \omega l_s\}$  an orthonormal frame of  $\omega D_2$ , and  $\{g_1, \psi g_1, \dots, g_t, \psi g_t\}$  an orthonormal frame of  $\mu$ .

**Lemma 4.4.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map with the semi-slant angles  $\theta$ . If tensor  $\omega$  is parallel, then*

$$\mathcal{T}_{\psi U} \psi U = -\cos^2 \theta \cdot \mathcal{T}_U U, \quad \text{for all } U \in \Gamma(D_2). \quad (4.6)$$

*Proof.* If the tensor  $\omega$  is parallel such that

$$(\nabla_U \omega)V = 0. \quad (4.7)$$

From equation (3.9), we have

$$\mathcal{T}_V \psi U = \mathcal{T}_U \psi V.$$

Replace  $V \rightarrow \psi U$ , in above equation and using Theorem (4.2), we have

$$\mathcal{T}_{\psi U} \psi U = -\cos^2 \theta \cdot \mathcal{T}_U U, \quad \text{for all } U \in \Gamma(D_2).$$

□

Similarly, from Theorem (3.6), we have proved the following Theorem.

**Theorem 4.5.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map. Then the slant distribution  $D_1$  is integrable if and only if,*

$$\omega(\widehat{\nabla}_U V - \widehat{\nabla}_V U) = 0, \quad \text{for all } U, V \in \Gamma(D_1).$$

Similarly, from Theorem (3.7) we have proved the following Theorem:

**Theorem 4.6.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map. Then the slant distribution  $D_2$  is integrable if and only if,*

$$P(\psi(\widehat{\nabla}_U V - \widehat{\nabla}_V U)) = 0, \quad \text{for all } U, V \in \Gamma(D_2).$$

**Theorem 4.7.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map such that  $D_1$  is integrable. Then  $f$  is harmonic if and only if  $\text{trace}(\nabla f_*) = 0$  on  $D_2$  and  $\overline{H}$ , where  $\overline{H}$  denotes the mean curvature vector field of  $\text{range } f_*$ .*

*Proof.* Using Lemma (2.2), we get  $\text{trace}(\nabla f_*)|_{\ker f_*} \in (\text{range } f_*)$  and  $\text{trace}(\nabla f_*)|_{(\ker f_*)^\perp} \in (\text{range } f_*)^\perp$  so that

$$\text{trace}(\nabla f_*) = 0 \Leftrightarrow \text{trace}(\nabla f_*)|_{\ker f_*} = 0, \quad \text{and} \quad \text{trace}(\nabla f_*)|_{(\ker f_*)^\perp} = 0.$$

Since  $D_1$  is invariant under  $\phi$ , we can choose locally orthonormal frame  $\{e_1, \phi e_1, \dots, \dots, e_k, \phi e_k\}$  of  $D_1$ . Using the integrability of the  $D_1$

$$\begin{aligned} (\nabla f_*)(\phi e_i, \phi e_i) &= -f_*(\nabla_{\phi e_i} \phi e_i), \\ &= f_*(\nabla_{e_i} e_i), \\ &= -(\nabla f_*)(e_i, e_i), \quad \text{for } 1 \leq i \leq k. \end{aligned}$$

Thus,

$$\text{trace}(\nabla f_*)|_{(\ker f_*)} = 0 \Leftrightarrow \text{trace}(\nabla f_*)|_{D_2} = 0, \quad \text{and} \quad \text{trace}(\nabla f_*)(\xi, \xi) = 0.$$

Moreover, it is easy to get that

$$\text{trace}(\nabla f_*)|_{(\ker f_*)^\perp} = l\overline{H}, \quad \text{for all } l = \dim(\ker f_*)^\perp,$$

so that

$$\text{trace}(\nabla f_*)|_{(\ker f_*)^\perp} = 0 \Leftrightarrow \overline{H} = 0.$$

□

**Theorem 4.8.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map. Then  $f$  is a totally geodesic map if and only if*

$$\omega(\widehat{\nabla}_U \psi V + \mathcal{T}_U \omega V) + C(\mathcal{T}_U \psi V + \mathcal{H} \nabla_U \omega V) = 0,$$

$$\omega(\widehat{\nabla}_U B X + \mathcal{T}_U C X) + C(\mathcal{T}_U B X + \mathcal{H} \nabla_U C X) = 0,$$

$$\overline{Q}(\nabla_Y^f f_* Z) = 0,$$

for all  $U, V \in \Gamma(\ker f_*)$  and  $X, Y, Z \in \Gamma((\ker f_*)^\perp)$ .

*Proof.* If  $X, Y, Z \in \Gamma((\ker f_*)^\perp)$ , then by Lemma (2.2), we get

$$(\nabla f_*)(Y, Z) = 0 \Leftrightarrow \overline{Q}((\nabla f_*)(Y, Z)) = \overline{Q}(\nabla_Y^f f_* Z) = 0.$$

For  $U, V \in \Gamma(\ker f_*)$ , using equations (2.1), (4.2), (4.3), (2.10) and (2.11), we have

$$\begin{aligned} (\nabla f_*)(U, V) &= f_*(\phi(\widehat{\nabla}_U \psi V + \mathcal{T}_U \psi V + \mathcal{T}_U \omega V + \mathcal{H} \nabla_U \omega V - \eta(\nabla_U V)\xi), \\ &= f_*(\psi \widehat{\nabla}_U \psi V + \omega \widehat{\nabla}_U \psi V + B \mathcal{T}_U \psi V + C \mathcal{T}_U \psi V + \psi \mathcal{T}_U \omega V \\ &\quad + \omega \mathcal{T}_U \omega V + B \mathcal{H} \nabla_U \omega V + C \mathcal{H} \nabla_U \omega V). \end{aligned}$$

Hence

$$(\nabla f_*)(U, V) = 0 \Leftrightarrow \omega(\widehat{\nabla}_U \psi V + \mathcal{T}_U \omega V) + C(\mathcal{T}_U \omega V + \mathcal{H} \nabla_U \omega V) = 0.$$

If  $U \in \Gamma(\ker f_*)$  and  $X \in \Gamma((\ker f_*)^\perp)$ , since the tensor  $(\nabla f_*)$  is symmetric. Using equations (2.1), (4.2), (4.3), (2.10) and (2.11), we get

$$\begin{aligned} (\nabla f_*)(U, X) &= f_*(\phi(\widehat{\nabla}_U B X + \mathcal{T}_U B X + \mathcal{T}_U C X + \mathcal{H} \nabla_U C X) - \eta(\nabla_U X)\xi), \\ &= f_*(\psi \widehat{\nabla}_U B X + \omega \widehat{\nabla}_U B X + B \mathcal{T}_U B X + C \mathcal{T}_U B X + \psi \mathcal{T}_U C X \\ &\quad + \omega \mathcal{T}_U C X + B \mathcal{H} \nabla_U C X + C \mathcal{H} \nabla_U C X). \end{aligned}$$

Thus,

$$(\nabla f_*)(U, X) = 0 \Leftrightarrow \omega(\widehat{\nabla}_U B X + \mathcal{T}_U C X) + C(\mathcal{T}_U B X + \mathcal{H} \nabla_U C X) = 0.$$

□

Similarly, from Lemma (3.10) we have proved the following Lemma.

**Lemma 4.9.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map. Then*

$$H \in \Gamma(\omega D_2)$$

**Theorem 4.10.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map. Then  $(M, \phi, \xi, \eta, g_M)$  is locally a Riemannian product manifold of the leaves of  $\Gamma(\ker f_*)$  and  $\Gamma(\ker f_*)^\perp$  if and only if*

$$\omega(\widehat{\nabla}_U \psi V + \mathcal{T}_U \omega V) + C(\mathcal{T}_U \psi V + \mathcal{H} \nabla_U \omega V) = 0,$$

$$\psi(\mathcal{V} \nabla_X B Z + \mathcal{A}_X C Z) + B(\mathcal{A}_X B Z + \mathcal{H} \nabla_X C Z) = 0,$$

for all  $U, V \in \Gamma(\ker f_*)$  and  $X, Z \in \Gamma(\ker f_*)^\perp$ .

*Proof.* For  $U, V \in \Gamma(\ker f_*)$ , using equation (2.1), (2.6), (4.2), (4.3), (2.10) and (2.11), we get

$$\begin{aligned} \nabla_U V &= -\phi(\widehat{\nabla}_U \psi V + \mathcal{T}_U \psi V + \mathcal{T}_U \omega V + \mathcal{H} \nabla_U \omega V) + \eta(\nabla_U V)\xi \\ &= -(\psi \widehat{\nabla}_U \psi V + \omega \widehat{\nabla}_U \psi V + B \mathcal{T}_U \psi V + C \mathcal{T}_U \psi V + \psi \mathcal{T}_U \omega V + \omega \mathcal{T}_U \omega V \\ &\quad + B \mathcal{H} \nabla_U \omega V + C \mathcal{H} \nabla_U \omega V). \end{aligned}$$



Thus,

$$\nabla_U V \in \Gamma(\ker f_*) \Leftrightarrow \omega(\widehat{\nabla}_U \psi V + \mathcal{T}_U \omega V) + C(\mathcal{T}_U \omega V + \mathcal{H} \nabla_U \omega V) = 0.$$

For all  $X, Z \in \Gamma((\ker f_*)^\perp)$ , using equations (2.1), (2.6), (4.2), (4.3), (2.10) and (2.11), we have

$$\begin{aligned} \nabla_X Z &= -\phi(\mathcal{V} \nabla_X BZ + \mathcal{A}_X BZ + \mathcal{A}_X CZ + \mathcal{H} \nabla_X CZ), \\ &= -(\psi \mathcal{V} \nabla_X BZ + \omega \mathcal{V} \nabla_X BZ + B \mathcal{A}_X BZ + C \mathcal{A}_X BZ \\ &\quad + \psi \mathcal{A}_X CZ + \omega \mathcal{A}_X CZ + B \mathcal{H} \nabla_X CZ + C \mathcal{H} \nabla_X CZ). \end{aligned}$$

Hence  $\nabla_X Z \in \Gamma((\ker f_*)^\perp) \Leftrightarrow \psi(\mathcal{V} \nabla_X BZ + \mathcal{A}_X CZ) + B(\mathcal{A}_X BZ + \mathcal{H} \nabla_X CZ)$ .  $\square$

**Theorem 4.11.** *Let  $(M, \phi, \xi, \eta, g_M)$  be a Cosymplectic manifold and  $(N, g_N)$  a Riemannian manifold. Let  $f : (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N)$  be a semi-slant Riemannian map. Then the fibers of  $f$  are locally Riemannian product manifolds of the leaves of  $D_1$  and  $D_2$  if and only if*

$$Q(\psi \widehat{\nabla}_U \psi V + B \mathcal{T}_U \psi V) = 0, \quad \text{and} \quad \omega \widehat{\nabla}_U \psi V + C \mathcal{T}_U \psi V = 0, \quad \text{for all } U, V \in \Gamma(D_1),$$

$$P(\psi(\widehat{\nabla}_X \psi Y + \mathcal{T}_X \omega Y) + B(\mathcal{T}_X \psi Y + \mathcal{H} \nabla_X \omega Y)) = 0,$$

$$\omega(\widehat{\nabla}_X \psi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \psi Y + \mathcal{H} \nabla_X \omega Y) = 0,$$

for all  $X, Y \in \Gamma(D_2)$ .

## 5. EXAMPLE

Note that given an Euclidean space  $R^{2n+1}$  with coordinates  $(x_1, x_2, \dots, x_{2n-1}, x_{2n}, x_{2n+1})$ , we can canonically choose an almost contact structure  $(\phi, \xi, \eta, g_{2n+1})$  on  $R^{2n+1}$  as follows:

$$\begin{aligned} &J(a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + \dots + a_{2n-1} \frac{\partial}{\partial x_{2n-1}} + a_{2n} \frac{\partial}{\partial x_{2n}} + a_{2n+1} \frac{\partial}{\partial x_{2n+1}}) \\ &= -a_2 \frac{\partial}{\partial x_1} + a_1 \frac{\partial}{\partial x_2} + \dots - a_{2n} \frac{\partial}{\partial x_{2n-1}} + a_{2n-1} \frac{\partial}{\partial x_{2n}}, \end{aligned}$$

where  $\xi = \frac{\partial}{\partial x_{2n+1}}$ ,  $\eta = dx_{2n+1}$ ,  $g_{2n+1}$  is usual inner product and  $a_1, a_2, \dots, a_{2n+1}$  are  $C^\infty$  functions defined in  $R^{2n+1}$ . In example 2 and 3 of this section, we will use this notation.

**Example 5.1.** Let  $R^{11}$  ( $R^{2n+1}$ ,  $n = 5$ ) has almost contact structure  $(\phi, \xi, \eta, g_{11})$ . Define a map  $f : R^{11} \rightarrow R^5$  by

$$f(x_1, x_2, \dots, x_{11}) = \left( \frac{x_3 + \sqrt{3}x_5}{2}, 0, x_6, \frac{\sqrt{3}x_7 + x_9}{2}, x_8 \right),$$

which is a semi-slant Riemannian map such that

$$\begin{aligned} \langle \xi \rangle &= \left\langle \frac{\partial}{\partial x_{11}} \right\rangle, D_1 = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle \\ \text{and } D_2 &= \left\langle \frac{1}{2}(\sqrt{3} \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}), \frac{1}{2}(\frac{\partial}{\partial x_7} - \sqrt{3} \frac{\partial}{\partial x_9}), \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_{10}} \right\rangle, \end{aligned}$$

with the semi-slant angle  $\theta = \frac{\pi}{6}$ .

**Example 5.2.** Let  $R^{11}$  ( $R^{2n+1}$ ,  $n = 5$ ) has almost contact structure  $(\phi, \xi, \eta, g_{11})$ . Define a map  $f : R^{11} \rightarrow R^8$  by

$$f(x_1, x_2, \dots, x_{11}) = (x_1, x_2, \frac{x_3 - x_5}{\sqrt{2}}, \frac{x_4 - x_6}{\sqrt{2}}, 0, \frac{x_7 - x_9}{\sqrt{2}}, x_{10}, x_{11}),$$

which is a semi-slant Riemannian map such that

$$D_1 = \left\langle \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5} \right), \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_6} \right) \right\rangle$$

$$\text{and } D_2 = \left\langle \left( \frac{\partial}{\partial x_8}, \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_9} \right) \right) \right\rangle$$

with the semi-slant angle  $\theta = \frac{\pi}{4}$ .

**Example 5.3.** Let  $R^7$  has got a Cosymplectic structure as in Example (2.1). Define a map  $f : R^7 \rightarrow R^3$  by

$$f(x_1, x_2, x_3, y_1, y_2, y_3, z) = \left( \frac{x_3 + y_2}{\sqrt{2}}, 0, x_2 \right)$$

which is a semi-slant Riemannian map such that

$$D_1 = \langle E_1, E_4 \rangle$$

$$\text{and } D_2 = \left\langle E_6, \frac{1}{\sqrt{2}}(E_3 - E_5) \right\rangle, \langle \xi \rangle = \langle E_7 \rangle$$

with the semi-slant angle  $\theta = \frac{\pi}{4}$ .

**Example 5.4.** Let  $R^9$  has got a Cosymplectic structure as in Example (2.1). Define a map  $f : R^9 \rightarrow R^6$  by

$$f(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z) = \left( \frac{x_2 - y_3}{\sqrt{2}}, x_3, 0, x_4, y_4, z \right)$$

which is a semi-slant Riemannian map such that

$$D_1 = \langle E_1, E_5 \rangle$$

$$\text{and } D_2 = \left\langle \frac{1}{\sqrt{2}}(E_2 + E_7), E_6 \right\rangle,$$

with the semi-slant angle  $\theta = \frac{\pi}{4}$ .

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