

ON THE EXISTENCE OF SOLUTIONS OF SOME FUNCTIONAL INTEGRAL EQUATIONS IN $L^p(\mathbb{R}_+)$

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ABSTRACT. In this paper, we study the existence of solutions for a functional integral equations by using Darbo's fixed point theorem and a measure of noncompactness on the spaces $L^p(\mathbb{R}_+)$, $1 \leq p < +\infty$. The obtained result generalizes and extends several one obtained earlier in many papers and monographs. An example which shows the applicability of our result is also included.

1. INTRODUCTION

Integral equation have a lot of applications in many branches of mathematical physics, engineering, mechanics, biology and economics see [22] and references therein. Several different techniques were proposed to study the existence of solutions of the functional integral equations in appropriate function spaces. Although all of these techniques have the same goal, they differ in the function spaces and the fixed point theorems to be applied.

Many papers in the field of functional integral equations give different sets of conditions for the existence of solutions of such equations, see for instance [13, 10, 2, 15, 17, 7]. A part from that, integral equations are often investigated in research papers and monographs (cf. [12, 11, 16, 14, 8, 6]) and the references cited therein.

Measures of noncompactness and Darbo's fixed point theorem play an important roles in fixed point theory and their applications. Measures of noncompactness were introduced by Kuratowski [18]. In 1955, Darbo presented a fixed point theorem [14], using this notion. This result was used to establish the existence and behaviour of solutions in some functional spaces to many classes of integral equations.

Agarwal and O'Regan [4] in 2004, proved the existence of the solutions for the nonlinear integral equation

$$x(t) = \int_0^{+\infty} k(t, s)f(t, x(s))ds, \quad t \in \mathbb{R}_+$$

Date: Received: Feb 23, 2020; Accepted: Apr 23, 2020.

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2010 *Mathematics Subject Classification.* 45G10, 45M99, 47H09.

Key words and phrases. the space of Lebesgue integrable functions, measure of noncompactness.

in $C_l[0, +\infty)$, where $C_l[0, +\infty)$, denotes the space of bounded and continuous functions on \mathbb{R}_+ which have limit at infinity.

In [21], the author gave the existence of an integrable solutions of the following functional integral equation

$$x(t) = f(t, x(t)) + g\left(t, \int_0^{+\infty} k(t, s)f(t, x(s))ds\right), \quad t \in \mathbb{R}_+$$

In [19], the authors discussed the solvability the functional integral equation of convolution type

$$x(t) = f(t, x(t)) + \int_0^{+\infty} k(t-s)(Qx)(s)ds$$

using a new construction of a measure of noncompactness in $L^p(\mathbb{R}_+)$.

Next, the authors in [3] study the existence of solutions to the following general functional integral equation

$$x(t) = f(t, x(t)) + g\left(t, \int_0^{+\infty} k(t-s)(Qx)(s)ds\right)$$

using the same new construction of a measure of noncompactness in $L^p(\mathbb{R}_+)$. Motivated by the work [19] and [3] in this paper, we will study the existence of solutions for the following more general integral equation

$$x(t) = f_1(t, x(t)) + f_2\left(t, \int_0^{+\infty} k_1(t-s)(Q_1x)(s)ds, \int_0^{+\infty} k_2(t, s)(Q_2x)(s)ds\right), \quad t \in \mathbb{R}_+ \quad (1.1)$$

Throughout f_1, f_2, k_2 are Carathéodory functions, $k_1 \in L^1(\mathbb{R})$ and $Q_i, i = 1, 2$ are operators which act continuously from the space $L^p(\mathbb{R}_+)$ onto itself.

2. NOTATION, DEFINITIONS AND AUXILIARY FACTS

Assume that a function $f(t, x, y) = f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions i.e. it is measurable in t for any $(x, y) \in \mathbb{R} \times \mathbb{R}$ and continuous in x, y for almost all $t \in \mathbb{R}_+$

Furthermore, we recall a few fact about the convolution operator (cf.[20]) Let $k_1 \in L^1(\mathbb{R})$ be a given function. Then for any function $x \in L^p(\mathbb{R}_+)$, the integral $(K_1x)(t) = \int_0^{+\infty} k_1(t-s)x(s)ds$ exists for almost $t \in \mathbb{R}_+$. Moreover, the function $(K_1x)(t)$ belongs to the space $L^p(\mathbb{R}_+)$. Thus K_1 is a linear operator which maps the space $L^p(\mathbb{R}_+)$ onto itself and K_1 is also bounded since

$$\|K_1x\| \leq \|K_1\|_{L^1(\mathbb{R})} \|x\|_{L^p(\mathbb{R}_+)}$$

for every $x \in L^p(\mathbb{R}_+)$, it will be continuous. Hence the norm $\|K_1\|$ of the convolution operator is majorized by $\|K_1\|_{L^1(\mathbb{R})}$.

Let K_2 be the linear Fredholm operator $K_2 : L^p(\mathbb{R}_+) \rightarrow L^p(\mathbb{R}_+)$ defined by $(K_2x)(t) = \int_0^{+\infty} k_2(t, s)x(s)ds$. It is a continuous operator, and $\|K_2x\|_p \leq \|K_2\| \|x\|_p$. The norm of the operator is majorized by

$$\|K_2\| = \sup\left(\|K_2x\|_{L^p(\mathbb{R}_+)}; \|x\|_{L^p(\mathbb{R}_+)} \leq 1\right) \text{ and hence } \|K_2\| < \infty.$$

Now, we will collect some definitions and basic results which will be used further on throughout the paper.

First, we denote by $L^p(\mathbb{R}_+)$ the space of Lebesgue integrable functions on \mathbb{R}_+ equipped with the standard norm, $x \in L^p(\mathbb{R}_+)$, $\|x\|_p^p = \int_0^{+\infty} |x(t)|^p dt$

Next, we recall some basic facts concerning measure of noncompactness. Assume that $(E, \|\cdot\|)$ is a real Banach space with zero element θ . Let $B(x, r)$ denote the closed ball centered at x and with radius r . The symbol B_r stands for the ball $B(\theta, r)$. If X is a subset of E , then \overline{X} and $ConvX$ denote the closure and convex closure of X , respectively. with the symbols λX and $X+Y$, we denote the standard algebraic operations on sets. Moreover, we denote by M_E the family of all nonempty and bounded subsets of E and N_E its subfamily consisting of all relatively compact subsets. The definition of the concept of a measure of noncompactness presented bellow comes from [9].

Definition 2.1. [9] A mapping $\mu : M_E \rightarrow \mathbb{R}_+ = [0, +\infty[$ is said to be a measure of noncompactness in E if it satisfies following conditions

- (1) The family $\ker \mu = \{X \in M_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset N_E$;
- (2) $X \subset Y \implies \mu(X) \leq \mu(Y)$
- (3) $\mu(\overline{X}) = \mu(ConvX) = \mu(X)$
- (4) $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$, for $\lambda \in [0, 1]$
- (5) If $\{X_n\}$ is a sequence of nonempty, bounded, closed subsets of E such that $X_{n+1} \subset X_n$, ($n = 1, 2, \dots$) and $\lim_{n \rightarrow \infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Observe that the intersection set X_∞ belongs to $\ker \mu$. Indeed, since $\mu(X_\infty) \leq \mu(X_n)$ for any n , then we infer $\mu(X_\infty) = 0$, so $X_\infty \in \ker \mu$. For other facts concerning measures of noncompactness we refer to [9], [18].

In the following, we give a nonempty $X \subset L^p(\mathbb{R}_+)$ bounded, $\varepsilon > 0$, and $T > 0$. For arbitrary function $x \in X$, we let

$$\omega(x, \varepsilon) = \sup \left\{ \left(\int_0^\infty |x(t+h) - x(t)|^p dt \right)^{\frac{1}{p}}, |h| < \varepsilon \right\},$$

$$\omega(X, \varepsilon) = \sup \{ \omega(x, \varepsilon) : x \in X \}$$

and

$$\omega_0(X) = \lim_{\varepsilon \rightarrow 0} \omega(X, \varepsilon)$$

Also, let

$$d_T(X) = \sup \left\{ \left(\int_0^\infty |x(t)|^p dt \right)^{\frac{1}{p}}, x \in X \right\}$$

and

$$d(X) = \lim_{T \rightarrow \infty} d_T(X)$$

Then, the function $\mu : M_{L^p(\mathbb{R}_+)} \rightarrow \mathbb{R}_+$ given by $\mu(X) = \omega_0(X) + d(X)$ is a measure of noncompactness on $L^p(\mathbb{R}_+)$, see [19]

Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem and includes the existence part of Banach's theorem

Theorem 2.2. *Schauder (see [5]) Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E , Then every compact continuous map $T : \Omega \rightarrow \Omega$ has at least one fixed point.*

In the following, we state a fixed point theorem of Darbo type proved by Banas and Goebel [9]

Theorem 2.3. *(See [14], [9]) Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E , and let $T : \Omega \rightarrow \Omega$ be a continuous mapping such that a constant $k \in [0, 1)$ exists with the property*

$$\mu(TX) \leq k\mu(X)$$

for any nonempty X of Ω . Then T has a fixed point in the set Ω .

Now, we need to characterize the compact subsets of $L^p(\mathbb{R}_+)$.

Theorem 2.4. [19] *Let \mathcal{F} be a bounded set in $L^p(\mathbb{R}^N)$ with $1 \leq p < +\infty$. Then, \mathcal{F} has a compact closure in $L^p(\mathbb{R}^N)$ if and only if $\lim_{h \rightarrow 0} \|\tau_h f - f\|_p = 0$ uniformly in $f \in \mathcal{F}$, where $\tau_h f(x) = f(x + h)$ for all $x \in \mathbb{R}^N$. In addition, for $\varepsilon > 0$, there is a bounded and measurable subset Ω of \mathbb{R}^N such that $\|f\|_{L^p(\mathbb{R}^N \setminus \Omega)} < \varepsilon$ for all $f \in \mathcal{F}$.*

Corollary 2.5. *Let \mathcal{F} be a bounded set in $L^p(\mathbb{R}^N)$ with $1 \leq p < +\infty$. The closure of \mathcal{F} in $L^p(\mathbb{R}^N)$ is compact if and only if $\lim_{h \rightarrow 0} \left(\int_0^\infty |f(x) - f(x + h)|^p dx \right)^{\frac{1}{p}} = 0$ uniformly in $f \in \mathcal{F}$. Also, for $\varepsilon > 0$, there is a constant $T > 0$ such that $\left(\int_T^\infty |f(x)|^p dx \right)^{\frac{1}{p}} < \varepsilon$ for all $f \in \mathcal{F}$.*

We shall study the existence of the solutions of Eq.(1.1) assuming some conditions are satisfied

3. MAIN RESULTS

Theorem 3.1. *Assume that the following conditions are satisfied.*

- (1) $f_1 : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions, and a constant $\lambda_1 \in [0, 1)$ and $a_1 \in L^p(\mathbb{R}_+)$ exist such that

$$|f(t, x) - f(s, y)| \leq |a_1(t) - a_1(s)| + \lambda_1 |x - y| \quad (3.1)$$

for any $x, y \in \mathbb{R}$ and almost all $s, t \in \mathbb{R}_+$

- (2) $f_1(\cdot, 0) \in L^p(\mathbb{R}_+)$
 (3) $k_2 : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions, and $g_1, g_2 \in L^p(\mathbb{R}_+)$ and $g \in L^q(\mathbb{R}_+)$ ($\frac{1}{p} + \frac{1}{q} = 1$) exist such that $|k_2(t, s)| \leq g_1(t)g(s)$ for all $t, s \in \mathbb{R}_+$ and

$$|k_2(t_1, s) - k_2(t_2, s)| \leq g(s) |g_2(t_1) - g_2(t_2)| \quad (3.2)$$

- (4) $f_2 : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions, and constants $\lambda_2, \lambda_3 \geq 0$ and $a_2 \in L^p(\mathbb{R}_+)$ exist such that

$$|f_2(t, x, y) - f_2(s, z, t)| \leq |a_2(t) - a_2(s)| + \lambda_2 |x - z| + \lambda_3 |y - t| \quad (3.3)$$

for any $x, y, z, t \in \mathbb{R}$ and almost all $s, t \in \mathbb{R}_+$.

- (5) $f_2(\cdot, 0, 0) \in L^p(\mathbb{R}_+)$
 (6) $k_1 \in L^1(\mathbb{R})$

Notice that, under this hypothesis, the linear operator $K_1 : L^p(\mathbb{R}_+) \rightarrow L^p(\mathbb{R}_+)$ is given by $(K_1x)(t) = \int_0^\infty k_1(t-s)x(s)ds$ and it is a continuous operator and $\|K_1x\|_p \leq \|K_1\|_{L^1(\mathbb{R})} \|x\|_p$.

- (7) The operators Q_i , $i = 1, 2$ act continuously from $L^p(\mathbb{R}_+)$ into itself and constants $b_i \in \mathbb{R}_+$, $i = 1, 2$ exist such that

$$\lambda_1 + \lambda_2 b_1 \|K_1\|_{L^1(\mathbb{R})} + \lambda_3 b_2 \|K_2\| < 1$$

and

$$\|Q_i x\|_{L^p[T, +\infty)} \leq b_i \|x\|_{L^p[T, +\infty)} \quad (3.4)$$

for any $x \in L^p(\mathbb{R}_+)$ and $T \in \mathbb{R}_+$.

Therefore, the nonlinear integral equation (1.1) have at least one solution in the space $L^p(\mathbb{R}_+)$.

Proof. First, we define the operator $F : L^p(\mathbb{R}_+) \rightarrow L^p(\mathbb{R}_+)$ by

$$\left. \begin{aligned} (Fx)(t) = & f_1(t, x(t)) + \\ & f_2\left(t, \int_0^{+\infty} k_1(t-s)Q_1(x)(s)ds, \int_0^{+\infty} k_2(t,s)Q_2(x)(s)ds\right) \end{aligned} \right\}$$

By considering the Carathéodory conditions, we infer that Fx is measurable for any $x \in L^p(\mathbb{R}_+)$. Now, we prove that $Fx \in L^p(\mathbb{R}_+)$ for any $x \in L^p(\mathbb{R}_+)$. By using conditions (1), -, (7), for a.e. $t \in \mathbb{R}_+$, then, we have the following inequality

$$\left. \begin{aligned} |(Fx)(t)| \leq & |f_1(t, x) - f_1(t, 0)| + |f_1(t, 0)| \\ & + \left| f_2\left(t, \int_0^{+\infty} k_1(t-s)Q_1(x)(s)ds, \int_0^{+\infty} k_2(t,s)Q_2(x)(s)ds\right) - f_2(t, 0, 0) \right| \\ & + |f_2(t, 0, 0)| \leq \lambda_1 \|x\|_p + \|f_1(\cdot, 0)\|_p + \|f_2(\cdot, 0, 0)\|_p \\ & + \lambda_2 b_1 \|K_1\|_{L^1(\mathbb{R})} \|x\|_p + \lambda_3 b_2 \|K_2\| \|x\|_p \end{aligned} \right\}$$

Therefore, we obtain

$$\left. \begin{aligned} \|Fx\|_p \leq & \|f_1(\cdot, 0)\|_p + \|f_2(\cdot, 0, 0)\|_p \\ & + \left(\lambda_1 + \lambda_2 b_1 \|K_1\|_{L^1(\mathbb{R})} + \lambda_3 b_2 \|K_2\| \right) \|x\|_p \end{aligned} \right\} \quad (3.5)$$

Hence, $F(x) \in L^p(\mathbb{R}_+)$ and F is well defined and also from (3.5), we have $F(\overline{B}_{r_0}) \subset \overline{B}_{r_0}$, where r_0 is

$$r_0 = \frac{\|f_1(\cdot, 0)\|_p + \|f_2(\cdot, 0, 0)\|_p}{1 - \left(\lambda_1 + \lambda_2 b_1 \|K_1\|_{L^1(\mathbb{R})} + \lambda_3 b_2 \|K_2\| \right)}$$

Also, F is continuous in $L^p(\mathbb{R}_+)$ because $f_1(t, \cdot)$, $f_2(t, \cdot, \cdot)$, K_1 , $K_2(t, \cdot)$ and Q_i , $i = 1, 2$ are continuous for a.e. $t \in \mathbb{R}_+$. Further, we will show that

$$\omega_0(FX) \leq \left(\lambda_1 + \lambda_2 b_1 \|K_1\|_{L^1(\mathbb{R})} + \lambda_3 b_2 \|K_2\| \right) \omega_0(X)$$

for any nonempty set $X \subset \overline{B_{r_0}}$. For this, we fix an arbitrary $\varepsilon > 0$. Let us choose $x \in X$ and $t, h \in \mathbb{R}_+$ with $|h| \leq \varepsilon$. we have

$$\left. \begin{aligned} & |(Fx)(t) - (Fx)(t+h)| \leq |f_1(t, x(t)) - f_1(t+h, x(t))| \\ & \quad + |f_1(t+h, x(t)) - f_1(t+h, x(t+h))| \\ & + \left| f_2 \left(t, \int_0^{+\infty} k_1(t-s)(Q_1x)(s)ds, \int_0^{+\infty} k_2(t,s)(Q_2x)(s)ds \right) \right. \\ & - \left. f_2 \left(t+h, \int_0^{+\infty} k_1(t-s)(Q_1x)(s)ds, \int_0^{+\infty} k_2(t,s)(Q_2x)(s)ds \right) \right| \\ & + \left| f_2 \left(t+h, \int_0^{+\infty} k_1(t-s)(Q_1x)(s)ds, \int_0^{+\infty} k_2(t,s)(Q_2x)(s)ds \right) \right. \\ & - \left. f_2 \left(t+h, \int_0^{+\infty} k_1(t+h-s)(Q_1x)(s)ds, \int_0^{+\infty} k_2(t,s)(Q_2x)(s)ds \right) \right| \\ & + \left| f_2 \left(t+h, \int_0^{+\infty} k_1(t+h-s)(Q_1x)(s)ds, \int_0^{+\infty} k_2(t,s)(Q_2x)(s)ds \right) \right. \\ & - \left. f_2 \left(t+h, \int_0^{+\infty} k_1(t+h-s)(Q_1x)(s)ds, \int_0^{+\infty} k_2(t+h,s)(Q_2x)(s)ds \right) \right| \\ & \leq |a_1(t) - a_1(t+h)| + \lambda_1 |x(t) - x(t+h)| + |a_2(t) - a_2(t+h)| \\ & \quad + \lambda_2 \left| \int_0^{+\infty} k_1(t-s) - k_1(t+h-s)(Q_1x)(s)ds \right| \\ & \quad + \lambda_3 \left| \int_0^{+\infty} k_2(t,s) - k_2(t+h,s)(Q_2x)(s)ds \right| \end{aligned} \right\}$$

Therefore

$$\left. \begin{aligned} & \left(\int_0^\infty |(Fx)(t) - (Fx)(t+h)|^p dt \right)^{\frac{1}{p}} \\ & \leq \left(\int_0^\infty |a_1(t) - a_1(t+h)|^p dt \right)^{\frac{1}{p}} + \lambda_1 \left(\int_0^\infty |x(t) - x(t+h)|^p dt \right)^{\frac{1}{p}} \\ & + \left(\int_0^\infty |a_2(t) - a_2(t+h)|^p dt \right)^{\frac{1}{p}} \\ & + \lambda_2 \left(\int_0^\infty \left| \int_0^{+\infty} [k_1(t-s) - k_1(t+h-s)](Q_1x)(s)ds \right|^p dt \right)^{\frac{1}{p}} \\ & + \lambda_3 \left(\int_0^\infty \left| \int_0^{+\infty} [k_2(t,s) - k_2(t+h,s)](Q_2x)(s)ds \right|^p dt \right)^{\frac{1}{p}} \\ & \leq \left(\int_0^\infty |a_1(t) - a_1(t+h)|^p dt \right)^{\frac{1}{p}} + \lambda_1 \left(\int_0^\infty |x(t) - x(t+h)|^p dt \right)^{\frac{1}{p}} \\ & + \left(\int_0^\infty |a_2(t) - a_2(t+h)|^p dt \right)^{\frac{1}{p}} \\ & + \lambda_2 \|Q_1x\|_p \left(\int_{\mathbb{R}} |k_1(t) - k_1(t+h)| dt \right) \\ & + \lambda_3 \|Q_2x\|_p \left(\int_0^{+\infty} \left(\int_0^{+\infty} |k_2(t,s) - k_2(t+h,s)|^q ds \right) dt \right)^{\frac{1}{p}} \end{aligned} \right\}$$

so

$$\left. \begin{aligned} & \left(\int_0^\infty |(Fx)(t) - (Fx)(t+h)|^p dt \right)^{\frac{1}{p}} \leq \\ & \left(\int_0^\infty |a_1(t) - a_1(t+h)|^p dt \right)^{\frac{1}{p}} + \lambda_1 \left(\int_0^\infty |x(t) - x(t+h)|^p dt \right)^{\frac{1}{p}} \\ & \quad + \left(\int_0^\infty |a_2(t) - a_2(t+h)|^p dt \right)^{\frac{1}{p}} \\ & \quad + \lambda_2 \|Q_1x\|_p \left(\int_0^{+\infty} |k_1(t-s) - k_1(t+h-s)| dt \right) \\ & \quad + \lambda_3 \|Q_2x\|_p \left(\int_0^{+\infty} \left(\int_0^{+\infty} |g_2(t) - g_2(t+h)|^q |g(s)|^q ds \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}} \end{aligned} \right\}$$

So, we have

$$\left. \begin{aligned} & \left(\int_0^\infty |(Fx)(t) - (Fx)(t+h)|^p dt \right)^{\frac{1}{p}} \leq \\ & \left(\int_0^\infty |a_1(t) - a_1(t+h)|^p dt \right)^{\frac{1}{p}} + \lambda_1 \left(\int_0^\infty |x(t) - x(t+h)|^p dt \right)^{\frac{1}{p}} \\ & \quad + \left(\int_0^\infty |a_2(t) - a_2(t+h)|^p dt \right)^{\frac{1}{p}} \\ & \quad + \lambda_2 \|Q_1 x\|_p \left(\int_0^{+\infty} |k_1(t) - k_1(t+h)| dt \right) \\ & \quad \lambda_3 \|Q_2 x\|_p \|g\|_q \left(\int_0^{+\infty} |g_2(t) - g_2(t+h)|^p \right)^{\frac{1}{p}} \end{aligned} \right\}$$

Then, from the above inequalities, we get

$$\left. \begin{aligned} & \omega(FX, \varepsilon) \leq \omega(a_1, \varepsilon) + \lambda_1 \omega(X, \varepsilon) + \omega(a_2, \varepsilon) \\ & + \lambda_2 b_1 r_0 \|k_1 - \tau_h k_1\|_{L^1(\mathbb{R})} + \lambda_3 b_2 r_0 \|g\|_{L^q(\mathbb{R}_+)} \omega(g_2, \varepsilon) \end{aligned} \right\}$$

Since $\{a_1\}, \{a_2\}, \{g_2\}$ are compacts set in $L^p(\mathbb{R}_+)$ and $\{k_1\}$ is a compact set in $L^1(\mathbb{R})$, we have $\omega(a_1, \varepsilon) \rightarrow 0, \omega(a_2, \varepsilon) \rightarrow 0, \|k_1 - \tau_h k_1\|_{L^1(\mathbb{R})} \rightarrow 0$ and $\omega(g_2, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then, we obtain

$$\left. \begin{aligned} & \omega_0(FX, \varepsilon) \leq \lambda_1 \omega_0(X, \varepsilon) \leq \\ & \left(\lambda_1 + \lambda_2 b_1 \|K_1\|_{L^1(\mathbb{R})} + \lambda_3 b_2 \|K_2\| \right) \omega_0(X) \end{aligned} \right\} \quad (3.6)$$

In the following, we fix an arbitrar number $T > 0$. Then, for an arbitrary function $x \in X$, we have

$$\left. \begin{aligned} & \left(\int_T^\infty |F(x)(t)|^p dt \right)^{\frac{1}{p}} \\ & \leq \left(\int_T^\infty |f_1(t, x) - f_1(t, 0)|^p dt \right)^{\frac{1}{p}} + \left(\int_T^\infty |f_1(t, 0)|^p dt \right)^{\frac{1}{p}} \\ & \left(\int_T^\infty |f_2(t, \int_0^\infty k_1(t-s)(Q_1 x)(s) ds, \int_0^\infty k_2(t, s)(Q_2 x)(s) ds) - f_2(t, 0, 0)|^p dt \right)^{\frac{1}{p}} \\ & \quad + \left(\int_T^\infty |f_2(t, 0, 0)|^p \right)^{\frac{1}{p}} \\ & \leq \lambda_1 \left(\int_T^\infty |x(t)|^p \right)^{\frac{1}{p}} + \left(\int_T^\infty |f_1(t, 0)|^p dt \right)^{\frac{1}{p}} \\ & \leq \lambda_2 b_1 \|K_1\|_{L^1(\mathbb{R})} \left(\int_T^\infty |x(t)|^p dt \right)^{\frac{1}{p}} + \lambda_3 b_2 \|g\|_{L^q(\mathbb{R}_+)} \|g_1\|_{L^p(T, +\infty)} \|x\|_p \\ & \quad + \left(\int_T^\infty |f_2(t, 0, 0)|^p \right)^{\frac{1}{p}} \end{aligned} \right\}$$

Since $\{f_1(t, 0)\}$ and $\{f_2(t, 0, 0)\}$ are compacts in $L^p(\mathbb{R}_+)$, then, as T goes to 0, we obtain $\left(\int_T^\infty |f_1(t, 0)|^p dt \right)^{\frac{1}{p}}$ goes to 0, $\left(\int_T^\infty |f_2(t, 0, 0)|^p \right)^{\frac{1}{p}}$ goes also to 0 and $\|g_1\|_{L^p(T, +\infty)}$ goes to zero. Therefore, we obtain

$$d(FX) \leq \left(\lambda_1 + \lambda_2 b_1 \|K_1\|_{L^1(\mathbb{R})} \right) d(X) \quad (3.7)$$

So, from (3.6) and (3.7), it follows,

$$\mu(FX) \leq \left(\lambda_1 + \lambda_2 b_1 \|K_1\|_{L^1(\mathbb{R})} + \lambda_3 b_2 \|K_2\| \right) \mu(X) \quad (3.8)$$

By (3.8) and Theorem 2.3, we deduce that the operator F has a fixed point x in B_{r_0} and consequently, Eq.(1.1) has at least one solution in $L^p(\mathbb{R}_+)$. \square

4. EXAMPLE

Consider the functional integral equation

$$x(t) = \frac{\cos x(t)}{t+2} + \frac{1}{3} \int_0^\infty (t-s) \exp[-(t-s)] \frac{ds}{1+[x(s)]^2} + \int_0^\infty \frac{x(s)}{e(t+3)^2(s+2)^2} e^{-|x(s)|} ds \quad (4.1)$$

Eq. (4.1) is a special case of Eq.(1.1) with

$$\left. \begin{aligned} f_1(t, x) &= \frac{\cos x}{t+2}, \quad f_2(t, x, y) = \frac{1}{3}x + y, \quad (Q_1x)(s) = \frac{1}{1+[x(s)]^2} \\ k_1(t) &= te^{-t}, \quad k_2(t, s) = \frac{1}{e(t+3)^2(s+2)^2}, \quad (Q_2x)(s) = e^{-|x(s)|}x(s) \end{aligned} \right\}$$

In this example, hypothesis 1, holds with $a(t) = \frac{1}{t+2}$ and $\lambda = \frac{1}{2}$, indeed, we have

$$\left. \begin{aligned} |f_1(t, x) - f_1(s, y)| &= \left| \frac{\cos x}{t+2} - \frac{\cos y}{s+2} \right| \\ &\leq \left| \frac{1}{t+2} - \frac{1}{s+2} \right| + \frac{1}{2} |x - y| \end{aligned} \right\}$$

In addition, $|f_1(t, 0)| = \frac{1}{t+2} \in L^p(\mathbb{R}_+)$, indeed, $\|f_1(t, 0)\|_{L^p(\mathbb{R}_+)}^p = \left(\int_0^{+\infty} \frac{dx}{(1+x)^p} \right) = \frac{1}{p-1}$, for all $p > 1$. Thus, we have $\|f_1(t, 0)\|_{L^p(\mathbb{R}_+)} = \left(\frac{1}{p-1} \right)^{\frac{1}{p}}$. Next,

$$|f_2(t, x, y) - f_2(s, z, w)| \leq \frac{1}{3} |x - z| + |y - w|$$

Hence, $a_2(t) = 0$, and $\lambda_2 = \frac{1}{3}$ and $\lambda_3 = 1$, $f_2(t, 0, 0) = 0 \in L^p(\mathbb{R}_+)$. In our example, the function $k_1(t) = te^{-t}$, by [3] satisfies $\|K_1\|_{L^1(\mathbb{R})} \leq \frac{1}{\sqrt{e}}$

$$\left. \begin{aligned} k_2(t, s) &\leq \frac{1}{e(t+3)^2} \times \frac{1}{(s+2)^2} \\ g_1(t) &= \frac{1}{e(t+3)^2} \\ g_2(s) = g(s) &= \frac{1}{(s+2)^2} \\ |k_2(t_1, s) - k_2(t_2, s)| &\leq \left| \frac{1}{e(t_1+3)^2} - \frac{1}{e(t_2+3)^2} \right| \left| \frac{1}{(s+2)^2} \right| \end{aligned} \right\}$$

Q_1 and Q_2 satisfy assumptions of theorem 3.1, with $b_2 = b_1 = 1$. By using theorem 3.4 in [1], we have $\|K_2\| \leq \frac{1}{e}$, therefore $\lambda_1 + \lambda_2 b_1 \|K_1\|_{L^1(\mathbb{R})} + \lambda_3 b_2 \|K_2\| \leq \frac{1}{2} + \frac{1}{3\sqrt{e}} + \frac{1}{e} < 1$. Now, by theorem 3.1, our functional integral equation (4.1) has a solution belonging to $L^p(\mathbb{R}_+)$.

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