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ON THE EQUIDISTANT DIMENSION OF THE GENERALIZED EDGE CORONA PRODUCT OF GRAPHS

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Abstract. In this work, we investigate the equidistant dimension problem of generalized edge corona product of graphs. We establish general bounds in terms of the parameters of a graph G and the associated attached graphs, and identify families of graphs for which these bounds are attained. Exact values are obtained for several important classes, including paths, cycles, complete graphs, and complete bipartite graphs.

Keywords. Metric dimension, distance-equalizer set, equidistant dimension, generalized edge corona product of graphs
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1. Introduction and preliminaries

A graph $G = (V(G), E(G))$ is finite, simple, connected, and undirected unless stated otherwise. The distance between two vertices $u, v \in V(G)$, denoted by $d_G(u, v)$ (or $d(u, v)$), is the length of a shortest path between them, and the diameter $\mathrm{diam}(G)$ is the maximum such distance over all pairs of vertices. For a vertex $y \in V(G)$, the open neighborhood is $N(y) = \{x \in V(G) : xy \in E(G)\}$.

In a cycle C_n, two vertices u and v are said to be antipodal if $d_{C_n}(u, v) = \lfloor n/2 \rfloor$. Similarly, two edges $e_i, e_j \in E(C_n)$ are called antipodal edges if each endpoint of e_i is at distance $\lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor - 1$ from some endpoint of e_j.

Given $x_1, x_2 \in V(G)$, let $R\{x_1, x_2\} = \{z \in V(G) : d(x_1, z) \neq d(x_2, z)\}$. A set $W \subseteq V(G)$ is a resolving set if $W \cap R\{x_1, x_2\} \neq \emptyset$ for all distinct $x_1, x_2 \in V(G)$. The minimum cardinality of such a set is the metric dimension $\dim(G)$. Introduced by Slater [19] and independently by Harary and Melter [10], the metric dimension problem has been extensively studied [4, 9, 11, 12, 14, 17], with NP-hardness established in [6] and applications in chemical graph theory [4], robot navigation [13], mastermind [5] and network design [1, 2, 3].

The equidistant dimension, introduced by González et al. [8], is a variant of the metric dimension in which resolving sets are replaced by distance-equalizer sets: a subset $D \subseteq V(G)$ such that for every two distinct vertices $x, y \in V(G) \setminus D$, there exists $w \in D$ with $d(x, w) = d(y, w)$. The minimum cardinality of such

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a set is denoted $\text{eqdim}(G)$. The equidistant dimension problem has been the subject of several recent studies. González et al. [8] introduced the concept, established general bounds, and determined exact values for classical families such as paths, cycles, complete graphs, and multipartite graphs. Gispert-Fernández et al. [7] proved that computing the equidistant dimension is NP-complete, analyzed lexicographic product graphs, and introduced the pairwise domination number as a tool for obtaining bounds and exact values. Further work has focused on highly symmetric or geometric graphs: Savić et al. [18] investigated graphs arising from convex polytopes, while Kratica et al. [15] examined Johnson and Kneser graphs, obtaining exact results and structural characterizations.

Most existing work focuses on standard families or a limited range of graph operations. The behavior of $\text{eqdim}(G)$ under more complex constructions, such as the generalized edge corona product [16], remains largely unexplored. In this product, G is a graph of order n and size m, and H_i, $1 \leq i \leq m$, are m simple graphs of orders at least two. The generalized edge corona product $G[H_i]_1^m$ is obtained by taking the graph G together with the H_i's by joining the two end vertices of the i-th edge of G to every vertex of H_i. This operation generalizes the standard edge corona and produces highly connected graphs with rich structure.

This paper investigates the equidistant dimension problem of generalized edge corona product of graphs. We establish general bounds for $\text{eqdim}(G[H_i]_1^m)$ in terms of $\text{eqdim}(G)$ and $\text{eqdim}(H_i)$, $1 \leq i \leq m$, and derive exact values for several important graph families for G such as paths, cycles, complete graphs and complete bipartite graphs. These results extend the study of the equidistant dimension to a broader class of graph products and unify earlier results obtained for special cases.

2. Main results

The remainder of the paper is organized as follows. Section 2.1 presents our main results on the equidistant dimension of the generalized edge corona product of graphs. In Section 2.2, we study this parameter for some classes of graphs, including paths, cycles, complete graphs, and complete bipartite graphs. Finally, Section 3 provides concluding remarks and possible directions for future research.

2.1. Equidistant dimension of $G[H_i]_1^m$.

In what follows, we adopt the following terminology. Let $\mathcal{P}$ be a $u-v$ path. If $d(u,v) = l$ is even, then $\mathcal{P}$ has an odd number of vertices. The vertex w of $\mathcal{P}$ such that $d(u,w) = d(v,w) = \frac{l}{2}$ will be called a central vertex of $\mathcal{P}$.

If $d(u,v) = l$ is odd, then $\mathcal{P}$ has an even number of vertices. In this case there is an edge xy in $\mathcal{P}$ such that $d(u,x) = d(v,y)$. This edge xy will be called a central edge of $\mathcal{P}$.

We now establish an upper bound for the equidistant dimension of the generalized edge corona product of graphs, supported by the following lemmas.

Lemma 2.1. *Let G be a connected graph and let $e_i = uv$ be a pendant edge of G such that* $\deg(v) = 1$. *Let H_i be the graph corresponding to the edge $e_i = uv$ in the* *generalized edge corona product $G[H_i]_1^m$. If D is a minimum distance-equalizer set of $G[H_i]_1^m$, then $v \notin D$.*

Proof. Consider an edge $e_j = uw$ in G (where $w \neq v$), and let H_j be the graph corresponding graph to e_j. Let $a, b \in V(H_i)$. Then a and b are equalized by both u and v.

Let $a \in V(H_i)$ and $b \in V(H_j)$. Then a and b are equalized only by u. This means that a minimum distance-equalizer set does not contain v. □

Lemma 2.2. *Let G be a connected graph and let uv be a pendant edge of G such that $\deg(v) = 1$. Let H_i be the graph corresponding to the edge $e_i = uv$ in the generalized edge corona product $G[H_i]_1^m$. If D is a minimum distance-equalizer set of $G[H_i]_1^m$, then u must belong to D.*

Proof. By the Lemma 2.1, $v \notin D$. Now, consider two vertices $a \in V(H_i)$ and $b \in V(H_j)$ (for some edge $e_j = uw$, $w \neq v$). Vertex u is adjacent to all vertices in both H_i and H_j. Clearly u equalizes a and b. As $v \notin D$, and u is the only vertex in G that equalizes a and b, we must have $u \in D$. □

Lemma 2.3. *Let G be a connected graph with a vertex v of degree 2. Let D be a minimum distance-equalizer set for $G[H_i]_1^m$. Then $v \in D$.*

Proof. Let the two edges incident with v in G be $e_i = uv$ and $e_{i+1} = vw$. Let H_i and H_{i+1} be the graphs associated with these edges in the generalized edge corona product. Choose vertices $a \in V(H_i)$ and $b \in V(H_{i+1})$. In $G[H_i]_1^m$, a shortest path from any vertex $k \neq v$ to a (or b) will pass through u(or w). Hence, $d_{G[H_i]_1^m}(k, a) \neq d_{G[H_i]_1^m}(k, b)$. However, since v is adjacent to all vertices of both H_i and H_{i+1}, it satisfies $d_{G[H_i]_1^m}(v, a) = d_{G[H_i]_1^m}(v, b) = 1$. No other vertex in $G[H_i]_1^m$ equalizes all such pairs, so v must be included in any minimum distance-equalizer set. Hence, $v \in D$. □

Lemma 2.4. *Let G be a connected graph without any pendant edge. Let H_i be the graph corresponding to the edge e_i of G. Then any distance-equalizer set for $G[H_i]_1^m$ must contain at least one vertex from each H_i.*

Proof. Consider any two distinct edges e_i and e_j in G, with their corresponding graphs H_i and H_j, respectively. Let $a \in V(H_i)$ and $b \in V(H_j)$. We consider a shortest path between a and b in $G[H_i]_1^m$.

Case 1: If the distance $d_{G[H_i]_1^m}(a, b)$ is even, then the shortest path from a to b has an odd number of vertices. If v is the central vertex on this path then $d_{G[H_i]_1^m}(a, v) = d_{G[H_i]_1^m}(b, v) = \frac{d_{G[H_i]_1^m}(a,b)}{2}$.

Case 1.1: If $\deg(v) = 2$ in G, then by Lemma 2.3, v must be included in the distance-equalizer set as it equalizes a and b.

Case 1.2: If $\deg(v) > 2$ in G, then there exists an edge $e_t = vt$ that is not part of the path from a to b. Let H_t be the graph corresponding to the edge e_t. A vertex $z \in V(H_t)$ adjacent to v satisfies $d_{G[H_i]_1^m}(a, z) = d_{G[H_i]_1^m}(b, z)$, so z equalizes a and b.

Case 2: If $d_{G[H_i]_1^m}(a, b)$ is odd, then a shortest path contains an even number of vertices. Let $e_k = v_k v_{k+1}$ be the central edge in the path. Then, the corresponding graph H_k contains a vertex $z \in V(H_k)$ such that $d_{G[H_i]_1^m}(a, z) = d_{G[H_i]_1^m}(b, z)$, and hence z equalizes a and b. This proves our claim. □

Theorem 2.5. *Let G be a graph of order $n \geq 2$ and size m. Let $H_1, H_2 \dots H_m$ be m simple graphs of orders at least two. Let D be a distance-equalizer set of $G[H_i]_1^m$. Define the following sets:*

(1) $A = \{x \in V(G) : xy \in E(G) \text{ and } deg(y) = 1\}$
(2) $B = \{w \in V(G) : \deg(w) = 2 \text{ and } w \notin A\}$
(3) $C = \{z \in V(H_i) : e_i = u_i u_{i+1} \text{ is a non-pendant edge of } G\}$

Then the following statements hold:

i: $D \subseteq A \cup B \cup C$
ii: $\operatorname{eqdim}(G[H_i]_1^m) \leq |A| + |B| + |C|$.

Proof. The proof follows from Lemmas 2.2, 2.3 and 2.4. □

2.2. Equidistant Dimension of $G[H_i]_1^m$ for Some Classes of Graphs.

Theorem 2.6. *Let P_n be a path of order $n \geq 4$ and size $n-1$. Let $H_1, H_2 \dots H_{n-1}$ be $n-1$ simple graphs of orders at least two. Then* $\operatorname{eqdim}(P_n[H_i]_1^{n-1}) = 2n - 5$.

Proof. We prove the equality by establishing both upper and lower bounds. We begin by obtaining an upper bound.

Let $v_1, v_2 \dots v_n$ be the vertices and $e_i = v_i v_{i+1}$ for $1 \leq i \leq n-1$ be the edges of P_n. Let H_i be the graph corresponding to the edge e_i. Fix $w_i^0 \in V(H_i)$, $2 \leq i \leq n-2$. Let $A = \{v \in V(P_n) : deg(v) = 2\}$ and $B = \{w_i^0 \in V(H_i) : 2 \leq i \leq n-2\}$. Let $D = A \cup B$. We claim that D is a distance-equalizer set for $P_n[H_i]_1^{n-1}$. Let $x, y \in V(P_n[H_i]_1^{n-1}) \setminus D$. Then we have the following cases.

Case 1: $x, y \in V(H_i)$, $1 \leq i \leq n-1$.
Since the set A contains the internal vertices of P_n, either v_i or v_{i+1} (one of the endpoints of the edge e_i) will equalize x and y.

Case 2: $x \in V(H_i)$ and $y \in V(H_j)$, $i \neq j$.
Without loss of generality, assume $i < j$.

Case 2.1: $j = i + 1$.
Then x and y are equalized by $v_{i+1} \in A$.

Case 2.2: $j \geq i + 2$.
Let $\mathcal{P}$ be a shortest path between x and y in $P_n[H_i]_1^{n-1}$.

If $d_{P_n[H_i]_1^{n-1}}(x, y)$ is odd, then $\mathcal{P}$ contains an even number of vertices, and there exists a central edge $e_t = v_t v_{t+1}$ on $\mathcal{P}$. In the corresponding copy H_t, there exists a vertex $z \in V(H_t)$ such that $d_{P_n[H_i]_1^{n-1}}(x, z) = d_{P_n[H_i]_1^{n-1}}(y, z)$. Thus, z equalizes x and y.

If $d_{P_n[H_i]_1^{n-1}}(x, y)$ is even, then $\mathcal{P}$ has an odd number of vertices, and there exists a central vertex v_t such that $d_{P_n[H_i]_1^{n-1}}(x, v_t) = d_{P_n[H_i]_1^{n-1}}(y, v_t) = \frac{d_{P_n[H_i]_1^{n-1}}(x,y)}{2}$. Hence v_t equalizes x and y.

It follows that D is a distance-equalizer set and $D \subset A \cup B$. Further, $|D| \leq |A| + |B| = (n-2) + (n-3) = 2n - 5$. Therefore, $\operatorname{eqdim}(P_n[H_i]_1^{n-1}) \leq 2n - 5$.

On the other hand, we prove minimality of $|D|$ by contradiction. Assume that D is not minimal. Then there exists a vertex $x \in D$ such that $D' = D \setminus \{x\}$ is also a distance-equalizer set of $P_n[H_i]_1^{n-1}$. We show that this leads to a contradiction by considering all possible types of vertices in $D = A \cup B$.

Case 1: $x \in A$.
Let $x = v_i$ for some i, $2 \leq i \leq n-1$ and $D' = D \setminus \{v_i\}$. It is clear that v_i equalizes any $u \in V(H_{i-1}) \setminus D'$ and any $v \in V(H_i) \setminus D'$. Since $v_i \notin D'$, there is no vertex in D' that equalizes u and v.

Case 2: $x \in B$.
Let $x = w_i^0$ for some i, $2 \leq i \leq n-1$ and $D' = D \setminus \{w_i^0\}$. Then $D' \cap V(H_i) = \varnothing$. Choose $u \in V(H_{i-1}) \setminus D'$ and $v \in V(H_{i+1}) \setminus D'$. Since $D' \cap V(H_i) = \varnothing$, for any $w' \in D'$, we have $d(u, w') \neq d(v, w')$. This implies that u and v are not equalized by any vertex in D'. This contradicts the assumption that D' is a distance-equalizer set. $\square$

Corollary 2.7. $\mathrm{eqdim}(P_3[H_i]_1^2) = 1$.

Proof. Consider P_3 with vertices v_1, v_2, v_3 and edges $e_1 = v_1v_2$ and $e_2 = v_2v_3$. Let H_1 and H_2 be the graphs corresponding to the edges e_1 and e_2 respectively. It is easy to see that the vertex v_2 is adjacent to all vertices in H_1 and H_2. Hence, $\mathrm{eqdim}(P_3[H_i]_1^2) = 1$ $\square$

Theorem 2.8. *Let K_n be a complete graph of order $n \geq 6$ and size m. Let $H_1, H_2 \ldots H_m$, where $m = \binom{n}{2}$ be simple graphs of orders at least two. Then $\mathrm{eqdim}(K_n[H_i]_1^m) = n - 1$.*

Proof. We begin by providing an upper bound for $\mathrm{eqdim}(K_n[H_i]_1^m)$. Let $v_1, v_2 \ldots v_n$ be the vertices of K_n. Let $D = \{v_1, v_2 \ldots v_{n-1}\}$. We show that D is a distance-equalizer set for $K_n[H_i]_1^m$. Let $x, y \in V(K_n[H_i]_1^m) \setminus D$.

Case 1: $x = v_n \in V(K_n)$ and $y \in V(H_i)$ for some i.
Without loss of generality, assume that H_i corresponds to the edge $e_i = v_iv_{i+1}$. Since every vertex of H_i is adjacent to v_i and v_{i+1}, yv_i and yv_{i+1} are edges in $K_n[H_i]_1^m$. Further xv_i and xv_{i+1} are edges of K_n. Consequently, $d_{K_n[H_i]_1^m}(x, v_i) = d_{K_n[H_i]_1^m}(y, v_i) = 1$ or $d_{K_n[H_i]_1^m}(x, v_{i+1}) = d_{K_n[H_i]_1^m}(y, v_{i+1}) = 1$. Hence, either v_i or v_{i+1} serves as an equalizer for the pair (x, y).

Case 2: $x, y \in V(H_i)$ for some i.
Let $e_i = v_iv_{i+1}$. Since $|D| = n - 1$ at least one of v_i or v_{i+1} is in D. Without loss of generality, let $v_i \in D$. Since v_i is adjacent to every vertex of H_i, x and y are equalized by v_i.

Case 3: $x \in V(H_i)$ and $y \in V(H_j)$ with $i \neq j$.
Suppose H_i and H_j correspond to the edges $e_i = v_iv_{i+1}$ and $e_j = v_jv_{j+1}$ of K_n, respectively.

If e_i and e_j share a common endpoint, say v_j, then every vertex of H_i and H_j is adjacent to v_j. Thus $d_{K_n[H_i]_1^m}(x, v_j) = d_{K_n[H_i]_1^m}(y, v_j) = 1$, so x and y are equalized by v_j.

If e_i and e_j do not share a common endpoint, then, since K_n is complete, there exists a vertex $v_t \in V(K_n)$ that is adjacent to both e_i and e_j. In this case, $d_{K_n[H_i]_1^m}(x, v_t) = d_{K_n[H_i]_1^m}(y, v_t) = 2$, and hence x and y are equalized by v_t.

It follows that D is a distance-equalizer set and therefore, $\mathrm{eqdim}(K_n[H_i]_1^m) \leq n-1$.

On the other hand, suppose that $\mathrm{eqdim}(K_n[H_i]_1^m) \leq n-2$. Then there exists a distance-equalizer set D' with $|D'| \leq n-2$. Let $v_1, v_2 \in V(K_n) \setminus D'$. Consider the edge $e = v_1v_2$ and $x \in V(H_e)$. Then for any $w \in D' \cap V(K_n)$, $d_{K_n[H_i]_1^m}(v_1, w) = 1$ (since K_n is complete) and $d_{K_n[H_i]_1^m}(x, w) = d_{K_n[H_i]_1^m}(x, v_1) + d_{K_n[H_i]_1^m}(v_1, w) = 2$. Hence $d_{K_n[H_i]_1^m}(v_1, w) \neq d_{K_n[H_i]_1^m}(x, w)$, contradicting D' is a distance-equalizer set. Therefore, a minimal distance-equalizer set has cardinality exactly $n-1$. $\square$

Theorem 2.9. *Let C_n be a cycle of order $n \geq 3$ and size n. Let $H_1, H_2 \dots H_n$ be n simple graphs of orders at least two. Then* $\mathrm{eqdim}(C_n[H_i]_1^n) = n$.

Proof. Let $v_1, v_2 \dots v_n$ be the vertices of C_n and let $e_i = v_iv_{i+1}$, where $v_{n+1} = v_1$. Let $D = V(C_n)$. We claim that D is a distance-equalizer set for $C_n[H_i]_1^n$.

Case 1: n is odd.
Let $x, y \in V(C_n[H_i]_1^n) \setminus D$. Then we have the following cases.

Case 1.1: $x, y \in V(H_i)$
Since the set D contains all vertices of C_n, either v_i or v_{i+1} (one of the endpoints of the edge e_i) will equalize x and y.

Case 1.2: $x \in V(H_i)$ and $y \in V(H_j)$, $i \neq j$.
Without loss of generality, assume $i < j$.

Case 1.2.1: $j = i+1$.
It is clear that x and y are equalized by $v_{i+1} \in D$.

Case 1.2.2: $j \geq i+2$.
Let $\mathcal{P}$ be a shortest path between x and y in $C_n[H_i]_1^n$.

If $d_{C_n[H_i]_1^n}(x, y)$ is even, then $\mathcal{P}$ has an odd number of vertices, and there exists a central vertex $v_t \in D$ such that $d_{C_n[H_i]_1^n}(x, v_t) = d_{C_n[H_i]_1^n}(y, v_t) = \frac{d_{G[H_i]_1^n}(x,y)}{2}$. Hence v_t equalizes x and y.

If $d_{C_n[H_i]_1^n}(x, y)$ is odd, then consider another path $\mathcal{P}^c$ between x and y in $C_n[H_i]_1^n$. This is possible because C_n is a cycle. Clearly, the length of the alternative path $\mathcal{P}^c$ is even and it has an odd number of vertices. Then there exists a central vertex v_s such that $d_{C_n[H_i]_1^n}(x, v_s) = d_{C_n[H_i]_1^n}(y, v_s) = \frac{d_{C_n[H_i]_1^n}(x,y)}{2}$. Hence v_s equalizes x and y.

It follows that D is a distance-equalizer set. Therefore, $\mathrm{eqdim}(C_n[H_i]_1^{n-1}) \leq n$.

Now suppose that D is not minimal. Then there exists a vertex $x \in D$ such that $D' = D \setminus \{x\}$ is also a distance-equalizer set of $C_n[H_i]_1^n$. Without loss of generality, assume that $x = v_i$ and $D' = D \setminus \{v_i\}$. Now consider the edges $e_{i-1} = v_{i-1}v_i$ and $e_i = v_iv_{i+1}$, together with the attached graphs H_{i-1} and H_i. Choose vertices $u \in V(H_{i-1})$ and $v \in V(H_i)$. For any vertex $w \in D' = D \setminus \{v_i\}$, the distances satisfy $d_{C_n[H_i]_1^n}(w, u) = \min\{d_{C_n}(w, v_{i-1}),\, d_{C_n}(w, v_i)\} + 1$, and $d_{C_n[H_i]_1^n}(w, v) = \min\{d_{C_n}(w, v_i),\, d_{C_n}(w, v_{i+1})\} + 1$. Since n is odd, a shortest path from w to v_i is unique, so exactly one of v_{i-1} and v_{i+1} lies on that path. Without loss of generality, assume that v_{i-1} lies on this path. Thus, $d_{C_n[H_i]_1^n}(w,u) = d_{C_n[H_i]_1^n}(w,v_{i-1})+1$ and $d_{C_n[H_i]_1^n}(w,v) = d_{C_n[H_i]_1^n}(w,v_i)+1$. Hence, no vertex in D' equalizes u and v. This contradicts the assumption that D' is a distance-equalizer set. Hence D is minimal.

Case 2: n is even.
Let $A = \{v_1, v_2 \dots v_{\frac{n}{2}}\} \subset V(C_n)$ and $B = \{w_i^0 \in V(H_i) : 1 \leq i \leq \frac{n}{2}\}$ be the set of all fixed vertices from each H_i. Let $D = A \cup B$. We claim that D is a distance-equalizer set for $C_n[H_i]_1^n$. Let $x, y \in V(C_n[H_i]_1^n) \setminus D$. Then we have the following cases.

Case 2.1: $x, y \in V(H_i)$.
Without loss of generality, assume that H_i corresponds to the edge $e_i = v_i v_{i+1}$. For $1 \leq i \leq \frac{n}{2}$, either v_i or v_{i+1} (one of the endpoints of the edge e_i) will equalize x and y.

For $\frac{n}{2}+1 \leq i \leq n$, consider the antipodal edge $e_{i'}$ of e_i in C_n. Let $H_{i'}$ be the graph corresponding to the edge $e_{i'}$. Clearly x and y are equalized by $w_{i'}^0 \in D \cap V(H_{i'})$.

Case 2.2: $x \in V(H_i)$ and $y \in V(H_j), i \neq j$.
Without loss of generality, assume $i < j$.

Case 2.2.1: $j = i+1$.
For $2 \leq j \leq \frac{n}{2}$, the vertices x and y are equalized by $v_j \in A \cap D$.

For $\frac{n}{2}+1 \leq j \leq n$, consider the antipodal vertex $v_{j'}$ of v_j in C_n. Since $v_j \notin D$, its corresponding antipodal vertex $v_{j'} \in D$. Hence x and y are equalized by $v_{j'}$.

Case 2.2.2: $j \geq i+2$.
Let $\mathcal{P}$ be a shortest path between x and y in $C_n[H]_i{}^m$.

If $d_{C_n[H]_i{}^m}(x,y)$ is even, then $\mathcal{P}$ has odd number of vertices and there exists a central vertex $v_t \in D$ such that $d_{C_n[H]_i{}^m}(x,v_t) = d_{C_n[H]_i{}^m}(y,v_t) = \frac{d_{C_n[H]_i{}^m}(x,y)}{2}$. Hence x and y are equalized by v_t. If $v_t \notin D$, then consider the antipodal vertex $v_{t'}$ of v_t. Since $v_t \notin D$, its corresponding antipodal vertex $v_{t'} \in D$. Thus, x and y are equalized by $v_{t'}$.

If $d_{C_n[H]_i{}^m}(x,y)$ is odd, then $\mathcal{P}$ has an even number of vertices and there exists a central edge e_t such that $w_t^0 \in D \cap V(H_t)$ equalizes x and y. If $w_t^0 \notin D \cap V(H_t)$, then consider the antipodal edge $e_{t'}$ of e_t such that $w_{t'}^0 \in D \cap V(H_{t'})$ equalizes x and y.

Case 2.3: $x \in V(C_n) \setminus A$ and $y \in V(H_i) \setminus B$.
The proof is similar to that of case 2.2.2.

It follows that D is a distance-equalizer set and $D \subset A \cup B$. Further, $|D| \leq |A| + |B| = n$. Therefore, $\text{eqdim}(C_n[H_i]_1^{n-1}) \leq n$.

Now suppose that D is not minimal. Then there exists a vertex $x \in D$ such that $D' = D \setminus \{x\}$ is a distance-equalizer of $C_n[H]_i{}^m$. If $x = v_i, 2 \leq i \leq \frac{n}{2}-1$, then consider the edges $e_{i-1} = v_{i-1}v_i$ and $e_i = v_i v_{i+1}$. Let H_{i-1} and H_i be the graphs corresponding to the edges e_{i-1} and e_i respectively. Choose vertices $a \in V(H_{i-1})$ and $b \in V(H_i)$. Let $v_{i'}$ be the antipodal vertex of v_i. Since both $v_i, v_{i'} \notin D'$, a and

b are not equalized by D'. If $x = w_i^0 \in V(H_i)$ for some i, then consider the edge e_i and its antipodal edge $e_{i'}$. Then, for any $a, b \in V(H_{i'})$, no vertex of D' equalizes a and b. This contradicts the assumption that D' is the distance-equalizer set. Hence D is minimal. □

Theorem 2.10. *Let $K_{m,n}$ be a complete bipartite graph of order $m, n \geq 3$ and size mn. Let $H_1, H_2 \ldots H_{mn}$ be mn simple graphs of orders at least two. Then* $\text{eqdim}(K_{m,n}[H_i]_1^{mn}) = min\{m, n\}$.

Proof. Let $X = \{a_1, a_2 \ldots a_m\}$ and $Y = \{b_1, b_2 \ldots b_n\}$ be the partite sets of $K_{m,n}$. Without loss of generality, assume $m \leq n$. Let H_{ij} be the graph corresponding to the edge $e_{ij} = a_i b_j$, $1 \leq i \leq m$, $1 \leq j \leq n$. We claim that $D = X$ is a distance-equalizer set for $K_{m,n}[H_i]_1^{mn}$. Let $x, y \in V(K_{m,n}[H_i]_1^{mn}) \setminus D$. Then we have the following cases.

Case 1: $x, y \in Y$.
For any $a_k \in X$, $d_{K_{m,n}[H_i]_1^{mn}}(x, a_k) = d_{K_{m,n}[H_i]_1^{mn}}(y, a_k) = 1$, since $K_{m,n}$ is a complete bipartite graph. Thus, a_k serves as an equalizer for the pair (x, y).

Case 2: $x, y \in V(H_{ij})$ for some i, j.
Since $a_i \in X$, $d_{K_{m,n}[H_i]_1^{mn}}(x, a_i) = d_{K_{m,n}[H_i]_1^{mn}}(y, a_i) = 1$.

Case 3: $x \in Y$ and $y \in V(H_{ij})$.
Since $a_i \in X$, $d_{K_{m,n}[H_i]_1^{mn}}(x, a_i) = d_{K_{m,n}[H_i]_1^{mn}}(y, a_i) = 1$. Clearly x and y are equalized by a_i.

Case 4: $x \in V(H_{ij})$ and $y \in V(H_{i'j'})$.

Case 4.1: $i = i'$ (same a_i).
Since $a_i \in A$, $d_{K_{m,n}[H_i]_1^{mn}}(x, a_i) = d_{K_{m,n}[H_i]_1^{mn}}(y, a_i) = 1$. Hence a_i equalizes x and y.

Case 4.2: $j = j'$ (same b_j).
Choose a_k with $k \neq i$ and $k \neq i'$. Then $d_{K_{m,n}[H_i]_1^{mn}}(x, a_k) = d_{K_{m,n}[H_i]_1^{mn}}(x, b_j) + d_{K_{m,n}[H_i]_1^{mn}}(b_j, a_k) = 2$ and $d_{K_{m,n}[H_i]_1^{mn}}(y, a_k) = d_{K_{m,n}[H_i]_1^{mn}}(y, b_j) + d_{K_{m,n}[H_i]_1^{mn}}(b_j, a_k) = 2$. Hence a_k equalizes x and y. Note that the existence of a_k is guaranteed because $|X| \geq 3$. If $k = i$, a_i does not equalize x and y as $d(a_i, x) = 1$ and $d(a_i, y) = d(a_i, b_j) + d(b_j, y) = 2$. A similar argument holds if $k = i'$.

Case 4.3: $i \neq i'$ and $j \neq j'$.
Choose a_k with $k \neq i$ and $k \neq i'$. Then $d_{K_{m,n}[H_i]_1^{mn}}(x, a_k) = d_{K_{m,n}[H_i]_1^{mn}}(x, b_j) + d_{K_{m,n}[H_i]_1^{mn}}(b_j, a_k) = 2$ and $d_{K_{m,n}[H_i]_1^{mn}}(y, a_k) = d_{K_{m,n}[H_i]_1^{mn}}(y, b_j) + d_{K_{m,n}[H_i]_1^{mn}}(b_j, a_k) = 2$. Hence x and y are equalized by a_k.

In all cases, there exists $w \in D$ with $d_{K_{m,n}[H_i]_1^{mn}}(x, w) = d_{K_{m,n}[H_i]_1^{mn}}(y, w)$. Thus, $D = X$ is a distance-equalizer set of cardinality m.

On the other hand, suppose that D' is a distance-equalizer set with $|D'| < m \leq n$. Then there exists $a_i \in X \setminus D'$. Consider the pair a_i and b_j. Both are not in D'. For any $w \in D'$, we have the following.

If $w \in X$, then $d_{K_{m,n}[H_i]_1^{mn}}(a_i, w) = d_{K_{m,n}[H_i]_1^{mn}}(a_i, b_j) + d_{K_{m,n}[H_i]_1^{mn}}(b_j, w) = 2$ and $d_{K_{m,n}[H_i]_1^{mn}}(b_j, w) = 1$.

If $w \in Y$, then $d_{K_{m,n}[H_i]_1^{mn}}(a_i, w) = 1$ and $d_{K_{m,n}[H_i]_1^{mn}}(b_j, w) = 2$ (if $w \neq b_j$) or $d_{K_{m,n}[H_i]_1^{mn}}(b_j, w) = 0$ (if $w = b_j$, but $b_j \notin D'$).

If $w \in V(H_i)$ for some i, then $d_{K_{m,n}[H_i]_1^{mn}}(a_i, w) = 1$ and $d_{K_{m,n}[H_i]_1^{mn}}(a_k, w) = d_{K_{m,n}[H_i]_1^{mn}}(a_k, b_j) + d_{K_{m,n}[H_i]_1^{mn}}(b_j, w) = 2$, for $i \neq k$.

In all subcases, $d_{K_{m,n}[H_i]_1^{mn}}(a_i, w) \neq d_{K_{m,n}[H_i]_1^{mn}}(b_j, w)$ for any $w \in D'$, contradicting that D' is a minimum distance-equalizer set for $K_{m,n}[H_i]_1^{mn}$. Hence, $\text{eqdim}(K_{m,n}[H_i]_1^{mn}) \geq m$. This completes the proof of the lower bound. □

3. Conclusion

Summing up, the results establish the behavior of distance-equalizer sets in generalized edge corona products, with bounds and exact values identified for several important graph families. This study enhances the understanding of resolvability parameters in complex graph constructions. As a future direction, ongoing work focuses on extending these techniques to generalized corona products, which are expected to reveal further structural insights into equidistant dimension.

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References

1. Anand, B. S., Dayap, J. A., Casinillo, L. F., Pepper, R., & Nair, R. S. (2025). On distance k-domination number of graphs under product operations. Gulf Journal of Mathematics, 19(1), 117-124. https://doi.org/10.56947/gjom.v19i1.2032
2. Beerliova, Z., Eberhard, F., Erlebach, T., Hall, A., Hoffmann, M., Mihalak, M., & Ram, L. (2006). Network discovery and verification. IEEE Journal on Selected Areas in Communications, 24, 2168-2181. https://doi.org/10.1109/JSAC.2006.884015
3. Caay, M., Maza, R., Duarte, R., Lana, S. R., Soriano, B., & Janer, A. (2025). Equitable isolate domination in graphs. Gulf Journal of Mathematics, 19(1), 337-346. https://doi.org/10.56947/gjom.v19i1.2603
4. Chartrand, G., Eroh, L., Johnson, M. A., & Oellermann, O. R. (2000). Resolvability in graphs and the metric dimension of a graph. Discrete Applied Mathematics, 105(1-3), 99-113. https://doi.org/10.1016/S0166-218X(00)00198-0
5. Chvátal, V. (1983). Mastermind. Combinatorica, 3(3), 325-329. https://doi.org/10.1007/BF02579188
6. Garey, M. R., & Johnson, D. S. (1979). Computers and intractability: A guide to the theory of NP-completeness. Freeman, New York. https://doi.org/10.1137/1024022
7. Gispert-Fernandez, A., & Rodrıguez-Velázquez, J. A. (2024). The equidistant dimension of graphs: NP-completeness and the case of lexicographic product graphs. Aims Mathematics, 9(6), 15325-15345. https://doi.org/10.3934/math.2024744
8. González, A., Hernando, C., & Mora, M. (2022). The equidistant dimension of graphs. Bulletin of the Malaysian Mathematical Sciences Society, 45(4), 1757-1775. https://doi.org/10.1007/s40840-022-01295-z
9. Grigorious, C., Manuel, P., Miller, M., Rajan, B., & Stephen, S. (2014). On the metric dimension of circulant and Harary graphs. Applied Mathematics and Computation, 248, 47-54. https://doi.org/10.1016/j.amc.2014.09.045
10. Harary, F., & Melter, R. A. (1976). On the metric dimension of a graph. Ars combin, 2(191-195), 1.

11. Hernando, C., Mora, M., Pelayo, I. M., Seara, C., Cáceres, J., & Puertas, M. L. (2005). On the metric dimension of some families of graphs. Electronic notes in discrete mathematics, 22, 129-133. https://doi.org/10.1016/j.endm.2005.06.023
12. Imran, M., Baig, A. Q., Shafiq, M. K., & Tomecu, L. (2014). On metric dimension of generalized Petersen graphs P (n, 3). Ars Combinatoria, 117, 113-130.
13. Khuller, S., Raghavachari, B., & Rosenfeld, A. (1996). Landmarks in graphs. Discrete applied mathematics, 70(3), 217-229. https://doi.org/10.1016/0166-218X(95)00106-2
14. Klein, D. J., & Yi, E. (2012). A comparison on metric dimension of graphs, line graphs, and line graphs of the subdivision graphs. European journal of pure and applied mathematics, 5(3), 302-316.
15. Kratica, J., Čangalović, M., & Kovačević-Vujčić, V. (2024). Equidistant dimension of Johnson and Kneser graphs. arXiv preprint arXiv:2406.17870. https://doi.org/10.48550/arXiv.2406.17870
16. Luo, Y., & Yan, W. (2018). Spectra of the generalized edge corona of graphs. Discrete Mathematics, Algorithms and Applications, 10(01), 1850002. https://doi.org/10.1142/S1793830918500027
17. Manuel, P., Bharati, R., Rajasingh, I., & Chris Monica, M. (2008). On minimum metric dimension of honeycomb networks. Journal of Discrete algorithms, 6(1), 20-27. https://doi.org/10.1016/j.jda.2006.09.002
18. Savić, A., Maksimović, Z., Bogdanović, M., & Kratica, J. (2024). The equidistant dimension of some graphs of convex polytopes. arXiv preprint arXiv:2407.15307. https://doi.org/10.48550/arXiv.2407.15307
19. Slater, P. J. (1975). Leaves of trees. Congr. Numer, 14(549-559), 37.

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