Gulf Journal of Mathematics
Vol 23, Issue 1 (2026) 1-15

ISSN: 2309-4966
https://doi.org/10.56947/t8ty3q80

COLLOCATION SCHEME FOR 2D-VIES WITH TWO CONSTANT DELAYS

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Abstract. This paper presents a numerical approach for solving linear two-dimensional Volterra integral equations (2D-VIEs) with two distinct time delays using the Taylor collocation method. The method approximates the unknown function by Taylor polynomials and enforces the integral equation at selected collocation points, which can be efficiently solved. A convergence analysis is provided to establish the accuracy and stability of the proposed algorithm. Several numerical examples are included to demonstrate the method's effectiveness in handling problems with multiple delays and smooth kernels. The results show that the Taylor collocation technique achieves high precision at minimal computational cost.

Keywords. Taylor collocation method, 2D Volterra integral equations, time delays, convergence analysis.
2020 Mathematics Subject Classification. Primary 45D05; Secondary 65R20

1. Introduction and preliminaries

Volterra integral equations with several distinct time delays present substantial analytical difficulties and arise in numerous scientific and engineering contexts. Such equations serve as effective models for processes in control theory, biology, physics, and population dynamics [1, 4, 7, 10, 12, 13, 14, 15, 16, 17]. Their formulation naturally incorporates memory effects and delayed responses, allowing the description of systems whose present state depends essentially on their past evolution.

In particular, two-dimensional Volterra integral equations with double delays provide enhanced flexibility by coupling temporal and spatial variables. This structure enables the modeling of complex mechanisms such as age-structured populations, hereditary diffusion, and delayed feedback processes.

Applications of delay-type Volterra equations extend further to control systems [8], biological and ecological modeling [5, 7], diffusion and physical processes [6, 13], and population dynamics [1, 14]. Correspondingly, a substantial

Date: Received: Mar 30, 2026; Revised: Apr 27, 2026; Accepted: May 11, 2026.
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body of work has addressed their analytical and numerical properties. Khajehnasiri *et al.* [11] employed wavelet techniques for delay Volterra-Hammerstein equations; Mansouri and Azimzadeh [13] developed spectral methods for fractional delay integro-differential equations; Laib *et al.* [12] proposed a Taylor-based collocation scheme for systems with delays; Efut *et al.* [9] developed a robust iterative method for nonlinear Volterra delay integro-differential equations. Bouzeraieb *et al.* [3] studied one-dimensional double-delay integral equations via the Taylor collocation method; and high-dimensional extensions have been considered through efficient collocation algorithms in [2]. Despite these contributions, the numerical treatment of two-dimensional Volterra integral equations with multiple delays remains comparatively limited, which motivates the present work.

This study examines the two-dimensional Volterra integral equation characterized by two distinct delays, expressed as follows:

$$u(t,y) = g(t,y) + \int_{t-\theta_2}^{t-\theta_1} K(t,y,s)\, u(s,y)\, ds, \quad (t,y) \in [\theta_2, T] \times [0, Y], \tag{1.1}$$

subject to the initial condition

$$u(t,y) = \psi(t,y), \quad (t,y) \in [0,\theta_2) \times [0, Y], \tag{1.2}$$

where $0 < \theta_1 < \theta_2$ are two constant delays. We assume that g and K are sufficiently smooth to ensure the method's applicability. An iterative computational scheme is developed based on the Taylor collocation principle, and a rigorous convergence analysis is provided to confirm the method's reliability.

The structure of the paper is as follows. Section 2 describes the implementation of the Taylor collocation method for the problem under consideration. Section 3 presents the convergence and error analyses. Section 4 reports several computational examples validating the precision and performance of the proposed method. The final section recapitulates the principal outcomes and identifies potential avenues for further investigation.

2. Formulation of the Taylor Collocation Method

We consider a linear two-dimensional Volterra integral equation with two constant delays θ_1, θ_2 of the form:

$$u(t,y) = g(t,y) + \int_{t-\theta_2}^{t-\theta_1} K(t,y,s)u(s,y)ds, \quad (t,y) \in [\theta_2, T] \times [0, Y], \tag{2.1}$$

and $u(t,y) = \phi(t,y)$ for $(t,y) \in [0,\theta_2[\times [0,Y]$, where the functions g and K are given (real-valued) continuous functions defined, respectively, on $D = [0,T] \times [0,Y] \subset \mathbb{R}^2$ and $S = \{(t,y,s) : 0 \le s \le t \le T, 0 \le y \le Y\}$.

According to the classical theory of Volterra integral equations, problem (2.1) admits a unique solution $u \in C(D)$.

We suppose that $T = (r+1)\theta_2$, where $r \in \{1, 2, 3...\}$.

Let us introduce the collection of temporal nodes

$$\Pi_{\mathfrak{N}} = \left\{ t_n^i = (i+1)\theta_2 + nh : 0 \le n \le \mathfrak{N},\ 0 \le i \le r-1 \right\},$$

which provides a uniform discretization of the interval $[\theta_2, T]$. The mesh width satisfies

$$h = t^i_{n+1} - t^i_n, \qquad h = \frac{\theta_1}{\mathfrak{N}_1} = \frac{\theta_2}{\mathfrak{N}},$$

where $\mathfrak{N}_1$ and $\mathfrak{N}$ are positive integers. Similarly, we define the spatial partition

$$\Pi_{\mathfrak{M}} = \{\, y_j = jk : 0 \le j \le \mathfrak{M} \,\}, \qquad k = \frac{Y}{\mathfrak{M}},$$

which is a uniform subdivision of the interval $[0, Y]$.

Taking the Cartesian product of these two grids yields a rectangular mesh over the domain D,

$$\Pi_{\mathfrak{N},\mathfrak{M}} = \Pi_{\mathfrak{N}} \times \Pi_{\mathfrak{M}} = \left\{\, (t^i_n, y_m) : 0 \le n \le \mathfrak{N},\ 0 \le i \le r-1,\ 0 \le m \le \mathfrak{M} \,\right\}.$$

The temporal and spatial subintervals are defined as

$$\sigma^i_n = [t^i_n,\, t^i_{n+1}), \quad n = 0, \ldots, \mathfrak{N}-2, \qquad \sigma^i_{\mathfrak{N}-1} = [t^i_{\mathfrak{N}-1},\, t^i_{\mathfrak{N}}],$$

$$\delta_m = [y_m,\, y_{m+1}), \quad m = 0, \ldots, \mathfrak{M}-2, \qquad \delta_{\mathfrak{M}-1} = [y_{\mathfrak{M}-1},\, y_{\mathfrak{M}}],$$

and each rectangular mesh cell takes the form

$$D^i_{n,m} = \sigma^i_n \times \delta_m, \qquad 0 \le n \le \mathfrak{N}-1,\ 0 \le i \le r-1,\ 0 \le m \le \mathfrak{M}-1.$$

Let π_{p-1} denote the set of all bivariate real polynomials in (t, y) of total degree at most $p-1$. We then introduce the piecewise polynomial space

$$S^{(-1)}_{p-1}(\Pi_{\mathfrak{N},\mathfrak{M}}) = \Big\{ v \ :\ v^i_{n,m} = v|_{D^i_{n,m}} \in \pi_{p-1},\ 0 \le n \le \mathfrak{N}-1,\ 0 \le i \le r-1,\ 0 \le m \le \mathfrak{M}-1 \Big\}. \tag{2.2}$$

Functions in this space are bivariate polynomial splines of degree at most $p-1$ in both variables. The dimension of $S^{(-1)}_{p-1}(\Pi_{\mathfrak{N},\mathfrak{M}})$ is $r\,\mathfrak{N}\,\mathfrak{M}\,p^2$, which coincides with the total number of coefficients of the local polynomials $v^i_{n,m}$. These coefficients are obtained by applying Taylor expansions within each rectangular cell.

2.1. **Approximate solution in the rectangles $D^0_{n,m}$.**

First, to approximate u by $v^0_{n,m}$ in the rectangles $D^0_{n,m}$, for $n \in \{0, \ldots, \mathfrak{N}-1\}$ and $m \in \{0, \ldots, \mathfrak{M}-1\}$, the function u must be successively approximated by $v^0_{k,m}$ for all $0 \leqslant k \leqslant n$ in each subdomain $D^0_{k,m}$. For $n = 0$, we set $v^0_{0,m} = u$. Accordingly, the approximation takes the form

$$v^0_{n,m}(t, y) = \sum_{i+j=0}^{p-1} \frac{1}{i!\, j!} \frac{\partial^{i+j} \hat{v}^0_{n,m}(t^0_n, y_m)}{\partial t^i \partial y^j} (t - t^0_n)^i (y - y_m)^j, \quad (t, y) \in D^0_{n,m}, \tag{2.3}$$

where $\hat{v}^0_{n,m}$ denotes the exact solution of the corresponding integral equation.

$$\hat{v}^0_{n,m}(t, y) = g(t, y) + \int_{t-\theta_2}^{t-\theta_1} K(t, y, s)\phi(s, y)ds, \quad n \in \{0, \ldots, \mathfrak{N}_1 - 1\} \tag{2.4}$$

and

$$\hat{v}^0_{n,m}(t,y) = g(t,y) + \int_{t-\theta_2}^{\theta_2} K(t,y,s)\phi(s,y)ds + \sum_{\epsilon=0}^{n-\mathfrak{N}_1-1} \int_{t^0_\epsilon}^{t^0_{\epsilon+1}} K(t,y,s)v^0_{\epsilon,m}(s,y)ds$$
$$+ \int_{t^0_n-\theta_1}^{t-\theta_1} K(t,y,s)v^0_{n-\mathfrak{N}_1,m}(s,y)ds, \quad n \in \{\mathfrak{N}_1,\dots,\mathfrak{N}-1\}. \tag{2.5}$$

For each $n \in \{0,\dots,\mathfrak{N}_1-1\}$, equation (2.4) is differentiated j times with respect to y and i times with respect to t, yielding

$$\frac{\partial^{i+j}\hat{v}^0_{n,m}(t,y)}{\partial t^i \partial y^j} = \partial_1^{(i)}\partial_2^{(j)} g(t,y) + \frac{\partial^{i+j}}{\partial t^i \partial y^j}\left[\int_{t-\theta_2}^{t-\theta_1} K(t,y,s)\,\phi(s,y)\,ds\right]. \tag{2.6}$$

For each $n \in \{\mathfrak{N}_1,\dots,\mathfrak{N}-1\}$, equation (2.5) is differentiated j times with respect to y and i times with respect to t, which yields

$$\frac{\partial^{i+j}\hat{v}^0_{n,m}(t,y)}{\partial t^i \partial y^j} = \partial_1^{(i)}\partial_2^{(j)} g(t,y) + \frac{\partial^{i+j}}{\partial t^i \partial y^j}\left[\int_{t-\theta_2}^{\theta_2} K(t,y,s)\,\phi(s,y)\,ds\right]$$
$$+ \sum_{\epsilon=0}^{n-\mathfrak{N}_1-1}\sum_{l=0}^{j}\binom{j}{l}\int_{t^0_\epsilon}^{t^0_{\epsilon+1}} \partial_1^{(i)}\partial_2^{(j-l)}K(t,y,s)\frac{\partial^l v^0_{\epsilon,m}(s,y)}{\partial y^l}\,ds$$
$$+ \sum_{l=0}^{j}\sum_{\mu=0}^{i-1}\sum_{\lambda=0}^{\mu}\binom{j}{l}\binom{\mu}{\lambda}\frac{\partial^{\mu-\lambda}}{\partial t^{\mu-\lambda}}\left[\partial_1^{(i-1-\mu)}\partial_2^{(j-l)}K(t,y,t-\theta_1)\right]\frac{\partial^{\lambda+l}v^0_{n-\mathfrak{N}_1,m}(t-\theta_1,y)}{\partial t^\lambda \partial y^l}$$
$$+ \sum_{l=0}^{j}\binom{j}{l}\int_{t^0_n-\theta_1}^{t-\theta_1} \partial_1^{(i)}\partial_2^{(j-l)}K(t,y,s)\frac{\partial^l v^0_{n-\mathfrak{N}_1,m}(s,y)}{\partial y^l}\,ds. \tag{2.7}$$

2.2. **Approximate solution in the rectangles $D^d_{n,m}$.** Second, to approximate u by $v^d_{n,m}$ in the rectangles $D^d_{n,m}$, $n \in \{0,\dots,\mathfrak{N}-1\}, m \in \{0,\dots,\mathfrak{M}-1\}$, $d \in \{1,\dots,r-1\}$, u must be approximated by $v^k_{n,m}$ $(0 \leqslant k \leqslant d)$ in each rectangle $D^d_{k,m}$, such that

$$v^d_{n,m}(t,y) = \sum_{i+j=0}^{p-1}\frac{1}{i!j!}\frac{\partial^{i+j}\hat{v}^d_{n,m}(t^d_n,y_m)}{\partial t^i \partial y^j}(t-t^d_n)^i(y-y_m)^j; \quad (t,y) \in D^d_{n,m}, \tag{2.8}$$

where $\hat{v}^d_{n,m}$ is the exact solution of integral equation

$$\hat{v}^d_{n,m}(t,y) = g(t,y) + \int_{t-\theta_2}^{t^{d-1}_{n+1}} K(t,y,s)v^{d-1}_{n,m}(s,y)ds$$
$$+ \sum_{\epsilon=n+1}^{\mathfrak{N}-\mathfrak{N}_1+n-1}\int_{t^{d-1}_\epsilon}^{t^{d-1}_{\epsilon+1}} K(t,y,s)v^{d-1}_{\epsilon,m}(s,y)ds$$
$$+ \int_{t^d_n-\theta_1}^{t-\theta_1} K(t,y,s)v^{d-1}_{\mathfrak{N}-\mathfrak{N}_1+n,m}(s,y)ds, \quad n \in \{0,\dots,\mathfrak{N}_1-1\}, \tag{2.9}$$

and

$$
\begin{aligned}
\hat{v}^d_{n,m}(t,y) = g(t,y) &+ \int_{t-\theta_2}^{t^{d-1}_{n+1}} K(t,y,s) v^{d-1}_{n,m}(s,y)ds \\
&+ \sum_{\epsilon=n+1}^{\mathfrak{N}-1} \int_{t^{d-1}_{\epsilon}}^{t^{d-1}_{\epsilon+1}} K(t,y,s) v^{d-1}_{\epsilon,m}(s,y)ds \\
&+ \sum_{\epsilon=0}^{n-\mathfrak{N}_1-1} \int_{t^{d}_{\epsilon}}^{t^{d}_{\epsilon+1}} K(t,y,s) v^{d}_{\epsilon,m}(s,y)ds \\
&+ \int_{t^d_n-\theta_1}^{t-\theta_1} K(t,y,s) v^{d}_{n-\mathfrak{N}_1,m}(s,y)ds, \quad n \in \{\mathfrak{N}_1, \dots, \mathfrak{N}-1\}.
\end{aligned}
\tag{2.10}
$$

For $n \in \{0, \dots, \mathfrak{N}_1 - 1\}$, we differentiate equation (2.9) j times with respect to y and i times with respect to t, we obtain

$$
\begin{aligned}
&\frac{\partial^{i+j}\hat{v}^d_{n,m}(t,y)}{\partial t^i \partial y^j} = \partial_1^{(i)}\partial_2^{(j)} g(t,y) \\
&- \sum_{l=0}^{j}\sum_{\mu=0}^{i-1}\sum_{\lambda=0}^{\mu} \binom{j}{l}\binom{\mu}{\lambda} \frac{\partial^{\mu-\lambda}}{\partial t^{\mu-\lambda}} \left[\partial_1^{(i-1-\mu)}\partial_2^{(j-l)} K(t,y,t-\theta_2)\right] \frac{\partial^{\lambda+l} v^{d-1}_{n,m}(t-\theta_2,y)}{\partial t^{\lambda}\partial y^l} \\
&+ \sum_{l=0}^{j}\binom{j}{l} \int_{t-\theta_2}^{t^{d-1}_{n+1}} \partial_1^{(i)}\partial_2^{(j-l)} K(t,y,s) \frac{\partial^l v^{d-1}_{n,m}(s,y)}{\partial y^l} ds \\
&+ \sum_{\epsilon=n+1}^{\mathfrak{N}-\mathfrak{N}_1+n-1}\sum_{l=0}^{j}\binom{j}{l} \int_{t^{d-1}_{\epsilon}}^{t^{d-1}_{\epsilon+1}} \partial_1^{(i)}\partial_2^{(j-l)} K(t,y,s) \frac{\partial^l v^{d-1}_{\epsilon,m}(s,y)}{\partial y^l} ds \\
&+ \sum_{l=0}^{j}\sum_{\mu=0}^{i-1}\sum_{\lambda=0}^{\mu} \binom{j}{l}\binom{\mu}{\lambda} \frac{\partial^{\mu-\lambda}}{\partial t^{\mu-\lambda}} \left[\partial_1^{(i-1-\mu)}\partial_2^{(j-l)} K(t,y,t-\theta_1)\right] \frac{\partial^{\lambda+l} v^{d-1}_{\mathfrak{N}-\mathfrak{N}_1+n,m}(t-\theta_1,y)}{\partial t^{\lambda}\partial y^l} \\
&+ \sum_{l=0}^{j}\binom{j}{l} \int_{t^d_n-\theta_1}^{t-\theta_1} \partial_1^{(i)}\partial_2^{(j-l)} K(t,y,s) \frac{\partial^l v^{d-1}_{\mathfrak{N}-\mathfrak{N}_1+n,m}(s,y)}{\partial y^l} ds.
\end{aligned}
\tag{2.11}
$$

For $n \in \{\mathfrak{N}_1, \ldots, \mathfrak{N}-1\}$, to find $\dfrac{\partial^{i+j}\hat{v}^d_{n,m}}{\partial t^i \partial y^j}$, we differentiate equation (2.10) j times with respect to y and i times with respect to t, we obtain

$$
\begin{aligned}
&\frac{\partial^{i+j}\hat{v}^d_{n,m}(t,y)}{\partial t^i \partial y^j} = \partial_1^{(i)}\partial_2^{(j)} g(t,y)\\
&-\sum_{l=0}^{j}\sum_{\mu=0}^{i-1}\sum_{\lambda=0}^{\mu}\binom{j}{l}\binom{\mu}{\lambda}\frac{\partial^{\mu-\lambda}}{\partial t^{\mu-\lambda}}\left[\partial_1^{(i-1-\mu)}\partial_2^{(j-l)}K(t,y,t-\theta_2)\right]\frac{\partial^{\lambda+l}v^{d-1}_{n,m}(t-\theta_2,y)}{\partial t^\lambda \partial y^l}\\
&+\sum_{l=0}^{j}\binom{j}{l}\int_{t-\theta_2}^{t^{d-1}_{n+1}}\partial_1^{(i)}\partial_2^{(j-l)}K(t,y,s)\frac{\partial^l v^{d-1}_{n,m}(s,y)}{\partial y^l}ds\\
&+\sum_{\epsilon=n+1}^{\mathfrak{N}-1}\sum_{l=0}^{j}\binom{j}{l}\int_{t^{d-1}_{\epsilon}}^{t^{d-1}_{\epsilon+1}}\partial_1^{(i)}\partial_2^{(j-l)}K(t,y,s)\frac{\partial^l v^{d-1}_{\epsilon,m}(s,y)}{\partial y^l}ds\\
&+\sum_{\epsilon=0}^{n-\mathfrak{N}_1-1}\sum_{l=0}^{j}\binom{j}{l}\int_{t^{d}_{\epsilon}}^{t^{d}_{\epsilon+1}}\partial_1^{(i)}\partial_2^{(j-l)}K(t,y,s)\frac{\partial^l v^{d}_{\epsilon,m}(s,y)}{\partial y^l}ds\\
&+\sum_{l=0}^{j}\sum_{\mu=0}^{i-1}\sum_{\lambda=0}^{\mu}\binom{j}{l}\binom{\mu}{\lambda}\frac{\partial^{\mu-\lambda}}{\partial t^{\mu-\lambda}}\left[\partial_1^{(i-1-\mu)}\partial_2^{(j-l)}K(t,y,t-\theta_1)\right]\frac{\partial^{\lambda+l}v^{d}_{n-\mathfrak{N}_1,m}(t-\theta_1,y)}{\partial t^\lambda \partial y^l}\\
&+\sum_{l=0}^{j}\binom{j}{l}\int_{t^d_n-\theta_1}^{t-\theta_1}\partial_1^{(i)}\partial_2^{(j-l)}K(t,y,s)\frac{\partial^l v^{d}_{n-\mathfrak{N}_1,m}(s,y)}{\partial y^l}ds. \qquad (2.12)
\end{aligned}
$$

3. Theoretical Convergence Results

In order to establish the convergence of the proposed Taylor collocation scheme, we recall several auxiliary inequalities.

Lemma 3.1 (Taylor's theorem for two variables)**.** *Let g be m times continuously differentiable on $D = I \times J$ and let $(s_0, t_0) \in D$. Then for all $(s,t) \in D$,*

$$g(s,t) = \sum_{a+b=0}^{m-1}\frac{1}{a!b!}\frac{\partial^{a+b}g(s_0,t_0)}{\partial s^a \partial t^b}(s-s_0)^a(t-t_0)^b + \sum_{a+b=m}\frac{1}{a!b!}\frac{\partial^{a+b}g(s_1,t_1)}{\partial s^a \partial t^b}(s-s_0)^a(t-t_0)^b,$$

for suitable (s_1, t_1) on the segment joining (s_0, t_0) and (s, t).

Lemma 3.2 (Discrete Gronwall inequality [4])**.** *Let $\{\epsilon_m\}$ be a non-negative sequence such that*

$$\epsilon_m \le \varrho_0 + \sum_{i=0}^{m-1} k_i \epsilon_i, \quad m \ge 1,$$

with $\varrho_0 > 0$, $k_i > 0$. Then

$$\epsilon_m \le \varrho_0 \exp\left(\sum_{j=0}^{m-1} k_j\right), \quad m \ge 1.$$

We consider the space $L^\infty(D)$ with the norm

$$\|\psi\|_{L^\infty(D)} = \inf\{c \in \mathbb{R} : |\psi(t,s)| \le c \text{ for a.e. } (t,s) \in D\} < \infty.$$

We now state a useful lemma required to establish the convergence of the proposed method.

Lemma 3.3. *Assume that the functions g and K possess continuous derivatives up to order p on their corresponding domains. Then there exists a constant $\gamma(p) > 0$ such that, for every $0 \le n \le \mathfrak{N}-1$, $0 \le m \le \mathfrak{M}-1$, and for all integers i,j satisfying $0 \le i+j \le p$, we have*

$$\left\| \frac{\partial^{i+j}\hat{v}^d_{n,m}}{\partial t^i \partial y^j} \right\|_{L^\infty(D^d_{n,m})} \le \gamma(p),$$

where the initialization on the first layer is given by $\hat{v}^0_{0,m}(t,y) = u(t,y)$ for all $(t,y) \in D^0_{0,m}$.

Proof. Set

$$a^{i+j,d}_{n,m} = \left\| \frac{\partial^{i+j}\hat{v}^d_{n,m}}{\partial t^i \partial y^j} \right\|_{L^\infty(D^d_{n,m})}.$$

The argument is established in two steps.

Step 1. Case $i+j=0$, $m = 0,\dots,\mathfrak{M}-1$. From (2.4), it follows that, for every $n = 0,\dots,\mathfrak{N}_1-1$,

$$a^{0+0,0}_{n,m} \le c_1.$$

For $n = \mathfrak{N}_1,\dots,\mathfrak{N}-1$, relation (2.5) gives

$$a^{0+0,0}_{n,m} \le c_1 + c_2 \sum_{\epsilon=0}^{n-\mathfrak{N}_1-1} \sum_{\vartheta+\varrho=0}^{p-1} \int_{t^0_\epsilon}^{t^0_{\epsilon+1}} a^{\vartheta+\varrho,0}_{\epsilon,m}\, ds + c_2 \sum_{\vartheta+\varrho=0}^{p-1} \int_{t^0_n-\theta_1}^{t-\theta_1} a^{\vartheta+\varrho,0}_{n-\mathfrak{N}_1,m}\, ds.$$

Now let $i+j = 1,\dots,p-1$ and $m = 0,\dots,\mathfrak{M}-1$. From (2.6), we have for all $n = 0,\dots,\mathfrak{N}_1-1$,

$$a^{i+j,0}_{n,m} \le c_1.$$

From (2.7), for $n = \mathfrak{N}_1,\dots,\mathfrak{N}-1$,

$$a^{i+j,0}_{n,m} \le c_1 + c_2 p \sum_{\epsilon=0}^{n-\mathfrak{N}_1-1} \sum_{\vartheta+\varrho=0}^{p-1} \int_{t^0_\epsilon}^{t^0_{\epsilon+1}} a^{\vartheta+\varrho,0}_{\epsilon,m}\, ds$$
$$+ c_3 p^3 h \sum_{\vartheta+\varrho=0}^{p-1} a^{\vartheta+\varrho,0}_{n-\mathfrak{N}_1,m} + c_2 p \sum_{\vartheta+\varrho=0}^{p-1} \int_{t^0_n-\theta_1}^{t-\theta_1} a^{\vartheta+\varrho,0}_{n-\mathfrak{N}_1,m}\, ds.$$

The constants above are defined by

$$c_1 = \max\left\{ \left\|\partial_1^{(i)}\partial_2^{(j)} g\right\|_{L^\infty(D)} + \left\| \frac{\partial^{i+j}}{\partial t^i \partial y^j}\Big(\int_0^{\theta_2} K(t,y,s)\phi(s,y)\,ds \Big) \right\|_{L^\infty(D)} \right\},$$

$$c_2 = \max\left\{ \binom{j}{l} \frac{1}{\vartheta!}\frac{1}{(\varrho-l)!} \left\|\partial_1^{(i)}\partial_2^{(j-l)} K(t,y,s)(s-t_n^0)^{\vartheta}(y-y_m)^{\varrho-l}\right\|_{L^\infty(D)} \right\},$$

$$c_3 = \max\left\{ \binom{j}{l}\binom{\mu}{\lambda} \frac{1}{(\vartheta-\lambda)!(\varrho-l)!} \left\| \frac{\partial^{\mu-\lambda}}{\partial t^{\mu-\lambda}} \big[\partial_1^{(i-1-\mu)}\partial_2^{(j-l)} K(t,y,t-\theta_1)\big](y-y_m)^{\varrho-l} \right\|_{L^\infty(D)} \right\},$$

where $c_1, c_2, c_3 > 0$ are independent of $\mathfrak{N}$ and $\mathfrak{M}$.

Combining the above estimates, for all $i+j = 0,\dots,p-1$, $n = 0,\dots,\mathfrak{N}-1$, $m = 0,\dots,\mathfrak{M}-1$, we obtain

$$\sum_{i+j=0}^{p} a_{n,m}^{i+j,0} \le c_1(p+1)^2 + c_2(p+1)^2 ph \sum_{\epsilon=0}^{n-1}\sum_{\vartheta+\varrho=0}^{p-1} a_{\epsilon,m}^{\vartheta+\varrho,0}. \tag{3.1}$$

Letting $s_n = \sum_{i+j=0}^{p} a_{n,m}^{i+j,0}$, we deduce

$$s_n \le c_1(p+1)^2 + c_2(p+1)^2 ph \sum_{\epsilon=0}^{n-1} s_\epsilon.$$

By Lemma 3.2, this yields

$$s_n \le c_1(p+1)^2 \exp\left(\sum_{\epsilon=0}^{n-1} c_2(p+1)^2 ph \right),$$

and hence

$$a_{n,m}^{i+j,0} \le c_1(p+1)^2 \exp\big(c_2(p+1)^2 p\theta_2\big) =: \gamma_1(p).$$

Step 2. Let $i+j=0$, $m = 0,\dots,\mathfrak{M}-1$, and $d = 1,\dots,r-1$. From (2.9), for all $n = 0,\dots,\mathfrak{N}_1-1$, we obtain

$$a_{n,m}^{0+0,d} \le c_1 + c_2 \sum_{\vartheta+\varrho=0}^{p-1} \int_{t_{n+1}^{d-1}}^{t-\theta_2} a_{n,m}^{\vartheta+\varrho,d-1}\,ds + c_2 \sum_{\epsilon=n+1}^{\mathfrak{N}-\mathfrak{N}_1+n-1}\sum_{\vartheta+\varrho=0}^{p-1} \int_{t_\epsilon^{d-1}}^{t_{\epsilon+1}^{d-1}} a_{\epsilon,m}^{\vartheta+\varrho,d-1}\,ds$$
$$+ c_2 \sum_{\vartheta+\varrho=0}^{p-1} \int_{t_n^d-\theta_1}^{t-\theta_1} a_{\mathfrak{N}-\mathfrak{N}_1+n,m}^{\vartheta+\varrho,d-1}\,ds.$$

From (2.10), for all $n = \mathfrak{N}_1,\dots,\mathfrak{N}-1$,

$$a_{n,m}^{0+0,d} \le c_1 + c_2 \sum_{\vartheta+\varrho=0}^{p-1} \int_{t_{n+1}^{d-1}}^{t-\theta_2} a_{n,m}^{\vartheta+\varrho,d-1}\,ds + c_2 \sum_{\epsilon=n+1}^{\mathfrak{N}-1}\sum_{\vartheta+\varrho=0}^{p-1} \int_{t_\epsilon^{d-1}}^{t_{\epsilon+1}^{d-1}} a_{\epsilon,m}^{\vartheta+\varrho,d-1}\,ds$$
$$+ c_2 \sum_{\epsilon=0}^{n-\mathfrak{N}_1-1}\sum_{\vartheta+\varrho=0}^{p-1} \int_{t_\epsilon^d}^{t_{\epsilon+1}^d} a_{\epsilon,m}^{\vartheta+\varrho,d}\,ds + c_2 \sum_{\vartheta+\varrho=0}^{p-1} \int_{t_n^d-\theta_1}^{t-\theta_1} a_{n-\mathfrak{N}_1,m}^{\vartheta+\varrho,d}\,ds.$$

Define

$\Gamma_d = \max\left\{a_{n,m}^{i+j,d} : \ i+j=0,\dots,p, \ m=0,\dots,\mathfrak{M}-1, \ n=0,\dots,\mathfrak{N}-1\right\}, d=0,\dots,r-1.$

For $i+j=1,\dots,p-1$, $m=0,\dots,\mathfrak{M}-1$, and $d=1,\dots,r-1$, relation (2.11) gives, for all $n=0,\dots,\mathfrak{N}_1-1$,

$$\begin{aligned} a_{n,m}^{i+j,d} &\le c_1 + (c_3p^3 + c_2p)h \sum_{\vartheta+\varrho=0}^{p-1} \Gamma_{d-1} + c_2ph \sum_{\epsilon=n+1}^{\mathfrak{N}-\mathfrak{N}_1+n-1} \sum_{\vartheta+\varrho=0}^{p-1} \Gamma_{d-1} \\ &\le c_1 + \left(c_2p^3\theta_2 + c_3p^5h + c_2p^3h\right)\Gamma_{d-1}. \end{aligned}$$

For $n=\mathfrak{N}_1,\dots,\mathfrak{N}-1$, relation (2.12) yields

$$a_{n,m}^{i+j,d} \le c_1 + \left(c_2p^3\theta_2 + c_3p^5h + c_2p^3h\right)\Gamma_{d-1} + c_2ph \sum_{\epsilon=0}^{n-1} \sum_{\vartheta+\varrho=0}^{p-1} a_{\epsilon,m}^{\vartheta+\varrho,d}.$$

As in Step 1, define $s_n = \sum_{i+j=0}^{p} a_{n,m}^{i+j,d}$. Then, for all $n=0,\dots,\mathfrak{N}-1$,

$$s_n \le c_4 + c_5\Gamma_{d-1} + c_6h \sum_{\epsilon=0}^{n-1} s_\epsilon, \tag{3.2}$$

where

$$c_4 = c_1(p+1)^2, \qquad c_5 = \left(c_2p^3\theta_2 + c_3p^5h + c_2p^3h\right)(p+1)^2, \qquad c_6 = c_2p(p+1)^2.$$

Applying Lemma 3.2 to (3.2), we obtain

$$s_n \le (c_4 + c_5\Gamma_{d-1})\exp\left(c_6\theta_2\right).$$

Therefore,

$$\Gamma_d \le c_7 + c_8 \sum_{\epsilon=0}^{d-1} \Gamma_\epsilon,$$

where $c_7 = c_4\exp(c_6\theta_2)$ and $c_8 = c_5\exp(c_6\theta_2)$.

Applying Lemma 3.2 once more gives

$$a_{n,m}^{i+j,d} \le \Gamma_d \le c_7\exp(c_8d) =: \gamma_2(p).$$

Finally, by defining $\gamma(p) = \max\{\gamma_1(p), \gamma_2(p)\}$, the proof of Lemma 3.3 is concluded.

□

The next theorem proves the convergence of the proposed method.

Theorem 3.4. *Suppose that the functions g and K have continuous derivatives up to order p on their corresponding domains. Then, the discrete problem generated by (2.3) and (2.8) has a unique solution*

$$v \in S_{p-1}^{(-1)}\left(\Pi_{\mathfrak{N},\mathfrak{M}}\right).$$

Moreover, if the error is defined by $e(t,y) = u(t,y) - v(t,y)$, then

$$\|e\|_{L^\infty(D)} \le C(h+k)^p,$$

where $C > 0$ is independent of the mesh sizes h and k.

Proof. Define the local error on each subdomain $D^d_{n,m}$ by

$e^d_{n,m}(t,y) = u(t,y) - v^d_{n,m}(t,y), \qquad n = 0,\dots,\mathfrak{N}-1,\ m = 0,\dots,\mathfrak{M}-1,\ d = 0,\dots,r-1.$

The proof is divided into two steps. *Step 1.* We show that there exists a constant $C_1 > 0$, independent of h and k, such that

$$\|e^0_{n,m}\|_{L^\infty(D^0_{n,m})} \le C_1(h+k)^p.$$

Let $(t,y) \in D^0_{n,m}$, with $n = 0,\dots,\mathfrak{N}_1 - 1$ and $m = 0,\dots,\mathfrak{M} - 1$. Using Lemma 3.1, we obtain

$$\|e^0_{n,m}\| \le \sum_{i+j=p} \frac{1}{i!\,j!} \left\| \frac{\partial^{i+j} \hat{v}^0_{n,m}}{\partial t^i \partial y^j} \right\| h^i k^j,$$

where $\hat{v}^0_{0,m} = u$ and, for $n \neq 0$,

$$\hat{v}^0_{n,m}(t,y) = g(t,y) + \int_{t-\theta_2}^{t-\theta_1} K(t,y,s)\,\phi(s,y)\,ds.$$

By Lemma 3.3, it follows that

$$\|e^0_{n,m}\| \le \gamma(p) \sum_{i+j=p} \frac{h^i k^j}{i!\,j!} = \underbrace{\frac{\gamma(p)}{p!}}_{C_1}(h+k)^p. \tag{3.3}$$

For $n = \mathfrak{N}_1,\dots,\mathfrak{N}-1$, relation (2.5) yields

$$\begin{aligned} u(t,y) - \hat{v}^0_{n,m}(t,y) = &\sum_{\epsilon=0}^{n-\mathfrak{N}_1-1} \int_{t^0_\epsilon}^{t^0_{\epsilon+1}} K(t,y,s)\big(u(s,y) - v^0_{\epsilon,m}(s,y)\big)\,ds \\ &+ \int_{t^0_n-\theta_1}^{t-\theta_1} K(t,y,s)\big(u(s,y) - v^0_{n-\mathfrak{N}_1,m}(s,y)\big)\,ds. \end{aligned}$$

Hence,

$$|u(t,y) - \hat{v}^0_{n,m}(t,y)| \le 2h\,\overline{K} \sum_{\epsilon=0}^{n-1} \|e^0_{\epsilon,m}\|_{L^\infty(D^0_{\epsilon,m})},$$

where $\overline{K} = \|K\|_{L^\infty(D)}$. Therefore,

$$\begin{aligned} \|u - v^0_{n,m}\| &\le \|u - \hat{v}^0_{n,m}\| + \|\hat{v}^0_{n,m} - v^0_{n,m}\| \\ &\le 2h\,\overline{K} \sum_{\epsilon=0}^{n-1} \|e^0_{\epsilon,m}\|_{L^\infty(D^0_{\epsilon,m})} + \frac{\gamma(p)}{p!}(h+k)^p. \end{aligned}$$

Applying Lemma 3.2, we obtain

$$\begin{aligned} \|e^0_{n,m}\|_{L^\infty(D^0_{n,m})} &\le \frac{\gamma(p)}{p!}(h+k)^p \exp\left(\sum_{\epsilon=0}^{n-1} 2h\,\overline{K}\right) \\ &\le \underbrace{\frac{\gamma(p)}{p!}\exp(2\theta_2\overline{K})}_{C_1}(h+k)^p. \end{aligned}$$

Step 2. We now prove that there exists a constant $C_2 > 0$, independent of h and k, such that

$$\|e^d_{n,m}\|_{L^\infty(D^d_{n,m})} \leq C_2(h+k)^p, \qquad d = 1, \ldots, r-1.$$

Let

$$\|e^d\| = \max\{\|e^d_{n,m}\|_{L^\infty(D^d_{n,m})} :\ n = 0, \ldots, \mathfrak{N}-1,\ m = 0, \ldots, \mathfrak{M}-1\}, \qquad d = 0, \ldots, r-1,$$

and fix $(t, y) \in D^d_{n,m}$.

For $n = 0, \ldots, \mathfrak{N}_1 - 1$, using (2.9) we have

$$\begin{aligned} u(t,y) - \hat{v}^d_{n,m}(t,y) &= \int_{t-\theta_2}^{t^{d-1}_{n+1}} K(t,y,s)\big(u(s,y) - v^{d-1}_{n,m}(s,y)\big)\, ds \\ &\quad + \sum_{\epsilon=n+1}^{\mathfrak{N}-\mathfrak{N}_1+n-1} \int_{t^{d-1}_\epsilon}^{t^{d-1}_{\epsilon+1}} K(t,y,s)\big(u(s,y) - v^{d-1}_{\epsilon,m}(s,y)\big)\, ds \\ &\quad + \int_{t^d_n-\theta_1}^{t-\theta_1} K(t,y,s)\big(u(s,y) - v^{d-1}_{\mathfrak{N}-\mathfrak{N}_1+n,m}(s,y)\big)\, ds, \end{aligned}$$

which yields

$$|u(t,y) - \hat{v}^d_{n,m}(t,y)| \leq 3\theta_2\, \overline{K}\, \|e^{d-1}\|. \tag{3.4}$$

For $n = \mathfrak{N}_1, \ldots, \mathfrak{N} - 1$, relation (2.10) gives

$$\begin{aligned} u(t,y) - \hat{v}^d_{n,m}(t,y) &= \int_{t-\theta_2}^{t^{d-1}_{n+1}} K(t,y,s)\big(u(s,y) - v^{d-1}_{n,m}(s,y)\big)\, ds \\ &\quad + \sum_{\epsilon=n+1}^{\mathfrak{N}-1} \int_{t^{d-1}_\epsilon}^{t^{d-1}_{\epsilon+1}} K(t,y,s)\big(u(s,y) - v^{d-1}_{\epsilon,m}(s,y)\big)\, ds \\ &\quad + \sum_{\epsilon=0}^{n-\mathfrak{N}_1-1} \int_{t^d_\epsilon}^{t^d_{\epsilon+1}} K(t,y,s)\big(u(s,y) - v^d_{\epsilon,m}(s,y)\big)\, ds \\ &\quad + \int_{t^d_n-\theta_1}^{t-\theta_1} K(t,y,s)\big(u(s,y) - v^d_{n-\mathfrak{N}_1,m}(s,y)\big)\, ds, \end{aligned}$$

implying

$$|u(t,y) - \hat{v}^d_{n,m}(t,y)| \leq 2\theta_2\, \overline{K}\, \|e^{d-1}\| \ +\ h\,\overline{K} \sum_{\epsilon=0}^{n-1} \|e^d_{\epsilon,m}\|. \tag{3.5}$$

Combining (3.4) and (3.5), for all $n = 0, \ldots, \mathfrak{N} - 1$,

$$\begin{aligned} \|u - v^d_{n,m}\| &\leq \|u - \hat{v}^d_{n,m}\| + \|\hat{v}^d_{n,m} - v^d_{n,m}\| \\ &\leq 2\theta_2\, \overline{K}\, \|e^{d-1}\| + h\,\overline{K} \sum_{\epsilon=0}^{n-1} \|e^d_{\epsilon,m}\| + \frac{\gamma(p)}{p!}(h+k)^p. \end{aligned}$$

Applying Lemma 3.2, we obtain

$$\|u - v^d_{n,m}\| \leq \left(2\theta_2\, \overline{K}\, \|e^{d-1}\| + \frac{\gamma(p)}{p!}(h+k)^p\right) \exp(\theta_2 \overline{K}),$$

which implies

$$\|e^d\| \le \lambda_1 \theta_2 \sum_{\epsilon=0}^{d-1} \|e^\epsilon\| + \lambda_2 (h+k)^p,$$

where

$$\lambda_1 = 2\overline{K}\exp(\theta_2 \overline{K}), \qquad \lambda_2 = \frac{\gamma(p)}{p!}\exp(\theta_2 \overline{K}).$$

Applying Lemma 3.2 once again yields

$$\|e^d\| \le \lambda_2 (h+k)^p \exp(\lambda_1 T),$$

and consequently,

$$\|e^d_{n,m}\|_{L^\infty(D^d_{n,m})} \le \underbrace{\lambda_2 \exp(\lambda_1 T)}_{C_2}(h+k)^p,$$

for all n, m, d. This completes Step 2 and thus the proof of the theorem. □

4. Computational Experiments

In this section, we present two test problems to illustrate the performance of the proposed Taylor collocation method.

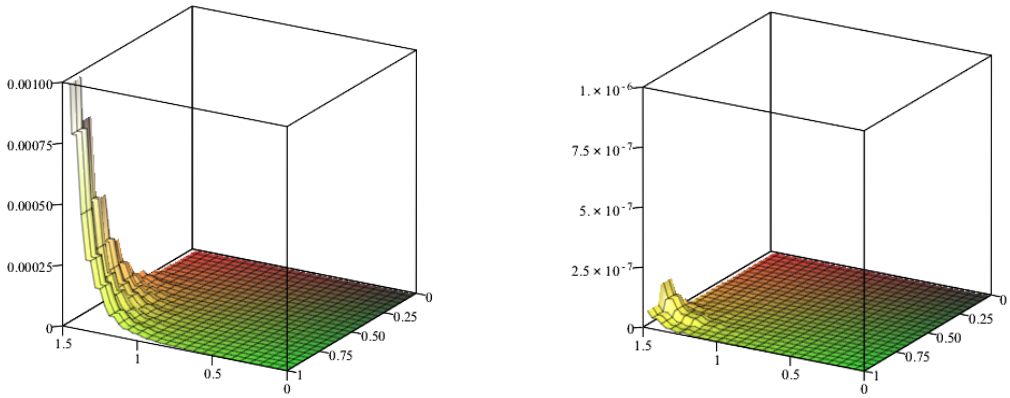
Example 4.1. The problem under consideration is a linear 2D Volterra integral equation of the form

$$u(t,y) = g(t,y) + \int_{t-0.5}^{t-0.1} (s + yt^5)\, u(s,y)\, ds,$$

for $(t,y) \in \left[\frac{1}{2}, \frac{3}{2}\right] \times [0,1]$. The function g is chosen such that the exact solution is given by $u(t,y) = y\sin(t)$, while on the initial interval $[0, \frac{1}{2}] \times [0,1]$ we set $u(t,y) = \Phi(t,y)$. We apply the Taylor collocation method to this equation. Table 1 reports the absolute errors at selected points for polynomial degrees $p = 3$ and $p = 6$, using the mesh parameters $\mathfrak{N} = \mathfrak{M} = 10$. The corresponding absolute error surfaces are depicted in Figure 1.

Table 1. Absolute errors for $(\mathfrak{N}, \mathfrak{M}) = (10, 10)$ of Example 4.1

(t,y)	$p = 3$	$p = 6$
(0.5, 0.0)	0	0
(0.6, 0.2)	3.65×10^{-11}	4.38×10^{-11}
(0.7, 0.4)	1.13×10^{-7}	6.44×10^{-13}
(0.8, 0.6)	4.24×10^{-7}	1.87×10^{-10}
(0.9, 0.8)	1.18×10^{-6}	3.40×10^{-10}
(1.1, 0.2)	3.84×10^{-7}	1.34×10^{-10}
(1.2, 0.4)	1.36×10^{-6}	1.54×10^{-10}
(1.3, 0.6)	4.00×10^{-6}	2.23×10^{-10}
(1.4, 0.8)	1.16×10^{-5}	2.62×10^{-9}
(1.5, 1.0)	5.10×10^{-4}	2.19×10^{-9}



(a) Error surface for $p = 3$ (b) Error surface for $p = 6$

FIGURE 1. Absolute error plots for Example 4.1

Example 4.2. Let us consider the following linear 2D VIE.

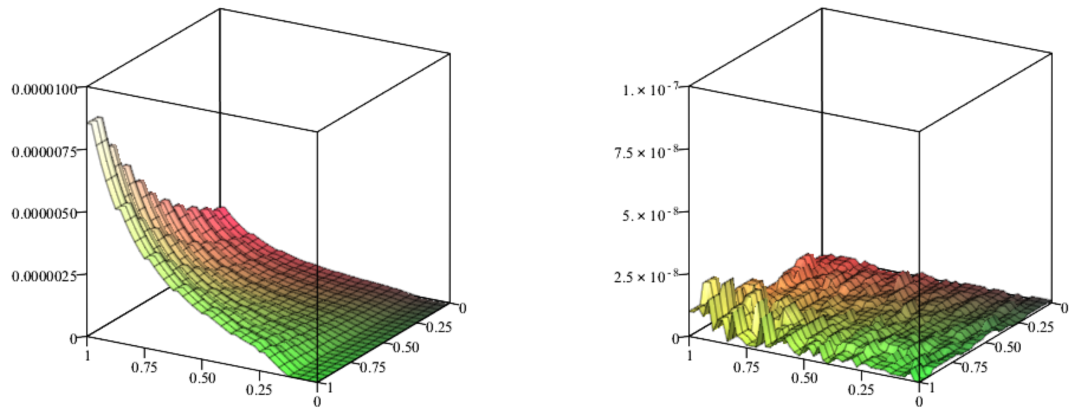
$$u(t, y) = g(t, y) + \int_{t-0.25}^{t-0.1} \big(t^5 + e^y + s\big)\, u(s, y)\, ds,$$

for $(t, y) \in \big[\frac{1}{4}, 1\big] \times [0, 1]$. The function g is selected such that the exact solution is $u(t, y) = e^{t+y}$. On the initial interval $[0, \frac{1}{4}] \times [0, 1]$, the solution is prescribed by $u(t, y) = \Phi(t, y)$.

We apply the Taylor collocation method to this equation. Table 2 lists the absolute errors at selected points for polynomial degrees $p = 3$ and $p = 6$ on a uniform mesh with $\mathfrak{N} = \mathfrak{M} = 10$. The corresponding error surfaces are displayed in Figure 2.

TABLE 2. Absolute errors for $(\mathfrak{N}, \mathfrak{M}) = (10, 10)$ of Example 4.2

(t, y)	$p = 3$	$p = 6$
(0.25, 0.0)	2.87×10^{-10}	1.38×10^{-10}
(0.30, 0.2)	5.80×10^{-11}	4.68×10^{-10}
(0.35, 0.4)	9.93×10^{-10}	2.69×10^{-10}
(0.50, 0.0)	1.84×10^{-7}	6.50×10^{-10}
(0.60, 0.4)	4.88×10^{-7}	2.04×10^{-10}
(0.70, 0.8)	1.40×10^{-6}	5.60×10^{-9}
(0.75, 0.0)	3.74×10^{-7}	6.85×10^{-10}
(0.80, 0.2)	6.01×10^{-7}	4.74×10^{-10}
(0.90, 0.6)	1.62×10^{-6}	7.53×10^{-11}
(0.95, 0.8)	2.73×10^{-6}	8.64×10^{-10}



(a) Error surface for $p = 3$ (b) Error surface for $p = 6$

FIGURE 2. Absolute error plots for Example 4.2

5. CONCLUSION

In this work, the Taylor collocation method was successfully extended to the numerical solution of linear 2D VIEs with two distinct delays. The method constructs local polynomial approximations of the unknown function on each subdomain and enforces the integral equation at collocation points, which leads to a solvable algebraic system. Theoretical analysis confirms the convergence and stability of the scheme under natural smoothness conditions on the data. Numerical experiments demonstrate that the method is accurate and efficient. Owing to its simplicity, the approach can be adapted to more complex delayed models, including nonlinear or higher-dimensional cases.

ACKNOWLEDGEMENTS

The authors sincerely thank the reviewers for their constructive remarks and helpful suggestions, which contributed to improving the quality and presentation of this manuscript.

REFERENCES

1. Bélair, J. (2020). Population models with state-dependent delays. *Mathematical Population Dynamics*, 165–176. https://doi.org/10.1201/9781003072706
2. Birem, F., Boulmerka, A., & Laib, H. (2025). Efficient collocation algorithm for solving three-dimensional linear and nonlinear Volterra integral equations. *International Journal of Computational Methods*, 22(05), 2450077. https://doi.org/10.1142/S0219876224500774
3. Bouzeraieb, H., Laib, H., & Boulmerka A.(2024). Numerical solution of neutral double delay Volterra integral equations using Taylor collocation method. *Nonlinear Dynamics and Systems Theory*, 24(3), 236–245. https://e-ndst.kiev.ua/v24n3/3(93).pdf
4. Brunner, H. (2004). Collocation methods for Volterra integral and related functional differential equations. *Cambridge University Press*. https://doi.org/10.1017/CBO9780511543234

5. Chettouh, A. (2025). Asymptotic behavior for the Lotka–Volterra equation with displacements and diffusion. *Ural Mathematical Journal*, 11(1), 46–62. https://doi.org/10.15826/umj.2025.1.003
6. Conti, M., Gatti, S., Grasselli, M., & Pata, V.(2010). Two-dimensional reaction–diffusion equations with memory. *Quarterly of Applied Mathematics*, 68(4), 607–643. https://doi.org/10.1090/S0033-569X-2010-01167-7
7. Cooke, K. L.(1976). An epidemic equation with immigration. *Mathematical Biosciences*, 29(1-2), 135–158. https://doi.org/10.1016/0025-5564(76)90033-X
8. Dolgii, Y. F.(2023). Canonical approximations in impulse stabilization for a system with aftereffect. *Ural Mathematical Journal*, 9(2 (17)), 77–85. https://doi.org/10.15826/umj.2023.2.006
9. Efut, O. A., Effiong, U. U., & Ikechi, I. D.(2021). A robust iterative approach for solving nonlinear Volterra delay integro-differential equations. *Ural Mathematical Journal*, 7(2 (13)), 59–85. https://doi.org/10.15826/umj.2021.2.005
10. Hafiane, N., Labed, B., & Yakhlef, S. (2026). Maximum principle for delayed Volterra jumps systems using anticipating adjoints. *Gulf Journal of Mathematics*, 22(1), 1–25. https://doi.org/10.56947/gjom.v22i1.3895
11. Khajehnasiri, A. A., Ezzati, R., & Kermani, M. A.(2021). Solving fractional two-dimensional nonlinear partial Volterra integral equation by using Bernoulli wavelet. *Iranian Journal of Science and Technology, Transactions A: Science*, 45(3), 983–995. https://doi.org/10.1007/s40995-021-01078-4
12. Laib, H., Bellour, A., & Boulmerka, A.(2022). Taylor collocation method for a system of nonlinear Volterra delay integro-differential equations with application to COVID-19 epidemic. *International Journal of Computer Mathematic*, 99(4), 852–876. https://doi.org/10.1080/00207160.2021.1938012
13. Mansouri, L., & Azimzadeh, Z.(2023). Numerical solution of fractional delay Volterra integro-differential equations by Bernstein polynomials. *Mathematical sciences*, 17(4), 455–466. https://doi.org/10.1007/s40096-022-00463-3
14. Ngwa, G. A., Niger, A. M., & Gumel, A. B. (2010). Mathematical assessment of the role of non-linear birth and maturation delay in the population dynamics of the malaria vector. *Applied Mathematics and Computation*, 217(7), 3286–3313. https://doi.org/10.1016/j.amc.2010.08.062
15. Pachpatte, B.G.(2006). Integral and finite difference inequalities and applications. *Elsevier*. https://shop.elsevier.com/books/integral-and-finite-difference-inequalities\-and-applications/pachpatte/978-0-444-52762-2
16. Paikaray, P., Beuria, S., & Parida, N. C. (2025). A numerical technique to solve linear stochastic Volterra integral equation. *Gulf Journal of Mathematics*, 20(2), 39–56. https://doi.org/10.56947/gjom.v20i2.3046
17. Smith, H. L., & Waltman, P.(1995). The theory of the chemostat: dynamics of microbial competition. *Cambridge University Press*. https://doi.org/10.1017/CBO9780511530043

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