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NORMALIZED B-SPLINE CONSTRUCTION FOR SMOOTH SPLINES AND QUASI-INTERPOLATION

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ABSTRACT. In this paper, we introduce a general framework for constructing univariate spline spaces defined on refined non-uniform partitions with variable knot insertion. The proposed approach is developed for spline spaces characterized by a prescribed level of smoothness and a corresponding range of polynomial degrees. Within this setting, we establish suitable continuity conditions at the inserted knots and develop a fully local Hermite interpolation scheme. We then construct a normalized B-spline-type basis consisting of non-negative, compactly supported functions that form a partition of unity. By means of blossoming techniques, we derive adapted B-spline-type representations and design quasi-interpolants exhibiting superconvergence properties. The proposed operators provide high-order accuracy for approximation on non-uniform meshes, as confirmed by theoretical analysis and numerical experiments.

Keywords. Splines, Hermite interpolation, Quasi-interpolation, Superconvergence, Polar form, Finite element
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1. INTRODUCTION

Spline quasi-interpolants are widely used in approximation theory and scientific computing because of their local character, simplicity, and low computational cost. In contrast to classical interpolation techniques, their construction does not require solving global linear systems, which makes them particularly suitable for practical applications. Their performance, however, depends strongly on the choice of the underlying basis functions. For stability and efficient local control, these functions are generally required to be non-negative, compactly supported, and to form a partition of unity. Classical B-splines satisfy these fundamental properties and are therefore commonly adopted as normalized bases in spline theory. Nevertheless, when higher smoothness is enforced or additional knots are introduced, the induced refinements often lead to non-uniform partitions, which may reduce the effectiveness of standard B-spline constructions in fully local settings or in the design of accurate quasi-interpolants.

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To overcome these limitations, several approaches have been proposed. Schumaker [10] introduced a C^1 convex interpolation scheme based on quadratic splines enriched by inserting an additional knot in each subinterval, illustrating how knot insertion can be used to incorporate Hermite-type data. Other related constructions have been investigated, although they do not always preserve key properties such as non-negativity or the partition of unity. Hermite spline interpolation has also been extensively studied within the classical framework of B-splines and uniform partitions (see, e.g., [4, 11, 6]), with particular attention devoted to specific cases such as cubic Hermite splines [2, 3]. More recently, Barrera et al. [1] proposed a geometric construction based on the Bernstein–Bézier representation, inspired by Powell–Sabin and Clough–Tocher refinements. Their method relies on inserting a single interior point in each subinterval, leading to a uniform subdivision into two micro-intervals and allowing the construction of normalized B-spline bases of degrees $2k$ and $2k+1$ with C^k smoothness. However, this approach remains limited to a fixed refinement pattern and does not naturally extend to general non-uniform partitions or to a broader range of spline degrees.

In this paper, we develop a general framework for constructing normalized spline bases over arbitrarily refined non-uniform partitions. More precisely, in each interval $[x_i, x_{i+1}]$, we allow the insertion of $L-1$ interior knots, where $L \geq 2$ may vary from one subinterval to another. This flexibility enables the definition of spline spaces adapted to local approximation requirements and significantly generalizes the classical uniform setting. Within this framework, we consider spline functions of global smoothness C^k and degree $2k+1-j$ $(j = 1, \dots, k)$, thereby extending the classical case where only the degree $2k+1$ is usually considered. An intermediate smoothness $C^{\mathcal{M}}$, with $k \leq \mathcal{M} \leq 2k-j$, is imposed at the inserted knots, ensuring that the associated Hermite interpolation problem is well-posed and admits a local formulation.

We then study the Hermite interpolation problem defined by prescribing function values and derivatives up to order k at the initial partition knots. A fully local construction is proposed to obtain Hermite interpolants in spline spaces of degree $2k+1-j$, taking advantage of the refined non-uniform structure and the imposed continuity conditions. This approach applies to the entire family of admissible degrees and to general non-uniform refinements, thus extending classical results that are restricted to uniform partitions and to the degree $2k+1$.

Building on our previous works [7, 8], where normalized B-spline bases were constructed in specific cases (namely C^1 quadratic and C^2 cubic splines), we extend these results to the general family of degrees $2k+1-j$ for $k \geq 1$. The proposed construction combines local Hermite-type arguments with Bernstein–Bézier techniques, leading to B-spline-type representations for piecewise polynomials with arbitrary C^k smoothness. Moreover, an explicit Marsden identity is derived in the context of refined non-uniform partitions.

Finally, we construct discrete quasi-interpolants exhibiting superconvergence properties. For spline spaces of degree $m \geq 2k+1-j$, the proposed operators achieve an approximation order of $m+1$ at the initial knots, while the approximation of the r-th derivative, for $1 \leq r \leq k$, is of order $m-r+1$. These results demonstrate the effectiveness of the proposed framework for high-accuracy numerical approximation, particularly in applications involving differential and integral equations.

The paper is organized as follows. Section 2 introduces the family of spline spaces defined on refined partitions. Section 3 is devoted to the construction of normalized B-spline bases. In Section 4, we derive a B-spline-type representation of the Hermite interpolant. Section 5 presents the construction of quasi-interpolants, while Section 6 is dedicated to the corresponding error estimates.

2. A set of smooth spline spaces defined across the partition τ_L

2.1. **Finite element of class C^k.** Let τ denote a non-uniform partition of the interval $I := [a, b]$,

$$\tau := (a = x_0 < x_1 < \cdots < x_n = b),$$

where $h_i = x_{i+1} - x_i$ denotes the length of the subinterval $[x_i, x_{i+1}]$. We assume that the values of a function f and of its derivatives up to order k are prescribed at the knots x_i, $i = 0, \ldots, n$.

Our objective is to construct a C^k spline of degree $2k + 1 - j$ $(1 \leq j \leq k)$ over the partition τ and to derive its local representation on each subinterval $[x_i, x_{i+1}]$. The local construction is based on the values of f and its derivatives at the endpoints x_i and x_{i+1}. Since these data alone are not sufficient to uniquely determine a polynomial of degree $2k+1-j$ on $[x_i, x_{i+1}]$, the local representation must be defined in a piecewise polynomial setting.

To this end, the partition τ is refined by inserting additional knots inside each interval (x_i, x_{i+1}) and by imposing a higher smoothness $C^{\mathcal{M}}$, with $k \leq \mathcal{M} \leq 2k - j$, at the inserted knots. The resulting refined partition is denoted by τ_L.

More precisely, we introduce $(L - 1)$ interior knots $x_{i,s}$ in $[x_i, x_{i+1}]$, for $s = 1, \ldots, L - 1$ and $L \in \mathbb{N}$, and define

$$\tau_{i,L} := (x_i < x_{i,1} < \cdots < x_{i,L-1} < x_{i+1})$$

as a subdivision of $[x_i, x_{i+1}]$ into L subintervals. Let $\mathcal{S}_i$ denote a spline of degree $2k+1-j$ and class C^k defined on $[x_i, x_{i+1}]$. At each inserted knot $x_{i,s}$, the spline $\mathcal{S}_i$ is assumed to satisfy $C^{\mathcal{M}}$ continuity, so that

$$\mathcal{S}_i \in V_{\mathcal{M}} \subset \mathbb{P}^k_{2k+1-j}([x_i, x_{i+1}], \tau_{i,L}).$$

Since the local spline $\mathcal{S}_i$ is required to be fully determined by the data prescribed at the endpoints x_i and x_{i+1}, it must be uniquely specified by $2k + 2$ degrees of freedom. Consequently, the dimension of the space $V_{\mathcal{M}}$ must satisfy

$$\dim V_{\mathcal{M}} = L(2k + 2 - j) - (L - 1)(\mathcal{M} + 1) = 2k + 2,$$

with $\mathcal{M} \neq 2k + 1 - j$. This condition leads to

$$L = \frac{2k + 1 - \mathcal{M}}{2k + 1 - \mathcal{M} - j}. \tag{2.1}$$

Hence, there exists a direct relationship between the imposed smoothness $\mathcal{M}$ at the inserted knots and the number of knots introduced in each interval $[x_i, x_{i+1}]$.

2.2. **Hermite interpolation problem.** In this paper, we focus on spline space $\mathbb{P}_d^k(I, \tau_L)$, of any desired level of smoothness $k \geq 1$ and

$$d = 2k + 1 - j.$$

We denote it by $\mathbb{P}_d^k(I, \tau_L)$. For $i = 0, \dots, n-1$ and $s = 1, \dots, L-1$ define

$$w_{i,1}^s := \frac{\mathcal{M}}{d}\, x_{i,s-1} + \frac{d-\mathcal{M}}{d}\, x_{i,s}, \qquad w_{i,2}^s := \frac{d-\mathcal{M}}{d}\, x_{i,s} + \frac{\mathcal{M}}{d}\, x_{i,s+1}.$$

Lemma 2.1. *The intervals* $[w_{i,1}^{s+1}, w_{i,2}^s]$ *are nonempty.*

Proof. The difference gives

$$\begin{aligned} w_{i,2}^s - w_{i,1}^{s+1} &= \frac{2k+1-j-\mathcal{M}}{2k+1-j} x_{i,s} + \frac{\mathcal{M}-2k-1+j}{2k+1-j} x_{i,s+1} \\ &= (x_{i,s+1} - x_{i,s}) \left(\frac{2\mathcal{M}-2k-1+j}{2k+1-j} \right). \end{aligned}$$

If $j = 1$, we obtain $L = 2$ where we insert a single knot that means we have just the interval $[w_{i,1}^1, w_{i,2}^1]$. Else for $j > 1$, we obtain

$$2k + 1 - j \leq 2k \leq 2\mathcal{M}. \tag{2.2}$$

Using (2.2) and $x_{i,s} < x_{i,s+1}$ to obtain $w_{i,2}^s > w_{i,1}^{s+1}$. □

Corollary 2.2. *The number of B-coefficients in the Bernstein-Bézier representation of* $\mathcal{S}$ *on* $[w_{i,1}^{s+1}, w_{i,2}^s]$ *is* $2(\mathcal{M} - k) + j$ *for* $j = 2, \dots, k$.

For a given data $f^{(r)}(x_i)$, $i = 0, \dots, n$, $r = 0, \dots, k$, the Hermite interpolation problem seeks to determine a piecewise polynomial function $\mathcal{S}$ with C^k continuity and degree d, such that

$$\mathcal{S}^{(r)}(x_i) = f^{(r)}(x_i). \tag{2.3}$$

Remark 2.3. For clarity, let k be a non-negative integer. For an interval $[w_1, w_2]$, we define the segments $\mathcal{J}_k^d(w_1)$ and $\mathcal{J}_k^d(w_2)$ as the sets of $k+1$ Bézier points having w_1 and w_2 as endpoints, respectively. More precisely, for $i \in \{1, 2\}$,

$$\mathcal{J}_k^d(w_i) := \left\{ \xi_t \;\middle|\; \xi_t = \frac{d-t}{d}\, w_i + \frac{t}{d}\, w_{3-i}, \quad t = 0, \dots, k \right\}.$$

The points ξ_t are referred to as Bézier points. A B-coefficient associated with a segment $\mathcal{J}_k^d(w_i)$ is a Bézier coefficient whose corresponding Bézier point lies in that segment.

In the subsequent theorem, we will address the Hermite interpolation problem (2.3) based on the number of knot insertions

To enhance clarity and readability, we begin by establishing that the problem is well-posed. Indeed, the existence and uniqueness of the spline solution are guaranteed by a dimension-counting argument, since the number of interpolation conditions coincides with the dimension of the spline space, as indicated by relation (2.1). It therefore remains to construct an explicit method for computing the B-coefficients.

Theorem 2.4. *For any given values $f^{(r)}(x_i)$, with $r = 0, \dots, k$ and $i = 0, \dots, n$, there exists a unique spline $\mathcal{S} \in \mathbb{P}_d^k(I, \tau_L)$ satisfying* (2.3).

Proof. We provide a clear construction of the spline $\mathcal{S}(x) \in \mathbb{P}_d^k(I, \tau_2)$ that satisfies (2.3) using its Bernstein-Bézier representation. To achieve this, we focus on each sub-interval $[x_i, x_{i+1}]$. Within each of these sub-intervals, the spline can be represented as the polynomial of degree d using the Bernstein-Bézier formulation. Figure 1 illustrates the Bernstein-Bézier representation of the spline $\mathcal{S}$ on $[x_i, x_{i+1}]$.

We now show that the B-coefficients in the Bernstein-Bézier representation on $[x_i, x_{i+1}]$ are uniquely determined by the interpolation conditions (2.3) and the smoothness conditions $C^{\mathcal{M}}$ at knot $x_{i,1}$. Based on the Bernstein-Bézier theory and the C^k smoothness at knots x_i and x_{i+1}, we can conclude that the derivatives (2.3) at these knots uniquely determine the B-coefficients that correspond to the Bézier points in the segments $\mathcal{J}_k^d(x_i)$ and $\mathcal{J}_k^d(x_{i+1})$. These segments have x_i and x_{i+1} as their endpoints in the intervals $[x_i, x_{i,1}]$ and $[x_{i,1}, x_{i+1}]$, respectively. In Figure 1, these Bézier points are denoted by red bullets (•).

$*$ **For $L = 2$:**
Due to the $C^{\mathcal{M}}$ smoothness at knot $x_{i,1}$, we can consider the B-coefficients of $\mathcal{S}$ in the segment $\mathcal{J}_{\mathcal{M}}^d(x_{i,1})$, having $x_{i,1}$ as endpoint in $[x_i, x_{i,1}]$ and $[x_{i,1}, x_{i+1}]$ as the B-coefficients (after subdivision) associated with a single polynomial $p_{i,\mathcal{M}}$ of degree $\mathcal{M}$ defined on the interval $[w_{i,1}^1, w_{i,2}^1]$ (see Figure 1).

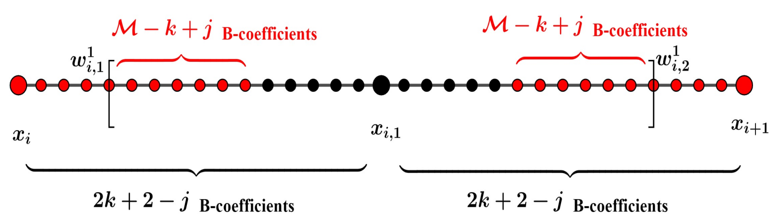

FIGURE 1. Representation of the B-coefficients of $\mathcal{S}(x) \in \mathbb{P}_d^k(I, \tau_2)$ on $[x_i, x_{i+1}]$.

We observe that the B-coefficients related to the Bézier points in $\mathcal{J}_{\mathcal{M}-k+j-1}^{\mathcal{M}}(w_{i,1}^1)$ and $\mathcal{J}_{\mathcal{M}-k+j-1}^{\mathcal{M}}(w_{i,2}^1)$, having $w_{i,1}^1$ and $w_{i,2}^1$ as endpoints in the intervals $[w_{i,1}^1, x_{i,1}]$ and $[x_{i,1}, w_{i,2}^1]$, respectively, are already determined by our construction. By considering only the Bézier points in the interval $[w_{i,1}, w_{i,2}^1]$, we have $2(\mathcal{M} - k + j)$ independent constraints that completely specify the polynomial $p_{i,\mathcal{M}}$ of degree $\mathcal{M}$, whose number of degrees of freedom is $\mathcal{M} + 1$. These constraints can be verified in Figure 1 by examining the intersection of the red bullets and the bullets in these intervals. Indeed, from (2.1), we have

$$\frac{2k + 1 - \mathcal{M}}{2k + 1 - \mathcal{M} - j} = 2.$$

Then $2k+1-\mathcal{M} = 2(2k+1-\mathcal{M}-j)$, which gives

$$\mathcal{M} - 2k + 2j - 1 = 0. \tag{2.4}$$

Therefore $2(\mathcal{M}-k+j) = \mathcal{M}+1$.

$*$ **For** $L = 3$ **:**

By the $C^{\mathcal{M}}$ smoothness at knot $x_{i,1}$ (resp. $x_{i,2}$), we can regard the B-coefficients of $\mathcal{S}$ in the segment $\mathcal{J}^d_{\mathcal{M}}(x_{i,1})$, with $x_{i,1}$ as an endpoint in both intervals $[x_i, x_{i,1}]$ and $[x_{i,1}, x_{i,2}]$ (resp. $\mathcal{J}^d_{\mathcal{M}}(x_{i,2})$, with $x_{i,2}$ as an endpoint in both intervals $[x_{i,1}, x_{i,2}]$ and $[x_{i,2}, x_{i+1}]$), as the B-coefficients (after subdivision) of a single polynomial $p^1_{i,\mathcal{M}}$ (resp. $p^2_{i,\mathcal{M}}$) of degree $\mathcal{M}$ defined on the interval $[w^1_{i,1}, w^1_{i,2}]$ (resp. $[w^2_{i,1}, w^2_{i,2}]$), considering only the Bézier points in these intervals.

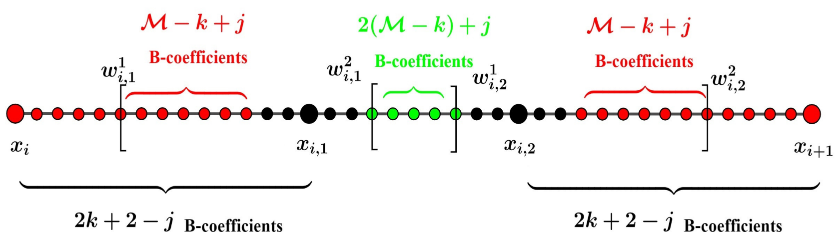

FIGURE 2. Representation of the B-coefficients of $\mathcal{S}(x) \in \mathbb{P}^k_d(I, \tau_3)$ on $[x_i, x_{i+1}]$.

We observe that the B-coefficients associated with the Bézier points within the segments :

- $\mathcal{J}^{\mathcal{M}}_{\mathcal{M}-k+j-1}(w^1_{i,1})$, having $w^1_{i,1}$ as an endpoint in the interval $[w^1_{i,1}, x_{i,1}]$,
- $\mathcal{J}^{\mathcal{M}}_{\mathcal{M}-k+j-1}(w^2_{i,2})$, having $w^2_{i,2}$ as an endpoint in the interval $[x_{i,2}, w^2_{i,2}]$,

are already established by our construction. This can be confirmed in Figure 2 by examining the intersection between the red bullets and the bullets within these intervals. This gives us $2(\mathcal{M}-k+j)$ independent constraints. They fully define polynomials $p^1_{i,\mathcal{M}}$ and $p^2_{i,\mathcal{M}}$, both of degree $\mathcal{M}$, each with a number of degrees of freedom equal to $\mathcal{M}+1$. Indeed, since $L=3$ and from (2.1) we have $2k+1-\mathcal{M} = 3(2k+1-\mathcal{M}-j)$, which gives

$$2\mathcal{M} - 2k + j = 2(k-j+1), \tag{2.5}$$

i.e., The number of B-coefficients in $[w^2_{i,1}, w^1_{i,2}]$ (indicated in Figure 2 by the green bullets $\bullet$) is equal to the number of B-coefficients required. Then

$$2(\mathcal{M}-k+j) + 2(k-j+1) = 2(\mathcal{M}+1).$$

$*$ **For** $L \geq 4$ **:**

By the $C^{\mathcal{M}}$ smoothness at knots $x_{i,s}$, $s = 1, \ldots, L-1$, we can regard the B-coefficients of $\mathcal{S}$ in each $\mathcal{J}^d_{\mathcal{M}}(x_{i,s})$, having $x_{i,s}$ as an endpoint in both intervals $[x_{i,s-1}, x_{i,s}]$ and $[x_{i,s}, x_{i,s+1}]$, as the B-coefficients associated with a single polynomial $p^s_{i,\mathcal{M}}$ each of degree $\mathcal{M}$ defined over the interval $[w^s_{i,1}, w^s_{i,2}]$ (considering only Bézier points in these intervals).

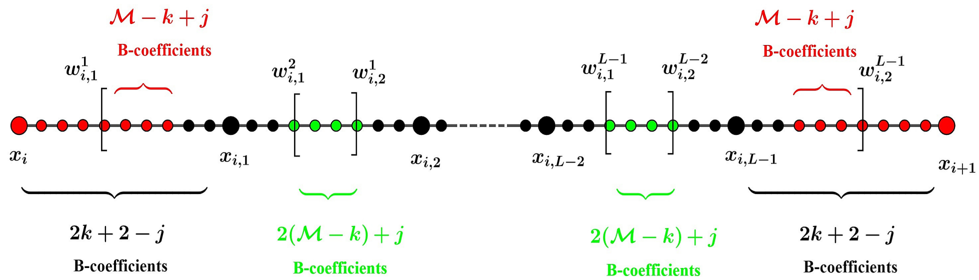

FIGURE 3. Representation of the B-coefficients of $\mathcal{S}(x) \in \mathbb{P}^k_d(I, \tau_L)$ on $[x_i, x_{i+1}]$ with $L \geq 4$.

The coefficients used to define each polynomial $p^s_{i,\mathcal{M}}$ for $s = 2, \dots, L-2$ are taken from the two intervals $[w^s_{i,1}, w^{s-1}_{i,2}]$ and $[w^{s+1}_{i,1}, w^s_{i,2}]$ (indicated by green bullets • in Figure 3). They fully determine the polynomial $p^s_{i,\mathcal{M}}$, with degree $\mathcal{M}$ and a number of degrees of freedom equal to $\mathcal{M}+1$. Indeed, since $L \geq 4$ and from (2.1), we have

$$\frac{2k+1-\mathcal{M}}{2k+1-\mathcal{M}-j} \geq 4,$$

and because $j \leq k$, we obtain

$$3\mathcal{M} - 4k - 1 + 2j \geq 0. \tag{2.6}$$

Then $2(2\mathcal{M}-2k+j) \geq \mathcal{M}+1$, i.e. the number of B-coefficients in $[w^s_{i,1}, w^{s-1}_{i,2}]$ and $[w^{s+1}_{i,1}, w^s_{i,2}]$ is greater than the number of degrees of freedom required to define the polynomial $p^s_{i,\mathcal{M}}$.
They completely specify $(L-1)$ polynomials $p^s_{i,\mathcal{M}}$, $s = 1, \dots, L-1$, which have each one degree $\mathcal{M}$ with the number of degrees of freedom equal to $(L-1)(\mathcal{M}+1)$ i.e.

$$2(\mathcal{M}-k+j) + 2(k-j+1) + (L-3)(\mathcal{M}+1) = (L-1)(\mathcal{M}+1).$$

Hence, we have shown that the B-coefficients for each subinterval of τ can be uniquely determined. Furthermore, due to the interpolation conditions (2.3) and the C^k smoothness at knots x_i and x_{i+1}, the C^k smoothness conditions at the shared knot of any neighboring subintervals of τ are also satisfied. This completes the proof. □

Based on the Bernstein-Bézier representation of $\mathcal{S}(x)$ provided above, we can infer that the conditions (2.3) are both sufficient and necessary to uniquely determine a spline $\mathcal{S}(x)$ on τ_L for an integer L greater than or equal to two. Hence, the dimension of the space $\mathbb{P}^k_d(I, \tau_L)$ is $(k+1)(n+1)$.
For $s = 0, \dots, L-1$, let

$$\mathcal{S}_i|_{[x_{i,s}, x_{i,s+1}]} = \mathcal{S}_{i,s},$$

denote the restrictions of the spline $\mathcal{S}_i$ within each subinterval $[x_{i,s}, x_{i,s+1}]$ with

$$x_{i,0} = x_i \quad \text{and} \quad x_{i,L} = x_{i+1}. \tag{2.7}$$

The polynomials $\mathcal{S}_{i,s}$ are expressed in the Bernstein basis as follows

$$\mathcal{S}_{i,s}(x) = \sum_{\theta=0}^{d} a_{\theta,s}\phi_{\theta,d}(t_s),\ t_s = \frac{x - x_{i,s}}{h_{i,s}},$$

where $h_{i,s} = x_{i,s+1} - x_{i,s}$ and $\phi_{\theta,d}$ are the Bernstein polynomials of degree d.

The determination of the unknown B-coefficients $a_{\theta,s}$ relies on the interpolation conditions at knots x_i and x_{i+1} (i.e., $\mathcal{S}_{i,0}^{(r)}(x_i) = f^{(r)}(x_i)$, $\mathcal{S}_{i,L-1}^{(r)}(x_{i+1}) = f^{(r)}(x_{i+1})$, $r = 0, \ldots, k$), as well as the $C^{\mathcal{M}}$ smoothness conditions at knots $x_{i,s}$, $s = 1, \ldots, L-1$.

The finite element S_i, of class C^k and degree d on $[x_i, x_{i+1}]$, is defined through its restrictions $S_{i,s}$. The corresponding spline space of degree d and class C^k on the interval I, associated with the partition τ_L, is given by

$$\mathbb{P}_d^k(I, \tau_L) := \{\mathcal{S} \in C^k(I) : \mathcal{S}_{i,s} \in \mathbb{P}_d(\mathbb{R}), \quad 0 \leq i \leq n-1 \text{ and } 0 \leq s \leq L-1\},$$

where $\mathbb{P}_d(\mathbb{R})$ is the polynomial space of degree d.

2.3. **Comparison with classical B-spline constructions.** Classical B-splines provide a well-established framework for constructing spline functions with desirable properties such as non-negativity, local support, and partition of unity. In particular, they offer a numerically stable basis for spline spaces defined on arbitrary knot sequences and are widely used in interpolation and approximation.

However, in the present setting, the spline spaces are defined on refined non-uniform partitions with a variable number of inserted knots and prescribed intermediate smoothness. In such a context, the direct use of classical B-spline bases becomes less straightforward, especially when aiming at fully local constructions driven only by endpoint interpolation data.

More precisely, classical B-spline representations typically rely on a global knot sequence with prescribed multiplicities, which may lead to constructions that are not explicitly local with respect to each subinterval. In contrast, the approach proposed in this work is based on a local Bernstein–Bézier representation, allowing the spline on each subinterval to be constructed independently from the others, using only the Hermite data at the endpoints.

Another important distinction lies in the flexibility of the construction. While classical B-splines are usually associated with fixed refinement patterns, the present framework allows for a variable number of inserted knots within each subinterval, together with a prescribed intermediate smoothness. This leads to a broader class of spline spaces adapted to local approximation requirements.

Finally, the proposed construction naturally leads to quasi-interpolants with superconvergence properties at the initial partition nodes. Such features are not directly obtained within the classical B-spline framework without additional modifications.

Therefore, while classical B-splines remain a fundamental tool in spline theory, the present approach provides an alternative framework that is particularly suited for local Hermite interpolation on refined non-uniform partitions.

3. Construction of normalized basis

Let $\psi_{i,s}$, for $i = 0, \ldots, n$ and $s = 0, \ldots, k$, represent the unique solution functions of the problem (2.3) within $\mathbb{P}_d^k(I, \tau_L)$, satisfying the following interpolation conditions:

$$\psi_{i,s}^{(r)}(x_t) = \delta_{it}\delta_{rs},\ \forall r = 0, \ldots, k,\ t = 0, \ldots, n, \tag{3.1}$$

Here, δ_{it} (or δ_{rs}) represents the Kronecker delta symbol.

The spline $\mathcal{S}$, the solution to the problem (2.3), can be represented as

$$\mathcal{S} := \sum_{i=0}^{n}\sum_{s=0}^{k} f^{(s)}(x_i)\psi_{i,s}.$$

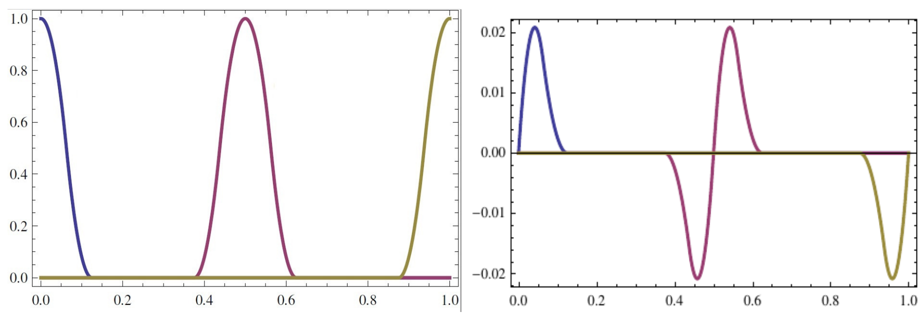

Figure 4. Hermite basis functions associated with the spline space $\mathbb{P}_2^1([0,1], \tau_1)$.

The set of functions $\psi_{i,s}$, where $i = 0, \ldots, n$ and $s = 0, \ldots, k$, constitutes the Hermite basis for the space $\mathbb{P}_d^k(I, \tau_L)$. However, the numerical instability resulting from the non-positivity of the basis elements renders it impractical for constructing approximants.

We construct the normalized B-splines $H_{i,s}$ as follows

$$H_{i,s}(x) := \sum_{\theta=0}^{k} \alpha_{i,s,\theta}\psi_{i,\theta}(x), \quad i = 0, \ldots, n,\ s = 0, \ldots, k.$$

It follows that

$$H_{i,s}^{(r)}(x_i) = \alpha_{i,s,r} \quad \text{for} \quad r = 0, \ldots, k. \tag{3.2}$$

One can observe that the vectors $\Gamma_{i,s} = (\alpha_{i,s,0}, \ldots, \alpha_{i,s,k})^T$ must be linearly independent for $H_{i,s}$, $s = 0, \ldots, k$, to be linearly independent. Thus, $H_{i,s}$ form a basis of $\mathbb{P}_d^k(I, \tau_L)$ and have local support $[x_{i-1}, x_{i+1}]$ $\big(H_{0,s}$ have a support $[x_0, x_1]$ (resp. $H_{n,s}$ have a support $[x_{n-1}, x_n])\big)$.

We will demonstrate that for every $x \in I$

$$H_{i,s}(x) \geq 0 \quad \text{and} \quad \sum_{i=0}^{n}\sum_{s=0}^{k} H_{i,s}(x) = 1.$$

The required conditions for the B-splines $H_{i,s}$ to fulfill the partition of unity are as follows

$$\sum_{s=0}^{k} \alpha_{i,s,0} = 1, \tag{3.3}$$

and

$$\sum_{s=0}^{k} \alpha_{i,s,r} = 0, \ \forall r = 1, \ldots, k. \tag{3.4}$$

We assume that the inserted knots $x_{i,\theta}$, $\theta = 1, \ldots, L-1$, possess the following barycentric coordinates in relation to $[x_{i,\theta-1}, x_{i,\theta+1}]$,

$$x_{i,\theta} = (\lambda_{i,\theta}, 1 - \lambda_{i,\theta}), \quad \lambda_{i,\theta} \in [0, 1]. \tag{3.5}$$

Lemma 3.1. *The B-coefficients of the normalized B-splines $H_{i,s}$, for $i = 0, \ldots, n$ and $s = 0, \ldots, k$, associated with Bézier points lying in the segments*

- $\mathcal{J}_k^d(x_{i-1})$, *having x_{i-1} as an endpoint of the interval $[x_{i-1}, x_{i-1,1}]$, for $i = 1, \ldots, n$,*
- $\mathcal{J}_k^d(x_{i+1})$, *having x_{i+1} as an endpoint of the interval $[x_{i,L-1}, x_{i+1}]$, for $i = 0, \ldots, n-1$,*

are equal to zero.

Proof. By construction, the B-coefficients of the Hermite basis elements (3.1) corresponding to the Bézier points in the neighborhood of x_{i-1} and x_{i+1} are equal to zero (refer to Figure 5).

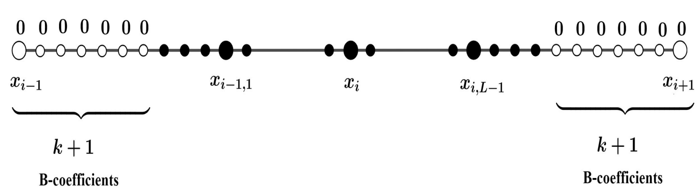

Figure 5. Representation of the zero B-coefficients in the Bernstein–Bézier form.

□

The following theorem establishes conditions ensuring the non-negativity of the normalized basis, based on the properties of the polar form.

Theorem 3.2. *The normalized B-splines $H_{i,s}$, $i = 0, \ldots, n$, $s = 0, \ldots, k$, are non-negative if*

1. *The B-coefficients corresponding to the Bézier points in the segment $\mathcal{J}_{d-\mathcal{M}}^d(x_i)$, having x_i as extremity in the intervals $[x_{i-1,L-1}, x_i]$, $i = 1, \ldots, n$, and $[x_i, x_{i,1}]$, $i = 0, \ldots, n-1$, are non-negative.*
2. *The B-coefficients corresponding to the Bézier points in the segment $\mathcal{J}_{\mathcal{M}}^{\mathcal{M}}(w_{i,1}^{\theta})$, having $w_{i,1}^{\theta}$ as extremity $[w_{i,1}^{\theta}, w_{i,2}^{\theta}]$, $\theta = 1, \ldots, L-1$, are non-negative.*

(3) *The B-coefficients corresponding to the Bézier points in the segment* $\mathcal{J}_{\mathcal{M}}^{\mathcal{M}}(w_{i-1,2}^{\theta})$, *having* $w_{i-1,2}^{\theta}$ *as extremity in* $[w_{i-1,1}^{\theta}, w_{i-1,2}^{\theta}]$, $\theta = 1, \ldots, L-1$, *are non-negative.*

Proof. From the properties of Bernstein-Bézier polynomials and from the $C^{\mathcal{M}}$ smoothness of the normalized B-splines $H_{i,s}$, at $x_{i,\theta}$, $\theta = 1, \ldots, L-1$, the B-coefficients of these B-splines corresponding to Bézier points in $[w_{i,1}^{\theta}, w_{i,2}^{\theta}]$ can be considered as the B-coefficients (after subdivision) of a polynomial $p_{i,\mathcal{M}}^{\theta}$ of degree $\mathcal{M}$. Let $a_{i,\theta,t}$ for $\theta = 1, \ldots, L-1$ and $t = 0, \ldots, \mathcal{M}$ be the B-coefficients of these polynomials in Bernstein-Bézier representation on $[w_{i,1}^{\theta}, w_{i,2}^{\theta}]$.
We first consider the case where θ equals 1, which corresponds to the first polynomial $p_{i,\mathcal{M}}^{1}$. From the assumption of non-negativity, we know that the B-coefficients $a_{i,1,t}$, $t = 0, \ldots, \mathcal{M}$, are non-negative.
For $t = 0, \ldots, \mathcal{M}$, let $b_{i,1,t}$ be the B-coefficients of the polynomial $p_{i,\mathcal{M}}^{1}|_{[w_{i,1}^{1}, x_{i,1}]}$ in Bernstein-Bézier representation on the interval $[w_{i,1}^{1}, x_{i,1}]$.
Let $\varrho_{i,l} = (\varrho_{i,l}^{1}, \varrho_{i,l}^{2})$, $l = 1, 2$, be the barycentric coordinates of the points $w_{i,1}^{1}$ and $x_{i,1}$ with respect to the interval $[w_{i,1}^{1}, w_{i,2}^{1}]$, where

$$\varrho_{i,1} = (1, 0), \quad \varrho_{i,2} = (\lambda_{i,1}, 1 - \lambda_{i,1}).$$

Then, the B-coefficients $b_{i,1,t}$ for $t = 0, \ldots, \mathcal{M}$, can be written as

$$b_{i,1,t} = \widehat{p}_{i,\mathcal{M}}^{1}(\underbrace{\varrho_{i,1}, \ldots, \varrho_{i,1}}_{t \text{ times}}, \underbrace{\varrho_{i,2}, \ldots, \varrho_{i,2}}_{(\mathcal{M}-t)\text{times}}).$$

Since any Bernstein basis polynomial is non-negative on its interval, and its polar form is non-negative whenever the barycentric coordinates are non-negative, then the B-coefficients $b_{i,1,t}$, $t = 0, \ldots, \mathcal{M}$, are non-negative.
By a similar argument, it can be shown that the B-coefficients of the polynomial $p_{i,\mathcal{M}}^{1}|_{[x_{i,1}, w_{i,2}^{1}]}$ in its Bernstein-Bézier representation on the interval $[x_{i,1}, w_{i,2}^{1}]$ are also non-negative.
Using a similar argument, we can prove the non-negativity of the B-coefficients of the polynomials $p_{i,\mathcal{M}}^{\theta}$, $\theta = 2, \ldots, L-1$.
By adding the assumption of non-negativity of the B-coefficients in $[x_i, w_{i,1}^{1}]$ and supplementing with zero B-coefficients (see Lemma 3.1), we have shown that all B-coefficients in $[x_i, x_{i+1}]$, $i = 0, \ldots, n-1$, are non-negative.
To show the non-negativity of the remaining B-coefficients in $[x_{i-1}, x_i]$, $i = 1, \ldots, n$, we use the same technique applied previously. Thus, we have verified that all the B-coefficients on the support of $H_{i,s}$ are non-negative. Therefore, the normalized B-splines $H_{i,s}$, $i = 0, \ldots, n$, $s = 0, \ldots, k$ are non-negative when all their B-coefficients are non-negative. □

Example 3.3. Let us define the partition X X with knots (X_j) as given by $X_{ki} := x_{i-1,k}$ and $X_{ki+s} := x_{i,s}$ for $s = 1, \ldots, k-1$ and $i = 0, \ldots, n$ with $x_{-1,k} = x_0$ and $x_{n,s} = x_n$, which are the inserted knots mentioned in section 2.1. The classical B-splines $B_{j,k}$ of degree k, $j = -k, \ldots, kn$ of the space $\mathbb{P}_k^{k-1}(I, \tau_{k+1} \setminus \tau)$ associated with the subdivision (X_j), $j = -k, \ldots, kn + k + 1$ with $X_{-k} = \ldots = X_{-1} = X_0$

and $X_{kn+1} = X_{kn+2} = \ldots = X_{kn+k+1}$. Each B-spline has support $\mathrm{supp}(B_{j,k}) = [X_j, X_{j+k+1}]$.

$$\alpha_{i,s,r} = \frac{k+1}{k+1-r} B^{(r)}_{ki-s,k}(x_i),\ r = 0, \ldots, k,\ s = 0, \ldots, k. \tag{3.6}$$

By this specific choice of $\alpha_{i,s,r}$, the normalized B-splines $H_{i,s}$, $i = 0, \ldots, n$, $s = 0, \ldots, k$ provide the classical B-splines of the space $\mathbb{P}^k_{k+1}(I, \tau_{k+1})$.

Figure 6 illustrates the C^2 cubic B-spline basis corresponding to the following choice.

$$\alpha_{i,s,0} = B_{2i-s,2}(x_i), \quad \alpha_{i,s,1} = \frac{3}{2} B'_{2i-s,2}(x_i) \text{ and } \alpha_{i,s,2} = 3B''_{2i-s,2}(x_i), \quad s = 0, 1, 2.$$

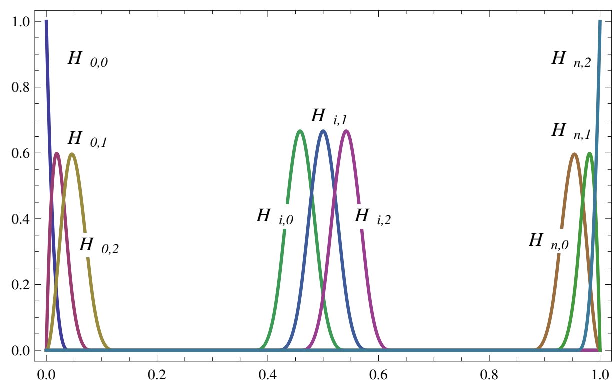

FIGURE 6. C^2 cubic normalized B-spline basis corresponding to the refined partition τ_2.

4. Representation of C^k Hermite interpolant in the normalized basis

4.1. **Marsden's Identity.** We denote by $x_i^{(r)}$ the knot x_i with multiplicity r, i.e., repeated r times. We assume that the vectors $(\alpha_{i,s,1}, \ldots, \alpha_{i,s,k})^\intercal$, for $s = 0, \ldots, k$, are given in such a way that the associated splines $H_{i,s}$ are non-negative and form a linearly independent set.

Let e_l denote the monomial of degree l, with $l \in \mathbb{N}$. For $l = 1, \ldots, k$, we denote by Σ_l the l-th elementary symmetric polynomial in k variables, defined by

$$\Sigma_l(X_1, \ldots, X_k) = \sum_{1 \le i_1 \le \cdots \le i_l \le k} X_{i_1} \cdots X_{i_l}.$$

Throughout the remainder of the paper, we assume that $m \geq d$.

Theorem 4.1. *For any monomial e_l of degree $l \leq m$, there exist k points $y_{i,s,t}$, $t = 1, \ldots, k$ such that*

$$\mathcal{H}e_l(x) = \sum_{i=0}^{n} \sum_{s=0}^{k} \widehat{e}_l \left(y_{i,s,1}, \ldots, y_{i,s,k}, x_i^{(m-k)} \right) H_{i,s}(x), \quad \forall x \in I.$$

Proof. By the uniqueness of the Hermite interpolant in $\mathbb{P}_d^k(I,\tau_L)$, it suffices to determine $y_{i,s,t}$, $t=1,\dots,k$, such that

$$e_l^{(r)}(x_i) \;=\; \sum_{s=0}^{k}\widehat{e}_l(y_{i,s,1},\dots,y_{i,s,k},x_i^{(m-k)})\alpha_{i,s,r}, \quad l=0,\dots,m. \tag{4.1}$$

$l=0$ is self-evident. For $l=1,\dots,k$ we have

$$\sum_{s=0}^{k}\Sigma_l(y_{i,s,1},\dots,y_{i,s,k})\alpha_{i,s,0} \;=\; \binom{k}{l}x_i^l, \quad \text{if } r=0, \tag{4.2}$$

$$\sum_{s=0}^{k}\Sigma_l(y_{i,s,1},\dots,y_{i,s,k})\alpha_{i,s,r} \;=\; \binom{k-r}{l-r}x_i^{l-r}\prod_{\theta=1}^{r}(m-\theta+1), \quad \text{if } 1\leqslant r\leqslant k, \tag{4.3}$$

The case $k+1\le l\le m$ is derived from the previous cases $l=1,\dots,k$. Certainly, utilizing Proposition 5 in [8], we obtain

$$\begin{aligned}
&\widehat{e}_l\left(y_{i,s,1},\dots,y_{i,s,k},x_i^{(m-k)}\right)\prod_{\theta=1}^{k}(m-\theta+1)\\
=\;& x_i^l\prod_{\theta=0}^{k-1}(m-l-\theta)\\
&+\sum_{t=1}^{k-1}x_i^{l-t}\prod_{\theta=0}^{t-1}(l-\theta)\prod_{\theta=0}^{k-t-1}(m-l-\theta)\Sigma_t(y_{i,s,1},\dots,y_{i,s,k})\\
&+x_i^{l-k}\prod_{\theta=0}^{k-1}(l-\theta)x_i^{l-k}\prod_{\theta=1}^{k}y_{i,s,\theta}.
\end{aligned}$$

Then, if Relations (4.2) and (4.3) hold, we have (4.1) for $k+1\le l\le m$.
Now for $l=1,\dots,k$, we put

$$\Sigma_l(y_{i,s,1},\dots,y_{i,s,k})=Q_{i,s,l}. \tag{4.4}$$

Then

$$\sum_{s=0}^{k}Q_{i,s,l}\,\alpha_{i,s,r}=\binom{k-r}{l-r}\,x_i^{l-r}\prod_{\theta=1}^{r}(m-\theta+1), \quad r=0,\dots,k.$$

Therefore we have

$$M_{i,l}\left(Q_{i,s,l}\right)_{0\le s\le k}=\left(\tbinom{k-r}{l-r}\,x_i^{l-r}\prod_{\theta=1}^{r}(m-\theta+1)\right)_{0\le r\le k}. \tag{4.5}$$

where $M_{i,l}=\left(\alpha_{i,s,r}\right)_{0\le r,s\le k}$. The constructed B-splines must be linearly independent; consequently, the matrix $M_{i,l}$ is nonsingular. Hence, systems (4.5) uniquely determine the unknowns $Q_{i,s,l}$.
Let

$$P(X)=\prod_{j=1}^{k}(X-y_{i,s,j})=X^k+\sum_{l=1}^{k}(-1)^lQ_{i,s,l}\,X^{k-l}.$$

Since $y_{i,s,t}$, $t = 1, \ldots, k$, are the roots of P, they satisfy

$$y_{i,s,t}^{k} + \sum_{l=1}^{k} (-1)^l Q_{i,s,l}\, y_{i,s,t}^{k-l} = 0,$$

and solving this system yields the values of $y_{i,s,t}$ for $t = 1, \ldots, k$.

□

Proposition 4.2. *For any $p \in \mathbb{P}_m(\mathbb{R})$, the Hermite interpolant $\mathcal{H}p$ of p in $\mathbb{P}_d^k(I, \tau_L)$ is expressed as*

$$\mathcal{H}p(x) = \sum_{i=0}^{n} \sum_{s=0}^{k} \widehat{p}\left(y_{i,s,1}, \ldots, y_{i,s,k}, x_i^{(m-k)}\right) H_{i,s}(x), \quad \forall x \in I. \tag{4.6}$$

Proof. We recall that $p(x) = \sum_{l=0}^{m} a_l e_l(x)$ where a_l, $l = 0, \ldots, m$, are real numbers. In accordance with Theorem 4.1 and the linearity of the blossom operator (For example, refer to [7]), we obtain

$$\begin{aligned}
\mathcal{H}p(x) \quad &= \quad \sum_{i=0}^{n} \sum_{s=0}^{k} \widehat{p}\left(y_{i,s,1}, \ldots, y_{i,s,k}, x_i^{(m-k)}\right) H_{i,s}(x) \\
&= \sum_{i=0}^{n} \sum_{s=0}^{k} \sum_{l=0}^{m} a_l \widehat{e}_l \left(y_{i,s,1}, \ldots, y_{i,s,k}, x_i^{(m-k)}\right) H_{i,s}(x) \\
&= \sum_{l=0}^{m} a_l \mathcal{H} e_l(x) \\
&= \mathcal{H}p(x).
\end{aligned}$$

□

Corollary 4.3. *We can establish Marsden's Identity for $p \in \mathbb{P}_d(\mathbb{R})$, then*

$$p(x) = \sum_{i=0}^{n} \sum_{s=0}^{k} \widehat{p}\left(y_{i,s,1}, \ldots, y_{i,s,k}, x_i^{(d-k)}\right) H_{i,s}(x), \quad \forall x \in I.$$

4.2. **Hermite interpolation in $\mathbb{P}_d^k(I, \tau_L)$.**

Theorem 4.4. *For any $f \in C^k(I)$, the Hermite interpolant $\mathcal{H}f$ of f in the space $\mathbb{P}_d^k(I, \tau_L)$ is given by*

$$\mathcal{H}f(x) = \sum_{i=0}^{n} \sum_{s=0}^{k} \widehat{T}_i \left(y_{i,s,1}, \ldots, y_{i,s,k}, x_i^{(m-k)}\right) H_{i,s}(x), \quad \forall x \in I, \tag{4.7}$$

where $T_i(x) := \sum_{t=0}^{k} \frac{(x - x_i)^t}{t!} f^{(t)}(x_i)$ and $m \geq d$.

Proof. For $s = 0\dots,k$, we have

$$\widehat{T}_i\left(y_{i,s,1},\dots,y_{i,s,k},x_i^{(m-k)}\right) = f(x_i) + \sum_{t=1}^{k} \frac{\displaystyle\sum_{\sigma\in\mathcal{P}_k}\prod_{\theta=1}^{t}(y_{i,s,\sigma(\theta)}-x_i)}{\displaystyle\prod_{\theta=1}^{t}(m-\theta+1)} f^{(t)}(x_i).$$

Define

$$\chi^{(r)}(x) := \sum_{i=0}^{n}\sum_{s=0}^{k}\widehat{T}_i\left(y_{i,s,1},\dots y_{i,s,k},x_i^{(m-k)}\right)H_{i,s}^{(r)}(x).$$

For $0\le j\le n$, we have

$$\begin{aligned}
\chi^{(r)}(x_j) &= \sum_{i=0}^{n}\sum_{s=0}^{k}\left(f(x_i) + \sum_{t=1}^{k} \frac{\displaystyle\sum_{\sigma\in\mathcal{P}_k}\prod_{\theta=1}^{t}(y_{i,s,\sigma(\theta)}-x_i)}{\displaystyle\prod_{\theta=1}^{t}(m-\theta+1)} f^{(t)}(x_i)\right)H_{i,s}^{(r)}(x_j)\\
&= \sum_{s=0}^{k}\left(f(x_j) + \sum_{t=1}^{k} \frac{\displaystyle\sum_{\sigma\in\mathcal{P}_k}\prod_{\theta=1}^{t}(y_{j,s,\sigma(\theta)}-x_j)}{\displaystyle\prod_{\theta=1}^{t}(m-\theta+1)} f^{(t)}(x_j)\right)\alpha_{j,s,r}\\
&= \sum_{s=0}^{k}\left(f(x_j) + \sum_{t=1}^{k} \frac{f^{(t)}(x_j)}{\displaystyle\prod_{\theta=1}^{t}(m-\theta+1)}\left(\sum_{l=1}^{t}(-x_j)^{t-l}\binom{k-l}{t-l}\Sigma_l(y_{i,s,1},\dots,y_{i,s,k})\right.\right.\\
&\quad + \left.\left.(-x_j)^t\binom{k}{t}\right)\right)\alpha_{j,s,r}.
\end{aligned}$$

- For $r=0$, we use (3.3) and (4.2) to get

$$\begin{aligned}
\chi(x_j) &= f(x_j) + \sum_{t=1}^{k}\frac{x_j^t f^{(t)}(x_j)}{\displaystyle\prod_{\theta=1}^{t}(m-\theta+1)}\left(\sum_{l=1}^{t}(-1)^{t-l}\binom{k-l}{t-l}\binom{k}{l} + (-1)^t\binom{k}{t}\right)\\
&= f(x_j) + \sum_{t=1}^{k}\frac{x_j^t f^{(t)}(x_j)}{\displaystyle\prod_{\theta=1}^{t}(m-\theta+1)}\sum_{l=0}^{t}(-1)^{t-l}\binom{k-l}{t-l}\binom{k}{l}.
\end{aligned}$$

Since

$$\sum_{l=0}^{t}(-1)^{t-l}\binom{k-l}{t-l}\binom{k}{l} = \frac{(-1)^{-t}k!}{t!(k-t)!}\sum_{l=0}^{t}(-1)^{l}\binom{t}{l} = 0, \tag{4.8}$$

it holds that

$$\chi(x_j) = f(x_j). \tag{4.9}$$

- For $r=1,\dots,k$, we use (3.4) to obtain

$$\chi^{(r)}(x_j) \;=\; \sum_{s=0}^{k}\sum_{t=1}^{k}\frac{f^{(t)}(x_j)}{\prod_{\theta=1}^{t}(m-\theta+1)}\sum_{l=1}^{t}(-x_j)^{t-l}\binom{k-l}{t-l}\Sigma_l(y_{i,s,1},\dots,y_{i,s,k})\alpha_{j,s,r}.$$

Since (4.3) we have

$$\chi^{(r)}(x_j) \;=\; \sum_{t=1}^{k}x_j^{t-r}f^{(t)}(x_j)\frac{\prod_{\theta=1}^{r}(m-\theta+1)}{\prod_{\theta=1}^{t}(m-\theta+1)}\sum_{l=1}^{t}(-1)^{t-l}\binom{k-l}{t-l}\binom{k-r}{l-r}. \tag{4.10}$$

When $t=r$, we have

$$\sum_{l=1}^{r}(-1)^{r-l}\binom{k-l}{r-l}\binom{k-r}{l-r} \;=\; \sum_{l=1}^{r}\frac{(-1)^{r-l}}{(r-l)!(l-r)!} \;=\; 1, \tag{4.11}$$

and when $t\neq r$, we have

$$\begin{aligned}\sum_{l=1}^{t}(-1)^{t-l}\binom{k-l}{t-l}\binom{k-r}{l-r} &= \frac{(k-r)!}{(k-t)!}\sum_{l=1}^{t}(-1)^{t-l}\binom{l}{r}\binom{t}{l}\\ &= \frac{(-1)^{-t}(k-r)!}{(k-t)!}\sum_{l=0}^{t}(-1)^{l}\binom{l}{r}\binom{t}{l}\\ &= \frac{(-1)^{-t}(k-r)!}{(k-t)!}\sum_{l=0}^{t}(-1)^{l}\binom{t}{r}\binom{t-r}{l-r},\end{aligned} \tag{4.12}$$

substitute $l=j+r$, then

$$\begin{aligned}\sum_{l=0}^{t}(-1)^{l}\binom{t-r}{l-r} &= \sum_{j=-r}^{t-r}(-1)^{j+r}\binom{t-r}{j}\\ &= \sum_{j=0}^{t-r}(-1)^{j+r}\binom{t-r}{j}\\ &= (-1)^{r}\sum_{j=0}^{t-r}(-1)^{j}\binom{t-r}{j}\\ &= 0.\end{aligned}$$

Thus

$$\sum_{l=1}^{t}(-1)^{t-l}\binom{k-l}{t-l}\binom{k-r}{l-r} \;=\; \delta_{tr}. \tag{4.13}$$

By using (4.10) and (4.13) we obtain

$$\chi^{(r)}(x_j)=f^{(r)}(x_j),\quad \forall r=1,\dots,k,$$

concluding the proof. ∎

Theorem 4.5. *For any polynomial $\mathcal{P}$ with degree $m \geq d$ and C^k continuity across the refinement τ_L, we obtain*

$$\mathcal{HP}(x) = \sum_{i=0}^{n}\sum_{s=0}^{k} \widehat{\mathcal{P}}_i\left(y_{i,s,1},\dots,y_{i,s,k},x_i^{(m-k)}\right) H_{i,s}(x), \quad \forall x \in I \tag{4.14}$$

where $\mathcal{P}_i$ denotes the restriction of $\mathcal{P}$ to the intervals $[x_{i-1,L-1}, x_i]$ or $[x_i, x_{i,1}]$.

Proof. Consider $\mathcal{P}$, a polynomial of degree m with $m \geq d$, belonging to the class C^k, defined on the interval I and equipped with the refinement τ_L. According to Theorem 4.4, the Hermite interpolant of $\mathcal{P}$ within $\mathbb{P}_d^k(I,\tau_L)$ can be expressed as follows

$$\mathcal{HP}(x) = \sum_{i=0}^{n}\sum_{s=0}^{k}\left(\mathcal{P}(x_i) + \sum_{t=1}^{k}\frac{\displaystyle\sum_{\sigma\in\mathcal{P}_k}\prod_{\theta=1}^{t}(y_{i,s,\sigma(\theta)}-x_i)}{\displaystyle\prod_{\theta=1}^{t}(m-\theta+1)}\mathcal{P}^{(t)}(x_i)\right)H_{i,s}(x).$$

$\mathcal{P}_i$ denotes the restriction of $\mathcal{P}$ to the intervals $[x_{i-1,L-1}, x_i]$ or $[x_i, x_{i,1}]$, we have

$$\mathcal{HP}(x) = \sum_{i=0}^{n}\sum_{s=0}^{k}\left(\mathcal{P}_i(x_i) + \sum_{t=1}^{k}\frac{\displaystyle\sum_{\sigma\in\mathcal{P}_k}\prod_{\theta=1}^{t}(y_{i,s,\sigma(\theta)}-x_i)}{\displaystyle\prod_{\theta=1}^{t}(m-\theta+1)}\mathcal{P}_i^{(t)}(x_i)\right)H_{i,s}(x).$$

From Taylor expansion we have

$$\mathcal{P}_i(x) = \sum_{t=0}^{m}\frac{(x-x_i)^t}{t!}\mathcal{P}_i^{(t)}(x_i).$$

For $t = k+1,\dots,m$, we put

$$\ell_{\alpha_j}(x) = \begin{cases} x - x_i, & \text{if} \quad 1 \leq j \leq t \\ 1, & \text{if} \quad t+1 \leq j \leq m. \end{cases}$$

Utilizing Proposition 5 from [8], we derive $(m \geq d)$

$$\widehat{p}\left(y_{i,s,1},\dots,y_{i,s,k},x_i^{(m-k)}\right) = \frac{\displaystyle\sum_{\substack{\alpha_j=1\\ j=1,\dots,k}}^{m}\prod_{\theta=1}^{k}\ell_{\alpha_\theta}(y_{i,s,\theta}) \prod_{\substack{\gamma=1\\ \gamma\neq\alpha_j,\ j=1,\dots,k}}^{m}\ell_\gamma(x_i)}{\displaystyle\prod_{\theta=1}^{k}(m-\theta+1)} = 0.$$

where $p(x) = \prod_{j=1}^{m} \ell_{\alpha_j}(x)$. Thus

$$\widehat{\mathcal{P}}_i\left(y_{i,s,1},\ldots,y_{i,s,k},x_i^{(m-k)}\right) = \mathcal{P}_i(x_i) + \sum_{t=1}^{k} \frac{\sum_{\sigma\in\mathcal{P}_k}\prod_{\theta=1}^{t}(y_{i,s,\sigma(\theta)}-x_i)}{\prod_{\theta=1}^{t}(m-\theta+1)} \mathcal{P}_i^{(t)}(x_i).$$

Therefore

$$\mathcal{H}\mathcal{P}(x) = \sum_{i=0}^{n}\sum_{s=0}^{k} \widehat{\mathcal{P}}_i\left(y_{i,s,1},\ldots,y_{i,s,k},x_i^{(m-k)}\right) H_{i,s}(x).$$

□

5. Superconvergent discrete quasi-interpolants

Our emphasis is on quasi-interpolants in the following format

$$\mathcal{Q}_m f = \sum_{i=0}^{n}\sum_{s=0}^{k} \mu_{i,s}(f) H_{i,s}, \tag{5.1}$$

Here, $\mu_{i,s}$ represents linear functionals constructed using values of the function f near the supports of the B-splines $H_{i,s}$. With these values, we construct a local linear polynomial operator $\mathcal{I}_{i,s}$ in the vicinity of supp $H_{i,s}$ to reproduce the space of polynomials with degrees at most m. It's worth mentioning that m is an integer greater than or equal to d.

Theorem 5.1. *Consider a function f defined on the interval I, with values available at discrete points in the vicinity of the support of $H_{i,s}$, $i = 0,\ldots,n$, $s = 0,\ldots,k$. If we represent $\mathcal{I}_{i,s}(f)$ as $p_{i,s}$, subsequently, the quasi-interpolant defined in (5.1) :*

$$\mu_{i,s}(f) = \widehat{p}_{i,s}\left(y_{i,s,1},\ldots,y_{i,s,k},x_i^{(m-k)}\right) \tag{5.2}$$

with

$$\mathcal{Q}_m p = \mathcal{H}p, \quad \forall p \in \mathbb{P}_m.$$

Proof. Given $f \in \mathbb{P}_m$, we determine that $p_{i,s} = \mathcal{I}_{i,s}(f)$. This result, derived from (5.1) and Theorem 4.2, leads us to

$$\begin{aligned}\mathcal{Q}_m f &= \sum_{i=0}^{n}\sum_{s=0}^{k} \mu_{i,s}(f) H_{i,s}\\ &= \sum_{i=0}^{n}\sum_{s=0}^{k} \widehat{p}_{i,s}\left(y_{i,s,1},\ldots,y_{i,s,k},x_i^{(m-k)}\right) H_{i,s}\\ &= \sum_{i=0}^{n}\sum_{s=0}^{k} \widehat{f}\left(y_{i,s,1},\ldots,y_{i,s,k},x_i^{(m-k)}\right) H_{i,s}.\end{aligned}$$

Hence,

$$\mathcal{Q}_m f = \mathcal{H} f.$$

$\square$

To create a superconvergent discrete quasi-interpolant, you only need to select $m+1$ distinct interpolation points near the support of $H_{i,s}$. Let $z_{i,s,t}$, where t ranges from 0 to m, represent these points. Given that $p_{i,s}$ is the Lagrange interpolant, meaning

$$p_{i,s} = \sum_{t=0}^{m} f(z_{i,s,t}) \mathbf{L}_{i,s,t}, \tag{5.3}$$

here, $\mathbf{L}_{i,s,t}$ represents the Lagrange basis functions of $\mathbb{P}_m$. Consequently,

$$\mathcal{Q}_m p = \mathcal{H} p, \ \forall p \in \mathbb{P}_m.$$

Consider $\eta^r_{i,s} \in \mathbb{R}$ for $r = 2, \ldots, k$. This entails

$$y_{i,s,r} = \eta^r_{i,s} y_{i,s,1} + (1 - \eta^r_{i,s}) y_{i,s,k} \quad \text{for} \quad r = 2, \ldots, k-1$$

and

$$x_i = \eta^k_{i,s} y_{i,s,1} + (1 - \eta^k_{i,s}) y_{i,s,k}.$$

In the following Theorem, we give an explicit formula of the coefficients $\mu_{i,s}(f)$ in terms of the data values $f(z_{i,s,t})$ for $t = 0, \ldots, m$.

Theorem 5.2. *Define $z_{i,s,t}$ as $z_{i,s,t} = \varepsilon_{i,s,t} y_{i,s,1} + (1 - \varepsilon_{i,s,t}) y_{i,s,k}$ for $0 \le t \le m$, constituting $(m+1)$ distinct points in the vicinity of the support of $H_{i,s}$. The quasi-interpolant with*

$$\mu_{i,s}(f) = \sum_{t=0}^{m} q_{i,s,t} f(z_{i,s,t}) \tag{5.4}$$

satisfies

$$\mathcal{Q}_m p = \mathcal{H} p, \quad \forall \ p \in \mathbb{P}_m,$$

if and only if

$$q_{i,s,t} = \begin{cases} \dfrac{\displaystyle\sum_{\substack{\alpha_j=0, j=1,2 \\ \alpha_j \neq t}}^{m} (\varepsilon_{i,s,\alpha_1} - 1) \varepsilon_{i,s,\alpha_2} \prod_{\substack{\gamma=0 \\ \gamma \neq t, \alpha_j}}^{m} (\eta^2_{i,s} - \varepsilon_{i,s,\gamma})}{\displaystyle\prod_{\theta=1}^{2} (m - \theta + 1) \prod_{\substack{\alpha=0 \\ \alpha \neq t}}^{m} (\varepsilon_{i,s,t} - \varepsilon_{i,s,\alpha})}, & \text{for} \quad k = 2 \\ \dfrac{\displaystyle\sum_{\substack{\alpha_j=0, j=1,\ldots,k \\ \alpha_j \neq t}}^{m} (\varepsilon_{i,s,\alpha_1} - 1) \prod_{r=2}^{k-1} (\eta^r_{i,s} - \varepsilon_{i,s,\alpha_r}) \varepsilon_{i,s,\alpha_k} \prod_{\substack{\gamma=0 \\ \gamma \neq t, \alpha_j}}^{m} (\eta^k_{i,s} - \varepsilon_{i,s,\gamma})}{\displaystyle\prod_{\theta=1}^{k} (m - \theta + 1) \prod_{\substack{\alpha=0 \\ \alpha \neq t}}^{m} (\varepsilon_{i,s,t} - \varepsilon_{i,s,\alpha})}, & \text{for} \quad k \geq 3, \end{cases} \tag{5.5}$$

where the α_j's are different from each other.

Proof. Let $\ell_{\alpha,t}(x) = x - z_{i,s,\alpha}$. Then

$$\mathbf{L}_{i,s,t}(x) = \frac{\displaystyle\prod_{\substack{\alpha=0\\ \alpha\neq t}}^{m} \ell_{\alpha,t}(x)}{\displaystyle\prod_{\substack{\alpha=0\\ \alpha\neq t}}^{m} \ell_{\alpha,t}(z_{i,s,t})}.$$

We set

$$q_{i,s,t} = \widehat{\mathbf{L}}_{i,s,t}(y_{i,s,1}, \ldots, y_{i,s,k}, x_i^{(m-k)}), \ \forall t = 0, \ldots, m.$$

Then the quasi-interpolant $\mathcal{Q}_m$ satisfies $\mathcal{Q}_m p = \mathcal{H} p, \quad \text{for} \quad p \in \mathbb{P}_m$. To calculate the value of $q_{i,s,t}$ for $t = 0, \ldots, m,$, we apply Proposition 5 in [8], and as a result, we derive

$$\widehat{\mathbf{L}}_{i,s,t}\left(y_{i,s,1}, \ldots, y_{i,s,k}, x_i^{(m-k)}\right) = \frac{\displaystyle\sum_{\substack{\alpha_j=0, j=1,\ldots,k\\ \alpha_j\neq t}}^{m} \left(\prod_{r=1}^{k} \ell_{\alpha_r,t}(y_{i,s,r}) \prod_{\substack{\gamma=0\\ \gamma\neq t,\alpha_j}}^{m} \ell_{\gamma,t}(x_i)\right)}{\displaystyle\prod_{\theta=1}^{k}(m-\theta+1) \prod_{\substack{\alpha=0\\ \alpha\neq t}}^{m} \ell_{\alpha,t}(z_{i,s,t})},$$

where the α_j's are different from each other. Since,

$$\ell_{\alpha_1,t}(y_{i,s,1}) = (1 - \varepsilon_{i,s,\alpha_1})(y_{i,s,1} - y_{i,s,k}), \quad \text{and} \quad \ell_{\alpha_k,t}(y_{i,s,k}) = -\varepsilon_{i,s,\alpha_k}(y_{i,s,1} - y_{i,s,k}).$$

For $r = 2, \ldots, k-1$ $(k \geq 3)$, we obtain

$$\ell_{\alpha_r,t}(y_{i,s,r}) \;=\; (\eta_{i,s}^{r} - \varepsilon_{i,s,\alpha_r})(y_{i,s,1} - y_{i,s,k}).$$

Moreover,

$$\ell_{\gamma,t}(x_i) = (\eta_{i,s}^{k} - \varepsilon_{i,s,\gamma})(y_{i,s,1} - y_{i,s,k}), \quad \text{and} \quad \ell_{\alpha,t}(z_{i,s,t}) = (\varepsilon_{i,s,t} - \varepsilon_{i,s,\alpha})(y_{i,s,1} - y_{i,s,k}).$$

Hence (6.4). □

Remark 5.3. The construction of discrete quasi-interpolants in the space $\mathbb{P}_2^1(I, \tau_2)$ (i.e., $k = j = 1$) was previously addressed in [7]. However, since the underlying basis in that work is built according to a different construction, it does not readily extend to the general framework developed in the present study.

6. Error estimate

Assume that $f \in \mathcal{C}^{d+1-r}(I)$. Since the operators $\mathcal{Q}_m$ for $m \geq d$ reproduce the polynomial space $\mathbb{P}_m$, as established in Theorem 5.2, there exist positive constants C_r, $r = 0, \ldots, k$, independent of m, such that the following estimate holds.

$$\| \mathcal{Q}_m^{(r)} f - f^{(r)} \|_{\infty,I} \leq C_r h^{d+1-r} \| f^{(d+1-r)} \|_{\infty,I}, \ r = 0, \ldots, k,$$

where $h = \max\limits_{i=0,\ldots,n-1} \max\limits_{s=0,\ldots,\mathcal{L}-1} h_{i,s}$ and $\| \, . \, \|_{\infty,I}$ denotes the infinity norm.

Lemma 6.1. *If p represents a polynomial of real coefficients with a degree of d, then*

$$\max_{a\le x\le b} \mid p^{(r)}(x) \mid \le \frac{K}{(b-a)^r} \max_{a\le x\le b} \mid p(x) \mid,$$

where K is a constant depending only on d.

Proof. According to Markov inequality [5] for any polynomial p of degree at most d we give

$$\max_{a\le x\le b} \mid p'(x) \mid \le \frac{2d^2}{b-a} \max_{a\le x\le b} \mid p(x) \mid .$$

By a similar argument, we obtain

$$\begin{aligned}\max_{a\le x\le b} \mid p''(x) \mid &\le \frac{2(d-1)^2}{b-a} \max_{a\le x\le b} \mid p'(x) \mid \\ &\le \frac{4d^2(d-1)^2}{(b-a)^2} \max_{a\le x\le b} \mid p(x) \mid .\end{aligned}$$

We can conclude that

$$\max_{a\le x\le b} \mid p^{(r)}(x) \mid \le \frac{K}{(b-a)^r} \max_{a\le x\le b} \mid p(x) \mid,$$

where $K = 2^r d^2(d-1)^2 \dots (d-r)^2$. □

Theorem 6.2. *For any function f belonging to $\mathcal{C}^{m+1}(I)$, the following holds*

$$\mid \mathcal{Q}_m^{(r)} f(x_i) - f^{(r)}(x_i) \mid = \mathcal{O}(h^{m+1-r}),$$

for $i = 0, \dots, n$ and $r = 0, \dots, k$

Proof. Consider $f \in \mathcal{C}^{m+1}(I)$. The Taylor expansion of f around each x_i, where i ranges from 0 to n, is expressed as

$$f(x) = \sum_{t=0}^{m} \frac{f^{(t)}(x_i)}{t!}(x-x_i)^t + \mathcal{O}((x-x_i)^{m+1}).$$

Let's define R_m as the polynomial component of the Taylor expansion. In this case, for every point x within the support of $H_{i,s}$, the following holds

$$f(x) = R_m(x) + \mathcal{O}((x-x_i)^{m+1}).$$

According to Theorem 5.1, we obtain $\mathcal{Q}_m R_m = \mathcal{H} R_m$. Utilizing the fact that $\mathcal{H} R_m(x_i) = R_m(x_i)$, we can then conclude

$$|\mathcal{Q}_m f(x_i) - f(x_i)| = |\mathcal{Q}_m f(x_i) - R_m(x_i)| = |\mathcal{Q}_m (f - R_m)(x_i)|.$$

By referencing (6.3) and considering the bounded nature of $q_{i,s,t}$, which is limited by a constant C, we derive

$$|\mu_{i,s}(f-R_m)| = \mid \sum_{t=0}^{m} q_{i,s,t}\left(f(z_{i,s,t}) - R_m(z_{i,s,t})\right) \mid \le C \sum_{k=0}^{m} |f(z_{i,s,t}) - R_m(z_{i,s,t})|.$$

Then

$$|\mu_{i,s}(f-R_m)| = \mathcal{O}((z_{i,s,t}-x_i)^{m+1})$$

and therefore

$$|\mathcal{Q}_m(f-R_m)(x_i)| = \mathcal{O}((z_{i,s,t}-x_i)^{m+1}).$$

Then

$$| \mathcal{Q}_m f(x_i) - f(x_i) |= \mathcal{O}(h^{m+1}). \tag{6.1}$$

Using Relation (3.2) and Lemma 6.1, we have

$$|\alpha_{i,s,r}| = |H_{i,s}^{(r)}(x_i)| \leq \frac{C}{(b-a)^r}|\alpha_{i,s,0}|, \quad r = 1, \dots, k,$$

where C is a constant that solely depends on d.
Since $0 < h \leq b - a$ then

$$|\alpha_{i,s,r}| \leq \frac{C}{h^r}|\alpha_{i,s,0}|, \quad r = 1, \dots, k. \tag{6.2}$$

In a similar way used to prove (6.1) using (6.2), we obtain

$$| \mathcal{Q}_m^{(r)} f(x_i) - f^{(r)}(x_i) |= \mathcal{O}(h^{m+1-r}), \quad for\ r = 1, \dots, k,$$

which completes the proof. □

Examples of spline quasi-interpolants illustrating the proposed construction

We provide discrete quasi-interpolation operators for $k = 2$ et $j = 2$. The linear functionals are defined by

$$\mu_{i,s}(f) = \sum_{k=0}^{m} q_{i,s,k} f(t_{i,s,k}) \tag{6.3}$$

where

$$t_{i,s,k} = \varepsilon_{i,s,k} y_{i,s,1}^{(m)} + (1 - \varepsilon_{i,s,k}) y_{i,s,2}^{(m)}$$

and

$$q_{i,s,k} = \frac{1}{m(m-1)} \frac{\displaystyle\sum_{\alpha,\theta=0,\alpha\neq k,\alpha\neq\theta}^{m} (\varepsilon_{i,s,\alpha} - 1)\varepsilon_{i,s,\theta} \prod_{\gamma=0,\ \gamma\neq\alpha,\theta,k}^{m} (\eta_{i,s} - \varepsilon_{i,s,\gamma})}{\displaystyle\prod_{\alpha=0,\ \alpha\neq k}^{m} (\varepsilon_{i,s,k} - \varepsilon_{i,s,\alpha})}. \tag{6.4}$$

We present numerical experiments corresponding to the choice of B-splines defined by (3.6) in Section 4. We consider the following test function by

$$f(x) = \exp(-3x) \sin(\frac{\pi}{2}x)$$

We define the local error between a function f and the quasi-interpolant $\mathcal{Q}_m f$ at the knots of τ by the following relation:

$$E_{m,n}^{(r)}(f) := \max_{0\leqslant i\leqslant n} \left|Q_m^{(r)} g(x_i) - f^{(r)}(x_i)\right|, \quad r = 0, 1, 2,$$

and the numerical convergence order by $\mathcal{NCO}_m^{(r)}$.

The observed convergence orders match the theoretical predictions, thus confirming the superconvergence properties established in the analysis.

n	$E_{3,n}^{(0)}(f)$	$\mathcal{NCO}_3^{(0)}$	$E_{4,n}^{(0)}(f)$	$\mathcal{NCO}_4^{(0)}$	$E_{5,n}^{(0)}(f)$	$\mathcal{NCO}_5^{(0)}$
16	3.2526×10^{-7}	–	2.9322×10^{-9}	–	7.7024×10^{-10}	–
32	2.2001×10^{-8}	3.88593	8.4598×10^{-11}	5.11523	1.2296×10^{-11}	5.96897
64	1.4290×10^{-9}	3.94444	2.5408×10^{-12}	5.05726	1.9301×10^{-13}	5.99342
128	9.1028×10^{-11}	3.97262	7.7883×10^{-14}	5.02783	3.0199×10^{-15}	5.99804
Theoretical value	–	04	–	05	–	06
n	$E_{3,n}^{(1)}(f)$	$\mathcal{NCO}_3^{(1)}$	$E_{4,n}^{(1)}(f)$	$\mathcal{NCO}_4^{(1)}$	$E_{5,n}^{(1)}(f)$	$\mathcal{NCO}_5^{(1)}$
16	1.3792×10^{-5}	–	3.5624×10^{-7}	–	1.0091×10^{-8}	–
32	1.7416×10^{-6}	2.98529	2.2456×10^{-8}	3.98765	1.7790×10^{-10}	5.82586
64	2.1881×10^{-7}	2.99266	1.4094×10^{-9}	3.99395	4.1873×10^{-12}	5.40892
128	2.7421×10^{-8}	2.99633	8.8271×10^{-11}	3.99701	1.2997×10^{-13}	5.00969
Theoretical value	–	03	–	04	–	05
n	$E_{3,n}^{(2)}(f)$	$\mathcal{NCO}_3^{(2)}$	$E_{4,n}^{(2)}(f)$	$\mathcal{NCO}_4^{(2)}$	$E_{5,n}^{(2)}(f)$	$\mathcal{NCO}_5^{(2)}$
16	5.8810×10^{-3}	–	1.7911×10^{-5}	–	1.1748×10^{-5}	–
32	1.4715×10^{-3}	1.99877	1.5738×10^{-6}	3.50853	6.8845×10^{-7}	4.09301
64	3.6801×10^{-4}	1.99948	1.5714×10^{-7}	3.32409	4.1129×10^{-8}	4.67034
128	9.2018×10^{-5}	1.99976	2.1328×10^{-8}	2.88130	2.5033×10^{-9}	4.03825
Theoretical value	–	02	–	03	–	04

TABLE 1. Approximation of function and first- and second-order derivatives.

Conclusion

We have presented a general framework for constructing univariate spline spaces on refined non-uniform partitions, together with a normalized basis and a fully local formulation of the Hermite interpolant leading to a B-spline-type representation. The resulting quasi-interpolants exhibit superconvergence properties, as confirmed by numerical experiments, which demonstrates the effectiveness of the proposed approach for high-accuracy approximation on non-uniform meshes.

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