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A STRONGLY CLEAN ENDOMORPHISM RING THROUGH PIERCE SHEAF

KAMALAKAR R. SURWADE[1*] and RAJENDRA P. DEORE[1]

Abstract. This paper aims to construct a Pierce sheaf of endomorphism rings for a module over an associative ring, establishing that if a module has local Pierce stalks, then its endomorphism ring is strongly clean. Furthermore, we prove that the endomorphism ring of a module is isomorphic to the ring of global sections of the sheaf.

Keywords. Endomorphism ring, Strongly clean ring, Pierce Sheaf.
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1. Introduction

In this paper, all rings under consideration are associative with unity (not necessarily commutative), and all modules are considered to be left modules. Let R be an associative ring with unity, and let $B(R)$ denote the Boolean algebra of central idempotents of R. Set $X = \operatorname{MaxSpec}(B(R))$, the Stone space of maximal ideals of $B(R)$.

For an R-module A, denote by A_x the Pierce stalk at $x \in X$. For each $x \in X$, let $\mathcal{E}_x = \operatorname{End}_{R_x}(A_x)$, and let $\mathcal{E}$ denote the disjoint union of the rings $\mathcal{E}_x$. We construct a sheaf $\mathcal{E}$ of rings on X with stalk $\mathcal{E}_x$, and establish an isomorphism

$$\Gamma(X, \mathcal{E}) \cong \operatorname{End}_R(A).$$

Moreover, it is proved that $\Gamma(X, \mathcal{E})$ is strongly clean whenever A is a finite-length R-module. This result is further extended to a direct sum A^n for every $n \in \mathbb{N}$.

For $e \in B(R)$, define

$$D(e) = \{x \in X : e \notin x\},$$

the corresponding basic clopen subset of X. The collection $\{D(e) : e \in B(R)\}$ forms a basis of clopen sets for the Stone topology on X. For an ideal I of $B(R)$, let $\overline{I}$ denote the ideal of R generated by I. We denote by $(X, \mathcal{R})$ the ringed space consisting of the topological space X equipped with a sheaf $\mathcal{R}$ of rings, and by $(X, \mathcal{A})$ the associated sheaf of $\mathcal{R}$-modules. In this setting, each stalk $\mathcal{A}_x$ at $x \in X$ is an $\mathcal{R}_x$-module [13]. Burgess and Stephenson [2] introduced a decomposition of any ring R as a subdirect product of its maximal indecomposable factors. For

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* Corresponding author

semi-perfect rings, this subdirect product becomes a direct product whenever the Jacobson radical is zero. However, this is because if $J(R) \neq 0$, the lifting of central idempotents of $R/J(R)$ may not give central idempotents of R.

In [11], Nicholson has introduced the class of *clean rings*, in which every element is the sum of a unit and an idempotent. If, in addition, a unit and an idempotent commute, the element is called *strongly clean*. Later, in [12], Nicholson related the module decompositions to the strong cleanness of endomorphism rings. In [3], Burgess and Stephenson showed that every Pierce stalk of a ring is local if and only if every element of a ring is strongly clean.

Further, Burkholder [4, Theorem 1.2] generalized this by proving that, for any finitely generated R-module A, the module A has local Pierce stalks if and only if there exists an element $a \in A$ such that for every $b \in A$, there exists a central idempotent $e \in B(R)$ with $b - ea$ generates A; equivalently, A is cyclic and generated by an element depending on $e \in B(R)$. In [10, Theorem 3], Monk proved that the category of commutative exchange rings is equivalent to the category of ringed spaces $(X, \mathcal{R})$, where X is a Boolean space and $\mathcal{R}$ is a sheaf of commutative local rings. Burgess and Stephenson [3] generalized Monk's theorem to show that the Pierce stalks of a ring R (not necessarily commutative) are local if and only if R is an exchange ring whose idempotents are central. Burkholder [4] further characterized a module with local Pierce stalks in Theorem 3.3. In our study, we prove that if A has local Pierce stalks, then $\operatorname{End}_R(A)$ also has local Pierce stalks on the Stone space X, implies that $\operatorname{End}_R(A)$ is strongly clean.

2. Sheaf of Rings and Modules

The investigations of rings and their modules are based on theory of sheaves. The background material on the Stone spaces has been taken from the authoritative work of R. S. Pierce [13]. Throughout the paper, the notations and terminologies are as given in [5, 13], *presheaf* and *sheaf* are as defined in [5].

Definition 2.1 (Pierce Sheaf)**.** [13] For $e \in B(R)$, define

$$R_e := \frac{R}{(1-e)R}, \qquad A_e := \frac{A}{(1-e)A}.$$

For a point $x \in X$, the *Pierce stalks* are given by

$$R_x = \varinjlim_{e \notin x} R_e, \qquad A_x = \varinjlim_{e \notin x} A_e.$$

Define $\overline{x} = \{eR \mid e \in x\}$ to be an ideal of R generated by all idempotents contained in the maximal ideal x of $B(R)$. It is a well-known fact that the Pierce stalks can be expressed as

$$R_x = \varinjlim_{e \notin x} R_e = \frac{R}{\overline{x}}, \qquad A_x = \varinjlim_{e \notin x} A_e = \frac{A}{\overline{x}A}.$$

Thus, R_x is a quotient ring of R, and A_x is a quotient R-module of A. These directed colimits are taken over the directed set of idempotents e with $e \notin x$.

For each basic open set $D(e)$, define the ring

$$\mathcal{E}(D(e)) := \mathrm{End}_{R_e}(A_e).$$

If $D(f) \subseteq D(e)$ (equivalently, $ef = f$, or $f \leq e$ in the sense of the directed set of idempotents), there are natural restriction maps

$$\rho_{e,f} : \mathrm{End}_{R_e}(A_e) \longrightarrow \mathrm{End}_{R_f}(A_f),$$

induced by the quotient maps $A_e \to A_f$ and $R_e \to R_f$. These maps are compatible with composition and yield a presheaf on the basis $\{D(e)\}$.

Since the sets $D(e)$ are clopen and form a basis closed under finite intersections, it suffices to verify that the sheaf axioms on finite coverings by basic open sets. The presheaf (defined on clopen sets) extends uniquely to a presheaf on all open subsets, and in fact to a sheaf. The existence of this extension is supported by the *partition property* of Boolean algebras.

A Boolean algebra is equipped with the *partition property* [13]: if X is a Boolean space and $\{N_i \mid i \in I\}$ is an open covering of X, then there exists a finite set of clopen subsets $\{M_1, \dots, M_r\}$ of X such that:

(i) for each $j \leq r$, there exists an $i \in I$ with $M_j \subseteq N_i$;
(ii) $M_j \cap M_k = \varnothing$ if $j \neq k$; and
(iii) $\displaystyle\bigcup_{j=1}^{r} M_j = X$.

The existence of pairwise orthogonal central idempotents is made possible precisely because of the partition property. One can extract the following simple fact from [13, Proof of Theorem 4.4]: if $e \in B(R)$ and $D(e) = \bigcup_{i=1}^{n} D(e_i)$ is a finite cover by basic open sets, then there exist pairwise orthogonal central idempotents $f_1, \dots, f_m \in B(R)$ such that $\sum_{j=1}^{m} f_j = e$ and each $f_j \leq e_k$ for some $k \in \{1, 2, \dots, n\}$. This assertion ensures the construction of a sheaf of abelian groups.

Proposition 2.2. *The presheaf $D(e) \mapsto \mathcal{E}(D(e)) = \mathrm{End}_{R_e}(A_e)$ is a sheaf on the basis $\{D(e)\}$; hence, it extends to a sheaf of rings on X.*

Proof. (Uniqueness) Suppose $\varphi \in \mathrm{End}_{R_e}(A_e)$ restricts to zero on a finite cover $D(e) = \bigcup_{i=1}^{n} D(e_i)$. By the partition property, there exist orthogonal idempotents $f_1, \dots, f_m$ with $\sum_{j=1}^{m} f_j = e$ and each $f_j \leq e_{i(j)}$ for some $i(j) \in \{1, 2, \dots, n\}$. Then

$$A_e \cong \bigoplus_{j=1}^{m} A_{f_j},$$

and the restriction of φ to each summand A_{f_j} is zero since $f_j \leq e_{i(j)}$ and φ vanishes on $D(e_{i(j)})$. Hence, $\varphi = 0$.

(Existence / Gluing) Suppose for each i, we have $\varphi_i \in \mathrm{End}_{R_{e_i}}(A_{e_i})$, and the family $\{\varphi_i\}$ satisfies the compatibility condition

$$\varphi_i|_{D(e_i)\cap D(e_j)} = \varphi_j|_{D(e_i)\cap D(e_j)} \quad \text{for all } i, j.$$

The cover $D(e) = \bigcup_i D(e_i)$ can be refined into a partition by basic open sets associated with orthogonal idempotents $f_1, \dots, f_m$ satisfying $e = \sum_{j=1}^m f_j$ and each $f_j \leq e_{i(j)}$. Each $\varphi_{i(j)}$ induces a well-defined endomorphism ψ_j of A_{f_j}. Define

$$\varphi\left(\sum_j a_j\right) := \sum_j \psi_j(a_j), \qquad a_j \in A_{f_j},$$

using the decomposition $A_e \cong \bigoplus_j A_{f_j}$. The compatibility on overlaps ensures that the ψ_j's are well-defined and that φ restricts to φ_i on each A_{e_i}. The centrality of the f_j's guarantees that φ is R_e-linear. Thus, the φ_i glue to a unique element $\varphi \in \mathrm{End}_{R_e}(A_e)$. So, the presheaf is a sheaf on the basis, which means it can be extended to a sheaf $\mathcal{E}$ of rings on X.

□

Remark 2.3. The stalk at a point $x \in X$ is, by construction, the directed limit

$$\mathcal{E}_x = \varinjlim_{e \notin x} \mathrm{End}_{R_e}(A_e).$$

Unwinding the definitions and using the fact that endomorphisms of a directed colimit are given by the compatible representatives at some stage, we obtain

$$\mathcal{E}_x \cong \mathrm{End}_{R_x}(A_x).$$

Concretely, any endomorphism of the stalk is represented by an endomorphism defined on some neighborhood $D(e)$; two representatives define the same element in the stalk if and only if they agree on a smaller neighborhood. This corresponds precisely to the directed colimit description.

Lemma 2.4. *Let R be a ring with unity, $B(R)$ its Boolean algebra of central idempotents, $X = \mathrm{MaxSpec}(B(R))$ its Stone space, and fix $x \in X$. For $e \in B(R)$, set $R_e := R/R(1-e)$ and $A_e := A/A(1-e)$. Then*

$$\varinjlim_{e \notin x} \mathrm{End}_{R_e}(A_e) = \mathrm{End}_{R_x}(A_x),$$

where $R_x = \varinjlim_{e \notin x} R_e$ and $A_x = \varinjlim_{e \notin x} A_e$.

Proof. For $x \in X$, let $J = \{e \mid e \notin x\}$ be the index category, and consider the category C consisting of objects $\mathrm{End}_{R_e}(A_e)$ for each $e \in J$. Define a functor $F : J \longrightarrow C$ by

$$F(e) = \mathrm{End}_{R_e}(A_e),$$

and for $f \leq e$, let the corresponding morphism be

$$\mathrm{End}_{R_e}(A_e) \longrightarrow \mathrm{End}_{R_f}(A_f).$$

Consider the diagonal functor $\delta : C \longrightarrow C^J$, which takes each object c to the constant functor δc. Let $r = \mathrm{End}_{R_x}(A_x)$, where $R_x = \varinjlim_{e \notin x} R_e$ and $A_x = \varinjlim_{e \notin x} A_e$.

For each $e \notin x$ and $\varphi \in \mathrm{End}_{R_e}(A_e)$, if $a \in (1-e)A$, then $\varphi(a) \in \overline{x}$. Define a morphism $u : F \longrightarrow \delta r$ such that for an object $e \in J$, u_e is the map

$$u_e : \mathrm{End}_{R_e}(A_e) \longrightarrow \mathrm{End}_{R_x}(A_x),$$

and for $f \leq e$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{End}_{R_e}(A_e) & \xrightarrow{u_e} & \mathrm{End}_{R_x}(A_x) \\ \big\downarrow & & \big\downarrow \\ \mathrm{End}_{R_f}(A_f) & \xrightarrow{u_f} & \mathrm{End}_{R_x}(A_x). \end{array}$$

This implies u is a natural transformation. Hence, (r, u) is the universal arrow, and we conclude that

$$\varinjlim_{e \notin x} \mathrm{End}_{R_e}(A_e) = \mathrm{End}_{R_x}(A_x).$$

□

Theorem 2.5. *The pair $(X, \mathcal{E})$, consisting of the Stone space X and the sheaf $\mathcal{E}_x$ of rings for $x \in X$, forms a ringed space.*

Proof. The assertion derives from Proposition 2.2 and Lemma 2.4. □

Remark 2.6. (1) The ring $\mathcal{E}_x = \mathrm{End}_{R_x}(A_x)$ is the stalk of the sheaf $\mathcal{E}$ at $x \in X$, which is obtained as the direct limit taken over all $e \notin x$.

(2) Lemma 2.5 together with Theorem 2.6 assert that the image of an open set $D(e)$ under the functor $\mathcal{E}$ is $\mathcal{E}_e = \mathrm{End}_{R_e}(A_e)$, where $R_e := R/R(1-e)$ and $A_e := A/A(1-e)$.

Theorem 2.7. *The endomorphism ring $\mathrm{End}_R(A)$ is isomorphic to the ring of global sections $\Gamma(X, \mathcal{E})$.*

Proof. Define a map

$$\Phi : \mathrm{End}_R(A) \longrightarrow \Gamma(X, \mathcal{E})$$

by setting $\Phi(\varphi)(x, a) = \varphi(a) + \overline{x}A \in \mathcal{E}_x$. Assume that for each $a \in A$, we have $\varphi(a) + \overline{x}A = \varphi_x(a) \in \mathcal{E}_x$. If $\varphi = \psi \in \mathrm{End}_R(A)$, then

$$\Phi(\varphi)(x, a) = \varphi_x(a) = \psi_x(a) = \Phi(\psi)(x, a)$$

for every $a \in A$ and $x \in X$. Define $\varphi_s : X \longrightarrow \mathcal{E}$ as the map of $\mathcal{E}$ over X given by $\varphi_s(x) = \varphi_x$. From [13, Definition 3.1], φ_s is continuous and hence a section. Thus, Φ is a well-defined map.

For the ring homomorphism property, we have

$$\Phi(\varphi + \psi)(x, a) = \varphi(a) + \psi(a) + \overline{x}A = \varphi_x(a) + \psi_x(a),$$

and hence Φ is a ring homomorphism.

Injectivity: Suppose $\varphi \in \mathrm{End}_R(A)$ maps to the zero global section. This means that $\Phi(\varphi)(x, a) = 0_x \in \mathcal{E}_x$ for all $x \in X$. Since X is a Stone space, by the finite partition property there exist pairwise orthogonal central idempotents f_j such that $1 = \sum_{j=1}^{m} f_j$. Then

$$A \cong \bigoplus_{j=1}^{m} A_{f_j}.$$

Since $\varphi_{f_j} = 0$ for every j, it follows that φ is zero on every summand and hence $\varphi = 0$. Therefore, Φ is injective.

Surjectivity: Let $s \in \Gamma(X, \mathcal{E})$ be a global section. By compactness of the Stone space X, we may represent s by finitely many compatible local endomorphisms $s_{e_i} \in \mathrm{End}_{R_{e_i}}(A_{e_i})$ for a finite cover $X = \bigcup_{i=1}^{n} D(e_i)$. Refine this cover to a finite partition $1 = \sum_{j=1}^{m} f_j$ with each $f_j \leq e_{i(j)}$. Then $A = \bigoplus_{j=1}^{m} A_{f_j}$. On each summand A_{f_j}, the local section (an element of $\mathcal{E}(D(f_j))$) yields a well-defined endomorphism $\psi_j \in \mathrm{End}_{R_{f_j}}(A_{f_j})$. Define

$$\psi : A = \bigoplus_{j=1}^{m} A_{f_j} \longrightarrow A, \qquad \psi\left(\sum_j a_j\right) = \sum_j \psi_j(a_j).$$

We now check that ψ is an R-module endomorphism. For $r \in R$ and $a_j \in A_{f_j}$,

$$\psi(r \cdot a_j) = \psi_j(ra_j) = r\psi_j(a_j),$$

since each ψ_j is R_{f_j}-linear, and multiplication by r preserves the decomposition after passing to the quotient by $1 - f_j$. Centrality of the f_j ensures that multiplication by r maps A_{f_j} to itself (indeed $rf_j = f_j r$, and $f_j(rr') = (f_j r)r'$, etc.). Thus, ψ is R-linear, and by construction, ψ induces the section s on each $D(e_i)$. Therefore, $\Phi(\psi) = s$, proving surjectivity.

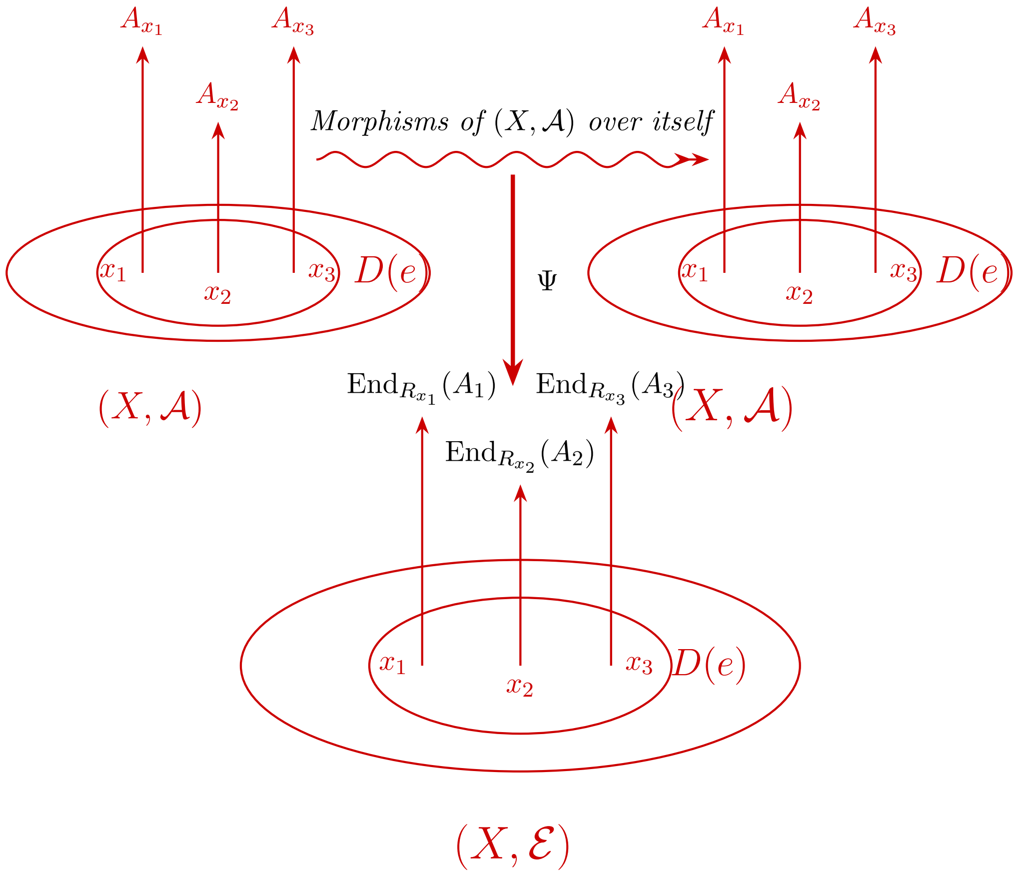
Combining injectivity and surjectivity, we obtain the desired isomorphism:

$$\Gamma(X, \mathcal{E}) \cong \mathrm{End}_R(A).$$

□

Remark 2.8. As shown in the figure, a morphism Ψ is a natural morphism from the set of all morphisms of the sheaf of modules into itself to the sheaf of endomorphism rings over the same Stone space. A section in $\Gamma(X, \mathcal{E})$ corresponds to a morphism from $(X, \mathcal{A})$ to itself. This correspondence yields an isomorphism

$$\Phi : \mathrm{End}_R(A) \longrightarrow \Gamma(X, \mathcal{E}).$$



Example 2.9. Let $R = \mathbb{Z}/6\mathbb{Z}$. The central idempotents in R are $0, 1, 3, 4$ (since $3^2 \equiv 3$ and $4^2 \equiv 4 \pmod 6$), and hence $B(R)$ is the four-element Boolean algebra generated by the idempotent $e \equiv 3 \pmod 6$ (equivalently by $f \equiv 4$). The Stone space X consists of two ultrafilters corresponding to the two primitive idempotents 3 and 4, which reflect the Chinese Remainder decomposition

$$\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$$

Let $A = R$ (the regular module). Then

$$A_3 \cong \frac{R}{R(1-3)} \cong \mathbb{Z}/2\mathbb{Z}, \qquad A_4 \cong \mathbb{Z}/3\mathbb{Z}.$$

The sheaf $\mathcal{E}$ has stalks

$$\mathcal{E}_{x_2} \cong \mathrm{End}_{\mathbb{Z}/2}(\mathbb{Z}/2) \cong \mathbb{Z}/2, \qquad \mathcal{E}_{x_3} \cong \mathrm{End}_{\mathbb{Z}/3}(\mathbb{Z}/3) \cong \mathbb{Z}/3.$$

The global section ring reconstructed from these local endomorphism rings (via the gluing technique) recovers

$$\mathrm{End}_R(A) \cong \mathbb{Z}/6\mathbb{Z}.$$

Proposition 2.10. *If A is a finite-length or Artinian or Noetherian local left R-module, then $\mathrm{End}_R(A)$ is a local ring.*

Proof. Every local left R-module is indecomposable. Furthermore, if A is a finite length module then by [6, Theorem 19.16], A is strongly indecomposable, hence $\mathrm{End}_R(A)$ is a local ring.

Assume A is a Noetherian local left R-module. Consider the following two chains of modules

$$0 \subseteq \ker(f) \subseteq \ker(f^2) \subseteq \cdots \quad (1)$$

$$M \supseteq \mathrm{im}(f) \supseteq \mathrm{im}(f^2) \supseteq \cdots \quad (2)$$

If A is Noetherian then series (1) stabilizes (and if it is Artinian then series (2) stabilizes). There exists an $r \in \mathbb{N}$ such that $\ker(f^r) = \ker(f^{2r})$. It gives $\mathrm{im}(f^r) \cong \dfrac{A}{\ker(f^r)} = \dfrac{A}{\ker(f^{2r})} \cong \mathrm{im}(f^{2r})$, which again implies $\mathrm{im}(f^r) = \mathrm{im}(f^{2r})$ because $\mathrm{im}(f^r) \supseteq \mathrm{im}(f^{2r})$. By [6, Theorem 19.16] $A = \ker(f^r) \oplus \mathrm{im}(f^r)$. But A is indecomposable, therefore f is either an isomorphism or a nilpotent. Hence, $\mathrm{End}_R(A)$ is a local ring. (Similar argument works for Artinian rings.) □

Proposition 2.11. [3, Proposition 1.2] *For a ring R, the Pierce stalk R_x is local for all $x \in X$ if and only if R is strongly clean.*

Burkholder in [4, Theorem 3.3] proved that a finitely generated quasi-projective R-module A has local Pierce stalks if and only if A has the exchange property and every idempotent of $\mathrm{End}_R(A)$ lies in the image of $B(R)$. The next theorem provides a generalization of [4, Theorem 3.3].

Theorem 2.12. *If a finite-length R-module A has a local stalk A_x for every $x \in X$, then $E = \mathrm{End}_R(A)$ is strongly clean.*

Proof. It is known that every local R-module is indecomposable and quotient of a finite-length module is of finite-length. Let $\varphi \in E$. For each $x \in X$, either φ_x or $\mathrm{Id}_x - \varphi_x$ is a unit, where Id_x denotes the identity R_x-endomorphism of A_x. There exists a clopen neighborhood U_x of x in X such that for all $y \in U_x$, either φ_y or $\mathrm{Id}_y - \varphi_y$ is a unit. Consider a partition of X into disjoint clopen sets C and D, such that for $x \in C$, φ_x is a unit and for $x \in D$, $\mathrm{Id}_x - \varphi_x$ is a unit. By the surjectivity of the map Φ from Theorem 2.8, there exists an idempotent endomorphism $e \in \mathrm{End}_R(A)$ such that $e_x = 0_x$ for all $x \in C$ and $e_x = \mathrm{Id}_x$ for all $x \in D$ (where $0_x(a + \overline{x}A) = \overline{x}A$ and $\mathrm{Id}_x(a + \overline{x}A) = a + \overline{x}A$). This implies that either φ or $\mathrm{Id} - \varphi$ is a unit in E, proving that E is strongly clean. □

We prove the following results of this section for a class of an R-module containing finite length R-module as well as Artinian and Noetherian modules.

Corollary 2.13. *If A is a local R-module, then* $\mathrm{End}_R(A)$ *is strongly clean.*

Proof. Since $A_x = A/\overline{x}A$ is a quotient module of A, and every quotient of a local module is local, it follows that A_x is local for all $x \in X$. Hence $\mathrm{End}_R(A)$ is strongly clean.

Alternative proof: Let $E = \mathrm{End}_R(A)$ be the endomorphism ring of a local R-module A. Then E has only two central idempotents, $B(E) = \{0, 1\}$. The Stone space of $B(E)$ contains a single point corresponding to the zero ideal $m = (0)$. Thus $\overline{m} = (0)$, and the Pierce stalk associated with m is

$$E_m = \frac{E}{\overline{m}E} = E,$$

which is a local ring. Hence, by Proposition 2.12 ([3, Proposition 1.2]), E is strongly clean. $\square$

Similarly, the endomorphism ring of a direct sum of finitely many local modules also admits local Pierce stalks.

Proposition 2.14. *If A is a local R-module and $B = A^n$ for some $n \in \mathbb{N}$, then $\mathrm{End}_R(B)$ is strongly clean.*

Proof. There is a ring isomorphism

$$E = \mathrm{End}_R(B) \cong \begin{pmatrix} \mathrm{End}_R(A) & \mathrm{End}_R(A) & \cdots & \mathrm{End}_R(A) \\ \mathrm{End}_R(A) & \mathrm{End}_R(A) & \cdots & \mathrm{End}_R(A) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{End}_R(A) & \mathrm{End}_R(A) & \cdots & \mathrm{End}_R(A) \end{pmatrix}_{n\times n},$$

where addition is entrywise and multiplication is given by the usual matrix product, $(C \cdot D)_{ij} = \sum_{k=1}^{n} C_{ik}D_{kj}$. Since A is a local R-module, $\mathrm{End}_R(A)$ is a local ring and has only trivial idempotents. By [6, Theorem 3.1], E is also a local ring, which means that the only central idempotents of E are 0 and 1. As E is local, its Pierce stalk E_x over the Boolean space of E is local. Therefore, by Proposition 2.12, $\mathrm{End}_R(B)$ is strongly clean. $\square$

Proposition 2.15. *If A and B are R-modules having local stalks, and the supports of A and B are disjoint (that is, if $A_x \neq 0$ then $B_x = 0$, and if $B_x \neq 0$ then $A_x = 0$), then $\mathrm{End}_R(A \oplus B)$ is strongly clean.*

Proof. For any $x \in X$, we have

$$(A \oplus B)_x \cong A_x \oplus B_x,$$

which is either A_x, B_x, or (0) each of which is local. Hence, $\mathrm{End}_R(A \oplus B)$ is strongly clean. $\square$

This result can be generalized as follows.

Proposition 2.16. *If $A_1, \ldots, A_n$ are R-modules having local stalks, and the supports of the A_i's are mutually disjoint, then*

$$\mathrm{End}_R\left(\bigoplus_{i=1}^{n} A_i\right)$$

is strongly clean.

3. Semi-perfect Rings

If R is a semi-perfect ring with $J(R) = 0$, then R is semi-simple. By the Wedderburn-Artin theorem [6],

$$R \cong \mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_r}(D_r),$$

for suitable division rings $D_1, \ldots, D_r$ and positive integers $n_1, \ldots, n_r$. Thus, R is a direct product of local rings, and each factor contributes a primitive central idempotent e_i in R.

Proposition 3.1. *If R is a semi-perfect ring with $J(R) = 0$ and*

$$A = D_1^{n_1} \oplus \cdots \oplus D_r^{n_r}$$

is an R-module where the D_i's are distinct division rings, then the endomorphism ring

$$E = \mathrm{End}_R(A)$$

is strongly clean.

Proof. Let $A = D_1^{n_1} \oplus \cdots \oplus D_r^{n_r}$ be an R-module where the D_i's are distinct division rings. Then

$$E = \mathrm{End}_R(A) \cong \prod_{i=1}^{r} \mathbb{M}_{n_i}(D_i).$$

The Boolean algebra $B(E)$ is finite, implying that the Stone space Y of $B(E)$ is also finite. The topology of a finite Hausdorff space is necessarily discrete, so every singleton of Y is clopen. Since the D_i's are distinct division rings, the rings $\mathbb{M}_{n_i}(D_i)$ are local and have only trivial idempotents.

Let $e_1, \ldots, e_r$ be the primitive central orthogonal idempotents of E. Without loss of generality, assume that $Y = \{y_1, \ldots, y_r\}$, that is, $B(E)$ has r maximal ideals such that each y_i is the ideal generated by $\{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_r\}$ in $B(E)$. This implies that the open set associated with e_i is

$$D(e_i) = \{y \in B(E) \mid e_i \notin y\} = \{y_i\}.$$

The Pierce stalk at y_i is

$$E_{y_i} = \frac{E}{\overline{y_i}E} = \frac{E}{(1 - e_i)E} = \mathbb{M}_{n_i}(D_i),$$

which is a local ring. Hence, the endomorphism ring $E = \mathrm{End}_R(A)$ is strongly clean. □

Corollary 3.2. *If R is semi-perfect with $J(R) = 0$, then R is strongly clean.*

Remark 3.3. If R is a semi-perfect ring, then

$$\frac{R}{J(R)} \cong \mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_r}(D_r).$$

The primitive central idempotents of $\frac{R}{J(R)}$ lift to idempotents in R, but the lifts need not be central in R. Thus, $\frac{R}{J(R)}$ may decompose non-trivially while R itself remains indecomposable.

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[1] Department of Mathematics, University of Mumbai, Mumbai, India.
Email address: kamalakar.surwade@mathematics.mu.ac.in

[2] Department of Mathematics, University of Mumbai, Mumbai, India.
Email address: rpdeore@mathematics.mu.ac.in