Gulf Journal of Mathematics ISSN: 2309-4966
Vol 23, Issue 1 (2026) 1-13 https://doi.org/10.56947/4yhmah32

INFINITE HORIZON OPTIMAL CONTROL OF FORWARD-BACKWARD STOCHASTIC VOLTERRA EQUATIONS WITH DELAY

IBTISSEM DJABER[1], NAWEL HAFIANE[2] and SAMIA YAKHLEF[1]

Abstract. This paper investigates an infinite-horizon optimal control problem for systems governed by coupled delayed forward-backward stochastic Volterra integral equations driven by Brownian motion and jump processes. Under partial information, we establish both necessary and sufficient stochastic maximum principles by combining localization arguments with Hida–Malliavin calculus techniques adapted to Volterra memory effects. We also study the associated infinite-horizon backward stochastic Volterra equations and prove existence and uniqueness in weighted spaces by means of a contraction mapping argument, introducing a free parameter $\psi(t)$ which is an $\mathcal{F}_\infty$-measurable stochastic process as in [6]. The obtained framework extends delayed stochastic control systems with memory and provides a rigorous basis for long-term optimization problems.

Keywords. Infinite horizon control, delayed FBSVIEs, stochastic Volterra systems, partial information, Hida–Malliavin calculus.

2020 Mathematics Subject Classification. Primary 60H20; Secondary 93E20, 45D05.

1. Introduction

Optimal control problems for forward–backward stochastic systems play a central role in mathematical finance, stochastic economics, and engineering applications, particularly in recursive utility, long-term investment, and risk-sensitive decision models. In contrast to finite-horizon settings, the infinite-horizon case requires additional analytical care because terminal conditions are replaced by suitable transversality constraints, and the long-time behavior of the coupled system must be controlled in weighted spaces.

Although the theory of infinite-horizon forward–backward stochastic differential equations is well developed, most existing contributions focus on Markovian systems without memory or delay effects. However, many realistic models in economics, finance, engineering, and population dynamics depend not only on the present state but also on the past trajectory of the system. This naturally leads to delayed stochastic Volterra models, where hereditary effects, aftereffects, and path dependence are explicitly represented through memory kernels.

Date: Received: Apr 1, 2026; Revised: Apr 27, 2026; Accepted: May 12, 2026.
* Corresponding author

The main objective of this work is to develop an optimal control framework for coupled delayed forward–backward stochastic Volterra integral equations on an infinite horizon under partial information. The Volterra structure introduces substantial technical challenges because both the state and adjoint equations inherit nonlocal dependence in time, while the delay component adds an additional layer of path dependence.

The contributions of this paper can be summarized as follows:

(1) We formulate an infinite-horizon stochastic control problem for delayed coupled forward–backward stochastic Volterra integral equations under partial information, extending classical delayed FBSDE models to a genuinely non-Markovian Volterra setting.
(2) By combining localization arguments with Hida–Malliavin calculus techniques, we derive both sufficient and necessary stochastic maximum principles. The resulting adjoint system captures simultaneously the memory effect, delay dependence, and infinite-horizon transversality structure.
(3) We establish existence and uniqueness results for a class of associated infinite-horizon backward stochastic Volterra integral equations with delay in exponentially weighted spaces, incorporating a free parameter $\psi(t)$ which is an $\mathcal{F}_\infty$-measurable stochastic process (see [6]). These solvability results provide the analytical foundation required for the optimality system.

The rest of the paper is organized as follows. Section 2 introduces the stochastic control problem, admissible controls, and the required analytical framework. Section 3 establishes a sufficient maximum principle under suitable concavity assumptions. Section 4 derives the necessary maximum principle by means of spike variation arguments. Section 5 is devoted to the well-posedness of the associated infinite-horizon BSVIEs with delay and a free stochastic parameter $\psi(t)$, which completes the mathematical basis of the control problem.

2. Setting of the Problem

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $B(t)$ and an independent compensated Poisson random measure $\tilde{N}(dt, de) = N(dt, de) - \nu(de)dt$ are defined. We assume that $\mathbb{F}$ is the natural filtration, made right continuous and complete, generated by the processes B and N.

We study infinite horizon coupled forward-backward stochastic Volterra integral equations with delay under partial information. The information available to the controller is given by a sub-filtration $\mathbb{G} = \{\mathcal{G}_t\}_{t\geq 0}$ such that $\mathcal{G}_t \subseteq \mathcal{F}_t$ for all $t \geq 0$. The set $U \subset \mathbb{R}$ is assumed to be convex. The set of admissible controls $\mathcal{A}_{\mathbb{G}}$ consists of càdlàg, U-valued and $\mathbb{G}$-adapted processes.

2.1. **Forward Equation.** The forward equation for the unknown process $X(t)$ is:

$$\begin{aligned} X(t) = \xi(t) + &\int_0^t b(t,s,X(s),X(s-\delta),u(s))\,ds \\ &+ \int_0^t \sigma(t,s,X(s),X(s-\delta),u(s))\,dB(s) \\ &+ \int_0^t \int_{\mathbb{R}_0} \theta(t,s,X(s),X(s-\delta),u(s),e)\,\tilde{N}(ds,de), \quad t \geq 0, \end{aligned} \tag{2.1}$$

with initial condition $X(t) = \xi(t)$ for $t \in [-\delta, 0]$.

2.2. **Backward Equation.** The backward equation for the unknown processes $(Y(t), Z(t,s), K(t,s,\cdot))$ includes a free stochastic process $\psi(t)$:

$$\begin{aligned} Y(t) = \psi(t) + &\int_t^\infty g(t,s,X(s),X(s-\delta),Y(s),Z(t,s),K(t,s,\cdot),u(s))\,ds \\ &- \int_t^\infty Z(t,s)\,dB(s) - \int_t^\infty \int_{\mathbb{R}_0} K(t,s,e)\,\tilde{N}(ds,de), \quad 0 \leq t < \infty. \end{aligned} \tag{2.2}$$

The function $\psi : [0,\infty) \times \Omega \to \mathbb{R}$ is a given (not necessarily adapted) $\mathcal{F}_\infty$-measurable stochastic process, called the free term, satisfying

$$\mathbb{E}\left[\int_0^\infty e^{\beta s} |\psi(s)|^2 ds\right] < \infty$$

for some $\beta > 0$ (see [6]). The presence of $\psi(\cdot)$ is essential for the well-posedness of the infinite-horizon problem; it acts as a free parameter that can be chosen to satisfy, e.g., a transversality condition or a fixed-point requirement. In this work, we consider the general case where ψ is an $\mathcal{F}_\infty$-measurable stochastic process, as in [6].

The quadruple (X, Y, Z, K) is said to be a solution of (2.1)–(2.2) if it satisfies both equations.

Throughout this paper, the coefficient functions are defined as:

$$\begin{aligned} b(t,s,x,x_1,u) &: [0,\infty)^2 \times \mathbb{R}^2 \times U \times \Omega \to \mathbb{R}, \\ \sigma(t,s,x,x_1,u) &: [0,\infty)^2 \times \mathbb{R}^2 \times U \times \Omega \to \mathbb{R}, \\ \theta(t,s,x,x_1,u,e) &: [0,\infty)^2 \times \mathbb{R}^2 \times U \times \mathbb{R}_0 \times \Omega \to \mathbb{R}, \\ g(t,s,x,x_1,y,z,k,u) &: [0,\infty)^2 \times \mathbb{R}^4 \times L^2(\nu) \times U \times \Omega \to \mathbb{R}. \end{aligned}$$

Let $\mathcal{U} = \mathcal{U}_{\mathbb{G}}$ be the family of admissible controls, required to be $\mathbb{G}$-predictable. We associate to the system the performance functional:

$$J(u) = \mathbb{E}\left[\int_0^\infty f(t,X(t),X(t-\delta),Y(t),u(t))\,dt + h(Y(0))\right], \quad u \in \mathcal{A}_{\mathbb{G}}, \tag{2.3}$$

where

$$\begin{aligned} f(t,x,x_1,y,u) &: [0,\infty) \times \mathbb{R}^3 \times U \times \Omega \to \mathbb{R}, \\ h &: \mathbb{R} \to \mathbb{R} \quad \text{(terminal cost function)}. \end{aligned}$$

2.3. **Assumptions H1.**

(1) The functions f and h are C^1 with respect to (x, x_1, y, u) and $Y(0)$ respectively. The function f is $\mathbb{F}$-adapted and C^1 in (x, y, u).
(2) b, σ, θ and g are continuously differentiable in their first variables, and for all t, x, x_1, y, z, k, u, e, the processes $s \mapsto b(t, s, x, x_1, u)$, $s \mapsto \sigma(t, s, x, x_1, u)$, $s \mapsto \theta(t, s, x, x_1, u, e)$ are $\mathcal{F}_s$-measurable for all $s \le t$ and $s \mapsto g(t, s, x, x_1, y, z, k, u)$ is $\mathcal{F}_s$-measurable for $t \le s$.

Our optimal control problem is to find $u^* \in \mathcal{U}_{\mathbb{G}}$ such that

$$\sup_{u \in \mathcal{U}} J(u) = J(u^*). \tag{2.4}$$

2.4. **Hamiltonian and Adjoint Equations.** Define the Hamiltonian functional:

$$\begin{aligned} H(t, x, x_1, y, z, k, u, \lambda, p, q, r) &:= H_0(t, x, x_1, y, z, k, u, p, q, \lambda) \\ &\quad + H_1(t, x, x_1, y, z, k, u, p, q, \lambda, r), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} H_0(t, x, x_1, y, z, k, u, p, q, \lambda) &:= f(t, x, x_1, y, u) + b(t, t, x, x_1, u)p(t) \\ &\quad + \sigma(t, t, x, x_1, u)q(t) + \int_{\mathbb{R}_0} \theta(t, t, x, x_1, u, e) r(t, e) \nu(de) \\ &\quad + g(t, t, x, x_1, y, z, k, u)\lambda(t), \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} H_1(t, x, x_1, y, z, k, u, p, q, \lambda, r) &:= \int_t^\infty \frac{\partial b}{\partial s}(s, t, x, x_1, u) p(s)\, ds \\ &\quad + \int_t^\infty \frac{\partial \sigma}{\partial s}(s, t, x, x_1, u) q(s, t)\, ds \\ &\quad + \int_t^\infty \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial s}(s, t, x, x_1, u, e) r(s, t, e) \nu(de)\, ds \\ &\quad - \int_0^t \frac{\partial g}{\partial s}(s, t, x, x_1, y, z, k, u)\lambda(s)\, ds \\ &\quad - \int_0^t \frac{\partial g}{\partial z}(s, t, x, x_1, y, z, k, u) \frac{\partial Z}{\partial s}(s, t)\lambda(s)\, ds \\ &\quad - \int_0^t \langle \nabla_k g(s, t, x, x_1, y, z, k, u), \frac{\partial K}{\partial s}(s, t, \cdot)\rangle \lambda(s)\, ds. \end{aligned} \tag{2.7}$$

The associated adjoint processes $\lambda(t)$, $(p(t), q(t), r(t, \cdot))$ satisfy:

$$\begin{cases} d\lambda(t) = \dfrac{\partial H}{\partial y}(t)\, dt + \dfrac{\partial H}{\partial z}(t)\, dB(t) + \displaystyle\int_{\mathbb{R}_0} \frac{d\nabla_k H}{d\nu}(t)\, \tilde{N}(dt, de), \quad t \ge 0, \\ \lambda(0) = h'(Y(0)), \end{cases} \tag{2.8}$$

and

$$
\begin{aligned}
dp(t) = \mathbb{E}[\varkappa(t) \mid \mathcal{F}_t]\, dt &- \left(\int_t^\infty \frac{\partial q}{\partial t}(t,s)\, dB(s) \right) dt \\
&- \left(\int_t^\infty \int_{\mathbb{R}_0} \frac{\partial r}{\partial t}(t,s,e)\, \tilde{N}(ds,de) \right) dt \\
&+ q(t,t)\, dB(t) + \int_{\mathbb{R}_0} r(t,t,e)\, \tilde{N}(dt,de), \qquad t \geq 0,
\end{aligned} \tag{2.9}
$$

where

$$
\varkappa(t) = -\frac{\partial H}{\partial x}(t) + \frac{\partial H}{\partial x_1}(t+\delta).
$$

Assumption H2 We suppose that $t \mapsto q(t,s)$ and $t \mapsto r(t,s,\cdot)$ are C^1 for all s, e, ω and that for all $T < \infty$,

$$
\mathbb{E}\left[\int_0^T \int_0^T \left(\frac{\partial q(t,s)}{\partial t} \right)^2 ds\, dt + \int_0^T \int_0^T \int_{\mathbb{R}_0} \left(\frac{\partial r(t,s,e)}{\partial t} \right)^2 \nu(de)\, ds\, dt \right] < \infty.
$$

Remark 2.1. Assumption H2 is verified in a class of linear BSVIE with jumps. For more details, we refer to [7].

2.5. **Function Spaces.** Throughout this work, we will use the following spaces:

- $\mathcal{S}^2$ is the set of $\mathbb{R}$-valued $\mathbb{F}$-adapted càdlàg processes $(X(t))_{t \in [0,\infty)}$ such that
$$
\|X\|_{\mathcal{S}^2}^2 = \mathbb{E}\left[\sup_{t \geq 0} |X(t)|^2 \right] < \infty.
$$
- $\mathbb{L}^2$ is the set of $\mathbb{R}$-valued $\mathbb{F}$-adapted processes $(Q(t,s))_{(t,s) \in [0,\infty)^2}$ such that
$$
\|Q\|_{\mathbb{L}^2}^2 := \mathbb{E}\left[\int_0^\infty \int_t^\infty |Q(t,s)|^2\, ds\, dt \right] < \infty.
$$
- $\mathbb{L}_\nu^2$ is the set of Borel functions $R : \mathbb{R}_0 := \mathbb{R} \setminus \{0\} \to \mathbb{R}$ such that
$$
\|R\|_{\mathbb{L}_\nu^2}^2 := \mathbb{E}\left[\int_0^\infty \int_t^\infty \int_{\mathbb{R}_0} |R(t,s,e)|^2\, \nu(de)\, ds\, dt \right] < \infty.
$$
- $\mathbb{H}_\nu^2$ is the set of $\mathbb{F}$-adapted predictable processes $R : [0,\infty)^2 \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$ such that
$$
\|R\|_{\mathbb{H}_\nu^2}^2 := \mathbb{E}\left[\int_0^\infty \int_t^\infty \int_{\mathbb{R}_0} |R(t,s,e)|^2\, \nu(de)\, ds\, dt \right] < \infty.
$$
  We equip $\mathbb{H}_\nu^2$ with this norm.

2.6. **Elements of Malliavin Calculus.** To handle the anticipating nature of Volterra equations, we employ the Hida-Malliavin calculus for Lévy processes as developed in [2]. Let D_t denote the Malliavin derivative with respect to the Brownian motion B, and let $F_{t,e}$ denote the Malliavin derivative with respect to the compensated Poisson random measure $\tilde{N}$. For a sufficiently regular random variable F, the following duality relations hold:

$$\mathbb{E}\Big[F\int_0^\infty \varphi(s)\,dB(s)\Big] = \mathbb{E}\Big[\int_0^\infty (D_sF)\varphi(s)\,ds\Big],$$

$$\mathbb{E}\Big[F\int_0^\infty\int_{\mathbb{R}_0} \psi(s,e)\,\tilde{N}(ds,de)\Big] = \mathbb{E}\Big[\int_0^\infty\int_{\mathbb{R}_0} (F_{s,e}F)\psi(s,e)\,\nu(de)ds\Big],$$

where φ and ψ are predictable processes satisfying appropriate integrability conditions. These identities allow us to transform certain stochastic integrals into deterministic ones, a key tool in deriving the adjoint equations for Volterra systems. For a comprehensive treatment, we refer the reader to [2] and the references therein.

3. Sufficient Maximum Principle

We formulate the optimal control problem for coupled forward-backward stochastic Volterra integral equations (SVIEs) with delay on an infinite horizon with partial information, generalizing the framework of [1] to the Volterra case.

From equations (2.1)–(2.2) we obtain the differential forms (see e.g. [2] or [5] for a rigorous derivation using Hida–Malliavin calculus). Note that the free stochastic process $\psi(t)$ does not affect the differential of Y except for the term $d\psi(t)$, which is a stochastic differential; however, because ψ is not necessarily adapted, the rigorous derivation requires anticipating calculus. For the purpose of the maximum principle, the crucial point is that ψ does not depend on the control, so its variation vanishes when considering differences. The formal differential $dY(t)$ includes $d\psi(t)$ instead of $\psi'(t)dt$; we write the latter only when ψ is absolutely continuous. In the general case, the term $d\psi(t)$ does not affect the adjoint equations.

$$\begin{aligned} dX(t) &= \xi'(t)dt + b(t,t,X(t),X_1(t),u(t))dt \\ &\quad + \left(\int_0^t \frac{\partial b}{\partial t}(t,s,X(s),X_1(s),u(s))ds\right)dt \\ &\quad + \sigma(t,t,X(t),X_1(t),u(t))dB(t) \\ &\quad + \left(\int_0^t \frac{\partial \sigma}{\partial t}(t,s,X(s),X_1(s),u(s))dB(s)\right)dt \\ &\quad + \int_{\mathbb{R}_0} \theta(t,t,X(t),X_1(t),u(t),e)\tilde{N}(dt,de) \\ &\quad + \left(\int_0^t\int_{\mathbb{R}_0} \frac{\partial \theta}{\partial t}(t,s,X(s),X_1(t),u(s),e)\tilde{N}(ds,de)\right)dt, \end{aligned} \tag{3.1}$$

and

$$
\begin{aligned}
dY(t) = d\psi(t) &- g(t,t,X(t),X_1(t),Y(t),Z(t,t),K(t,t,\cdot),u(t))dt \\
&+ \left(\int_t^\infty \frac{\partial g}{\partial t}(t,s,X(s),X_1(s),Y(s),Z(t,s),K(t,s,\cdot),u(s))ds\right)dt \\
&+ \left(\int_t^\infty \frac{\partial g}{\partial z}(t,s,\dots)\frac{\partial Z}{\partial t}(t,s)ds\right)dt \\
&+ \left(\int_t^\infty \left\langle \nabla_k g(t,s,\dots), \frac{\partial K}{\partial t}(t,s,\cdot)\right\rangle ds\right)dt \\
&+ Z(t,t)dB(t) - \left(\int_t^\infty \frac{\partial Z}{\partial t}(t,s)dB(s)\right)dt \\
&+ \int_{\mathbb{R}_0} K(t,t,e)\tilde{N}(dt,de) - \left(\int_t^\infty \int_{\mathbb{R}_0} \frac{\partial K}{\partial t}(t,s,e)\tilde{N}(ds,de)\right)dt.
\end{aligned}
\tag{3.2}
$$

Theorem 3.1 (Sufficient Maximum Principle)**.** *Let $\hat{u} \in \mathcal{U}$ with corresponding solutions $\hat{X}(t)$, $(\hat{Y}(t), \hat{Z}(t,\cdot), \hat{K}(t,\cdot,\cdot))$, $(\hat{p}(t), \hat{q}(t,\cdot), \hat{r}(t,\cdot,\cdot))$ and $\hat{\lambda}(t)$ of equations (2.1), (2.2), (2.8) and (2.9) respectively. Suppose that:*

Concavity: *The function $y \mapsto h(y)$ (terminal cost) and the Hamiltonian*

$$(x, x_1, y, z, k(\cdot), u) \mapsto H(t, x, x_1, y, z, k, u, \hat{\lambda}(t), \hat{p}(t), \hat{q}(t,\cdot), \hat{r}(t,\cdot,\cdot))$$

are concave for all $t \geq 0$.

Conditional Maximum Principle:

$$
\begin{aligned}
&\max_{v\in\mathcal{U}} \mathbb{E}\left[H(t,\hat{X}(t),\hat{X}_1(t),\hat{Y}(t),\hat{Z}(t,\cdot),\hat{K}(t,\cdot,\cdot),v,\hat{\lambda}(t),\hat{p}(t),\hat{q}(t,\cdot),\hat{r}(t,\cdot,\cdot)) \mid \mathcal{G}_t\right] \\
&= \mathbb{E}\left[H(t,\hat{X}(t),\hat{X}_1(t),\hat{Y}(t),\hat{Z}(t,\cdot),\hat{K}(t,\cdot,\cdot),\hat{u},\hat{\lambda}(t),\hat{p}(t),\hat{q}(t,\cdot),\hat{r}(t,\cdot,\cdot)) \mid \mathcal{G}_t\right].
\end{aligned}
$$

Transversality Conditions:

$$\overline{\lim_{T\to\infty}}\, \mathbb{E}[\hat{\lambda}(T)(\hat{Y}(T) - Y(T))] \geq 0 \quad \textit{and} \quad \overline{\lim_{T\to\infty}}\, \mathbb{E}[\hat{p}(T)(\hat{X}(T) - X(T))] \leq 0.$$

Then, $\hat{u}$ is an optimal control for our problem.

Proof. The proof follows the same steps as in the classical case; the presence of the free process $\psi(t)$ does not affect the concavity or the conditional maximum principle because it does not depend on the control or the state. The term $d\psi(t)$ cancels when considering differences $(\hat{Y}(t) - Y(t))$ (since $\psi(t)$ is the same for both trajectories). Thus the proof remains valid; we refer to [2] and [5] for the detailed algebra. □

Remark 3.2. The transversality conditions are essential in infinite horizon problems. Sufficient conditions for them to hold are, for example, that the processes $\hat{X}$, $\hat{Y}$ and the adjoint processes have at most polynomial growth and that the coefficients satisfy suitable integrability conditions; see [8] for a discussion.

4. Necessary Maximum Principle

A drawback of the previous section is that the concavity condition is not always satisfied in applications. In view of this, it is of interest to obtain conditions for the existence of an optimal control with partial information where concavity is not needed.

We assume the following:

- **(A1):** For all $\hat{u} \in \mathcal{U}$ and all bounded $\beta \in \mathcal{U}$,

$$\hat{u} + \epsilon\beta \in \mathcal{U},$$

for all such β, and all nonzero ϵ sufficiently small.

- **(A2):** For all $t_0 > 0$, $h > 0$ and all bounded $\mathcal{G}_{t_0}$-measurable random variables α, the control process $\beta(t)$ defined by

$$\beta(t) = \alpha \mathbf{1}_{[t_0, t_0+h]}(t)$$

belongs to $\mathcal{U}$.

For a bounded $\mathcal{G}_t$-adapted variation β, we define the derivative processes as:

$$\begin{aligned}
\dot{X}(t) &:= \left.\frac{d}{d\varepsilon} X^{\hat{u}+\varepsilon\beta}(t)\right|_{\varepsilon=0}, \\
\dot{Y}(t) &:= \left.\frac{d}{d\varepsilon} Y^{\hat{u}+\varepsilon\beta}(t)\right|_{\varepsilon=0}, \\
\dot{Z}(t,s) &:= \left.\frac{d}{d\varepsilon} Z^{\hat{u}+\varepsilon\beta}(t,s)\right|_{\varepsilon=0}, \\
\dot{K}(t,s,\cdot) &:= \left.\frac{d}{d\varepsilon} K^{\hat{u}+\varepsilon\beta}(t,s,\cdot)\right|_{\varepsilon=0}.
\end{aligned}$$

4.1. **Linearized Equations.** The variational processes satisfy the following linearized equations (see [2] or [5] for a detailed derivation using Hida–Malliavin calculus). The free process $\psi(t)$ does not appear because it is independent of the control.

Lemma 4.1 (Linearized forward equation)**.** *The process $\dot{X}$ satisfies*

$$\begin{aligned}
\dot{X}(t) = &\int_0^t \left[\frac{\partial b}{\partial x}(t,s)\dot{X}(s) + \frac{\partial b}{\partial x_1}(t,s)\dot{X}_1(s) + \frac{\partial b}{\partial u}(t,s)\beta(s)\right] ds \\
&+ \int_0^t \left[\frac{\partial \sigma}{\partial x}(t,s)\dot{X}(s) + \frac{\partial \sigma}{\partial x_1}(t,s)\dot{X}_1(s) + \frac{\partial \sigma}{\partial u}(t,s)\beta(s)\right] dB(s) \\
&+ \int_0^t \int_{\mathbb{R}_0} \left[\frac{\partial \theta}{\partial x}(t,s,e)\dot{X}(s) + \frac{\partial \theta}{\partial x_1}(t,s,e)\dot{X}_1(s) + \frac{\partial \theta}{\partial u}(t,s,e)\beta(s)\right] \tilde{N}(ds,de),
\end{aligned}$$

where $\dot{X}_1(t) = \dot{X}(t-\delta)$.

Lemma 4.2 (Linearized backward equation)**.** *The processes* $(\dot{Y}, \dot{Z}, \dot{K})$ *satisfy*

$$\begin{aligned}\dot{Y}(t) = \int_t^\infty \Bigg[&\frac{\partial g}{\partial x}(t,s)\dot{X}(s) + \frac{\partial g}{\partial x_1}(t,s)\dot{X}_1(s) + \frac{\partial g}{\partial y}(t,s)\dot{Y}(s) \\ &+ \frac{\partial g}{\partial z}(t,s)\dot{Z}(t,s) + \langle \nabla_k g(t,s), \dot{K}(t,s,\cdot)\rangle + \frac{\partial g}{\partial u}(t,s)\beta(s)\Bigg] ds \\ &- \int_t^\infty \dot{Z}(t,s)dB(s) - \int_t^\infty \int_{\mathbb{R}_0} \dot{K}(t,s,e)\tilde{N}(ds,de).\end{aligned}$$

The differential forms of these equations (obtained by differentiating with respect to t) involve second-order derivatives of the coefficients; they are omitted here for brevity (see [2] for the complete derivation).

4.2. **First Variation of the Cost Functional.**

Lemma 4.3 (First variation formula)**.** *Under assumptions (A1)-(A2) and the differentiability conditions on the coefficients, we have*

$$\frac{d}{d\epsilon} J(\hat{u} + \epsilon\beta)\bigg|_{\epsilon=0} = \mathbb{E}\left[\int_0^\infty \frac{\partial H}{\partial u}(t)\beta(t)dt\right].$$

Proof. The proof is identical to the case without $\psi(t)$ because $\psi(t)$ does not depend on the control; its contribution to the variation of Y cancels when considering differences. We refer to [2] or [5] for the detailed calculations. □

Theorem 4.4 (Necessary Maximum Principle)**.** *Let* $\hat{u} \in \mathcal{U}$ *be an optimal control with corresponding solutions* $\hat{X}(t)$, $(\hat{Y}(t), \hat{Z}(t,\cdot), \hat{K}(t,\cdot,\cdot))$, $\hat{p}(t)$, $\hat{q}(t,\cdot)$, $\hat{r}(t,\cdot,\cdot)$ *and* $\hat{\lambda}(t)$ *of the state and adjoint equations respectively. Suppose the following transversality conditions hold:*

$$\overline{\lim_{T\to\infty}}\, \mathbb{E}[\hat{p}(T)\dot{X}(T)] = 0 \quad \text{and} \quad \overline{\lim_{T\to\infty}}\, \mathbb{E}[\hat{\lambda}(T)\dot{Y}(T)] = 0.$$

Then,

$$\mathbb{E}\left[\frac{\partial H}{\partial u}(t)\bigg|\mathcal{G}_t\right]_{u=\hat{u}} = 0 \quad \text{for a.e. } t \geq 0.$$

Proof. By Lemma 4.3, optimality implies

$$\mathbb{E}\left[\int_0^\infty \frac{\partial H}{\partial u}(t)\beta(t)dt\right] = 0$$

for all bounded $\mathcal{G}_t$-adapted β. In particular, for any $t_0 \geq 0$ and $h > 0$, choose $\beta(t) = \alpha \mathbf{1}_{[t_0,t_0+h]}(t)$ with α bounded $\mathcal{G}_{t_0}$-measurable. Then

$$\mathbb{E}\left[\alpha \int_{t_0}^{t_0+h} \frac{\partial H}{\partial u}(t)dt\right] = 0.$$

Dividing by h and letting $h \downarrow 0$, Lebesgue's differentiation theorem yields

$$\mathbb{E}\left[\alpha \frac{\partial H}{\partial u}(t_0)\right] = 0 \quad \text{a.e. } t_0.$$

Since α is arbitrary, we conclude $\mathbb{E}[\frac{\partial H}{\partial u}(t_0)|\mathcal{G}_{t_0}] = 0$ for almost every t_0. □

5. Existence and uniqueness for infinite-horizon BSVIEs with a free parameter

5.1. **Problem formulation.** We consider the infinite-horizon backward stochastic Volterra integral equation with jumps and a free stochastic process $\psi(t)$:

$$\begin{aligned} Y(t) = \psi(t) + \int_t^\infty g\big(t, s, Y(s), Z(t,s), K(t,s,\cdot)\big)\, ds \\ - \int_t^\infty Z(t,s)\, dB(s) - \int_t^\infty \int_{\mathbb{R}_0} K(t,s,e)\, \tilde{N}(ds, de), \qquad t \geq 0. \end{aligned} \tag{5.1}$$

The function $\psi : [0,\infty) \times \Omega \to \mathbb{R}$ is a given (not necessarily adapted) $\mathcal{F}_\infty$-measurable stochastic process satisfying

$$\mathbb{E}\left[\int_0^\infty e^{\beta s} |\psi(s)|^2 ds\right] < \infty$$

for some $\beta > 0$ (see [6]). The presence of $\psi(\cdot)$ is essential for the well-posedness of the infinite-horizon problem; it acts as a free parameter that can be chosen to satisfy, e.g., a transversality condition or a fixed-point requirement.

Following the approach in [1], we transform the problem into a family of standard BSDEs and use a fixed-point argument. Our analysis extends the techniques of [6] to incorporate jumps.

5.2. **Functional framework.** For any $\beta > 0$, let $\triangle_\infty := \{(t,s) \in [0,\infty)^2 : t \leq s\}$ and define $H^{2,\beta}_\triangle[0,\infty)$ as the space of all processes (Y, Z, K) such that:

- $Y : [0,\infty) \times \Omega \to \mathbb{R}$ is $\mathbb{F}$-adapted,
- $Z : \triangle_\infty \times \Omega \to \mathbb{R}$ with $s \mapsto Z(t,s)$ being $\mathbb{F}$-adapted on $[t,\infty)$,
- $K : \triangle_\infty \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$ with $s \mapsto K(t,s,\cdot)$ being $\mathbb{F}$-adapted on $[t,\infty)$,

equipped with the norm

$$\begin{aligned} \|(Y,Z,K)\|^2_{H^{2,\beta}_\triangle[0,\infty)} := \mathbb{E}\int_0^\infty \Big[e^{\beta t}|Y(t)|^2 + \int_t^\infty e^{\beta s}|Z(t,s)|^2 ds \\ + \int_t^\infty \int_{\mathbb{R}_0} e^{\beta s} |K(t,s,e)|^2 \nu(de) ds\Big] dt. \end{aligned}$$

5.3. **Assumptions (A).** We make the following assumptions:

(1) The function $g : [0,\infty)^2 \times \mathbb{R}^3 \times L^2(\nu) \times \Omega \to \mathbb{R}$ satisfies:
   (a) $\mathbb{E}\left[\int_0^\infty \left(\int_t^\infty e^{\beta s} |g(t,s,0,0,0)| ds\right)^2 dt\right] < \infty$ for some $\beta > 0$,
   (b) There exists a constant $L > 0$ such that for all $t, s \in [0,\infty)$,

$$|g(t,s,y,z,k) - g(t,s,y',z',k')| \leq L\left(|y-y'| + |z-z'| + \|k-k'\|_{L^2(\nu)}\right).$$

(2) The free process ψ satisfies $\mathbb{E}\left[\int_0^\infty e^{\beta s}|\psi(s)|^2 ds\right] < \infty$.
(3) The parameter β is chosen sufficiently large (specifically, $\beta > 6L^2$ as will be shown in the proof).

Theorem 5.1. *Under assumptions (A), there exists a unique solution* $(Y, Z, K) \in H^{2,\beta}_{\triangle}[0, \infty)$ *to the infinite-horizon BSVIE* (5.1) *with the free stochastic parameter* ψ*.*

Proof. We prove the theorem using a fixed-point argument in the Banach space $H^{2,\beta}_{\triangle}[0, \infty)$.

Step 1: Setup of the fixed-point mapping. For a given triple $(y, z, k) \in H^{2,\beta}_{\triangle}[0, \infty)$, consider the family of infinite-horizon BSDEs parameterized by $t \geq 0$:

$$\begin{aligned}\widetilde{Y}^t(r) = \psi(r) + \int_r^\infty g(t, s, y(s), z(t, s), k(t, s, \cdot))\, ds \\ - \int_r^\infty \widetilde{Z}^t(s) dB(s) - \int_r^\infty \int_{\mathbb{R}_0} \widetilde{K}^t(s, e)\tilde{N}(ds, de), \quad r \geq t.\end{aligned} \tag{5.2}$$

For each fixed t, this is a standard infinite-horizon BSDE with jumps and a source term $\psi(r)$ (which is $\mathcal{F}_\infty$-measurable but not necessarily adapted; however, because the equation is interpreted in the forward sense and the filtration is generated by the noise, the existence and uniqueness still hold under the weighted square-integrability condition – see [6]). Under our assumptions, it admits a unique adapted solution $(\widetilde{Y}^t(\cdot), \widetilde{Z}^t(\cdot), \widetilde{K}^t(\cdot, \cdot))$ satisfying the estimate:

$$\begin{aligned}&\mathbb{E}\left[\sup_{r\geq t} e^{\beta r}|\widetilde{Y}^t(r)|^2 + \int_t^\infty e^{\beta s}|\widetilde{Z}^t(s)|^2 ds + \int_t^\infty \int_{\mathbb{R}_0} e^{\beta s}|\widetilde{K}^t(s, e)|^2 \nu(de) ds\right] \\ &\quad\leq C\mathbb{E}\left[\left(\int_t^\infty e^{\beta s/2}|g(t, s, y(s), z(t, s), k(t, s, \cdot))| ds\right)^2\right] + C\mathbb{E}\int_t^\infty e^{\beta s}|\psi(s)|^2 ds,\end{aligned}$$

for some constant $C > 0$ independent of t.

Now define:

$$Y(t) = \widetilde{Y}^t(t), \quad Z(t, s) = \widetilde{Z}^t(s), \quad K(t, s, \cdot) = \widetilde{K}^t(s, \cdot), \quad \text{for } (t, s) \in \triangle_\infty.$$

Then (Y, Z, K) satisfies (5.1) for the given (y, z, k).

Step 2: A priori estimate. Applying Itô's formula to $e^{\beta s}|\widetilde{Y}^t(s)|^2$ and using Young's inequality $2ab \leq \frac{\beta}{2}a^2 + \frac{2}{\beta}b^2$, we obtain after taking expectations and letting the upper limit go to infinity:

$$\begin{aligned}\mathbb{E}\left[e^{\beta t}|Y(t)|^2 + \int_t^\infty e^{\beta s}\Big(\tfrac{\beta}{2}|\widetilde{Y}^t(s)|^2 + |\widetilde{Z}^t(s)|^2 + \|\widetilde{K}^t(s, \cdot)\|^2_{L^2(\nu)}\Big) ds\right] \\ \leq \frac{2}{\beta}\,\mathbb{E}\int_t^\infty e^{\beta s}|g(t, s, y(s), z(t, s), k(t, s, \cdot))|^2 ds \\ + 2\,\mathbb{E}\int_t^\infty e^{\beta s}|\psi(s)|^2 ds.\end{aligned}$$

Using the Lipschitz condition and the inequality $(a + b)^2 \leq 2(a^2 + b^2)$, we have:

$$|g(t, s, y, z, k)|^2 \leq 2|g(t, s, 0, 0, 0)|^2 + 2L^2\big(|y|^2 + |z|^2 + \|k\|^2_{L^2(\nu)}\big).$$

Substituting this bound into the previous inequality and integrating over t from 0 to ∞ (using Fubini's theorem) yields:

$$\mathbb{E}\int_0^\infty e^{\beta t}|Y(t)|^2dt + \mathbb{E}\int_0^\infty\int_t^\infty e^{\beta s}\big(\tfrac{\beta}{2}|\widetilde{Y}^t(s)|^2 + |\widetilde{Z}^t(s)|^2 + \|\widetilde{K}^t(s,\cdot)\|^2_{L^2(\nu)}\big)dsdt$$
$$\leq \frac{4}{\beta}\mathbb{E}\int_0^\infty\int_t^\infty e^{\beta s}|g(t,s,0,0,0)|^2dsdt + \frac{4L^2}{\beta}\|(y,z,k)\|^2_{H^{2,\beta}_\triangle} + 2\,\mathbb{E}\int_0^\infty e^{\beta s}|\psi(s)|^2ds.$$

In particular, this shows that

$$\|(Y,Z,K)\|^2_{H^{2,\beta}_\triangle} \leq \frac{4}{\beta}\mathbb{E}\int_0^\infty\int_t^\infty e^{\beta s}|g(t,s,0,0,0)|^2dsdt + \frac{4L^2}{\beta}\|(y,z,k)\|^2_{H^{2,\beta}_\triangle} + 2\,\mathbb{E}\int_0^\infty e^{\beta s}|\psi(s)|^2ds,$$

so the mapping $\Phi : (y,z,k) \mapsto (Y,Z,K)$ is well defined on $H^{2,\beta}_\triangle[0,\infty)$.

Step 3: Contraction property. For two triples (y_i, z_i, k_i), $i = 1,2$, set $\Delta y = y_1 - y_2$, etc. The difference of the corresponding images satisfies an analogous estimate with

$$\Delta g(t,s) = g(t,s,y_1,z_1,k_1) - g(t,s,y_2,z_2,k_2).$$

By the Lipschitz condition, $|\Delta g| \leq L(|\Delta y| + |\Delta z| + \|\Delta k\|_{L^2(\nu)})$. Repeating the same computation as in Step 2 for the differences (note that $\psi(s)$ cancels because it is the same for both) gives

$$\|\Delta(Y,Z,K)\|^2_{H^{2,\beta}_\triangle} \leq \frac{6L^2}{\beta}\|\Delta(y,z,k)\|^2_{H^{2,\beta}_\triangle}.$$

Step 4: Existence and uniqueness via Banach fixed-point theorem. Choosing $\beta > 6L^2$ makes Φ a contraction on the Banach space $H^{2,\beta}_\triangle[0,\infty)$. By the Banach fixed-point theorem, Φ has a unique fixed point $(Y,Z,K) \in H^{2,\beta}_\triangle[0,\infty)$. This fixed point satisfies (5.1) by construction, proving existence. Uniqueness follows from the contraction property. □ □

Acknowledgements

This work is partially supported by PRFU project No: C00L03UN070120220004

References

[1] Agram, N., & Øksendal, B. (2014). Infinite horizon optimal control of forward-backward stochastic differential equations with delay. *Journal of Computational and Applied Mathematics*, 259, 336–349.

[2] Agram, N., Øksendal, B., & Yakhlef, S. (2018). Optimal control of forward-backward stochastic Volterra equations. In F. Gesztesy et al. (Eds.), *Non-linear partial differential equations, mathematical physics, and stochastic analysis* (pp. 3–35). European Mathematical Society.

[3] Bahlali, S., & Labed, B. (2006). Necessary and sufficient conditions of optimality for optimal control problem with initial and terminal costs. *Random Operators and Stochastic Equations*, 14(3), 291–301.

[4] Duffie, D., & Epstein, L. G. (1992). Stochastic differential utility. *Econometrica*, 60(2), 353–394.

[5] Hafiane, N., Labed, B., & Yakhlef, S. (2026). Maximum principle for delayed Volterra jumps systems using anticipating adjoints. *Gulf Journal of Mathematics*, 22(1), 1–25. https://doi.org/10.56947/gjom.v22i1.3895

[6] Hamaguchi, Y. (2021). Infinite horizon backward stochastic Volterra integral equations and discounted control problems. *Systems & Control Letters*, 156, 105000.
[7] Hu, Y., & Øksendal, B. (2017). Linear backward stochastic Volterra equations. *Stochastics and Dynamics*, 18(6), 1850042.
[8] Haadem, S., Øksendal, B., & Proske, F. (2012). A maximum principle for jump diffusion processes with infinite horizon. *Systems & Control Letters*, 61(9), 955–961.
[9] Maslowski, B., & Veverka, P. (2011). Sufficient stochastic maximum principle for discounted control problem. *Stochastic Processes and their Applications*, 121(11), 2559–2575.
[10] Mohammed, S. E. A. (1996). Stochastic differential equations with memory: Theory, examples and applications. In *Stochastic Analysis and Related Topics VI* (pp. 1–77). Birkhäuser Boston.
[11] Øksendal, B., & Sulem, A. (2019). *Applied Stochastic Control of Jump Diffusions* (4th ed.). Springer.
[12] Pardoux, E. (1999). BSDEs, weak convergence and homogenization of semilinear PDEs. In *Nonlinear Analysis, Differential Equations and Control* (pp. 503–549). Springer.
[13] Peng, S. (1990). A general stochastic maximum principle for optimal control problems. *SIAM Journal on Control and Optimization*, 28(4), 966–979.
[14] Peng, S., & Shi, Y. (2000). Infinite horizon forward-backward stochastic differential equations. *Stochastic Processes and their Applications*, 85(1), 75–92.
[15] Papapantoleon, A., Possamaï, D., & Saplaouras, A. (2016). Existence and uniqueness results for BSDEs with jumps: the whole nine yards. *Electronic Journal of Probability*, 21(131), 1–68.
[16] Tang, S., & Li, X. (1994). Necessary conditions for optimal control of stochastic systems with random jumps. *SIAM Journal on Control and Optimization*, 32(5), 1447–1475.
[17] Wang, T., & Shi, Y. (2010). Solvability of general backward stochastic Volterra integral equations with non-Lipschitz conditions. *Stochastic Processes and their Applications*, 120(9), 1599–1616.
[18] Yin, J. (2008). On solutions of a class of infinite horizon FBSDEs. *Statistics & Probability Letters*, 78(15), 2412–2419.
[19] Yong, J. (2006). Backward stochastic Volterra integral equations and some related problems. *Stochastic Processes and their Applications*, 116(5), 779–795.
[20] Yakhlef, S., & Ardjan, H. (2026). Explicit solution of infinite-horizon linear backward stochastic Volterra integral equations. Preprint.
[21] Hamed, I., & Chala, A. (2025). Pontryagin's approach to the risk-sensitive control problem for fractional backward stochastic differential equations. *Gulf Journal of Mathematics*, 21(2), 62–78. https://doi.org/10.56947/gjom.v21i2.3782

[1] Department of Mathematics, University of Mohamed Khider, Biskra, Algeria.

[2] Department of Mathematics, University of Batna, Algeria.

Email address: djaber.ibtissem.djeber@univ-biskra.dz, s.yakhlef@univ-biskra.dz
Email address: hafiane.nawel@yahoo.fr