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NON-ABSORBING PRODUCT GRAPH OF COMPLETELY SIMPLE SEMIGROUPS

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Abstract. We introduce the non-absorbing product graph of a semigroup $\mathbb{S}$, an undirected graph whose vertices are the elements of $\mathbb{S}$ and where two distinct vertices are adjacent precisely when their product differs from both factors. Focusing on completely simple semigroups, we establish a structural decomposition of this graph as the edge-union of a complete graph on the non-idempotent elements, a regular bipartite graph connecting idempotents to non-idempotents, and a regular multipartite graph on the idempotents. Moreover, we analyze the absorbing product graph, the complement of the non-absorbing product graph, which decomposes into a regular bipartite graph together with complete subgraphs arising from Green's classes of idempotents. Unlike its counterpart, it is always connected yet fails to be Hamiltonian in general. This natural partition of the vertex set into idempotents and non-idempotents reflects the core algebraic structure in a purely graphical way.

Keywords. Non-absorbing product graph, Completely simple semigroups, Hamiltonian graph, Planar graph.
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1. Introduction

Graphs associated with algebraic systems, such as semigroups and rings have long served as a means of translating algebraic properties into combinatorial form [1, 2, 7, 10, 11, 13]. In particular, graphs defined via product-based adjacency, in which vertices represent algebraic elements, have attracted considerable attention due to their ability to capture factorization phenomena and structural features of ideals and substructures [5, 8, 12, 14, 15].

Motivated by this interplay, we introduce a new graph construction that distinguishes between trivial and non-trivial multiplicative behaviour in a semigroup. Let $\mathbb{S}$ be a semigroup. The *non-absorbing product graph*, denoted by $G_{NA}(\mathbb{S})$, is the simple undirected graph with vertex set $\mathbb{S}$ in which two distinct elements a and b are adjacent if and only if $ab \neq a$ and $ab \neq b$. Thus, adjacency excludes absorbing interactions, namely those in which the product coincides with one of the factors, and instead captures genuinely non-trivial multiplicative interactions.

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Complementing this construction, we define the *absorbing product graph*, denoted by $G_A(\mathbb{S})$, as the graph-theoretic complement of $G_{NA}(\mathbb{S})$. In this graph, adjacency records precisely those pairs of elements whose product coincides with one of the factors. Such behaviour is closely tied to idempotent elements: if $e^2 = e$, then products of the form ea or ae may reproduce one of the operands under suitable conditions. This naturally induces a partition of the vertex set into idempotents and non-idempotents, reflecting two distinct types of multiplicative behaviour. While the absorbing product graph emphasizes exceptional interactions governed by idempotents, the non-absorbing product graph captures the generic structure arising from non-idempotent elements.

The present work investigates these graph constructions in the setting of *completely simple semigroups*, a fundamental class characterized by the Rees Theorem [9]. Our main result shows that the non-absorbing product graph admits a canonical structural decomposition into the edge-union of three components: a complete graph on the non-idempotent elements, a regular bipartite graph connecting idempotents and non-idempotents, and a regular multipartite graph induced by the idempotents. Moreover, we prove that the isomorphism type of this graph depends only on the cardinalities of the index sets and the order of the maximal subgroup, and is independent of the sandwich matrix.

We further analyse several graph-theoretic invariants, including connectivity, degree distribution, clique number, Hamiltonicity, and planarity. The absorbing product graph is also studied, where we establish a complementary structural decomposition governed by Green's $\mathscr{L}$- and $\mathscr{R}$-classes of idempotents. In contrast to the non-absorbing case, we show that this graph is always connected, but fails to be Hamiltonian in general.

This article is organized as follows. In Section 2, we present the necessary preliminaries. Section 3 is devoted to the structural analysis of the non-absorbing product graph and its invariants. Section 4 focuses on the absorbing product graph and its decomposition, highlighting the role of idempotents in governing absorbing behaviour.

2. Preliminaries

A semigroup $\mathbb{S}$ is said to be *completely simple* if it has no proper ideals and contains a primitive idempotent. By the classical *Rees Theorem* [9], every completely simple semigroup is isomorphic to a Rees matrix semigroup $\mathbb{S} \cong \mathcal{M}(\mathbb{G}; \mathbb{I}, \mathbb{J}; \mathbb{P})$, where $\mathbb{G}$ is a group, $\mathbb{I}$ and $\mathbb{J}$ are nonempty index sets, and $\mathbb{P} = (p_{ji})$ is a $\mathbb{J} \times \mathbb{I}$ sandwich matrix over $\mathbb{G}$ such that each row and each column of $\mathbb{P}$ contains at least one non-zero entry of $\mathbb{G}$.

The underlying set $\mathbb{S} = \{(i, g, j) \mid i \in \mathbb{I},\ g \in \mathbb{G},\ j \in \mathbb{J}\}$, with multiplication defined by $(i_1, a, j_1)(i_2, b, j_2) = (i_1, a\, p_{j_1 i_2}\, b, j_2)$. An element e of a semigroup $\mathbb{S}$ is called an idempotent element if $e^2 = e$. Green's relations on $\mathbb{S}$ admit a simple description. For $(i_1, a, j_1), (i_2, b, j_2) \in \mathbb{S}$,

$$(i_1, a, j_1)\, \mathscr{L}\, (i_2, b, j_2) \iff j_1 = j_2, \qquad (i_1, a, j_1)\, \mathscr{R}\, (i_2, b, j_2) \iff i_1 = i_2.$$

Consequently, for fixed i and j, each $\mathscr{H}$-class is of the form $\mathscr{H}_{i,j} = \{(i, g, j) \mid g \in \mathbb{G}\}$, and is isomorphic to the group $\mathbb{G}$. In particular, an element $(i, g, j) \in \mathbb{S}$

is idempotent precisely when $g = p_{ji}^{-1}$, so each $\mathscr{H}$-class contains exactly one idempotent.

All graphs considered in this article are finite, undirected, and simple. A graph $G = (V, E)$ is said to be *bipartite* if there exist disjoint subsets $U, W \subseteq V$ such that $U \cup W = V$ and every edge in E has one endpoint in U and the other in W. A bipartite graph with parts of cardinalities μ and ν is denoted by $B_{\mu,\nu}$.

A graph is *k-regular* if every vertex has degree k. A *k-regular μ-partite graph with part size ν* is a graph whose vertex set can be partitioned into μ disjoint subsets, each containing ν vertices, such that edges occur only between distinct parts and each vertex has degree k. For convenience, we denote such a graph using the symbol $R^k_{\mu,\nu}$.

A *clique* in a graph is a subset of vertices such that every pair of distinct vertices is adjacent. A clique of size n is called *complete graph* and is denoted by K_n. A graph G is said to be planar if it can be drawn in the plane in such a way that no two edges intersect except at their endpoints. By Kuratowski's theorem, a graph G is planar if and only if it contains no subgraph that is a subdivision of K_5 or $K_{3,3}$. A *star graph* $K_{1,m}$ consists of a single central vertex adjacent to m pendant vertices.

Given a graph G, its *complement* $\overline{G}$ is the graph on the same vertex set in which two distinct vertices are adjacent if and only if they are not adjacent in G. We write $\gamma_1 \sim \gamma_2$ to indicate that the vertices γ_1 and γ_2 are adjacent.

For undefined notation and terminology related to semigroup theory, graph theory, and algebraic graph theory, we refer the reader to [9], [4], and [3], respectively. Unless stated otherwise, throughout this article $\mathbb{S} = \mathcal{M}(\mathbb{G}; \mathbb{I}, \mathbb{J}; \mathbb{P})$ denotes a completely simple semigroup with $|\mathbb{G}| = \rho$, $|\mathbb{I}| = \mu$, and $|\mathbb{J}| = \nu$.

3. Non-absorbing product graph of completely simple semigroups

The structural properties of the non-absorbing product graph of completely simple semigroups are analysed in this section. We initiate the analysis by applying the definition of the non-absorbing product graph to determine the conditions under which two elements in $\mathbb{S}$ are non-adjacent, and subsequently derive the adjacency criteria therefrom.

Two elements γ_1 and γ_2 are non-adjacent in $G_{NA}(\mathbb{S})$ if and only if $\gamma_1\gamma_2 = \gamma_1$ or $\gamma_1\gamma_2 = \gamma_2$. Consequently, if we choose $\gamma_1 = (i_1, a, j_1)$ and $\gamma_2 = (i_2, b, j_2)$, then,

$$(i_1, a\, p_{j_1 i_2} b, j_2) = (i_1, a, j_1) \quad \text{or} \quad (i_1, a\, p_{j_1 i_2} b, j_2) = (i_2, b, j_2).$$

These are equivalent to $j_1 = j_2 \quad \text{and} \quad b = p_{j_1 i_2}^{-1} \quad \text{or} \quad i_1 = i_2 \quad \text{and} \quad a = p_{j_1 i_2}^{-1}$.

If $(i_2, b\, p_{j_2 i_1} a, j_1) = (i_1, a, j_1) \quad \text{or} \quad (i_2, b, j_2)$, then

$$i_1 = i_2 \quad \text{and} \quad b = p_{j_2 i_1}^{-1} \quad \text{or} \quad j_1 = j_2 \quad \text{and} \quad a = p_{j_2 i_1}^{-1}$$

Although completely simple semigroups are non-commutative, interchanging the roles of γ_1 and γ_2 affects neither adjacency nor non-adjacency up to graph isomorphism. Specifically, the graphs defined by $\gamma_1 \sim \gamma_2 \iff \gamma_1\gamma_2 \notin \{\gamma_1, \gamma_2\}$ and $\gamma_1 \sim \gamma_2 \iff \gamma_2\gamma_1 \notin \{\gamma_1, \gamma_2\}$ are isomorphic via the inversion map $\Phi((i, g, j)) = (i, g^{-1}, j)$, which preserves both absorbing products $\gamma_1\gamma_2 \in \{\gamma_1, \gamma_2\}$

and $\gamma_2\gamma_1 \in \{\gamma_1, \gamma_2\}$. Thus, $G_{NA}(\mathbb{S})$ remains well-defined with invariant adjacency and non-adjacency structures under this reversal.

Following our discussion of the non-adjacency conditions for two elements in $G_{NA}(\mathbb{S})$, we now present the adjacency relations.

Proposition 3.1. *Two elements $\gamma_1 = (i_1, a, j_1)$, $\gamma_2 = (i_2, b, j_2)$ are adjacent in $G_{NA}(\mathbb{S})$ if and only if one of the following holds:*

(i) $i_1 \neq i_2$ *and* $j_1 \neq j_2$*;*
(ii) $i_1 = i_2 = i$*,* $a \neq p^{-1}_{j_1 i}$ *and* $b \neq p^{-1}_{j_2 i}$*;*
(iii) $j_1 = j_2 = j$*,* $a \neq p^{-1}_{j i_1}$ *and* $b \neq p^{-1}_{j i_2}$

Proof. Let $\gamma_1 = (i_1, a, j_1)$ and $\gamma_2 = (i_2, b, j_2)$. Then $\gamma_1\gamma_2 = (i_1, a\, p_{j_1 i_2} b, j_2)$. Thus, γ_1 and γ_2 are non-adjacent if and only if $\gamma_1\gamma_2 \in \{\gamma_1, \gamma_2\}$, which holds if and only if

$$(j_1 = j_2 \text{ and } b = p^{-1}_{j_1 i_2}) \quad \text{or} \quad (i_1 = i_2 \text{ and } a = p^{-1}_{j_1 i_2}).$$

Adjacency follows by negating these conditions, yielding the stated cases. □

If $\gamma = (i, a, j)$ is an idempotent in $\mathbb{S}$, then $(i, ap_{ji}a, j) = (i, a, j)$. Thus, the idempotents are of the form (i, p^{-1}_{ji}, j). The above adjacency conditions can be reformulated in terms of idempotents as follows.

Proposition 3.2. *Two elements $\gamma_1 = (i_1, a, j_1)$, $\gamma_2 = (i_2, b, j_2)$ in $\mathbb{S}$ are adjacent in $G_{NA}(\mathbb{S})$ if and only if one of the following holds:*

(i) γ_1 *and* γ_2 *are non-idempotents;*
(ii) *at least one of* γ_1 *or* γ_2 *is idempotent with* $i_1 \neq i_2$ *and* $j_1 \neq j_2$*.*

In terms of Green's equivalences, these conditions can be expressed in the following way.

Corollary 3.3. *Let $\gamma_1 = (i_1, a, j_1)$, $\gamma_2 = (i_2, b, j_2) \in \mathbb{S}$. Then γ_1 and γ_2 are adjacent in $G_{NA}(\mathbb{S})$ if and only if one of the following holds:*

(i) γ_1 *and* γ_2 *lie in neither the same* $\mathcal{L}$*-class nor the same* $\mathcal{R}$*-class;*
(ii) γ_1 *and* γ_2 *lie in the same* $\mathcal{R}$*-class, and* $a \neq p^{-1}_{j_1 i}$*,* $b \neq p^{-1}_{j_2 i}$*;*
(iii) γ_1 *and* γ_2 *lie in the same* $\mathcal{L}$*-class, and* $a \neq p^{-1}_{j i_1}$*,* $b \neq p^{-1}_{j i_2}$*.*

Corollary 3.4. *Let $\gamma_1 = (i, a, j)$ and $\gamma_2 = (i, b, j)$ lie in the same $\mathcal{H}$-class. Then γ_1 and γ_2 are adjacent in $G_{NA}(\mathbb{S})$ if and only if neither γ_1 nor γ_2 is an idempotent.*

The non-absorbing product graph of a completely simple semigroup exhibits a strong form of structural invariance. Specifically, the isomorphism type of the non-absorbing product graph depends only on the order ρ of the underlying group and on the cardinalities μ and ν of the index sets in its Rees matrix representation. It is therefore independent of the particular choice of the group $\mathbb{G}$ and of the specific entries of the sandwich matrix $\mathbb{P}$. Consequently, for fixed values of μ, ν, and ρ, all non-absorbing product graphs arising from completely simple semigroups are mutually isomorphic.

Theorem 3.5. *Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be finite completely simple semigroups. Suppose that, under their Rees representations, $\mathbb{S}_k \cong \mathcal{M}(\mathbb{G}_k; \mathbb{I}, \mathbb{J}; \mathbb{P}_k)$, $\quad k = 1, 2$, where*

$|\mathbb{G}_1| = |\mathbb{G}_2| = \rho$, $|\mathbb{I}| = \mu$, and $|\mathbb{J}| = \nu$. Then the non-absorbing product graphs $G_{NA}(\mathbb{S}_1)$ and $G_{NA}(\mathbb{S}_2)$ are isomorphic.

Proof. Let $\mathbb{S}_k = \{(i, a, j) : i \in \mathbb{I},\ j \in \mathbb{J},\ a \in \mathbb{G}_k\}, \mathbb{P}_k = [p_{ji}^{(k)}] \quad k = 1, 2$. For each pair $(i, j) \in \mathbb{I} \times \mathbb{J}$, choose a bijection $\phi_{ij} : \mathbb{G}_1 \longrightarrow \mathbb{G}_2$ satisfying

$$\phi_{ij}\Big((p_{ji}^{(1)})^{-1}\Big) = (p_{ji}^{(2)})^{-1}.$$

Such a bijection exists since both groups have the same finite order and only one prescribed image is fixed. Define $\Phi : \mathbb{S}_1 \to \mathbb{S}_2$ by $\Phi(i, a, j) = (i, \phi_{ij}(a), j)$. Then Φ is clearly a bijection.

Let $\gamma_1 = (i_1, a, j_1)$ and $\gamma_2 = (i_2, b, j_2)$ be vertices of $G_{NA}(\mathbb{S}_1)$. By Proposition 3.1, γ_1 and γ_2 are adjacent if and only if one of the following holds:

(i) $i_1 \neq i_2$ and $j_1 \neq j_2$;
(ii) $i_1 = i_2 = i$ and $a \neq (p_{j_1 i}^{(1)})^{-1}$, $b \neq (p_{j_2 i}^{(1)})^{-1}$;
(iii) $j_1 = j_2 = j$ and $a \neq (p_{j i_1}^{(1)})^{-1}$, $b \neq (p_{j i_2}^{(1)})^{-1}$.

In case (i), adjacency depends only on the index positions and is therefore preserved by Φ. In cases (ii) and (iii), the defining inequalities are preserved by the choice of the bijections ϕ_{ij}, which map the unique excluded group element $(p_{ji}^{(1)})^{-1}$ to $(p_{ji}^{(2)})^{-1}$. Hence γ_1 and γ_2 are adjacent in $G_{NA}(\mathbb{S}_1)$ if and only if $\Phi(\gamma_1)$ and $\Phi(\gamma_2)$ are adjacent in $G_{NA}(\mathbb{S}_2)$.

Therefore, Φ induces a graph isomorphism from $G_{NA}(\mathbb{S}_1)$ to $G_{NA}(\mathbb{S}_2)$. □

We next describe the internal structure of the non-absorbing product graph within an equivalence class determined by a fixed $\mathscr{L}$-relation.

Proposition 3.6. *For $\gamma \in \mathbb{S}$, let $\mathscr{L}_\gamma$ denotes the class of elements $\mathscr{L}$ related to γ. Then the subgraph induced by $\mathscr{L}_\gamma$ decomposes into $\mu + 1$ components, a $\mu(\rho - 1)$-clique together with μ isolated vertices. That is, for a fixed $j \in \mathbb{J}$*

$$G_{NA}(\mathscr{L}_\gamma) \cong K_{\mu(\rho-1)} \cup \bigcup_{i \in \mathbb{I}} K_1,$$

where the isolated vertices are precisely the idempotent elements (i, p_{ji}^{-1}, j) for all $i \in \mathbb{I}$.

Proof. For a fixed $j \in \mathbb{J}$, let $\gamma = (i, a, j)$ for any $a \in \mathbb{G}$ and $i \in \mathbb{I}$. Then $\mathscr{L}_\gamma = \{(i, b, j) : b \in \mathbb{G},\ i \in \mathbb{I}\}$. Let $i_1 \in \mathbb{I}$. By Proposition 3.1, $(i_1, p_{ji_1}^{-1}, j)$ is non-adjacent to all elements in $\mathscr{L}_\gamma$. Thus, it is an isolated vertex. Now, consider (i_1, c, j) for all $c \neq p_{ji_1}^{-1}$. By Proposition 3.1, it is adjacent to all others in $\mathscr{L}_\gamma$ except $(i_1, p_{ji_1}^{-1}, j)$. Since i_1 is arbitrary, for all $i \in \mathbb{I}$, (i, p_{ji}^{-1}, j) form μ isolated vertices. All other $\mu(\rho - 1)$ vertices form a clique for the subgraph induced by $\mathscr{L}_\gamma$. □

Following the same reason as in Proposition 3.6, we obtain an analogous characterization for the subgraph induced by the $\mathscr{R}$-class containing γ.

Proposition 3.7. *For $\gamma \in \mathbb{S}$, the subgraph induced by $\mathscr{R}_\gamma$ is a disconnected graph with $\nu+1$ components, a $\nu(\rho-1)$- clique together with ν isolated vertices. That is, for a fixed $i \in \mathbb{I}$*

$$G_{NA}(\mathscr{R}_\gamma) \cong K_{\nu(\rho-1)} \cup \bigcup_{j\in\mathbb{J}} K_1,$$

where the isolated vertices are precisely the idempotent elements (i, p_{ji}^{-1}, j) for all $j \in \mathbb{J}$.

We next examine the subgraph of $G_{NA}(\mathbb{S})$ induced by a single $\mathscr{H}$-class.

Proposition 3.8. *For $\gamma \in \mathbb{S}$, the subgraph induced by $\mathscr{H}_\gamma$ is a disconnected graph with 2 components, a $(\rho-1)$-clique together with an isolated vertex. That is, for a fixed $i \in \mathbb{I}$ and $j \in \mathbb{J}$*

$$G_{NA}(\mathscr{H}_\gamma) \cong K_{(\rho-1)} \cup K_1,$$

where the isolated vertex is precisely the idempotent (i, p_{ji}^{-1}, j).

Proof. For a fixed $i \in \mathbb{I}$ and $j \in \mathbb{J}$, let $\gamma = (i, a, j)$ for any $a \in \mathbb{G}$. Then $\mathscr{H}_\gamma = \{(i, b, j) : b \in \mathbb{G}\}$. By Corollary 3.4, if $a \neq p_{ji}^{-1}$, then γ is adjacent to all other elements in $\mathscr{H}_\gamma$ except (i, p_{ji}^{-1}, j), which is an idempotent in $\mathbb{S}$. Thus the subgraph induced by $\mathscr{H}_\gamma$ decomposes into a $(\rho-1)$-clique together with the isolated vertex (i, p_{ji}^{-1}, j). □

A vertex $\gamma = (i, a, j) \in \mathbb{S} = \mathcal{M}(\mathbb{G}; \mathbb{I}, \mathbb{J}; \mathbb{P})$ is absorbing in the non-absorbing product graph $G_{NA}(\mathbb{S})$ if and only if γ is idempotent. Equivalently, each $\mathscr{H}$-class $\mathscr{H}_{ij} = \{(i, g, j) : g \in \mathbb{G}\}$ contains a unique absorbing vertex (i, p_{ji}^{-1}, j), while all remaining elements of $\mathscr{H}_{ij}$ are non-absorbing. From the perspective of Green's relations, the set of absorbing vertices forms a transversal of the $\mathscr{H}$-classes: each $\mathscr{L}$-class $\mathscr{L}_j$ and each $\mathscr{R}$-class $\mathscr{R}_i$ contains exactly one absorbing vertex in every $\mathscr{H}$-class it meets. Since $\mathbb{S}$ consists of a single $\mathscr{D}$-class, the absorbing vertices are uniformly distributed across $\mathbb{S}$, and every non-absorbing vertex lies in the same $\mathscr{D}$-class as some absorbing vertex.

Now, we determine the conditions under which $G_{NA}(\mathbb{S})$ is connected. When $\mu = 1$, there is exactly one $\mathscr{R}$-class of ν vertices. Then, by Proposition 3.7, $G_{NA}(\mathbb{S})$ is a disconnected graph with $\nu+1$ components. Now, when $\nu = 1$, there forms exactly one $\mathscr{L}$-class of μ vertices and Proposition 3.6 yields that $G_{NA}(\mathbb{S})$ is a disconnected graph with $\mu+1$ components. In the following, we prove that $G_{NA}(\mathbb{S})$ is connected when both μ and ν are greater than 1.

Theorem 3.9. *Let $\mathbb{S} = \mathcal{M}(\mathbb{G}; \mathbb{I}, \mathbb{J}; \mathbb{P})$ be a completely simple semigroup with $|\mathbb{I}| = \mu > 1$ and $|\mathbb{J}| = \nu > 1$. Then the non-absorbing product graph $G_{NA}(\mathbb{S})$ is connected.*

Proof. Let $\gamma_1 = (i_1, a, j_1)$ and $\gamma_2 = (i_2, b, j_2)$ be arbitrary vertices in $\mathbb{S}$.

If γ_1 and γ_2 are non-idempotents, then by Proposition 3.2, they are adjacent in $G_{NA}(\mathbb{S})$.

If at least one of γ_1 or γ_2 is idempotent, with $i_1 \neq i_2$ and $j_1 \neq j_2$, then by Proposition 3.2, they are adjacent in $G_{NA}(\mathbb{S})$. Otherwise, if $i_1 = i_2 = i$ with γ_1 is idempotent, then γ_1 and γ_2 are non-adjacent and we can find a non-idempotent $\gamma_3 = (i_3, c, j_3)$ with $i_3 \neq i$, $j_3 \neq j_2$ such that there is a path $\gamma_1 \sim \gamma_3 \sim \gamma_2$ in $G_{NA}(\mathbb{S})$. The same is the case when $j_1 = j_2 = j$.

When γ_1 and γ_2 are non-adjacent idempotents, we can choose two non-idempotents $\gamma_3 = (i_3, c, j_3)$ with $i_3 \neq i_1$ and $j_3 \neq j_1$ and $\gamma_4 = (i_4, d, j_4)$ with $i_4 \neq i_2$ and $j_4 \neq j_2$ such that there exists a path $\gamma_1 \sim \gamma_3 \sim \gamma_4 \sim \gamma_2$.

Thus, we see that there is always a path of length at most 3 connecting γ_1 and γ_2 and hence $G_{NA}(\mathbb{S})$ is connected. □

From the preceding discussion, the following corollary is immediate.

Corollary 3.10. *Let $\mathbb{S}$ be a completely simple semigroup with $\mu, \nu > 1$. Then $G_{NA}(\mathbb{S})$ has diameter at most 3.*

Now we proceed to describe an important structural feature of $G_{NA}(\mathbb{S})$. The next result shows that $G_{NA}(\mathbb{S})$ contains a ν-partite subgraph, with part size μ, which is $k = (\mu - 1)(\nu - 1)$-regular, denoted by $R^k_{\nu,\mu}$.

Proposition 3.11. *Let A be the set of all idempotents in $G_{NA}(\mathbb{S})$. Then the subgraph of $G_{NA}(\mathbb{S})$ induced by A satisfies $G_{NA}(A) \cong R^k_{\nu,\mu}$, where $k = (\mu - 1)(\nu - 1)$.*

Proof. For each $\gamma \in A$, $\mathscr{L}_\gamma(A)$ contains μ elements and there are ν such $\mathscr{L}_\gamma$'s. By Corollary 3.3, there is no edge within each $\mathscr{L}_\gamma$. Thus $G_{NA}(A)$ is partitioned in to ν of $\mathscr{L}_\gamma$ sets which yields that $G_{NA}(A)$ is a ν partite graph with part size μ. By case (iii) of Theorem 3.9, γ is adjacent to $(\mu - 1)(\nu - 1)$ elements in A. □

The above argument is also valid if we consider $\mathscr{R}_\gamma(A)$ instead of $\mathscr{L}_\gamma(A)$. In that case, $G_{NA}(A)$ is a regular μ-partite graph with part size ν. Then we have the following:

Proposition 3.12. *Let A be the set of all idempotents in $G_{NA}(\mathbb{S})$. Then the subgraph of $G_{NA}(\mathbb{S})$ induced by A, $G_{NA}(A) \cong R^k_{\mu,\nu}$ where $k = (\mu - 1)(\nu - 1)$.*

From all our discussions above, we see that the non-absorbing product graph $G_{NA}(\mathbb{S})$ of a completely simple semigroup $\mathbb{S} \cong \mathcal{M}(\mathbb{G}; \mathbb{I}, \mathbb{J}; \mathbb{P})$ with $|\mathbb{I}| = \mu$, $|\mathbb{J}| = \nu$, and $|\mathbb{G}| = \rho$ decomposes as the edge-union of three induced subgraphs. These are: (i) a complete graph $K_{\mu\nu(\rho-1)}$ on the non-absorbing elements, (ii) a bipartite graph between the $\mu\nu(\rho-1)$ non-absorbing elements and the $\mu\nu$ absorbing elements, and (iii) a $[(\mu-1)(\nu-1)]$-regular μ-partite graph with part sizes ν. This structure is based on the partition of $\mathbb{S}$ into absorbing vertices $V_A = \{(i, p^{-1}_{ji}, j) \mid i \in \mathbb{I}, j \in \mathbb{J}\}$ (one per $\mathscr{H}$-class) and the non-absorbing vertices $V_{NA} = \mathbb{S} \setminus V_A$.

Theorem 3.13. *Let $\mathbb{S} = \mathcal{M}(\mathbb{G}; \mathbb{I}, \mathbb{J}; \mathbb{P})$ be a completely simple semigroup with $|\mathbb{I}| = \mu$, $|\mathbb{J}| = \nu$, and $|\mathbb{G}| = \rho$. Then*

$$G_{NA}(\mathbb{S}) \cong K_{\mu\nu(\rho-1)} \cup B_{\mu\nu,\,(\rho-1)\mu\nu} \cup R^{(\mu-1)(\nu-1)}_{\mu,\nu},$$

where the bipartite graph $B_{\mu\nu,(\rho-1)\mu\nu}$ has one part of size $\mu\nu$, each vertex of which is $(\rho-1)(\mu-1)(\nu-1)$-regular, and the other part of size $(\rho-1)\mu\nu$, each vertex of which is $(\mu-1)(\nu-1)$-regular.

Proof. The case $\mu = 1$ and $\nu = 1$ follows immediately from Propositions 3.6 and 3.7.

Now suppose $\mu, \nu > 1$. Let $\gamma_1 = (i_1, a, j_1)$ and $\gamma_2 = (i_2, b, j_2)$ be arbitrary vertices in $\mathbb{S}$.

If γ_1 and γ_2 are non-idempotent, then proposition 3.2, they are adjacent. The set of all such vertices has size $\mu\nu(\rho - 1)$, forming a complete subgraph $K_{\mu\nu(\rho-1)}$.

Let γ_1 be an idempotent vertex. Then by Theorem 3.9, γ_1 is adjacent to γ_2 with $b \neq p_{j_2 i_2}^{-1}$, provided $i_1 \neq i_2$ and $j_1 \neq j_2$. There are $(\rho-1)(\mu-1)(\nu-1)$ such neighbors for each γ_1, giving a bipartite graph with one part of size $\mu\nu$ (vertices like γ_1) and the other part of size $(\rho-1)\mu\nu$ (vertices like γ_2). Each vertex in the first part has degree $(\rho-1)(\mu-1)(\nu-1)$, and each vertex in the second part has degree $(\mu-1)(\nu-1)$.

Finally, vertices with $b = p_{j_2 i_2}^{-1}$ form a $(\mu-1)(\nu-1)$-regular μ-partite subgraph with part size ν, as established in Proposition 3.12.

Combining these three induced subgraphs yields the stated decomposition of $G_{NA}(\mathbb{S})$. □

The following are the immediate consequences of Theorem 3.13.

Corollary 3.14. *The clique number of $G_{NA}(\mathbb{S})$ is*

$$\omega(G_{NA}(\mathbb{S})) = \mu\nu(\rho - 1).$$

Corollary 3.15. *For any vertex $\gamma = (i, a, j) \in G_{NA}(\mathbb{S})$,*

$$\deg(i, a, j) = \begin{cases} \rho(\mu-1)(\nu-1), & \text{if } a = p_{ji}^{-1} (idempotent), \\ \rho\mu\nu - (\mu+\nu), & \text{if } a \neq p_{ji}^{-1} (non-idempotent). \end{cases}$$

Corollary 3.16. *The number of edges in $G_{NA}(\mathbb{S})$ is*

$$|E(G_{NA}(\mathbb{S}))| = \frac{\mu\nu}{2}\Big[(\mu+\nu)(1-2\rho) + \rho(1+\rho\mu\nu)\Big].$$

We now characterize the Hamiltonian property of $G_{NA}(\mathbb{S})$.

Theorem 3.17. *The non-absorbing product graph $G_{NA}(\mathbb{S})$ is Hamiltonian whenever it is connected.*

Proof. Decompose the vertex set of $G_{NA}(\mathbb{S})$ into the set V_{NA} of non-absorbing elements and the set V_A of absorbing elements. By Theorem 3.13, V_A induces a $(\mu-1)(\nu-1)$-regular μ-partite graph with part sizes ν. In [6] it is proved that a balanced μ-partite graph with part size ν is Hamiltonian if its minimum degree satisfies

$$\delta(G) > \begin{cases} \left(\dfrac{\mu}{2} - \dfrac{1}{\mu+1}\right)\nu, & \text{if } \mu \text{ is odd,} \\ \left(\dfrac{\mu}{2} - \dfrac{2}{\mu+2}\right)\nu, & \text{if } \mu \text{ is even.} \end{cases}$$

A direct calculation shows that

$$(\mu-1)(\nu-1) > \begin{cases} \left(\dfrac{\mu}{2} - \dfrac{1}{\mu+1}\right)\nu, & \mu \text{ odd}, \\ \left(\dfrac{\mu}{2} - \dfrac{2}{\mu+2}\right)\nu, & \mu \text{ even}, \end{cases} \tag{3.1}$$

for all (μ, ν) with $\mu > 1$ and $\nu > 2$. By Proposition 3.11, V_A is also viewed as a $(\mu-1)(\nu-1)$-regular ν-partite graph with part sizes μ. Thus (μ, ν) satisfies the condition 3.1 for $\nu = 2$ and $\mu > 2$. Hence V_A is Hamiltonian except $\mu = \nu = 2$.

Let P_1 denote a Hamiltonian path in V_A with endpoints u_1 and u_2. By Theorem 3.13, each absorbing vertex is adjacent to at least two non-absorbing vertices in the bipartite subgraph $B_{\mu\nu,(\rho-1)\mu\nu}$. Hence, there exist distinct vertices $v_1, v_2 \in V_{NA}$ such that $v_1 \sim u_1$ and $v_2 \sim u_2$. The subgraph induced by V_{NA} is the complete graph $K_{\mu\nu(\rho-1)}$ and therefore Hamiltonian. Let P_2 be a Hamiltonian path in V_{NA} from v_1 to v_2. Then concatenating the paths gives a Hamiltonian cycle in $G_{NA}(\mathbb{S})$: $u_1 P_1 u_2\, v_2 P_2 v_1 u_1$.

For $\mu = \nu = 2$, a Hamiltonian cycle can be explicitly constructed as follows: Let $\mathbb{I} = \{a, b\}$ and $\mathbb{J} = \{x, y\}$, and set

$$V_A = \{\alpha_1 = (a, P_{xa}^{-1}, x),\ \alpha_2 = (a, P_{ya}^{-1}, y),\ \alpha_3 = (b, P_{xb}^{-1}, x),\ \alpha_4 = (b, P_{yb}^{-1}, y)\}.$$

Start at a non-absorbing vertex in $\mathscr{H}_{\alpha_1}$, move to α_4 then to α_1, then traverse all non-absorbing elements in $\mathscr{H}_{\alpha_4}$ and $\mathscr{H}_{\alpha_2}$ sequentially. Next, move to α_3, then to α_2, continue through the remaining non-absorbing elements in $\mathscr{H}_{\alpha_3}$, and finally visit the remaining non-absorbing vertices in $\mathscr{H}_{\alpha_1}$, returning to the starting vertex. This forms a Hamiltonian cycle. □

Example 3.18. Let $\mathbb{S} = \mathcal{M}(\mathbb{G}; \mathbb{I}, \mathbb{J}; \mathbb{P})$ be a completely simple semigroup with $\mathbb{I} = \{a, b\}$, $\mathbb{J} = \{x, y, z\}$, $\mathbb{G} = \mathbb{Z}_3 = \{0, 1, 2\}$ and $\mathbb{P} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \end{bmatrix}$. Then $G_{NA}(\mathbb{S})$ can be viewed as shown in Figure 1. It is the union of a complete graph of order 12, a bipartite graph of part sizes 6 and 12, and a 2- regular 2 partite graph with part size 3, shown in Figure 2.

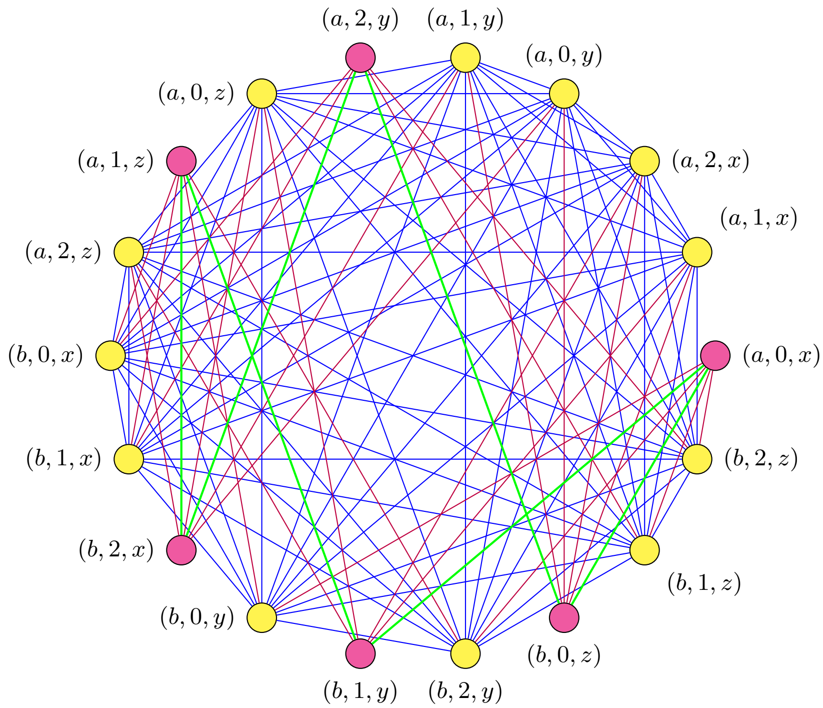

FIGURE 1. The non-absorbing product graph of $G_{NA}(\mathbb{S})$

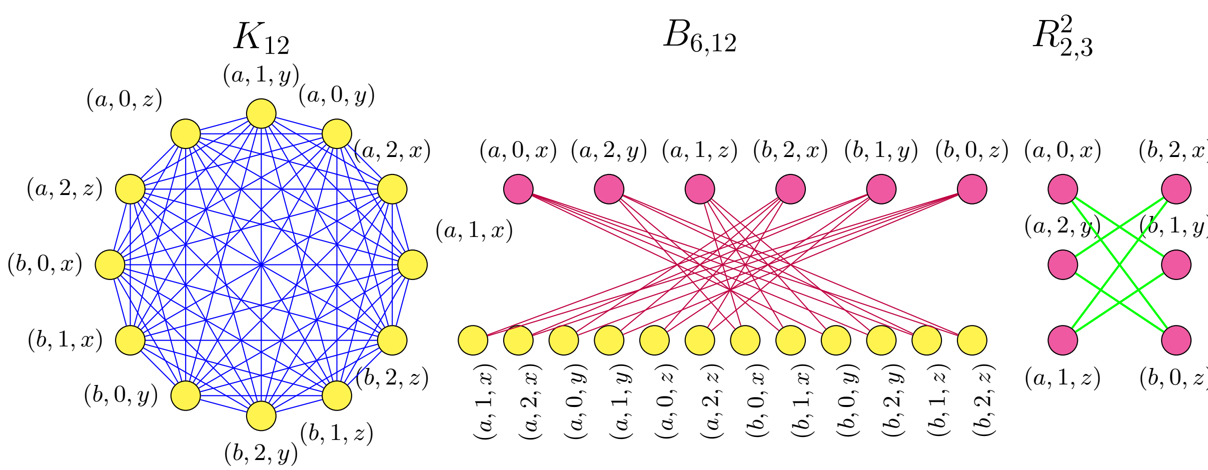

FIGURE 2. Decomposition of $G_{NA}(\mathbb{S})$

Theorem 3.19. *Let* $\mathbb{S} = \mathcal{M}(\mathbb{G}; \mathbb{I}, \mathbb{J}; \mathbb{P})$ *be a completely simple semigroup with* $|\mathbb{I}| = \mu$, $|\mathbb{J}| = \nu$ *and* $|\mathbb{G}| = \rho$. *Then the non-absorbing product graph* $G_{NA}(\mathbb{S})$ *is planar if and only if one of the following holds:*

(i) $\mu = \rho = 1$ *or* $\nu = \rho = 1$;
(ii) $\mu = \nu = 1$ *and* $\rho < 6$;
(iii) *For* $\mu = 1$, $(\nu, \rho) = (2, 2), (3, 2), (4, 2), (2, 3)$;
(iv) *For* $\nu = 1$, $(\mu, \rho) = (2, 2), (3, 2), (4, 2), (2, 3)$;

(v) *For* $\rho = 1$, $(\mu, \nu) = (2,2), (2,3), (3,2)$;
(vi) $\mu = \nu = \rho = 2$.

Proof. The graph $G_{NA}(\mathbb{S})$ is disconnected for $\mu = 1$ or $\nu = 1$. For $\mu = 1$, by Proposition 3.7,

$$G_{NA}(\mathbb{S}) \cong K_{\nu(\rho-1)} \cup \bigcup_{j \in \mathbb{J}} K_1$$

For $\nu = 1$, by Proposition 3.6,

$$G_{NA}(\mathbb{S}) \cong K_{\mu(\rho-1)} \cup \bigcup_{i \in \mathbb{I}} K_1$$

Since the complete graph K_n is planar if and only if $n < 5$, (i) to (iv) are immediate. The remaining disconnected $G_{NA}(\mathbb{S})$ are non-planar.
For $\mu, \nu > 1$, $G_{NA}(\mathbb{S})$ is connected and by Theorem 3.13,

$$G_{NA}(\mathbb{S}) \cong K_{\mu\nu(\rho-1)} \cup B_{\mu\nu,\,(\rho-1)\mu\nu} \cup R_{\mu,\nu}^{(\mu-1)(\nu-1)} \tag{3.2}$$

Case(i): $\rho = 1$. Then,

$$G_{NA}(\mathbb{S}) \cong R_{\mu,\nu}^{(\mu-1)(\nu-1)}$$

For $(\mu, \nu) = (2,2)$, it becomes $K_2 \cup K_2$ so is planar. When $(\mu, \nu) = (2,3)$ or $(3,2)$, $G_{NA}(\mathbb{S}) \cong R_{2,3}^2 \cong C_6$, is planar. All other cases $G_{NA}(\mathbb{S})$ contain a subdivision of $K_{3,3}$ or K_5, and therefore are non-planar by Kuratowski's theorem.

Case(ii): $\rho > 1$. From Equation 3.2, in this case, the $G_{NA}(\mathbb{S})$ is non-planar except $\mu = \nu = \rho = 2$.

To confirm the planarity at $\mu = \nu = \rho = 2$, consider $\mathbb{I} = \{a, b\}$, $\mathbb{J} = \{x, y\}$, $\mathbb{G} = \mathbb{Z}_3 = \{0, 1\}$ and $\mathbb{P}=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. By Theorem 3.5, $G_{NA}(\mathbb{S})$ from this particular case is isomorphic to all graphs under consideration. Figure 3 depicts the structure and its redrawing, respectively, making it clear that the graph is planar.

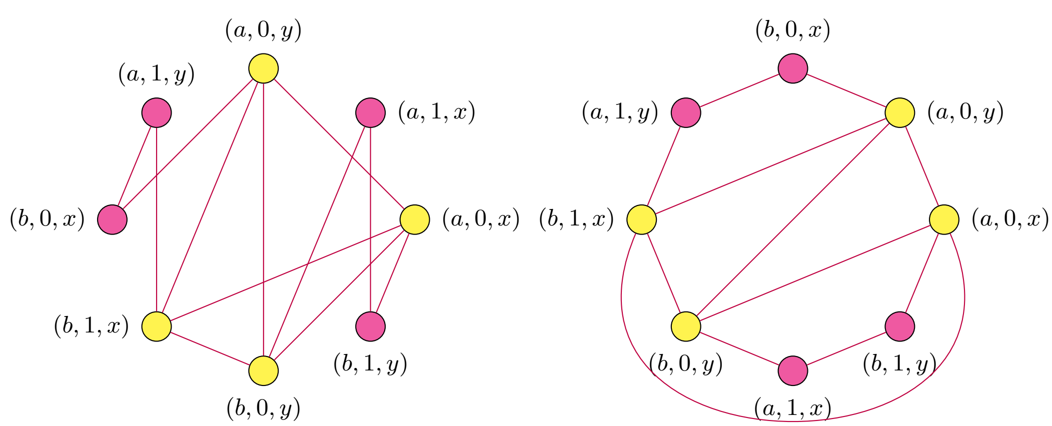

FIGURE 3. $G_{NA}(\mathbb{S})$ and its planar representation for $\mu = \nu = \rho = 2$

∎

An interesting question in the study of non-absorbing product graphs is the inverse problem: given a graph exhibiting the structural patterns described in Theorem 3.13—namely, a complete graph, a bipartite graph, and a regular multipartite graph—does there exist a completely simple semigroup whose non-absorbing product graph is isomorphic to it?

The answer is affirmative, provided the graph satisfies the numeric constraints imposed by the cardinalities of the group and the index sets. Since the structure of $G_{NA}(\mathbb{S})$ depends only on the order ρ of the underlying group and the sizes μ and ν of the index sets, one can construct a completely simple semigroup with any group of order ρ and index sets of sizes μ and ν such that the resulting non-absorbing product graph realizes the given decomposition. This establishes a precise correspondence between graphs with the specified combinatorial structure and completely simple semigroups, complementing the forward structural theorem.

Theorem 3.20. *Let $\mathbb{G}$ be any finite group of order ρ, and let $\mu, \nu \geq 1$. Suppose that a graph Γ admits a disjoint vertex partition $V(\Gamma) = A \sqcup N$ with $|A| = \mu\nu$ and $|N| = (\rho - 1)\mu\nu$ such that:*

1. *N induces a complete graph;*
2. *A induces a μ-partite $(\mu - 1)(\nu - 1)$-regular graph with each part of size ν;*
3. *the bipartite graph between A and N is such that vertices in A have degree $(\rho - 1)(\mu - 1)(\nu - 1)$ and vertices in N have degree $(\mu - 1)(\nu - 1)$.*

Then there exists a completely simple semigroup $\mathbb{S} \cong \mathcal{M}(\mathbb{G}; \mathbb{I}, \mathbb{J}; \mathbb{P})$, with $|\mathbb{I}| = \mu$, $|\mathbb{J}| = \nu$, and arbitrary sandwich matrix $\mathbb{P}$, such that $G_{NA}(\mathbb{S}) \cong \Gamma$.

Proof. Let $\mathbb{G}$ be any group of order ρ, and let $\mathbb{I}$ and $\mathbb{J}$ be index sets of sizes μ and ν, respectively. Construct a Rees matrix semigroup $\mathbb{S} = \mathcal{M}(\mathbb{G}; \mathbb{I}, \mathbb{J}; \mathbb{P})$, where $\mathbb{P} = (p_{ji})$ is an arbitrary $\nu \times \mu$ matrix with entries in $\mathbb{G}$ and each row and column of $\mathbb{P}$ contains at least one nonzero entry. Then $\mathbb{S}$ is completely simple with $|\mathbb{S}| = \mu\nu\rho$, and its elements are all triples $(i, a, j) \in \mathbb{I} \times \mathbb{G} \times \mathbb{J}$.

Define the subset of *absorbing elements*

$$A = \{(i, p_{ji}^{-1}, j) \mid i \in \mathbb{I}, j \in \mathbb{J}\},$$

and let $N = \mathbb{S} \setminus A$ denote the *non-absorbing elements*. By the definition of the non-absorbing product graph $G_{NA}(\mathbb{S})$, two distinct vertices (i_1, a, j_1) and (i_2, b, j_2) are adjacent if and only if their product is distinct from both factors.

From the structure of Rees matrix semigroups:

1. The vertices in N form a *clique of size* $|N| = \mu\nu(\rho - 1)$, since for any $a \neq p_{j_1 i_1}^{-1}$ and $b \neq p_{j_2 i_2}^{-1}$, the product $(i_1, a, j_1)(i_2, b, j_2)$ differs from both factors.
2. The adjacency between A and N produces a *bipartite graph* with the prescribed degrees: each vertex in A has degree $(\rho - 1)(\mu - 1)(\nu - 1)$, and each vertex in N has degree $(\mu - 1)(\nu - 1)$.
3. The vertices in A form a *μ-partite regular graph* with part sizes ν and degree $(\mu - 1)(\nu - 1)$, as described in Theorem 3.13.

Since the adjacency structure depends *only* on ρ, μ, and ν, and not on the specific choice of group $\mathbb{G}$ or the entries of $\mathbb{P}$, the constructed semigroup realizes any graph Γ satisfying the given conditions. Therefore, $G_{NA}(\mathbb{S}) \cong \Gamma$. $\square$

4. Absorbing product graph of completely simple semigroups

We now turn our attention to the *absorbing product graph* $G_A(\mathbb{S})$ of a completely simple semigroup $\mathbb{S} \cong \mathcal{M}(\mathbb{G}; \mathbb{I}, \mathbb{J}; \mathbb{P})$. By definition, two elements of $\mathbb{S}$ are adjacent in $G_A(\mathbb{S})$ if and only if they are non-adjacent in the non-absorbing product graph $G_{NA}(\mathbb{S})$. In particular, for $\gamma_1 = (i_1, a, j_1)$ and $\gamma_2 = (i_2, b, j_2)$, adjacency in $G_A(\mathbb{S})$ occurs exactly when either γ_1 or γ_2 is an idempotent with $i_1 = i_2$ or $j_1 = j_2$.

Proposition 4.1. *Let $\gamma_1 = (i_1, a, j_1)$ and $\gamma_2 = (i_2, b, j_2)$ be elements of $\mathbb{S}$. Then γ_1 and γ_2 are adjacent in $G_A(\mathbb{S})$ if and only if $\gamma_1 \mathscr{L} \gamma_2$ or $\gamma_1 \mathscr{R} \gamma_2$ with either γ_1 or γ_2 is an idempotent.*

Proof. Suppose γ_1 and γ_2 are adjacent in $G_A(\mathbb{S})$. Then, by the adjacency condition in $G_A(\mathbb{S})$, one of them is idempotent and $i_1 = i_2$ or $j_1 = j_2$. For $j_1 = j_2$, they share a common $\mathscr{L}$-class and $i_1 = i_2$ they share a common $\mathscr{R}$-class.

Conversely suppose $\gamma_1 \mathscr{L} \gamma_2$ or $\gamma_1 \mathscr{R} \gamma_2$ with either γ_1 or γ_2 is an idempotent. Without loss of generality, assume $\gamma_1 \mathscr{R} \gamma_2$ with γ_1 is an idempotent. Then $i_1 = i_2$ and $a = p^{-1}_{j_1 i_1}$. Now, $\gamma_1\gamma_2 = (i_1, p^{-1}_{j_1 i_1}, j_1)(i_2, b, j_2) = (i_1, p^{-1}_{j_1 i_1} p_{j_1 i_2} b, j_2) = (i_2, b, j_2) = \gamma_2$. Thus γ_1 and γ_2 are adjacent in $G_A(\mathbb{S})$.
If $\gamma_1 \mathscr{L} \gamma_2$ with either γ_1 or γ_2 is an idempotent, the proof is analogous to the previous one. $\square$

Proposition 4.2. *If two elements in $\mathbb{S}$ are $\mathscr{L}$-related or $\mathscr{R}$-related, then they are adjacent in $G_A(\mathbb{S})$.*

Proof. If two elements are $\mathscr{L}$-related, they belong to the same row in the Rees matrix. By the adjacency condition, at least one of them is of the form (i, p^{-1}_{ji}, j) (absorbing), and hence they are adjacent in $G_A(\mathbb{S})$. The argument for $\mathscr{R}$-related elements is analogous, using the column structure. $\square$

Using the preceding results on non-absorbing subgraphs (Theorems 3.6–3.8), we can describe the structure of subgraphs of $G_A(\mathbb{S})$ induced by $\mathscr{L}_\gamma$, $\mathscr{R}_\gamma$, and $\mathscr{H}_\gamma$. These structural descriptions are obtained directly as the complements of the corresponding subgraphs in $G_{NA}(\mathbb{S})$. They also make explicit the role of absorbing (idempotent) elements in forming cliques and star-like connections in $G_A(\mathbb{S})$.

Proposition 4.3. *The subgraph of $G_A(\mathbb{S})$ induced by a $\mathscr{L}_\gamma$-class is the edge-union of a μ-clique and a complete bipartite graph connecting it to $(\rho-1)\mu$ non-absorbing elements. In particular, for fixed $j \in \mathbb{J}$,*

$$G_A(\mathscr{L}_\gamma) \cong K_\mu \cup K_{\mu,(\rho-1)\mu}.$$

Proposition 4.4. *The subgraph of $G_A(\mathbb{S})$ induced by a $\mathscr{R}_\gamma$-class is the edge-union of a ν-clique and a complete bipartite graph connecting it to $(\rho-1)\nu$ non-absorbing elements. For fixed $i \in \mathbb{I}$,*

$$G_A(\mathscr{R}_\gamma) \cong K_\nu \cup K_{\nu,(\rho-1)\nu}.$$

Proposition 4.5. *The subgraph of $G_A(\mathbb{S})$ induced by a $\mathscr{H}_\gamma$-class is a star graph on ρ vertices, with the center at the absorbing element (i, p_{ji}^{-1}, j). That is, for fixed $i \in \mathbb{I}$ and $j \in \mathbb{J}$,*

$$G_A(\mathscr{H}_\gamma) \cong K_{1,\,\rho-1}.$$

For the non-absorbing product graph $G_{NA}(\mathbb{S})$, connectivity holds when the index sets satisfy $\mu, \nu > 1$, as shown in Theorem 3.9. In contrast, the absorbing product graph $G_A(\mathbb{S})$ is connected for *all* values of μ and ν, without restriction, as established in Theorem 4.6.

Theorem 4.6. *Let $\mathbb{S} = \mathcal{M}(\mathbb{G}; \mathbb{I}, \mathbb{J}; \mathbb{P})$ be a completely simple semigroup with $|\mathbb{G}| = \rho$, $|\mathbb{I}| = \mu$ and $|\mathbb{J}| = \nu$. Then the absorbing product graph $G_A(\mathbb{S})$ is connected.*

Proof. Let $\gamma_1 = (i_1, a, j_1)$ and $\gamma_2 = (i_2, b, j_2)$ be arbitrary vertices of $G_A(\mathbb{S})$.

If γ_1 and γ_2 are $\mathscr{L}$-related or $\mathscr{R}$-related and either one of them is an idempotent, then by Theorem 4.1, they are adjacent in $G_A(\mathbb{S})$.

Suppose γ_1 and γ_2 are $\mathscr{L}$-related and both are non-idempotent. Choose an idempotent element γ_3 from the same $\mathscr{L}$- class. Such an element exists, since every $\mathscr{L}$- class contains an idempotent. Then by Theorem 4.1, the edges $\gamma_1 \sim \gamma_3$ and $\gamma_3 \sim \gamma_2$ are present in $G_A(\mathbb{S})$. Thus, there exists a path $\gamma_1 \sim \gamma_3 \sim \gamma_2$ of length 2 connecting γ_1 and γ_2.

Suppose γ_1 and γ_2 are $\mathscr{R}$-related and both are non-idempotent. Choose an idempotent element γ_3 from the same $\mathscr{R}$- class. Thus by the same line of reasoning, $\gamma_1 \sim \gamma_3 \sim \gamma_2$.

Suppose now that γ_1 and γ_2 are neither $\mathscr{L}$- nor $\mathscr{R}$-related. Consider an idempotent vertex γ_3 which is $\mathscr{L}$-related to γ_1 and $\mathscr{R}$-related to γ_2 or $\mathscr{R}$-related to γ_1 and $\mathscr{L}$-related to γ_2. By Theorem 4.1, both the cases the edges $\gamma_1 \sim \gamma_3$ and $\gamma_3 \sim \gamma_2$ are present in $G_A(\mathbb{S})$.

Thus, every pair of vertices in $G_A(\mathbb{S})$ is connected by a path of length at most 2, and therefore $G_A(\mathbb{S})$ is connected. ☐

Corollary 4.7. *Let $\mathbb{S}$ be a completely simple semigroup. Then $G_A(\mathbb{S})$ has diameter at most 2.*

We have seen that the non-absorbing product graph $G_{NA}(\mathbb{S})$ of a completely simple semigroup admits a highly regular decomposition, consisting of a clique, a bipartite graph, and a regular multipartite subgraph. This decomposition reflects the uniform behavior of products that avoid absorption by either factor.

Similarly, the absorbing product graph $G_A(\mathbb{S})$, being the complement of $G_{NA}(\mathbb{S})$, decomposes into the edge-disjoint union of a bipartite graph together with complete subgraphs arising from the $\mathscr{L}$- and $\mathscr{R}$-classes of absorbing elements. This well-defined structure is formalized in the following theorem.

Theorem 4.8. *Let $\mathbb{S} = \mathcal{M}(\mathbb{G}; \mathbb{I}, \mathbb{J}; \mathbb{P})$ be a completely simple semigroup with $|\mathbb{G}| = \rho$, $|\mathbb{I}| = \mu$, and $|\mathbb{J}| = \nu$. Then the absorbing product graph $G_A(\mathbb{S})$ is the edge-union of a bipartite graph and two families of complete subgraphs arising from the $\mathscr{L}$- and $\mathscr{R}$-classes of absorbing elements. In particular,*

$$G_A(\mathbb{S}) = B_{\mu\nu,\,(\rho-1)\mu\nu} \;\cup\; \bigcup_{i=1}^{\mu} K_\nu \;\cup\; \bigcup_{j=1}^{\nu} K_\mu,$$

where, in the bipartite subgraph, vertices in the part of size $\mu\nu$ have degree $(\rho - 1)(\mu + \nu - 1)$ and vertices in the part of size $(\rho - 1)\mu\nu$ have degree $\mu + \nu - 1$.

Proof. By definition, two elements of $\mathbb{S}$ are adjacent in $G_A(\mathbb{S})$ precisely when they are non-adjacent in $G_{NA}(\mathbb{S})$. By Theorem 3.13, the structure of $G_A(\mathbb{S})$ is determined by the complements of the components of $G_{NA}(\mathbb{S})$.

From Theorem 4.1, each $\mathscr{L}$-class of absorbing elements forms a complete subgraph of order ν, and analogously, each $\mathscr{R}$-class forms a complete subgraph of order μ. Thus, $G_A(\mathbb{S})$ contains $\mu + \nu$ complete subgraphs: μ of order ν and ν of order μ, together with a bipartite subgraph with one part of size $\mu\nu$ that is $(\rho-1)(\mu+\nu-1)$-regular and the other part of size $(\rho-1)\mu\nu$ that is $(\mu+\nu-1)$-regular. □

As immediate consequences, we have the following corollaries:

Corollary 4.9. *For any vertex $(i, a, j) \in G_A(\mathbb{S})$,*

$$\deg(i, a, j) = \begin{cases} \rho(\mu + \nu - 1) - 1, & \text{if } a = p_{ji}^{-1} (\text{idempotent}), \\ \mu + \nu - 1, & \text{if } a \neq p_{ji}^{-1} (\text{non} - \text{idempotent}). \end{cases}$$

Corollary 4.10. *The number of edges in $G_A(\mathbb{S})$ is*

$$|E(G_A(\mathbb{S}))| = \frac{\mu\nu}{2}\Big[(\mu + \nu)(2\rho - 1) - 2\rho\Big].$$

Remark 4.11. In Theorem 3.17, we showed that the non-absorbing product graph $G_{NA}(\mathbb{S})$ is Hamiltonian whenever $\mu, \nu > 1$. In contrast, the corresponding absorbing product graph $G_A(\mathbb{S})$ need not be Hamiltonian. Indeed, if $\mathbb{S} = \mathcal{M}(\mathbb{G}; \mathbb{I}, \mathbb{J}; \mathbb{P})$ is a completely simple semigroup with $\mu, \nu > 1$, then $G_A(\mathbb{S})$ contain cut vertices namely, the absorbing elements (i, p_{ji}^{-1}, j), and hence $G_A(\mathbb{S})$ is not Hamiltonian in general.

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