

NEW RESULTS OF GLOBAL EXISTENCE AND UNIQUENESS FOR FUZZY FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. Fractional calculus and fuzzy set theory together provide a powerful framework for modeling complex systems with memory and uncertainty. However, existing studies on fuzzy fractional differential equations often suffer from an improperly defined solution space and rely on restrictive assumptions such as Lipschitz continuity. In this work, we correct these issues by first establishing existence and uniqueness for fuzzy fractional integral equations under weaker conditions. We then introduce a general fuzzy fractional integro-differential equation combining Riemann-Liouville derivatives with Volterra and Fredholm operators. Using the monotone iterative technique with upper and lower solutions, we obtain global existence and uniqueness results that significantly relax previous hypotheses, providing a more solid theoretical foundation for this emerging field.

Keywords. Fuzzy fractional differential equations; Riemann-Liouville derivative; Monotone iterative technique; Upper and lower solutions; Existence and uniqueness

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1. INTRODUCTION

Over recent years, fractional calculus has become a widely adopted framework for modeling intricate phenomena across various disciplines. The unique properties of fractional operators, such as their non-local nature and ability to capture memory and hereditary effects make them particularly suitable for describing systems in electrical networks, control theory, fractal theory, chemistry, and other fields where classical integer-order models prove inadequate [2, 6, 9]. This led to the independent development of a systematic theory for fractional differential equations, comparable to the well-developed theory of ordinary differential equations, where existence results for initial value problems have received particular attention [4, 6, 7].

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Parallel to these developments, the notion of fuzzy sets and fuzzy numbers introduced in [28] and subsequently enriched by Mizumoto and Tanaka [18] has revolutionized how we handle uncertainty and imprecision in mathematical modeling. The extension of differential calculus to fuzzy-valued functions began with the introduction of the Hukuhara derivative for fuzzy-valued functions and the investigation of fuzzy initial value problems [11, 23]. Significant advances were made by Stefanini and Bede [7, 26], who introduced the generalizations of Hukuhara difference and developed generalized differentiability concepts for both interval-valued and fuzzy-valued functions, establishing connections between various differentiability notions and investigating their properties [7, 14].

The intersection of these two important fields fractional calculus and fuzzy set theory has given rise to fuzzy fractional differential equations (FFDEs for short). Agarwal et al. introduced in [2] the concept of Riemann-Liouville fractional differential equations involving Hukuhara differentiability for solving uncertain fractional differential equations. Subsequently, researchers have explored various aspects of this emerging field. For instance, Arshad et al. [5] demonstrated that solutions exist and are unique for FFDEs of Riemann-Liouville type.

Despite these advances, results for FFDEs remain relatively scarce compared to their integer-order counterparts [5, 8, 11, 12, 13, 14, 24, 25]. Authors in [13, 14] proved existence results for fractional fuzzy differential equations of the form

$$\begin{cases} \mathfrak{D}^\sigma \mathfrak{U}(\tau) = \vartheta(\tau, \mathfrak{U}(\tau)), & \tau \in (0, b], \\ [I^{1-\sigma} \mathfrak{U}](0^+) = u_0, \end{cases} \quad (1.1)$$

where $u_0 \in \mathbb{R}_{\mathcal{F}}$, $(0, b]$ is the interval, and $\vartheta \in \mathcal{C}((0, b] \times \mathbb{R}_{\mathcal{F}}, \mathbb{R}_{\mathbb{F}})$. While the ideas in these works are valuable, we identify a fundamental issue: the definition of the solution space. The space $\mathcal{C}([0, b], \mathbb{R}_{\mathcal{F}})$ is commonly employed as the solution space, which is inappropriate except when $u_0 = 0$, since for any $\mathfrak{U} \in \mathcal{C}([0, b], \mathbb{R}_{\mathcal{F}})$, we have $\tau^{1-\sigma} \mathfrak{U}(\tau)|_{\tau=0} = 0$ (See [14, 13]). Although [13] correctly introduced the solution space concept, the integral solution definition remained problematic except for the zero initial condition, affecting the corresponding solution space for FFDEs. Thus, the first objective is to establish existence and uniqueness results for the fuzzy fractional integral equation given by:

$$\begin{cases} ({}_gH \mathfrak{D}_*^r \mathfrak{U})(\tau) = \vartheta(\tau, \mathfrak{U}(\tau), K\mathfrak{U}(\tau), \mathfrak{H}\mathfrak{U}(\tau)), & \mathcal{J} = (0, b] \\ \mathfrak{U}(0) = u_0 \in \mathbb{R}_{\mathcal{F}}, \end{cases} \quad (1.2)$$

where $0 < r \leq 1$, ${}_gH \mathfrak{D}_*^r$ denotes the Caputo fractional generalized derivative, and $\vartheta : \mathcal{J} \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{\mathcal{F}}$ is continuous in τ . The operators K and $\mathfrak{H}$ are defined as

$$\begin{aligned} K\mathfrak{U}(\tau) &= \int_0^\tau \mathcal{K}(\tau, s) \mathfrak{U}(s) ds, \\ \mathfrak{H}\mathfrak{U}(\tau) &= \int_0^b \mathfrak{p}(\tau, s) \mathfrak{U}(s) ds, \end{aligned} \quad (1.3)$$

with $\mathcal{K}, \mathfrak{p} \in \mathcal{C}(\mathcal{J}, \mathbb{R}_+)$, representing Volterra and Fredholm integral operators respectively. This formulation introduces a fully nonlinear fuzzy fractional integro-differential equation that generalizes previously studied problems [8, 11, 12, 13] and extends to the fuzzy Volterra–Fredholm type. Combining the monotone iterative technique with the method of upper and lower solutions (ULs for short), our approach is significant because it allows us to establish existence and uniqueness of global solutions under conditions on the nonlinear function ϑ that are weaker than those required in previous studies on fractional differential equations [11, 12, 13]. Importantly, our approach can be readily applied to fuzzy integro-differential equations studied in [8], where stronger conditions such as boundedness and Lipschitz continuity were previously required. This represents a substantial improvement in the theoretical framework for fuzzy fractional differential equations and provides a solid foundation for further developments in this emerging field. The remainder of this article is organized as follows. In Section 2 we recall some essential notions regarding fuzzy numbers, fractional calculus, and generalized differentiability concepts. Sections 3 and 4 present the main existence and uniqueness results using the monotone iterative technique.

2. PRELIMINARIES

This section recalls essential notions and results concerning fuzzy functions that will be used throughout the paper.

Let $\mathbb{R}_{\mathcal{F}}$ stands for the set of all fuzzy numbers on $\mathbb{R}$. For $\sigma \in (0, 1]$, the σ -level set of $V \in \mathbb{R}_{\mathcal{F}}$ is defined as $[V]^{\sigma} = \{\tau \in \mathbb{R} \mid V(\tau) \geq \sigma\}$, and $[V]^0$ is the closure of $\{\tau \in \mathbb{R} \mid V(\tau) > 0\}$. For every $\sigma \in [0, 1]$, the level set is a closed bounded interval, written as $[V]^{\sigma} = [V_{\sigma l}, V_{\sigma r}]$. The zero fuzzy number is denoted by $\hat{0}$, with $\hat{0}(\tau) = 1$ if $\tau = 0$ and $\hat{0}(\tau) = 0$ otherwise.

If $V, \mathfrak{V} \in \mathbb{R}_{\mathcal{F}}$ and there exists a unique $Z \in \mathbb{R}_{\mathcal{F}}$ satisfying $\mathfrak{V} + Z = V$, then Z denotes the H-difference of V and $\mathfrak{V}$ (see, e.g., [18]). Triangular fuzzy numbers are of the form $V = (a, b, c)$, where a and c are the endpoints of the 0-level set and the 1-level set reduces to $\{b\}$ (cf. [20, 23]).

The distance between $V, \mathfrak{V} \in \mathbb{R}_{\mathcal{F}}$ is given by

$$\mathfrak{B}(V, \mathfrak{V}) = \sup_{0 < \sigma \leq 1} \max\{|V_{\sigma l} - \mathfrak{V}_{\sigma l}|, |V_{\sigma r} - \mathfrak{V}_{\sigma r}|\}.$$

In this work we employ the fuzzy Riemann integral. For an integrable fuzzy function $\mathfrak{G} : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ with level sets $[\mathfrak{G}(\tau)]^{\sigma} = [(\mathfrak{G}(\tau))_{\sigma l}, (\mathfrak{G}(\tau))_{\sigma r}]$, the boundary functions are integrable and

$$\begin{aligned} \sigma &\in [0, 1], \\ \left[\int_a^b \mathfrak{G}(\tau) d\tau \right]^{\sigma} &= \left[\int_a^b (\mathfrak{G}(\tau))_{\sigma l} d\tau, \int_a^b (\mathfrak{G}(\tau))_{\sigma r} d\tau \right]. \end{aligned}$$

A function $\mathfrak{G} \in \mathcal{C}(\mathcal{J}, \mathbb{R}_{\mathcal{F}})$ is an element of $L^1((0, b], \mathbb{R}_{\mathcal{F}})$ if $\mathfrak{B}(\int_a^b \mathfrak{G}(s) ds, \hat{0}) < \infty$.

Definition 2.1. [7] Let $\mathfrak{G} : (a, b) \rightarrow \mathbb{R}_{\mathcal{F}}$ and $\tau_0 \in (a, b)$. $\mathfrak{G}$ is said to be generalized differentiable at τ_0 (see, e.g., [26]) whenever there exists an element $\mathfrak{G}'(\tau_0) \in \mathbb{R}_{\mathcal{F}}$ such that either

- (i) for sufficiently small $h > 0$, the H-differences $\mathfrak{G}(\tau_0 + h) \ominus \mathfrak{G}(\tau_0)$ and $\mathfrak{G}(\tau_0) \ominus \mathfrak{G}(\tau_0 - h)$ exist, and

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\mathfrak{G}(\tau_0 + h) \ominus \mathfrak{G}(\tau_0)}{h} &= \lim_{h \rightarrow 0} \frac{\mathfrak{G}(\tau_0) \ominus \mathfrak{G}(\tau_0 - h)}{h} \\ &= \mathfrak{G}'(\tau_0); \end{aligned}$$

- (ii) for all sufficiently small $h > 0$, the H-differences $\mathfrak{G}(\tau_0) \ominus \mathfrak{G}(\tau_0 + h)$ and $\mathfrak{G}(\tau_0 - h) \ominus \mathfrak{G}(\tau_0)$ exist, and

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\mathfrak{G}(\tau_0) \ominus \mathfrak{G}(\tau_0 + h)}{-h} &= \lim_{h \rightarrow 0} \frac{\mathfrak{G}(\tau_0 - h) \ominus \mathfrak{G}(\tau_0)}{-h} \\ &= \mathfrak{G}'(\tau_0). \end{aligned}$$

(The denominators $-h$ indicate multiplication by $-1/h$.)

Definition 2.2. For $V, \mathfrak{Y} \in \mathbb{R}_{\mathcal{F}}$, we write $V \leq \mathfrak{Y}$ if and only if

$$V_{\sigma l} \leq \mathfrak{Y}_{\sigma l} \quad \text{and} \quad V_{\sigma r} \leq \mathfrak{Y}_{\sigma r} \quad \forall \sigma \in [0, 1].$$

The fuzzy interval between V and $\mathfrak{Y}$ is defined as

$$[V, \mathfrak{Y}] := \{Z \in \mathbb{R}_{\mathcal{F}} : V \leq Z \leq \mathfrak{Y}\}.$$

For two fuzzy functions $\mathfrak{G}_1, \mathfrak{G}_2 : \mathcal{J} \rightarrow \mathbb{R}_{\mathcal{F}}$, we say $\mathfrak{G}_1 \leq \mathfrak{G}_2$ if $\mathfrak{G}_1(\tau) \leq \mathfrak{G}_2(\tau)$ for every $\tau \in \mathcal{J}$. The corresponding function interval is

$$[\mathfrak{G}_1, \mathfrak{G}_2] := \{\mathfrak{G} \in \mathcal{C}(\mathcal{J}, \mathbb{R}_{\mathcal{F}}) : \mathfrak{G}_1 \leq \mathfrak{G} \leq \mathfrak{G}_2\}.$$

This partial ordering on fuzzy functions plays a key role in our analysis.

Lemma 2.3. Let $V, \mathfrak{Y}, Z, W \in \mathbb{R}_{\mathcal{F}}$ and $c > 0$. Then:

- $V = \mathfrak{Y}$ is equivalent to the simultaneous satisfaction of $V \leq \mathfrak{Y}$ and $V \geq \mathfrak{Y}$.
- $V \leq \mathfrak{Y}$ implies $V + Z \leq \mathfrak{Y} + Z$.
- The conjunction $V \leq \mathfrak{Y}$ and $Z \leq W$ yields $V + Z \leq \mathfrak{Y} + W$.
- From $V \leq \mathfrak{Y}$ it follows that $cV \leq c\mathfrak{Y}$.

Lemma 2.4. Suppose $\mathfrak{G}_1, \mathfrak{G}_2 \in \mathcal{C}(\mathcal{J}, \mathbb{R}_{\mathcal{F}}) \cap L^1(\mathcal{J}, \mathbb{R}_{\mathcal{F}})$ satisfy $\mathfrak{G}_1(\tau) \leq \mathfrak{G}_2(\tau)$ for every $\tau \in \mathcal{J}$. Then

$$\int_0^\tau \mathfrak{G}_1(s) ds \leq \int_0^\tau \mathfrak{G}_2(s) ds, \quad \forall \tau \in \mathcal{J}.$$

Lemma 2.5. Let $0 < q < 1$ and let $\mathfrak{H} \in \mathcal{C}(\mathcal{J}, \mathbb{R}_{\mathcal{F}})$. Assume there exist $\mathfrak{H}_1, \mathfrak{H}_2 \in \mathcal{C}(\mathcal{J}, \mathbb{R}_{\mathcal{F}}) \cap L^1(\mathcal{J}, \mathbb{R}_{\mathcal{F}})$ such that

$$\mathfrak{H}_1(\tau) \leq \mathfrak{H}(\tau) \leq \mathfrak{H}_2(\tau), \quad \tau \in \mathcal{J}.$$

Then the fractional integral $\int_0^\tau (\tau - s)^{q-1} \mathfrak{H}(s) ds$ is well-defined on $\mathcal{J}$.

Proof. From the ordering assumption we obtain

$$\begin{aligned} & \int_0^\tau (\tau - s)^{q-1} \mathfrak{H}_1(s) ds \\ & \leq \int_0^\tau (\tau - s)^{q-1} \mathfrak{H}(s) ds \\ & \leq \int_0^\tau (\tau - s)^{q-1} \mathfrak{H}_2(s) ds, \quad \tau \in \mathcal{J}. \end{aligned}$$

Since $\mathfrak{H}_1, \mathfrak{H}_2$ belong to $\mathcal{C}(\mathcal{J}, \mathbb{R}_{\mathcal{F}}) \cap L^1(\mathcal{J}, \mathbb{R}_{\mathcal{F}})$, the left and rightmost integrals are well-defined on $\mathcal{J}$. Using the properties of the distance $\mathfrak{B}$, for each $\tau \in \mathcal{J}$,

$$\begin{aligned} & \mathfrak{B}\left(\int_0^\tau (\tau - s)^{q-1} \mathfrak{H}(s) ds, \hat{0}\right) \\ & \leq \mathfrak{B}\left(\int_0^\tau (\tau - s)^{q-1} \mathfrak{H}_1(s) ds, \hat{0}\right) \\ & + \mathfrak{B}\left(\int_0^\tau (\tau - s)^{q-1} \mathfrak{H}_2(s) ds, \hat{0}\right) < \infty, \end{aligned}$$

which shows that the integral is well-defined. $\square$

Lemma 2.6. (See [22]) Let $\{\mathfrak{G}_n\} \subset \mathcal{C}(\mathcal{J}, \mathbb{R}_{\mathcal{F}})$ and $\mathfrak{H} \in \mathcal{C}(\mathcal{J}, \mathbb{R}_{\mathcal{F}})$. Assume $\mathfrak{G}_n(\tau) \leq \mathfrak{H}(\tau)$ for every $n \in \mathbb{N}$ and all $\tau \in \mathcal{J}$. If the sequence $\{\mathfrak{G}_n\}$ converges pointwise (with respect to the metric $\mathfrak{B}$) to some function $\mathfrak{G} : \mathcal{J} \rightarrow \mathbb{R}_{\mathcal{F}}$, then necessarily $\mathfrak{G}(\tau) \leq \mathfrak{H}(\tau)$ for each $\tau \in \mathcal{J}$.

Lemma 2.7. (See [22]) Consider a monotone sequence $\{\mathfrak{G}_n\} \subset \mathcal{C}(\mathcal{J}, \mathbb{R}_{\mathcal{F}})$ that admits a subsequence converging pointwise (in the sense of the metric $\mathfrak{B}$) to a function $\mathfrak{G}$. Then the whole sequence $\{\mathfrak{G}_n\}$ converges pointwise to the same limit $\mathfrak{G}$.

Lemma 2.8. [Generalized Gronwall inequality, see e.g. [27]] Let $\beta > 0$ and let $\delta : \bar{\mathcal{J}} \rightarrow \mathbb{R}_+$ be locally integrable. Suppose $\kappa : \bar{\mathcal{J}} \rightarrow \mathbb{R}_+$ is continuous, nondecreasing, and bounded above by a constant M . If $\mathfrak{R} : \mathcal{J} \rightarrow \mathbb{R}_+$ is locally integrable and satisfies

$$\mathfrak{R}(\tau) \leq \delta(\tau) + \kappa(\tau) \int_0^\tau (\tau - s)^{\beta-1} \mathfrak{R}(s) ds \quad (\tau \in \mathcal{J}),$$

then for all $\tau \in \mathcal{J}$,

$$\mathfrak{R}(\tau) \leq \delta(\tau) + \int_0^\tau \left[\sum_{n=1}^{\infty} \frac{(\kappa(\tau) \Gamma(\beta))^n}{\Gamma(n\beta)} (\tau - s)^{n\beta-1} \delta(s) \right] ds.$$

3. PRINCIPAL FINDINGS

In this section, existence and uniqueness results are established for two general classes of fuzzy fractional integral equations using the method of ULs. These results will later be employed to prove existence for the IVP corresponding to the FFIDEs in (1.2).

We consider two equivalent formulations of fuzzy fractional integral equations that arise from the fractional differential problem (1.2):

$$\begin{aligned} \mathbf{u}(\tau) &= \mathfrak{G}(\tau) \\ &+ \frac{1}{\Gamma(r)} \int_0^\tau (\tau - s)^{r-1} \vartheta(s, \mathbf{u}(s), K\mathbf{u}(s), \mathfrak{H}\mathbf{u}(s)) ds, \end{aligned} \quad (3.1)$$

$$\begin{aligned} \mathbf{u}(\tau) &= \mathfrak{G}(\tau) \\ &\ominus \frac{-1}{\Gamma(r)} \int_0^\tau (\tau - s)^{r-1} \vartheta(s, \mathbf{u}(s), K\mathbf{u}(s), \mathfrak{H}\mathbf{u}(s)) ds, \end{aligned} \quad (3.2)$$

as long as the right-hand sides are pointwise defined on $\mathcal{J}$. Hereafter we assume $0 < r < 1$, $\mathcal{J}, \mathfrak{G} \in \mathcal{C}(\mathcal{J}, \mathbb{R}_{\mathcal{F}})$, $\vartheta \in \mathcal{C}(\mathcal{J} \times \mathbb{R}_{\mathcal{F}}^3, \mathbb{R}_{\mathcal{F}})$, and the operators $K, \mathfrak{H}$ are defined as in (1.3) with kernels $\mathcal{K}, \mathfrak{p} \in \mathcal{C}(\mathcal{J}, \mathbb{R}_+)$.

Definition 3.1. Let $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \underline{\mathbf{u}}_1, \underline{\mathbf{u}}_2 \in \mathcal{C}(\mathcal{J}, \mathbb{R}_{\mathcal{F}})$ and assume that the expressions $\vartheta(\tau, \underline{\mathbf{u}}_i(\tau), (K\underline{\mathbf{u}}_i)(\tau), \mathfrak{H}\underline{\mathbf{u}}_i(\tau))$ and $\vartheta(\tau, \bar{\mathbf{u}}_i(\tau), (K\bar{\mathbf{u}}_i)(\tau), \mathfrak{H}\bar{\mathbf{u}}_i(\tau))$ are integrable on $\mathcal{J}$ for $i = 1, 2$.

(i) The pair $(\bar{\mathbf{u}}_1, \underline{\mathbf{u}}_1)$ is called a pair of ULs for the equation (3.1) if

$$\begin{aligned} \bar{\mathbf{u}}_1(\tau) &\geq \mathfrak{G}(\tau) + \frac{1}{\Gamma(r)} \int_0^\tau (\tau - s)^{r-1} \\ &\times \vartheta(s, \bar{\mathbf{u}}_1(s), K\bar{\mathbf{u}}_1(s), \mathfrak{H}\bar{\mathbf{u}}_1(s)) ds, \quad \tau \in \mathcal{J}, \end{aligned}$$

and

$$\begin{aligned} \underline{\mathbf{u}}_1(\tau) &\leq \mathfrak{G}(\tau) + \frac{1}{\Gamma(r)} \int_0^\tau (\tau - s)^{r-1} \\ &\times \vartheta(s, \underline{\mathbf{u}}_1(s), K\underline{\mathbf{u}}_1(s), \mathfrak{H}\underline{\mathbf{u}}_1(s)) ds, \quad \tau \in \mathcal{J}. \end{aligned}$$

(ii) The pair $(\bar{\mathbf{u}}_2, \underline{\mathbf{u}}_2)$ is called a pair of ULs for (3.2) if

$$\begin{aligned} \bar{\mathbf{u}}_2(\tau) &\geq \mathfrak{G}(\tau) \ominus \frac{-1}{\Gamma(r)} \int_0^\tau (\tau - s)^{r-1} \\ &\times \vartheta(s, \bar{\mathbf{u}}_2(s), K\bar{\mathbf{u}}_2(s), \mathfrak{H}\bar{\mathbf{u}}_2(s)) ds, \quad \tau \in \mathcal{J}, \end{aligned}$$

and

$$\begin{aligned} \underline{\mathbf{u}}_2(\tau) &\leq \mathfrak{G}(\tau) \ominus \frac{-1}{\Gamma(r)} \int_0^\tau (\tau - s)^{r-1} \\ &\times \vartheta(s, \underline{\mathbf{u}}_2(s), K\underline{\mathbf{u}}_2(s), \mathfrak{H}\underline{\mathbf{u}}_2(s)) ds, \quad \tau \in \mathcal{J}, \end{aligned}$$

provided all occurring H -differences are well defined.

Definition 3.2. Let $\bar{\mathbf{u}}_1, \underline{\mathbf{u}}_1$ be ULs for (3.1). A function $\mathbf{u} \in [\underline{\mathbf{u}}_1, \bar{\mathbf{u}}_1]$ is called a solution of (3.1) if it satisfies the equation. A solution $\mathbf{u}_M \in [\underline{\mathbf{u}}_1, \bar{\mathbf{u}}_1]$ is said to be maximal if

$$\mathbf{u}_M(\tau) \geq \mathbf{u}(\tau), \quad \tau \in \mathcal{J},$$

for every solution $\mathbf{u} \in [\underline{\mathbf{u}}_1, \bar{\mathbf{u}}_1]$. Minimal solutions are defined analogously. The same terminology applies to (3.2) using the corresponding ULs.

Theorem 3.3. *Let $\mathfrak{G}, \vartheta, \mathcal{K}, \mathfrak{p}$ be as in (3.1) and let $\bar{\mathbf{u}}_1, \underline{\mathbf{u}}_1 \in \mathcal{C}(\mathcal{J}, \mathbb{R}_{\mathcal{F}})$ be ULs of (3.1) with $\underline{\mathbf{u}}_1(\tau) \leq \bar{\mathbf{u}}_1(\tau)$ for all $\tau \in \mathcal{J}$. Suppose $\vartheta(\tau, x, y, z)$ is nondecreasing in $x \in [\underline{\mathbf{u}}_1(\tau), \bar{\mathbf{u}}_1(\tau)]$, in $y \in [K\underline{\mathbf{u}}_1(\tau), K\bar{\mathbf{u}}_1(\tau)]$ and in $z \in [\mathfrak{H}\underline{\mathbf{u}}_1(\tau), \mathfrak{H}\bar{\mathbf{u}}_1(\tau)]$ for each $\tau \in \mathcal{J}$.*

Then (3.1) possesses maximal and minimal solutions $\mathbf{u}_M, \mathbf{u}_L \in [\underline{\mathbf{u}}_1, \bar{\mathbf{u}}_1]$. Moreover, if there exist constants $l_1, l_2, l_3 \geq 0$ (with $l_1 = l_2 = l_3 = 0$ only when ϑ is constant) such that

$$\begin{aligned} & \vartheta(\tau, x_1, x_2, x_3) + l_1 y_1 + l_2 y_2 + l_3 y_3 \\ & \leq \vartheta(\tau, y_1, y_2, y_3) + l_1 x_1 + l_2 x_2 + l_3 x_3, \quad \tau \in \mathcal{J}, \end{aligned} \quad (3.3)$$

for all $x_1, y_1 \in [\underline{\mathbf{u}}_1(\tau), \bar{\mathbf{u}}_1(\tau)]$, $x_2, y_2 \in [K\underline{\mathbf{u}}_1(\tau), K\bar{\mathbf{u}}_1(\tau)]$, $x_3, y_3 \in [\mathfrak{H}\underline{\mathbf{u}}_1(\tau), \mathfrak{H}\bar{\mathbf{u}}_1(\tau)]$ with $x_1 \geq y_1$, $x_2 \geq y_2$, $x_3 \geq y_3$, then the solution in $[\underline{\mathbf{u}}_1, \bar{\mathbf{u}}_1]$ is unique.

Proof. We apply the monotone iterative technique. Set $\mathbf{v}_0 = \underline{\mathbf{u}}_1$ and define the sequence $\{\mathbf{v}_n\}$ by

$$\begin{aligned} \mathbf{v}_{n+1}(\tau) &= \mathfrak{G}(\tau) + \frac{1}{\Gamma(r)} \int_0^\tau (\tau - s)^{r-1} \\ &\quad \times \vartheta(s, \mathbf{v}_n(s), K\mathbf{v}_n(s), \mathfrak{H}\mathbf{v}_n(s)) ds, \quad \tau \in \mathcal{J}. \end{aligned} \quad (3.4)$$

Similarly, set $\mathbf{w}_0 = \bar{\mathbf{u}}_1$ and define

$$\begin{aligned} \mathbf{w}_{n+1}(\tau) &= \mathfrak{G}(\tau) + \frac{1}{\Gamma(r)} \int_0^\tau \\ &\quad \times (\tau - s)^{r-1} \vartheta(s, \mathbf{w}_n(s), K\mathbf{w}_n(s), \mathfrak{H}\mathbf{w}_n(s)) ds, \quad \tau \in \mathcal{J}. \end{aligned} \quad (3.5)$$

We claim that the sequences are well-defined and satisfy

$$\begin{aligned} \underline{\mathbf{u}}_1 &= \mathbf{v}_0 \leq \mathbf{v}_1 \leq \cdots \leq \mathbf{v}_n \\ &\leq \cdots \leq \mathbf{w}_n \leq \cdots \leq \mathbf{w}_1 \leq \mathbf{w}_0 = \bar{\mathbf{u}}_1, \quad \tau \in \mathcal{J}. \end{aligned} \quad (3.6)$$

First, from the definition of lower solution, $\mathbf{v}_0 \leq \mathbf{v}_1$. By Lemma 2.4 and the monotonicity of ϑ , we obtain $\mathbf{v}_1 \leq \mathbf{w}_0$. Assuming inductively that $\mathbf{v}_{n-1} \leq \mathbf{v}_n \leq \mathbf{w}_0$, the same reasoning yields $\mathbf{v}_n \leq \mathbf{v}_{n+1} \leq \mathbf{w}_0$. An analogous induction shows $\mathbf{w}_0 \geq \mathbf{w}_1 \geq \cdots \geq \mathbf{w}_n \geq \mathbf{v}_n$. Thus the sequences are monotone and bounded, so pointwise limits exist:

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbf{v}_n(\tau) &= \mathbf{u}_L(\tau), \\ \lim_{n \rightarrow \infty} \mathbf{w}_n(\tau) &= \mathbf{u}_M(\tau), \quad \tau \in \mathcal{J}. \end{aligned}$$

Using the continuity and monotonicity of ϑ , the dominated convergence theorem allows us to pass to the limit in (3.4) and (3.5), showing that $\mathbf{u}_L$ and $\mathbf{u}_M$ are solutions of (3.1). To prove extremality, let $\mathbf{u}$ be any solution with $\underline{\mathbf{u}}_1 \leq \mathbf{u} \leq \bar{\mathbf{u}}_1$. A simple induction gives $\mathbf{v}_n \leq \mathbf{u} \leq \mathbf{w}_n$ for all n ; taking limits yields $\mathbf{u}_L \leq \mathbf{u} \leq \mathbf{u}_M$. Hence $\mathbf{u}_L$ and $\mathbf{u}_M$ are the minimal and maximal solutions, respectively.

Now assume condition (3.3) holds. For a fixed $\sigma \in [0, 1]$, write the level sets:

$$\begin{aligned} [\mathbf{u}_L(\tau)]^\sigma &= [(\mathbf{u}_L(\tau))_{\sigma l}, (\mathbf{u}_L(\tau))_{\sigma r}], \\ [\mathbf{u}_M(\tau)]^\sigma &= [(\mathbf{u}_M(\tau))_{\sigma l}, (\mathbf{u}_M(\tau))_{\sigma r}]. \end{aligned}$$

Since $\mathbf{u}_L \leq \mathbf{u}_M$, there exist nonnegative functions $R_l(\tau, \sigma), R_r(\tau, \sigma)$ such that

$$(\mathbf{u}_M)_{\sigma l} = (\mathbf{u}_L)_{\sigma l} + R_l(\tau, \sigma),$$

$$(\mathbf{u}_M)_{\sigma r} = (\mathbf{u}_L)_{\sigma r} + R_r(\tau, \sigma).$$

Using (3.3) and the same estimates as in the proof of Theorem 3.3 (but now with the left-hand side level), we obtain

$$R_l(\tau, \sigma) \leq \kappa \int_0^\tau (\tau - s)^{r-1} R_l(s, \sigma) ds,$$

where $\kappa = \frac{l_1 + (l_2 \mathcal{K}^* + l_3 \mathbf{p}^*)b}{\Gamma(r)}$ with $\mathcal{K}^* = \max\{\mathcal{K}(\tau, s) : (\tau, s) \in \bar{\mathcal{J}} \times \bar{\mathcal{J}}\}$ and $\mathbf{p}^* = \max\{\mathbf{p}(\tau, s) : (\tau, s) \in \bar{\mathcal{J}} \times \bar{\mathcal{J}}\}$. By the generalized Gronwall inequality (Lemma 2.8), $R_l(\tau, \sigma) \leq 0$ for all $\tau \in \mathcal{J}$. The same argument applied to the right endpoints yields $R_r(\tau, \sigma) \leq 0$. Hence $R_l = R_r = 0$, i.e., $\mathbf{u}_M = \mathbf{u}_L$. $\square$

Theorem 3.4. *Let $\mathfrak{G}, \vartheta, \mathcal{K}, \mathbf{p}$ be as in (3.2) and let $\bar{\mathbf{u}}_2, \underline{\mathbf{u}}_2 \in \mathcal{C}(\mathcal{J}, \mathbb{R}_{\mathcal{F}})$ be ULs of (3.2) with $\underline{\mathbf{u}}_2(\tau) \leq \bar{\mathbf{u}}_2(\tau)$ for $\tau \in \mathcal{J}$. Assume $\vartheta(\tau, x, y, z)$ is nondecreasing in each argument over the corresponding intervals, and that the iterative sequences*

$$\mathbf{v}_0 = \underline{\mathbf{u}}_2,$$

$$\begin{aligned} \mathbf{v}_{n+1}(\tau) &= \mathfrak{G}(\tau) \ominus \frac{-1}{\Gamma(r)} \int_0^\tau \\ &\times (\tau - s)^{r-1} \vartheta(s, \mathbf{v}_n(s), K\mathbf{v}_n(s), \mathfrak{H}\mathbf{v}_n(s)) ds, \end{aligned}$$

$$\mathbf{w}_0 = \bar{\mathbf{u}}_2,$$

$$\begin{aligned} \mathbf{w}_{n+1}(\tau) &= \mathfrak{G}(\tau) \ominus \frac{-1}{\Gamma(r)} \int_0^\tau \\ &\times (\tau - s)^{r-1} \vartheta(s, \mathbf{w}_n(s), K\mathbf{w}_n(s), \mathfrak{H}\mathbf{w}_n(s)) ds, \end{aligned}$$

are well defined. Then (3.2) possesses maximal and minimal solutions $\mathbf{u}_M, \mathbf{u}_L \in [\underline{\mathbf{u}}_2, \bar{\mathbf{u}}_2]$. If, moreover, condition (3.3) holds (with the intervals replaced by those of $\underline{\mathbf{u}}_2, \bar{\mathbf{u}}_2$), then the solution in $[\underline{\mathbf{u}}_2, \bar{\mathbf{u}}_2]$ is unique.

Proof. The existence part follows by a monotone iterative argument similar to Theorem 3.3, using Lemma 2.3 to handle the H -difference. For uniqueness, let $\mathbf{u}_L, \mathbf{u}_M$ be the minimal and maximal solutions. As before, define $R_l(\tau, \sigma), R_r(\tau, \sigma)$ from the level sets. Using condition (3.3) we derive the system

$$R_l(\tau, \sigma) \leq \kappa \int_0^\tau (\tau - s)^{r-1} R_r(s, \sigma) ds,$$

$$R_r(\tau, \sigma) \leq \kappa \int_0^\tau (\tau - s)^{r-1} R_l(s, \sigma) ds,$$

with κ as before. Iterating these inequalities yields

$$R_l(\tau, \sigma) \leq \kappa^2 b^r B(r, r) \int_0^\tau (\tau - s)^{r-1} R_l(s, \sigma) ds,$$

where B denotes the Beta function. Applying the generalized Gronwall inequality gives $R_l \leq 0$; similarly $R_r \leq 0$. Hence $\mathbf{u}_L = \mathbf{u}_M$. $\square$

4. FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS IN THE FUZZY SETTING

We now consider the Volterra type fuzzy fractional integro-differential equation of order $0 < r < 1$:

$$\begin{aligned} \mathfrak{D}^r \mathbf{u}(\tau) &= \vartheta(\tau, \mathbf{u}(\tau), K\mathbf{u}(\tau), \mathfrak{H}\mathbf{u}(\tau)), \quad \tau \in \mathcal{J}, \\ \lim_{\tau \rightarrow 0^+} \tau^{1-r} \mathbf{u}(\tau) &= \mathbf{u}_0 \in \mathbb{R}_{\mathcal{F}}, \end{aligned} \quad (4.1)$$

where $J = (0, b]$, $\vartheta \in \mathcal{C}(\mathcal{J} \times \mathbb{R}_{\mathcal{F}}^3, \mathbb{R}_{\mathcal{F}})$, the operators $K, \mathfrak{H}$ are defined as in (1.3) with kernels $\mathcal{K}, \mathfrak{p} \in \mathcal{C}(\mathcal{J}, \mathbb{R}_+)$, and $\mathfrak{D}^r$ denotes the fuzzy fractional derivative introduced below.

Definition 4.1. The fuzzy fractional integral of order $r > 0$ for a function $\mathbf{u} \in \mathcal{C}(\mathcal{J}, \mathbb{R}_{\mathcal{F}}) \cap L^1(\mathcal{J}, \mathbb{R}_{\mathcal{F}})$ is expressed as

$$I^r \mathbf{u}(\tau) = \frac{1}{\Gamma(r)} \int_0^\tau (\tau - s)^{r-1} \mathbf{u}(s) ds, \quad \tau \in \mathcal{J},$$

with $I^0 \mathbf{u}(\tau) = \mathbf{u}(\tau)$.

Remark 4.2. (see e.g. [7, 13]) If $0 < r < 1$ and $\mathbf{u} \in \mathcal{C}_\rho(\bar{\mathcal{J}}, \mathbb{R}_{\mathcal{F}})$ with $\rho < r$, then $I^r \mathbf{u} \in \mathcal{C}(\bar{\mathcal{J}}, \mathbb{R}_{\mathcal{F}})$ and $I^r \mathbf{u}(0) = \hat{0}$. If $\mathbf{u} \in \mathcal{C}_r(\bar{\mathcal{J}}, \mathbb{R}_{\mathcal{F}})$, then $I^r \mathbf{u}$ remains bounded at 0; for $0 < r < \rho$ the integral may be unbounded at 0.

Definition 4.3. Let $\mathbf{u} \in \mathcal{C}(\mathcal{J}, \mathbb{R}_{\mathcal{F}}) \cap L^1(\mathcal{J}, \mathbb{R}_{\mathcal{F}})$ and $0 < r < 1$.

- (i) $\mathbf{u}$ is said to be (i)-fractionally differentiable of order r at $\tau \in \mathcal{J}$ if there exists $\mathfrak{D}^r \mathbf{u}(\tau) \in \mathbb{R}_{\mathcal{F}}$ such that

$$\mathfrak{D}^r \mathbf{u}(\tau) = \frac{1}{\Gamma(1-r)} \frac{d}{d\tau} \int_0^\tau (\tau - s)^{-r} \mathbf{u}(s) ds,$$

where the outer derivative is understood in the (i)-sense.

- (ii) $\mathbf{u}$ is said to be (ii)-fractionally differentiable of order r at $\tau \in \mathcal{J}$ if there exists $\mathfrak{D}^r \mathbf{u}(\tau) \in \mathbb{R}_{\mathcal{F}}$ such that

$$\mathfrak{D}^r \mathbf{u}(\tau) = \frac{1}{\Gamma(1-r)} \frac{d}{d\tau} \int_0^\tau (\tau - s)^{-r} \mathbf{u}(s) ds,$$

where the outer derivative is understood in the (ii)-sense.

The fuzzy fractional IVP (4.1) is associated with the following two integral formulations, which are instrumental in establishing existence but are not equivalent to the original problem:

$$\begin{aligned} \mathbf{u}(\tau) &= \mathbf{u}_0 \tau^{r-1} + \frac{1}{\Gamma(r)} \int_0^\tau \\ &\quad \times (\tau - s)^{r-1} \vartheta(s, \mathbf{u}(s), K\mathbf{u}(s), \mathfrak{H}\mathbf{u}(s)) ds, \quad \tau \in \mathcal{J}, \end{aligned} \quad (4.2)$$

$$\begin{aligned} \mathbf{u}(\tau) &= \mathbf{u}_0 \tau^{r-1} \ominus (-1) \frac{1}{\Gamma(r)} \int_0^\tau \\ &\quad \times (\tau - s)^{r-1} \vartheta(s, \mathbf{u}(s), K\mathbf{u}(s), \mathfrak{H}\mathbf{u}(s)) ds, \quad \tau \in \mathcal{J}. \end{aligned} \quad (4.3)$$

Definition 4.4. A function $\mathbf{u} \in \mathcal{C}_{1-r}(\mathcal{J}, \mathbb{R}_{\mathcal{F}})$ qualifies as a solution of (4.1) whenever it satisfies both the fractional differential equation and the initial condition.

The primary existence findings of this section are stated next. As the arguments follow a similar pattern, a full demonstration is given solely for the second scenario, which demands extra attention.

Theorem 4.5. *Let $\bar{\mathbf{u}}_1, \underline{\mathbf{u}}_1 \in \mathcal{C}_{1-r}(\bar{\mathcal{J}}, \mathbb{R}_{\mathcal{F}})$ be upper and lower solutions for (4.2) such that*

$$\begin{aligned} \underline{\mathbf{u}}_1(\tau) &\leq \bar{\mathbf{u}}_1(\tau), \quad \tau \in \mathcal{J}, \\ \tau^{1-r} \underline{\mathbf{u}}_1(\tau) \Big|_{\tau=0} &\leq \tau^{1-r} \bar{\mathbf{u}}_1(\tau) \Big|_{\tau=0}. \end{aligned}$$

Assume there exists $0 < \rho < 1$ with the property that $\tau^\rho \vartheta(\tau, \underline{\mathbf{u}}_1, K\underline{\mathbf{u}}_1, \mathfrak{H}\underline{\mathbf{u}}_1)$ and $\tau^\rho \vartheta(\tau, \bar{\mathbf{u}}_1, K\bar{\mathbf{u}}_1, \mathfrak{H}\bar{\mathbf{u}}_1)$ are continuous on $\bar{\mathcal{J}}$. Suppose $\vartheta(\tau, x, y, z)$ is nondecreasing in $x \in [\underline{\mathbf{u}}_1(\tau), \bar{\mathbf{u}}_1(\tau)]$, in $y \in [K\underline{\mathbf{u}}_1(\tau), K\bar{\mathbf{u}}_1(\tau)]$ and in $z \in [\mathfrak{H}\underline{\mathbf{u}}_1(\tau), \mathfrak{H}\bar{\mathbf{u}}_1(\tau)]$ for each $\tau \in \mathcal{J}$. Then (4.1) admits at least one solution $\mathbf{u} \in [\underline{\mathbf{u}}_1, \bar{\mathbf{u}}_1]$ which is (i)-fractionally differentiable of order r and satisfies $\lim_{\tau \rightarrow 0^+} \tau^{1-r} \mathbf{u}(\tau) = \mathbf{u}_0$.

Theorem 4.6. *Let $\bar{\mathbf{u}}_2, \underline{\mathbf{u}}_2 \in \mathcal{C}_{1-r}(\bar{\mathcal{J}}, \mathbb{R}_{\mathcal{F}})$ be ULs for (4.3) such that*

$$\begin{aligned} \underline{\mathbf{u}}_2(\tau) &\leq \bar{\mathbf{u}}_2(\tau), \quad \tau \in \mathcal{J}, \\ \tau^{1-r} \underline{\mathbf{u}}_2(\tau) \Big|_{\tau=0} &\leq \tau^{1-r} \bar{\mathbf{u}}_2(\tau) \Big|_{\tau=0}. \end{aligned}$$

Let $0 < \rho < r < 1$ be such that $\tau^\rho \vartheta(\tau, \underline{\mathbf{u}}_2, K\underline{\mathbf{u}}_2, \mathfrak{H}\underline{\mathbf{u}}_2)$ and $\tau^\rho \vartheta(\tau, \bar{\mathbf{u}}_2, K\bar{\mathbf{u}}_2, \mathfrak{H}\bar{\mathbf{u}}_2)$ are continuous on $\bar{\mathcal{J}}$. Suppose $\vartheta(\tau, x, y, z)$ is nondecreasing in $x \in [\underline{\mathbf{u}}_2(\tau), \bar{\mathbf{u}}_2(\tau)]$, in $y \in [K\underline{\mathbf{u}}_2(\tau), K\bar{\mathbf{u}}_2(\tau)]$ and in $z \in [\mathfrak{H}\underline{\mathbf{u}}_2(\tau), \mathfrak{H}\bar{\mathbf{u}}_2(\tau)]$ for each $\tau \in \mathcal{J}$. Moreover, assume the iterative sequences

$$\begin{aligned} \mathbf{v}_0 &= \underline{\mathbf{u}}_2, \\ \mathbf{v}_{n+1}(\tau) &= \mathbf{u}_0 \tau^{r-1} \ominus (-1) \frac{1}{\Gamma(r)} \int_0^\tau (\tau-s)^{r-1} \\ &\quad \times \vartheta(s, \mathbf{v}_n(s), K\mathbf{v}_n(s), \mathfrak{H}\mathbf{v}_n(s)) ds, \\ \mathbf{w}_0 &= \bar{\mathbf{u}}_2, \\ \mathbf{w}_{n+1}(\tau) &= \mathbf{u}_0 \tau^{r-1} \ominus (-1) \frac{1}{\Gamma(r)} \int_0^\tau (\tau-s)^{r-1} \\ &\quad \times \vartheta(s, \mathbf{w}_n(s), K\mathbf{w}_n(s), \mathfrak{H}\mathbf{w}_n(s)) ds, \end{aligned}$$

are well defined on J . Then (4.1) possesses at least one solution $\mathbf{u} \in [\underline{\mathbf{u}}_2, \bar{\mathbf{u}}_2]$ which is (ii)-fractionally differentiable of order r and satisfies $\lim_{\tau \rightarrow 0^+} \tau^{1-r} \mathbf{u}(\tau) = \mathbf{u}_0$.

Proof. All hypotheses of Theorem 3.4 with $\mathfrak{G}(\tau) = \mathbf{u}_0 \tau^{r-1}$ are satisfied, hence (4.3) has at least one solution, namely $\mathbf{u}$. We shall verify that this $\mathbf{u}$ is indeed a solution of (4.1).

Define an auxiliary function

$$\Phi(\tau) = \int_0^\tau (\tau-s)^{-r} \mathbf{u}(s) ds, \quad \tau \in \mathcal{J}.$$

Let $\tau \in \mathcal{J}$ and choose $h > 0$ such that $\tau + h \leq b$. An easy computation using the representation of $\mathbf{u}$ from (4.3) yields

$$\begin{aligned} \Phi(\tau) \ominus \Phi(\tau + h) \\ = \frac{(-1)}{\Gamma(1-r)} \int_{\tau}^{\tau+h} \vartheta(s, \mathbf{u}(s), K\mathbf{u}(s), \mathfrak{H}\mathbf{u}(s)) ds. \end{aligned}$$

Indeed, substituting the expression of $\mathbf{u}$ and interchanging the order of integration gives

$$\begin{aligned} \Phi(\tau + h) + \frac{(-1)}{\Gamma(1-r)} \int_{\tau}^{\tau+h} \vartheta(s, \mathbf{u}, K\mathbf{u}, \mathfrak{H}\mathbf{u}) ds \\ = \int_0^{\tau+h} (\tau + h - s)^{-r} s^{r-1} \mathbf{u}_0 ds \\ \ominus \frac{(-1)}{\Gamma(r)} \int_0^{\tau+h} \int_0^s (\tau + h - s)^{-r} (s - \xi)^{r-1} \\ \times \vartheta(\xi, \mathbf{u}, K\mathbf{u}, \mathfrak{H}\mathbf{u}) d\xi ds \\ + \frac{(-1)}{\Gamma(1-r)} \int_{\tau}^{\tau+h} \vartheta(s, \mathbf{u}, K\mathbf{u}, \mathfrak{H}\mathbf{u}) ds. \end{aligned}$$

Changing the order of integration in the double integral and simplifying using the Beta function leads to

$$\begin{aligned} \Phi(\tau + h) \ominus \frac{(-1)}{\Gamma(r)} \int_0^{\tau+h} \int_0^{\xi} (\tau + h - s)^{-r} (s - \xi)^{r-1} \\ \times \vartheta(\xi, \mathbf{u}, K\mathbf{u}, \mathfrak{H}\mathbf{u}) ds d\xi \\ + \frac{(-1)}{\Gamma(1-r)} \int_{\tau}^{\tau+h} \vartheta(s, \mathbf{u}, K\mathbf{u}, \mathfrak{H}\mathbf{u}) ds = \Phi(\tau). \end{aligned}$$

A careful manipulation (similar to the classical case) then establishes the claimed identity.

Dividing by $-h$ and passing to the limit $h \rightarrow 0^+$ gives

$$\lim_{h \rightarrow 0^+} \frac{\Phi(\tau) \ominus \Phi(\tau + h)}{-h} = \Gamma(1-r) \vartheta(\tau, \mathbf{u}(\tau), K\mathbf{u}(\tau), \mathfrak{H}\mathbf{u}(\tau)).$$

A completely analogous argument with a backward difference shows

$$\lim_{h \rightarrow 0^+} \frac{\Phi(\tau - h) \ominus \Phi(\tau)}{-h} = \Gamma(1-r) \vartheta(\tau, \mathbf{u}(\tau), K\mathbf{u}(\tau), \mathfrak{H}\mathbf{u}(\tau)).$$

Therefore $\mathbf{u}$ is (ii)-fractionally differentiable of order r and satisfies

$$\mathfrak{D}^r \mathbf{u}(\tau) = \vartheta(\tau, \mathbf{u}(\tau), K\mathbf{u}(\tau), \mathfrak{H}\mathbf{u}(\tau)), \quad \tau \in \mathcal{J}.$$

It remains to verify the initial condition. Because $0 < \rho < 1$ and the continuity assumptions on $\tau^\rho \vartheta(\tau, \underline{\mathbf{u}}_2, K\underline{\mathbf{u}}_2, \mathfrak{H}\underline{\mathbf{u}}_2)$ and $\tau^\rho(\vartheta(\tau, \bar{\mathbf{u}}_2, K\bar{\mathbf{u}}_2, \mathfrak{H}\bar{\mathbf{u}}_2))$ imply that there exist constants $M_1, M_2 > 0$ with

$$\begin{aligned} D(\vartheta(\tau, \underline{\mathbf{u}}_2(\tau), K\underline{\mathbf{u}}_2(\tau), \mathfrak{H}\underline{\mathbf{u}}_2(\tau)), \hat{0}) &\leq M_1 \tau^{-\rho}, \\ D(\vartheta(\tau, \bar{\mathbf{u}}_2(\tau), K\bar{\mathbf{u}}_2(\tau), \mathfrak{H}\bar{\mathbf{u}}_2(\tau)), \hat{0}) &\leq M_2 \tau^{-\rho}. \end{aligned}$$

Set $M = \max\{M_1, M_2\}$. Using the representation (4.3) we estimate

$$\begin{aligned}
& D(\tau^{1-r}\mathbf{u}(\tau), \mathbf{u}_0) \\
&= D\left(\mathbf{u}_0 \ominus (-1)\frac{\tau^{1-r}}{\Gamma(r)} \int_0^\tau (\tau-s)^{r-1} \right. \\
&\quad \left. \times \vartheta(s, \mathbf{u}(s), K\mathbf{u}(s), \mathfrak{H}\mathbf{u}(s)) ds, \mathbf{u}_0\right) \\
&\leq \frac{\tau^{1-r}}{\Gamma(r)} \int_0^\tau (\tau-s)^{r-1} D(\vartheta(s, \mathbf{u}(s), K\mathbf{u}(s), \mathfrak{H}\mathbf{u}(s)), \hat{0}) ds \\
&\leq \frac{\tau^{1-r}}{\Gamma(r)} \int_0^\tau (\tau-s)^{r-1} D(\vartheta(s, \underline{\mathbf{u}}_2(s), K\underline{\mathbf{u}}_2(s), \mathfrak{H}\underline{\mathbf{u}}_2(s)), \hat{0}) ds \\
&\quad + \frac{\tau^{1-r}}{\Gamma(r)} \int_0^\tau (\tau-s)^{r-1} D(\vartheta(s, \bar{\mathbf{u}}_2(s), K\bar{\mathbf{u}}_2(s), \mathfrak{H}\bar{\mathbf{u}}_2(s)), \hat{0}) ds \\
&\leq \frac{2M\tau^{1-r}}{\Gamma(r)} \int_0^\tau (\tau-s)^{r-1} s^{-\rho} ds = \frac{2M\tau^{1-r}\Gamma(1-\rho)}{\Gamma(1-\rho+r)}.
\end{aligned}$$

Since $1-r > 0$ and $r-\rho > 0$, the right hand side goes to 0 as $\tau \rightarrow 0^+$. Hence $\lim_{\tau \rightarrow 0^+} \tau^{1-r}\mathbf{u}(\tau) = \mathbf{u}_0$, completing the proof. $\square$

CONCLUSION

We introduced in this current paper a novel method of ULs to establish existence and uniqueness results for fuzzy fractional integral equations under weaker assumptions, relying on partial ordering to compare fuzzy functions. By constructing a lower solution $\underline{u}$ and an upper solution $\bar{u}$ with $\underline{u} \leq \bar{u}$, we demonstrated that a solution u exists verifying $\underline{u} \leq u \leq \bar{u}$. The method was successfully applied to fuzzy initial value problems for fractional integro-differential equations via Riemann–Liouville operators, which are crucial for modeling uncertainty in physical phenomena such as electromagnetics, viscoelasticity, and fluid dynamics. Notably, our approach also extends to crisp equations under milder assumptions than previously required (e.g., without Lipschitz continuity) and offers a practical numerical approximation when exact solutions are inaccessible. The flexibility of the method suggests promising future extensions to fractional partial integro-differential equations. Overall, this work provides a unified framework for handling fractional systems under uncertainty, combining analytical rigor with broad applicability.

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