

EDGE-BASED FIRST ZAGREB INDEX FOR FUZZY GRAPHS

SK RABIUL ISLAM¹, SHO VAN DOGRA^{2*}, QIN XIN³, SABNAM SWOMIN⁴ and
MADHUMANGAL PAL⁵

ABSTRACT. Topological indices have gained much importance in fuzzy graph (FG) theory and provide valuable information regarding the structure of graphs. This paper provides an innovative edge-based version of the first Zagreb index (ZI) for FGs. The proposed ZI on FG is comprehensively studied in case of different classes of FGs, such as fuzzy path, fuzzy cycle, and fuzzy star graph. It is also discussed how this index's behaves when the FG is isomorphic with one of its fuzzy subgraphs. Moreover, some sharp bounds on the proposed index for various operations on FGs are derived. It should be mentioned that the presented results not only contribute to the general theory of FGs but are also useful for various practical purposes, such as determining properties of molecular graphs in molecular chemistry.

Keywords. Fuzzy graph, first Zagreb index, edge based first Zagreb index.

2020 Mathematics Subject Classification. Primary 05C09, 05C72.

1. INTRODUCTION

FGs have come a long way over the years and have proved to be very useful in handling uncertainties in the context of classical graph theories. Because of their nature, FGs have been employed in various fields like chemistry, computer sciences, biology, and social networks. Many developments have been made in FGs.

Recently, there has been substantial interest in TIs in FGs because of their significance in modeling and decision-making processes. Classical indices including Wiener and Zagreb indices were investigated in a fuzzy environment by Kalathian et al. [13]. The concepts of Islam and Pal [7, 8, 9, 10, 11] also extended these ideas. Some other noteworthy contributions are those made in the field of cyclic connectivity index [14], studies related to picture fuzzy graphs [4] and advancements in the areas of cyber security [1, 2, 3] and biological/nanotechnology systems [17, 18]. Demir et al. [5] investigated several degree-based TIs for a generalized

Date: Received: Apr 3, 2026; Revised: Apr 27; Accepted: May 10, 2026.

* Corresponding author

© The Author(s) 2025. This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License. To view a copy of the licence, visit <https://creativecommons.org/licenses/by-nc-nd/4.0/>.

*Corresponding author.

graph family linked to number-theoretic realizations and derived explicit formulas along with asymptotic relationships. Hosamani and Krzywkowski (2016) [6] studied the differences of Zagreb co-indices under various graph operations, providing insights into their structural behavior and transformations. Neutrosophic planar graphs is proposed by Mahapatra et al. [16], together with the RSM index used for link prediction [15].

However, despite all these advancements, little importance has been accorded to the concept of the edge form of the first ZI under fuzzy conditions. Being inspired by this problem area, the current paper will introduce the idea of the first ZI in edge form for FGs, analyzing it in terms of its nature in basic graph classes like fuzzy paths, fuzzy cycles, and fuzzy star graphs. This research is significant because of its application in fields such as molecular chemistry, where FG indices play a crucial role. This study not only makes a contribution to theory but also to practice.

Some words have been used in abbreviated form throughout this research article. List of these words and their abbreviated forms are given in the Table 1.

Abbreviation	Full Form
FS	Fuzzy set
FG	Fuzzy graph
FSG	Fuzzy subgraph
CP	Cartesian product
DP	Direct product
MSV	Membership value
MSF	Membership function
SP	Strong product
SSP	Semi-strong product
ZI	Zagreb index
TI	Topological index

TABLE 1. List of Abbreviations

2. PRELIMINARIES

It is Zadeh who established the groundwork of FS theory back in 1965. After a decade, Rosenfeld brought the idea to graph theory, giving birth to the concept of FG in 1975. From the time of their invention, FGs have had many applications in different areas due to their capability of modeling uncertain and imprecise situations.

The definition of FG is as follows:

Definition 2.1. [12, 19] (FG) A FG is of the form $G = (V(G), \xi, \eta)$, in which case V is the vertex set of FG with the vertex MSF $\xi : V(G) \rightarrow [0, 1]$ and the edge MSF $\eta : V(G) \times V(G) \rightarrow [0, 1]$ accompanied by the condition: $\eta(x, y) \leq \xi(x) \wedge \xi(y), \forall x, y \in V(G)$.

Definition 2.2. The degree of a vertex v in a FG G is denoted as $d_G(v)$ or $d(v)$ and computed by summing the MSVs of all edges that are adjacent to v i.e.

$$d(u) = \sum_{v \in V(G)} \eta(uv).$$

Δ, δ denotes the maximum and minimum degree of G , respectively.

Definition 2.3. For a graph (crisp) $G = (V(G), E(G))$, the first ZI is given by:

$$M_1(G) = \sum_{v \in V(G)} [d(v)]^2 \text{ or equivalently, } M_1(G) = \sum_{uv \in E(G)} [d(u) + d(v)].$$

The first ZI formulation in the context of FGs was presented by Islam and Pal [7] in 2021.

Definition 2.4. For a FG $G = (V(G), \xi, \eta)$, the first ZI of G is defined as:

$$ZI_1(G) = \sum_{v \in V(G)} [\xi(v) d(v)]^2.$$

CP, composition, join, DP, SP, and SSP of two FGs are defined in [12, 19].

3. EDGE-BASED FIRST ZI FOR FGs

The section introduces the concept of edge-based first ZI and investigates some bounds for this index in case of different types of FGs like fuzzy path, fuzzy cycle, fuzzy star graph, partial FSG, complete FG and isomorphic FGs.

Let us start by defining our targeted index.

Definition 3.1. For a FG G , the edge-based first ZI, written as $ZI_1^*(G)$, is defined as:

$$ZI_1^*(G) = \sum_{uv \in E(G)} [\xi(u)d(u) + \xi(v)d(v)].$$

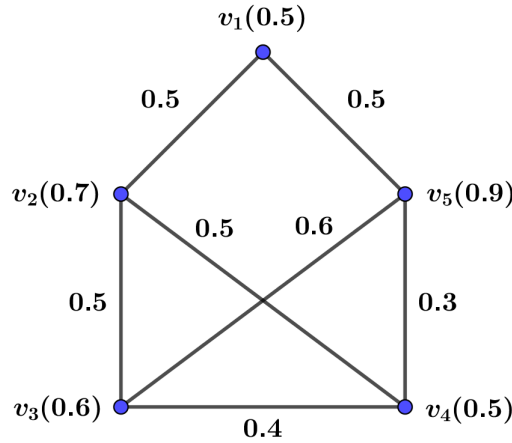


FIGURE 1. A FG G where $ZI_1(G) \neq ZI_1^*(G)$.

Example 3.2. Let us consider a FG $G = (V(G), \xi, \eta)$ as depicted in Figure 1. MSVs of vertex and edges are given in the figure. One can easily calculate that $ZI_1(G) = 4.1101$ and $ZI_1^*(G) = 11.92$. Here, $ZI_1(G) \neq ZI_1^*(G)$.

Theorem 3.3. Suppose P represents a fuzzy path with n vertices, then $ZI_1^*(P) \leq 2(2n - 3)$.

Proof. Let us take $P = (u_1, u_2, \dots, u_n)$ as a fuzzy path.

Then the end vertices u_1 and u_n have degree ≤ 1 , rest vertices u_i (for $2 \leq i \leq n-1$) are of degree ≤ 2 .

Now, we have,

$$\begin{aligned} ZI_1^*(P) &= \xi(u_1)d(u_1) + \xi(u_n)d(u_n) + 2 \sum_{i=2}^{n-1} \xi(u_i)d(u_i). \\ &\leq \xi(u_1) \cdot 1 + \xi(u_n) \cdot 1 + 2 \sum_{i=2}^{n-1} \xi(u_i) \cdot 2 \\ &= \xi(u_1) + \xi(u_n) + 4 \sum_{i=2}^{n-1} \xi(u_i). \end{aligned}$$

Using the fact $\xi(u_i) \in [0, 1], \forall i = 1(1)n$, we obtain, $ZI_1^*(P) \leq 2(2n - 3)$. Thus, we get $ZI_1^*(P) \leq 2(2n - 3)$. $\square$

Theorem 3.4. Suppose C represents a fuzzy cycle with n vertices, then $ZI_1^*(C) \leq 4n$.

Proof. Let us take $C = (u_1, u_2, \dots, u_n, u_{n+1} = u_1)$ as a fuzzy cycle.

Then each vertex of C is of degree ≤ 2 .

Since each vertex is adjacent to two edges, therefore we can write

$$ZI_1^*(C) = 2 \sum_{i=1}^n \xi(u_i)d(u_i).$$

$$\text{Thus, } ZI_1^*(C) \leq 2 \sum_{i=1}^n \xi(u_i) \cdot 2 = 4 \sum_{i=1}^n \xi(u_i) \leq 4n. \quad \square$$

Theorem 3.5. Suppose S represents a fuzzy star graph with $(n+1)$ vertices, then $ZI_1^*(S) \leq n(n+1)$.

Proof. Let u_0 be the central node and u_i are the leaves for $i = 1(1)n$

Then $d(u_0) \leq n$ and $d(u_i) \leq 1$ for $i = 1(1)n$.

Now, we have

$$\begin{aligned} ZI_1^*(S) &\leq \sum_{i=1}^n [\xi(u_0) \cdot n + \xi(u_i) \cdot 1] \\ &= n \cdot \xi(u_0) \cdot n + \sum_{i=1}^n \xi(u_i) \\ &= n^2 \xi(u_0) + \sum_{i=1}^n \xi(u_i). \end{aligned}$$

Using the fact $0 \leq \xi(u_i) \leq 1$ for $i = 0(1)n$, we get $ZI_1^*(S) \leq n^2 + n = n(n+1)$. $\square$

Theorem 3.6. Suppose H represents a FSG H of a FG G , then $ZI_1^*(H) \leq ZI_1^*(G)$.

Proof. As $d_H(v) \leq d_G(v)$, then

$$\begin{aligned} ZI_1^*(H) &= \sum_{uv \in E(H)} [\xi_H(u)d_H(u) + \xi_H(v)d_H(v)] \\ &\leq \sum_{uv \in E(G)} [\xi_G(u)d_G(u) + \xi_G(v)d_G(v)] \\ &\leq ZI_1^*(G). \end{aligned}$$

□

Theorem 3.7. Suppose G represents a n -vertex complete FG, then $ZI_1^*(G) \leq n(n-1)^2$.

Proof. Clearly, degree of each vertex $\leq (n-1)$, then

$$\begin{aligned} ZI_1^*(G) &= (n-1) \sum_{i=1}^n \xi(u_i)d(u_i) \text{ (as each vertex is incident to } (n-1) \text{ edges)} \\ &\leq n(n-1)^2. \end{aligned}$$

□

Theorem 3.8. Suppose G_1 and G_2 represents two isomorphic FGs, then $ZI_1^*(G_1) = ZI_1^*(G_2)$.

Proof. Clearly, $d_{G_1}(v) = d_{G_2}(\psi(v))$, where ψ is the isomorphic map. Thus, $ZI_1^*(G_1) = ZI_1^*(G_2)$. □

4. BOUNDS ON EDGE-BASED FIRST ZAGREB FOR FG DURING OPERATIONS

This section discusses the bounds on the edge-based first Zagreb index of FGs obtained by various graph operations.

Theorem 4.1. Suppose $G_1 = (V(G_1), \xi_1, \eta_1)$ and $G_2 = (V(G_2), \xi_2, \eta_2)$ represent two FGs and let $G_1 \times G_2 = G = (V(G), \xi, \eta)$ represent their CP, then $ZI_1^*(G) \leq n[ZI_1^*(G_1) + ZI_1^*(G_2)] + 2[n(n-1)]^2$.

Proof. From Theorem 4.1 ([9]), we have

$$d_G(u, v) \leq d_{G_1}(u) + d_{G_2}(v).$$

Now,

$$\begin{aligned} ZI_1^*(G) &= \sum_{(u_1, v_1)(u_2, v_2) \in E(G)} [\xi(u_1, v_1)d(u_1, v_1) + \xi(u_2, v_2)d(u_2, v_2)] \\ &= K_1 + K_2, \end{aligned}$$

where

$$\begin{aligned} K_1 &= \sum_{u \in V(G_1), v_1 v_2 \in E(G_2)} [\xi(u, v_1)d(u, v_1) + \xi(u, v_2)d(u, v_2)], \\ K_2 &= \sum_{v \in V(G_2), u_1 u_2 \in E(G_1)} [\xi(u_1, v)d(u_1, v) + \xi(u_2, v)d(u_2, v)]. \end{aligned}$$

Based on the above inequalities, we get

$$\begin{aligned}
K_1 &\leq \sum_{u \in V(G_1), v_1 v_2 \in E(G_2)} \left[2\xi_1(u)d_{G_1}(u) + \xi_2(v_1)d_{G_2}(v_1) + \xi_2(v_2)d_{G_2}(v_2) \right] \\
&\leq 2m_2 \sum_{u \in V_1} \xi_1(u)d_{G_1}(u) + n_1 ZI_1^*(G_2) \\
&\leq 2n_1 m_2 \Delta_1 + n_1 ZI_1^*(G_2),
\end{aligned}$$

where Δ_1 represents the maximum degree of G_1 .

In the same way

$$K_2 \leq n_2 ZI_1^*(G_1) + 2n_2 m_1 \Delta_2.$$

$$\text{Hence, } ZI_1^*(G) \leq n_2 ZI_1^*(G_1) + n_1 ZI_1^*(G_2) + 2(n_1 m_2 \Delta_1 + n_2 m_1 \Delta_2).$$

With the use of inequalities $m_i \leq \frac{n_i(n_i-1)}{2}$ and $\Delta_i \leq n_i - 1$ (for $i = 1, 2$), we get

$$n_1 m_2 \Delta_1 + n_2 m_1 \Delta_2 \leq [n(n-1)]^2.$$

As $n_1, n_2 \leq n$, we get $ZI_1^*(G) \leq n[ZI_1^*(G_1) + ZI_1^*(G_2)] + 2[n(n-1)]^2$. $\square$

Theorem 4.2. Suppose $G_1 = (V(G_1), \xi_1, \eta_1)$ and $G_2 = (V(G_2), \xi_2, \eta_2)$ represent two FGs and $G_1[G_2] = G = (V(G), \xi, \eta)$ represent their composition, then $ZI_1^*(G) \leq n^3 ZI_1^*(G_1) + n ZI_1^*(G_2) + 2n^3(n-1)^2$.

Proof. From Theorem 4.3 ([9]) in case of composition of two FGs G_1 and G_2 , $d_G(u, v) \leq n_2 d_{G_1}(u) + d_{G_2}(v)$.

$$\begin{aligned}
\text{Now, } ZI_1^*(G) &= \sum_{(u_1, v_1)(u_2, v_2) \in E(G)} [\xi(u_1, v_1)d_G(u_1, v_1) + \xi(u_2, v_2)d_G(u_2, v_2)] \\
&= K_1 + K_2,
\end{aligned}$$

$$\begin{aligned}
\text{where } K_1 &= \sum_{u \in V(G_1), v_1 v_2 \in E(G_2)} [\xi(u, v_1)d_G(u, v_1) + \xi(u, v_2)d_G(u, v_2)] \\
&\leq \sum_{u \in V(G_1), v_1 v_2 \in E(G_2)} [2n_2 \xi_1(u)d_{G_1}(u) + \xi_2(v_1)d_{G_2}(v_1) + \xi_2(v_2)d_{G_2}(v_2)] \\
&\leq 2n_1 n_2 m_2 \Delta_1 + n_1 ZI_1^*(G_2).
\end{aligned}$$

$$\begin{aligned}
\text{and } K_2 &= \sum_{u_1 u_2 \in E(G_1), v_1, v_2 \in V(G_2)} [\xi(u_1, v_1)d_G(u_1, v_1) + \xi(u_2, v_2)d_G(u_2, v_2)] \\
&\leq n_2 \sum_{u_1 u_2 \in E(G_1), v_1, v_2 \in V(G_2)} \{ \xi_1(u_1)d_{G_1}(u_1) + \xi_1(u_2)d_{G_1}(u_2) \} \\
&\quad + \sum_{u_1 u_2 \in E(G_1), v_1, v_2 \in V(G_2)} \{ \xi_2(v_1)d_{G_2}(v_1) + \xi_2(v_2)d_{G_2}(v_2) \} \\
&\leq n_2^3 ZI_1^*(G_1) + 2n_2^2 m_1 \Delta_2.
\end{aligned}$$

In this way,

$$\begin{aligned} ZI_1^*(G) &= K_1 + K_2 \\ &\leq n_2^3 ZI_1^*(G_1) + n_1 ZI_1^*(G_2) + 2n_1 n_2 m_2 \Delta_1 + 2n_2^2 m_1 \Delta_2 \\ &\leq n^3 ZI_1^*(G_1) + n ZI_1^*(G_2) + 2n^3(n-1)^2. \end{aligned}$$

□

Theorem 4.3. Suppose $G_1 = (V(G_1), \xi_1, \eta_1)$ and $G_2 = (V(G_2), \xi_2, \eta_2)$ represent two FGs and $G_1 + G_2 = G = (V(G), \xi, \eta)$ represent their join, then $ZI_1^*(G) \leq ZI_1^*(G_1) + ZI_1^*(G_2) + 2n^2(3n-2)$.

Proof. From Theorem 4.5 ([9]) in case of join of two FGs G_1 and G_2 ,

$$d_G(u) \leq \begin{cases} d_{G_1}(u) + n_2, & u \in V(G_1) \\ d_{G_2}(u) + n_1, & u \in V(G_2) \end{cases}$$

$$\begin{aligned} \text{Now, } ZI_1^*(G) &= \sum_{uv \in E(G)} [\xi(u)d_G(u) + \xi(v)d_G(v)] \\ &= K_1 + K_2 + K_3, \end{aligned}$$

$$\begin{aligned} \text{where, } K_1 &= \sum_{uv \in E(G_1)} [\xi(u)d_G(u) + \xi(v)d_G(v)] \\ &\leq \sum_{uv \in E(G_1)} [\xi_1(u)\{d_G(u) + n_2\} + \xi_1(v)\{d_{G_1}(v) + n_2\}] \\ &\leq ZI_1^*(G_1) + (n_1 + n_2)m_1. \end{aligned}$$

$$K_2 = \sum_{uv \in E(G_2)} [\xi(u)d_G(u) + \xi(v)d_G(v)]$$

In the same way, $K_2 \leq ZI_1^*(G_2) + (n_1 + n_2)m_2$,

and $K_3 \leq n_1 n_2(n_1 + n_2 + \Delta_1 + \Delta_2)$.

Thus,

$$ZI_1^*(G) \leq ZI_1^*(G_1) + ZI_1^*(G_2) + (n_1 + n_2)(m_1 + m_2 + n_1 n_2) + n_1 n_2(\Delta_1 + \Delta_2)$$

With the use of the inequalities

(i) $n_1, n_2 \leq n$

(ii) $m_1, m_2 \leq \frac{n(n-1)}{2}$

(iii) $\Delta_1, \Delta_2 \leq n-1$

we finally get $ZI_1^*(G) \leq ZI_1^*(G_1) + ZI_1^*(G_2) + 2n^2(3n-2)$. □

Theorem 4.4. Suppose $G_1 = (V(G_1), \xi_1, \eta_1)$ and $G_2 = (V(G_2), \xi_2, \eta_2)$ represent two FGs and $G_1 + G_2 = G = (V(G), \xi, \eta)$ represent their join, then $ZI_1^*(G_1 + G_2) \geq ZI_1^*(G_1) + ZI_1^*(G_2)$.

Proof. As G is join of G_1 and G_2 , then, we have (Theorem 4.5 ([9])),

$$d_G(u) \geq \begin{cases} d_{G_1}(u), & u \in V(G_1) \\ d_{G_2}(u), & u \in V(G_2) \end{cases}$$

$$\begin{aligned} \text{Hence, } ZI_1^*(G) &= \sum_{uv \in E(G)} [\xi(u)d_G(u) + \xi(v)d_G(v)] \\ &= K_1 + K_2 + K_3, \end{aligned}$$

$$\text{where, } K_1 = \sum_{uv \in E(G_1)} [\xi(u)d_G(u) + \xi(v)d_G(v)] \geq ZI_1^*(G_1),$$

$$K_2 = \sum_{uv \in E(G_2)} [\xi(u)d_G(u) + \xi(v)d_G(v)] \geq ZI_1^*(G_2),$$

$$\text{Also, } K_3 = \sum_{u \in V(G_1), v \in V(G_2)} [\xi(u)d_G(u) + \xi(v)d_G(v)] \geq 0.$$

$$\therefore ZI_1^*(G) \geq ZI_1^*(G_1) + ZI_1^*(G_2).$$

□

Theorem 4.5. Suppose $G_1 = (V(G_1), \xi_1, \eta_1)$ and $G_2 = (V(G_2), \xi_2, \eta_2)$ represent two FGs and $G_1 \Pi G_2 = G = (V(G), \xi, \eta)$ represent their DP, then $ZI_1(G) \leq \frac{1}{2}n^2(n-1)^3$.

Proof. As G is DP of G_1 and G_2 , then $\xi(u, v) = \xi_1(u) \wedge \xi_2(v)$ and $\eta((u_1, v_1)(u_2, v_2)) = \eta_1(u_1 u_2) \wedge \eta_2(v_1 v_2)$. Hence,

$$\begin{aligned} d_G(u, v) &= \sum_{(u_1, v_1) \in V(G)} \eta((u, v)(u_1, v_1)) \\ &= \sum_{(u_1, v_1) \in V(G)} \eta_1(u, u_1) \wedge \eta_2(v, v_1) \\ &\leq d_{G_1}(u) \wedge d_{G_2}(v) \leq \max\{\Delta_1, \Delta_2\}. \end{aligned}$$

Therefore,

$$\begin{aligned} ZI_1^*(G) &= \sum_{(u_1, v_1)(u_2, v_2) \in E(G)} [\xi(u_1, v_1)d_G(u_1, v_1) + \xi(u_2, v_2)d_G(u_2, v_2)] \\ &\leq 2m_1 m_2 \max\{\Delta_1, \Delta_2\} \leq \frac{1}{2}n^2(n-1)^3. \end{aligned}$$

□

Theorem 4.6. Suppose $G_1 = (V(G_1), \xi_1, \eta_1)$ and $G_2 = (V(G_2), \xi_2, \eta_2)$ represent two FGs and $G_1 \bullet G_2 = G = (V(G), \xi, \eta)$ represent their SSP, then $ZI_1^*(G) \leq \frac{1}{2}n(n-1)ZI_1^*(G_1) + n(n+1)ZI_1^*(G_2) + [n(n-1)]^2$.

Proof. As G is SSP of G_1 and G_2 , then $\xi(u, v) = \xi_1(u) \wedge \xi_2(v)$ and

$$\eta((u_1, v_1)(u_2, v_2)) = \begin{cases} \eta_1(u_1 u_2) \wedge \eta_2(v_1 v_2), & \text{if } u_1 u_2 \in E(G_1) \text{ and } v_1 v_2 \in E(G_2) \\ \xi_1(u) \wedge \eta_2(v_1 v_2), & \text{if } u_1 = u_2 = u \in V(G_1) \text{ and } v_1 v_2 \in E(G_2). \end{cases}$$

Hence,

$$d_G(u, v) \leq \sum_{uu_1 \in E(G_1)} \eta_1(uu_1) + \sum_{vv_1 \in E(G_2)} \eta_2(vv_1) + \sum_{vv_1 \in E(G_2)} \eta_2(vv_1) \leq d_{G_1}(u) + 2d_{G_2}(v).$$

Therefore,

$$\begin{aligned} ZI_1^*(G) &= \sum_{(u_1, v_1)(u_2, v_2) \in E(G)} [\xi(u_1, v_1)d_G(u_1, v_1) + \xi(u_2, v_2)d_G(u_2, v_2)] \\ &= K_1 + K_2, \end{aligned}$$

$$\begin{aligned} \text{where } K_1 &= \sum_{u_1 u_2 \in E(G_1), v_1 v_2 \in E(G_2)} [\xi(u_1, v_1)d_G(u_1, v_1) + \xi(u_2, v_2)d_G(u_2, v_2)] \\ &\leq \sum_{u_1 u_2 \in E(G_1), v_1 v_2 \in E(G_2)} \left[\{ \xi_{G_1}(u_1)d_{G_1}(u_1) + 2\xi_{G_2}(v_1)d_{G_2}(v_1) \} \right. \\ &\quad \left. + \{ \xi_{G_1}(u_2)d_{G_1}(u_2) + 2\xi_{G_2}(v_2)d_{G_2}(v_2) \} \right] \\ &\leq m_2 ZI_1^*(G_1) + 2m_1 ZI_1^*(G_2), \end{aligned}$$

$$\begin{aligned} \text{and } K_2 &= \sum_{u \in V(G_1), v_1 v_2 \in E(G_2)} [\xi(u, v_1)d_G(u, v_1) + \xi(u, v_2)d_G(u, v_2)] \\ &\leq \sum_{u \in V(G_1), v_1 v_2 \in E(G_2)} \left[\{ \xi_{G_1}(u)d_{G_1}(u) + 2\xi_{G_2}(v_1)d_{G_2}(v_1) \} \right. \\ &\quad \left. + \{ \xi_{G_1}(u)d_{G_1}(u) + 2\xi_{G_2}(v_2)d_{G_2}(v_2) \} \right] \\ &\leq 2n_1 m_2 \Delta_1 + 2n_1 ZI_1^*(G_2). \end{aligned}$$

$$\begin{aligned} \therefore ZI_1^*(G) &\leq m_2 ZI_1^*(G_1) + 2(m_1 + n_1) ZI_1^*(G_2) + 2n_1 m_2 \Delta_1 \\ &\leq \frac{1}{2}n(n-1) ZI_1^*(G_1) + n(n+1) ZI_1^*(G_2) + [n(n-1)]^2. \end{aligned}$$

□

Theorem 4.7. Suppose $G_1 = (V(G_1), \xi_1, \eta_1)$ and $G_2 = (V(G_2), \xi_2, \eta_2)$ represent two FGs and $G_1 \otimes G_2 = G = (V(G), \xi, \eta)$ represent their SP, then $ZI_1^*(G) \leq n(n+1)[ZI_1^*(G_1) + ZI_1^*(G_2)] + 4[n(n-1)]^2$.

Proof. As G is SP of G_1 and G_2 , then $\xi(u, v) = \xi_1(u) \wedge \xi_2(v)$ and

$$\eta((u_1, v_1)(u_2, v_2)) = \begin{cases} \eta_1(u_1 u_2) \wedge \eta_2(v_1 v_2), & \text{if } u_1 u_2 \in E(G_1) \text{ and } v_1 v_2 \in E(G_2) \\ \xi_1(u) \wedge \eta_2(v_1 v_2), & \text{if } u_1 = u_2 = u \in V(G_1) \text{ and } v_1 v_2 \in E(G_2) \\ \xi_2(v) \wedge \eta_1(u_1 u_2), & \text{if } u_1 u_2 \in E(G_1) \text{ and } v_1 = v_2 = v \in V(G_2) \end{cases}$$

Hence, $d_G(u, v) \leq 2[d_{G_1}(u) + d_{G_2}(v)]$.

Therefore,

$$\begin{aligned} ZI_1^*(G) &= \sum_{(u_1, v_1)(u_2, v_2) \in E(G)} [\xi(u_1, v_1)d_G(u_1, v_1) + \xi(u_2, v_2)d_G(u_2, v_2)] \\ &= K_1 + K_2 + K_3, \end{aligned}$$

$$\begin{aligned} \text{where, } K_1 &= \sum_{u_1 u_2 \in E(G_1), v_1 v_2 \in E(G_2)} [\xi(u_1, v_1)d_G(u_1, v_1) + \xi(u_2, v_2)d_G(u_2, v_2)] \\ &\leq 2m_2 ZI_1^*(G_1) + 2m_1 ZI_1^*(G_2). \end{aligned}$$

$$\begin{aligned} K_2 &= \sum_{u \in V(G_1), v_1 v_2 \in E(G_2)} [\xi(u, v_1)d_G(u, v_1) + \xi(u, v_2)d_G(u, v_2)] \\ &\leq 4n_1 m_2 \Delta_1 + 2n_1 ZI_1^*(G_2), \end{aligned}$$

$$\begin{aligned} \text{and } K_3 &= \sum_{u_1 u_2 \in E(G_1), v \in V(G_2)} [\xi(u_1, v)d_G(u_1, v) + \xi(u_2, v)d_G(u_2, v)] \\ &\leq 4n_2 m_1 \Delta_2 + 2n_2 ZI_1^*(G_1). \end{aligned}$$

Hence,

$$\begin{aligned} ZI_1^*(G) &\leq 2(n_2 + m_2)ZI_1^*(G_1) + 2(n_1 + m_1)ZI_1^*(G_2) + 4(n_1 m_2 \Delta_1 + n_2 m_1 \Delta_2) \\ &\leq n(n+1)[ZI_1^*(G_1) + ZI_1^*(G_2)] + 4[n(n-1)]^2. \end{aligned}$$

□

5. APPLICATION

In order to assess the usefulness of the TI, it is important to find its relationship with at least one of the physicochemical properties of a chemical compound. Benzenoid hydrocarbons, which are very diverse in nature, can be effectively used for this investigation. In this connection, FGs are drawn for these chemical compounds, where each carbon atom acts as a vertex, while the edge is represented by the bond between two carbon atoms. Formulae for calculating MSVs of the vertices and edges are given by:

$$\begin{aligned} \xi(v) &= \frac{\text{Atomic energy for the vertex } v}{\vee\{\text{Atomic energy of all vertices of } G\}} \\ \eta(uv) &= \wedge\left\{\frac{\text{Bond energy for the edge } uv}{\vee\{\xi(u), \xi(v)\}}, \xi(u), \xi(v)\right\}. \end{aligned}$$

In our study, 22 benzenoid hydrocarbons are analyzed. The following six physicochemical properties are considered for these hydrocarbons:

(i) Boiling point (BP), (ii) Entropy (E), (iii) Acentric factor (AF), (iv) Octanol-water partition coefficient (OWPC), (v) Kovats retention index (KRI) and (vi) Enthalpy of formation (EF).

The values of these physico-chemical properties for benzenoid hydrocarbons and edge based first ZI are listed in Table 2. From the data of Table 2, the linear

TABLE 2. The values of some physico-chemical properties and edge based first ZI for benzenoid hydrocarbons.

Name	BP	E	AF	OWPC	KRI	EF	Edge first ZI
Naphthalene	218	79.38	0.302	3.3	200	150.6	21.28
Phenanthrene	338	93.79	0.394	4.46	300	209.1	32
Anthracene	340.05	92.43	0.402	4.45	301.69	218.3	32
Chrysene	431	106.83	0.46	5.81	400	267.7	42.72
Tetraphene	425	108.22	0.46	5.76	398.5	276.9	42.72
Triphenylene	429	104.66	0.46	5.49	400	258.5	42.72
Naphthacene	440	105.47	0.46	5.76	408.3	286.1	42.72
Benzo[a]pyrene	496	111.85		6.13	453.44	279.9	49.92
Benzo[e]pyrene	493	110.46		6.44	450.73	289.1	49.92
Perylene	497	109.1	0.49	6.25	456.22	279.9	49.92
Anthanthrene	547	114.1		7.04	503.89	310.5	57.12
Benzo[ghi]perylene	542	114.1		6.63	501.32	301.3	57.12
Dibenz[a,h]anthracene	536	119.87		6.75	495.45	335.5	53.44
Dibenz[a,j]anthracene	531	119.87		6.54	489.8	335.5	53.44
Picene	519	119.87	0.54	7.11	500	326.3	53.44
Coronene	590	116.36	0.54	7.64	549.67	322.7	64.32
Benzo[c]phenanthrene	448	113.61		5.7	391.12	280.5	42.72
Pyrene	404	96.06	0.41	4.88	351.22	230.5	39.2
Dibenzo[a,e]pyrene	592	124.89		7.28	551.53	338.5	60.64
Dibenzo[a,h]pyrene	596	123.5		7.28	559.9	347.7	60.64
Dibenzo[a,i]pyrene	594	123.5		7.28	556.47	347.7	60.64
Dibenzo[a,l]pyrene	595	131.69		7.71	553	351.2	60.32

regression lines are determined for different parameters and also the corresponding correlation coefficients (R). All these linear regression lines are written below:

$$\begin{aligned}
 \text{BP} &= 8.8307(\text{Edge first ZI}) + 52.792, R = 0.993365, \\
 \text{E} &= 1.0266(\text{Edge first ZI}) + 61.007, R = 0.926841, \\
 \text{AF} &= 0.0057(\text{Edge first ZI}) + 0.2065, R = 0.96111, \\
 \text{OWPC} &= 0.1013(\text{Edge first ZI}) + 1.2446, R = 0.983751, \\
 \text{KRI} &= 8.6031(\text{Edge first ZI}) + 26.177, R = 0.993488, \\
 \text{EF} &= 4.4007(\text{Edge first ZI}) + 74.536, R = 0.944626.
 \end{aligned}$$

Since the correlation coefficients are greater than 0.9, the edge based first ZI could be used to predict the physico-chemical properties of benzenoid hydrocarbons effectively. These findings have demonstrated the effectiveness of this index. Linear regressions between this index and these physico-chemical properties of benzenoid hydrocarbons have been illustrated in Figure 2. In Figure 2, the regression line has been plotted as the blue line. This figure clearly represents the strength of the relationship between this index and these physico-chemical properties of benzenoid hydrocarbons. From Figure 2 and seeing the values of R , it is clear that the data points between the Kovats retention index and this index are closer to

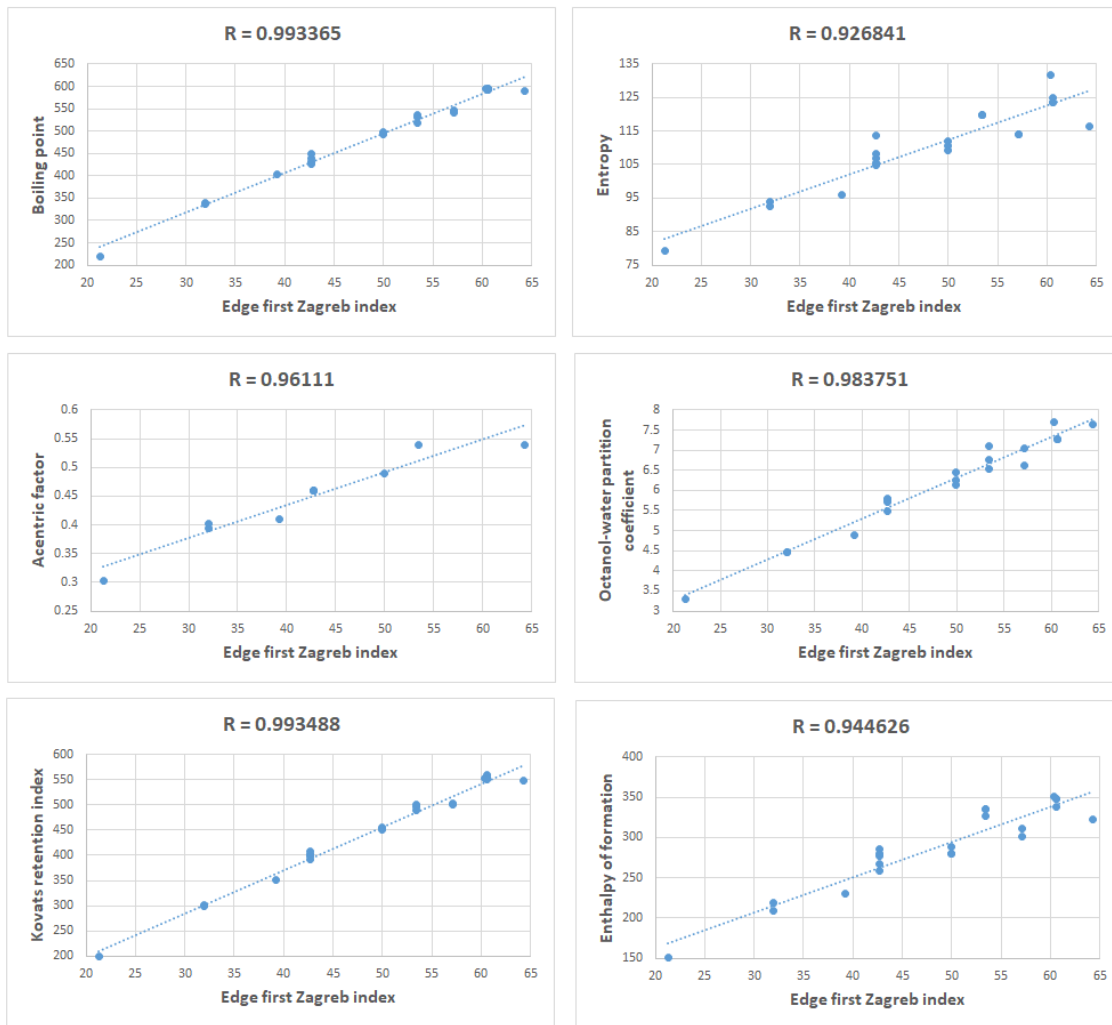


FIGURE 2. Linear fitting of Edge based first ZI with some physico-chemical properties of benzenoid hydrocarbons.

the regression line compared to other data points. Therefore, the linear regression model between this index and the Kovats retention index is more precise than the other models. The value of correlation coefficients signifies that this index is strongly associated with boiling point, OWPC, and moderately associated with acentric factor, entropy, and enthalpy of formation.

6. CONCLUSION

In this study, we have proposed the edge-based formulation of the first ZI tailored for FGs and derived analytical bounds for this index. The proposed ZI on FG has been comprehensively studied in case of different classes of FGs, such as fuzzy path, fuzzy cycle, and fuzzy star graph. Furthermore, we have established bounds for the index under various operations on FGs. To highlight its practical relevance, we have applied this index to a dataset comprising 22

benzenoid hydrocarbons. The analysis has demonstrated a strong correlation between the proposed index and multiple physicochemical properties, including BP, KRI, and OWPC. Additional noteworthy associations have been observed with parameters like AF, E, and EF, confirming the potential utility of this index in molecular chemistry.

REFERENCES

1. Ahmad, U., Khan, N. K., & Saeid, A. B. (2023). Fuzzy topological indices with application to cybercrime problem. *Granular Computing*, 8(5), 967–980. <https://doi.org/10.1007/s41066-023-00365-2>
2. Ahmad, U., & Sabir, M. (2023). Multicriteria decision-making based on the degree and distance-based indices of fuzzy graphs. *Granular Computing*, 8(4), 793–807. <https://doi.org/10.1007/s41066-022-00354-x>
3. Ahmad, U., & Nawaz, I. (2023). Wiener index of a directed rough fuzzy graph and application to human trafficking. *Journal of Intelligent & Fuzzy Systems*, 44(1), 1479–1495. <https://doi.org/10.3233/JIFS-221627>
4. Arif, W., Khan, W. A., Khan, A., & Rashmanlou, H. (2023). Some indices of picture fuzzy graphs and their applications. *Computational and Applied Mathematics*, 42(6), 253. <https://doi.org/10.1007/s40314-023-02393-9>
5. Demir, B., Sarp, Ü., & Emin, A. (2026). On the topological indices of a generalized graph family used in number-theoretic graph realizations. *Gulf Journal of Mathematics*, 22(2), 1–14. <https://doi.org/10.56947/gjom.v22i2.4197>
6. Hosamani, S. M., & Krzywkowski, M. (2016). On the difference of Zagreb coindices of graph-operation. *Gulf Journal of Mathematics*, 4(3), 36–41. <https://doi.org/10.56947/gjom.v4i3.79>
7. Islam, S. R., & Pal, M. (2021). First Zagreb index on a fuzzy graph and its application. *Journal of Intelligent & Fuzzy Systems*, 40(6), 10575–10587. <https://doi.org/10.3233/JIFS-201293>
8. Islam, S. R., & Pal, M. (2021). Hyper-Wiener index for fuzzy graph and its application in share market. *Journal of Intelligent & Fuzzy Systems*, 41(1), 2073–2083. <https://doi.org/10.3233/JIFS-210736>
9. Islam, S. R., & Pal, M. (2023). Second Zagreb index for fuzzy graphs and its application in mathematical chemistry. *Iranian Journal of Fuzzy Systems*, 20(1), 119–136. <https://doi.org/10.22111/ijfs.2023.7350>
10. Islam, S. R., & Pal, M. (2023). Further development of F-index for fuzzy graph and its application in Indian railway crime. *Journal of Applied Mathematics and Computing*, 69(1), 321–353. <https://doi.org/10.1007/s12190-022-01748-5>
11. Islam, S. R., & Pal, M. (2023). An investigation of edge F-index on fuzzy graphs and application in molecular chemistry. *Complex & Intelligent Systems*, 9(2), 2043–2063. <https://doi.org/10.1007/s40747-022-00896-2>
12. Islam, S. R., & Pal, M. (2024). Multiplicative version of first Zagreb index in fuzzy graph and its application in crime analysis. *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, 94(1), 127–141. <https://doi.org/10.1007/s40010-023-00868-2>
13. Kalathian, S., Ramalingam, S., Raman, S., & Srinivasan, N. (2020). Some topological indices in fuzzy graphs. *Journal of Intelligent & Fuzzy Systems*, 39(5), 6033–6046. <https://doi.org/10.3233/JIFS-189077>
14. Lu, J., Zhu, L., & Gao, W. (2022). Cyclic connectivity index of bipolar fuzzy incidence graph. *Open Chemistry*, 20(1), 331–341. <https://doi.org/10.1515/chem-2022-0149>
15. Mahapatra, R., Samanta, S., Pal, M., & Xin, Q. (2019). RSM index: A new way of link prediction in social networks. *Journal of Intelligent and Fuzzy Systems*, 37(2), 2137–2151. <https://doi.org/10.3233/JIFS-181452>

16. Mahapatra, R., Samanta, S., & Pal, M. (2021). Generalized neutrosophic planar graphs and its application. *Journal of Applied Mathematics and Computing*, 65(2), 693–712. <https://doi.org/10.1007/s12190-020-01411-x>
17. Mondal, S., De, N., & Pal, A. (2021). Multiplicative degree based topological indices of nanostar dendrimers. *Biointerface Research in Applied Chemistry*, 11(1), 7700–7711. <https://doi.org/10.33263/BRIAC111.7700771>
18. Mondal, S., Barik, S., De, N., & Pal, A. (2022). A note on neighborhood first Zagreb energy and its significance as a molecular descriptor. *Chemometrics and Intelligent Laboratory Systems*, 222, 104494. <https://doi.org/10.1016/j.chemolab.2022.104494>
19. Pal, M., Samanta, S., & Ghorai, G. (2020). *Modern trends in fuzzy graph theory*. Springer, Singapore. <https://doi.org/10.1007/978-981-15-8803-7>

¹ DEPARTMENT OF MATHEMATICS, BRAINWARE UNIVERSITY,
398, RAMKRISHNAPUR ROAD, JAGADIGHATA MARKET, BARASAT, KOLKATA-700125, IN-
DIA.

Email address: islamskrabiul61@gmail.com

ORCID iD: <https://orcid.org/0000-0003-0486-3412>

² DEPARTMENT OF SCIENCE AND HUMANITIES, KALIMPONG GOVT. POLYTECHNIC,
KALIMPONG, W.B. 734301, INDIA.

Email address: shovansd39@gmail.com

ORCID iD: <https://orcid.org/0000-0002-0768-1764>

³ FACULTY OF SCIENCE AND TECHNOLOGY, UNIVERSITY OF THE FAROE ISLANDS,
VESTARABRYGGJA 15, FO 100 TORSHAVN, FAROE ISLANDS (DENMARK).

Email address: qinx@setur.fo

ORCID iD: <https://orcid.org/0000-0002-6178-8538>

⁴ DEPARTMENT OF MATHEMATICS, BRAINWARE UNIVERSITY,
398, RAMKRISHNAPUR ROAD, JAGADIGHATA MARKET, BARASAT, KOLKATA-700125, IN-
DIA.

Email address: sabnam.puremath@gmail.com

ORCID iD: <https://orcid.org/0000-0002-5897-3695>

⁵ DEPARTMENT OF APPLIED MATHEMATICS, VIDYASAGAR UNIVERSITY,
MIDNAPORE 721102, INDIA.

Email address: mmpalvu@gmail.com

ORCID iD: <https://orcid.org/0000-0002-6709-836X>