

SOLVING NONLINEAR QUADRATIC FREDHOLM INTEGRAL EQUATIONS NUMERICALLY

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ABSTRACT. The two numerical techniques that will be introduced in this article to solve the Nonlinear Quadratic Fredholm Integral Equations (NQFIEs) are the Vieta-Lucas wavelets method and the Picard iteration method combined with the Legendre-Gauss quadrature rule. First, the existence and uniqueness of solutions to the main problem are established under certain assumptions. Then, we use these two methods to solve the NQFIEs in two cases. Additionally, we determine the convergence and estimate the error bound of the Vieta-Lucas wavelets method. Three numerical examples are given to illustrate the efficiency and accuracy of solving NQFIEs using the two methods.

Keywords. Nonlinear quadratic Fredholm integral equations, wavelets method, Picard method, Gauss quadrature rule

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1. INTRODUCTION

One important mathematical tool is the quadratic integral equation. Different natural phenomena can be described using these kinds of equations. Numerous engineering and scientific problems, including gas kinetic theory, traffic theory, transport theory, etc., are frequently described by this kind of equation. [7, 3, 11]. During the last decade, the attention of the scientific community was concentrated on the issues of existence, uniqueness, positive solutions, and monotonic solutions of such equations [5, 8, 10, 6], as well as the numerical techniques for solving them. The considered type of the equation possesses a certain complexity and is sometimes unsolvable. Hence, many methods were developed in order to obtain the approximated solutions of the main problem, including the fixed-point method [14], homotopy perturbation methods [1], the Muntz Legendre wavelet method [18], and others [4, 8].

The aim of this work is to study numerical approximation solutions of NQFIEs utilizing the Picard iteration method and the Vieta-Lucas wavelets approach combined with the Legendre-Gauss quadrature rule.

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The second kind of NQFIE to be studied is considered as follows:

$$\mathcal{Y}(x) = f(x) + g(x, \mathcal{Y}(x)) \int_0^1 \Psi(x, t, \mathcal{Y}(t)) dt, \quad x, t \in \Omega = [0, 1]. \quad (1.1)$$

Where $\mathcal{Y}(x)$ is the unidentified function, $f(x)$, $g(x, \mathcal{Y}(x))$, and $\Psi(x, t, u(t))$ are given functions.

The rest of the paper is structured as the **existence and uniqueness** section. In the **basic definitions** section, we define basic notions related to Legendre-Gauss quadrature rule, Vieta-Lucas polynomials, and Vieta-Lucas wavelets. The **proposed method** section provides two approaches to solving NQVIEs via the Vieta-Lucas wavelets method and Picard iteration method using Legendre-Gauss quadrature rule. **Convergence analysis** follows this section. Then, **numerical examples** are used to demonstrate the efficiency of the proposed technique. The paper ends with conclusions based on the numerical results.

2. EXISTENCE AND UNIQUENESS

Existence and uniqueness solutions of NQFIEs have been studied, as in [5, 6]. Under specific conditions, these equations admit a unique solution. We set $\Omega = [0, 1]$ and $\mathcal{Y} \in C(\Omega)$ with sup norm $\|\mathcal{Y}\|_\infty = \sup_{x \in \Omega} |\mathcal{Y}(x)|$. We consider the NQFIEs of the form

$$\mathcal{Y}(x) = f(x) + g(x, \mathcal{Y}(x)) \int_0^1 \Psi(x, t, \mathcal{Y}(t)) dt, \quad x, t \in \Omega. \quad (2.1)$$

We study two cases of $g(x, \mathcal{Y}(x))$:

case 1 $g(x, \mathcal{Y}(x))$ explicitly given ,

case 2 $g(x, \mathcal{Y}(x)) = \int_0^1 v(x, t, \mathcal{Y}(t)) dt$.

The specific assumptions in **case 2** and **case 3** are

- (i) $f : \Omega \rightarrow \mathbb{R}$ is a continuous function.
- (ii) $g : \Omega \times C(\Omega) \subset \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.
- (iii) $\Psi, v : \Omega \times \Omega \times C(\Omega) \subset \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.
- (iv) $\lambda_1 = \sup\{|\Psi(x, t, \mathcal{Y})| : x, t \in \Omega, \mathcal{Y} \in C(\Omega)\}$.
- (v) $\lambda_2 = \sup\{|g(x, \mathcal{Y})| : x \in \Omega, \mathcal{Y} \in C(\Omega)\}$ (respectively, $\lambda_2 = \sup\{|v(x, t, \mathcal{Y})| : x, t \in \Omega, \mathcal{Y} \in C(\Omega)\}$) for **case 1** (respectively, for **case 2**).
- (vi) g, v, Ψ are Lipschitz functions; there exist constants $L_1 > 0$ and $L_2 > 0$ such that $|\Psi(x, t, \mathcal{Y}_1) - \Psi(x, t, \mathcal{Y}_2)| \leq L_1 |\mathcal{Y}_1 - \mathcal{Y}_2|$ and $|g(x, \mathcal{Y}_1) - g(x, \mathcal{Y}_2)| \leq L_2 |\mathcal{Y}_1 - \mathcal{Y}_2|$ (respectively, $|v(x, t, \mathcal{Y}_1) - v(x, t, \mathcal{Y}_2)| \leq L_2 |\mathcal{Y}_1 - \mathcal{Y}_2|$) for **case 1** (respectively, for **case 2**) for all $x, t \in \Omega$ and $\mathcal{Y}_1, \mathcal{Y}_2 \in C(\Omega)$.

Theorem 2.1. *If the conditions (i) – (vi) and $(L_2\lambda_1 + L_1\lambda_2) < 1$ are satisfied. Then, the NQFIEs (2.1) has a unique solution $\mathcal{Y} \in C(\Omega)$ for **case 1** and **case 2**.*

Proof. Suppose F is a mapping that has been defined as $F : C(\Omega) \rightarrow C(\Omega)$, where

$$F(\mathcal{Y})(x) = f(x) + g(x, \mathcal{Y}(x)) \int_0^1 \Psi(x, t, \mathcal{Y}(t)) dt.$$

For the **case 1**: Let $\mathcal{Y}_1, \mathcal{Y}_2 \in C(\Omega)$. Then $F(\mathcal{Y}_1)(x) - F(\mathcal{Y}_2)(x) =$

$$\begin{aligned}
&= g(x, \mathcal{Y}_1(x)) \int_0^1 \Psi(x, t, \mathcal{Y}_1(t)) dt - g(x, \mathcal{Y}_2(x)) \int_0^1 \Psi(x, t, \mathcal{Y}_2(t)) dt, \\
&= g(x, \mathcal{Y}_1(x)) \int_0^1 \Psi(x, t, \mathcal{Y}_1(t)) dt - g(x, \mathcal{Y}_2(x)) \int_0^1 \Psi(x, t, \mathcal{Y}_2(t)) dt \\
&+ g(x, \mathcal{Y}_2(x)) \int_0^1 \Psi(x, t, \mathcal{Y}_1(t)) dt - g(x, \mathcal{Y}_2(x)) \int_0^1 \Psi(x, t, \mathcal{Y}_1(t)) dt, \quad (2.2) \\
&= (g(x, \mathcal{Y}_1(x)) - g(x, \mathcal{Y}_2(x))) \int_0^1 \Psi(x, t, \mathcal{Y}_1(t)) dt \\
&+ g(x, \mathcal{Y}_2(x)) \left(\int_0^1 \Psi(x, t, \mathcal{Y}_1(t)) - \Psi(x, t, \mathcal{Y}_2(t)) dt \right),
\end{aligned}$$

so, we have

$$\begin{aligned}
\|F(\mathcal{Y}_1)(x) - F(\mathcal{Y}_2)(x)\|_\infty &\leq \sup_{x \in \Omega} |g(x, \mathcal{Y}_1(x)) - g(x, \mathcal{Y}_2(x))| \int_0^1 |\Psi(x, t, \mathcal{Y}_1(t))| dt \\
&+ \sup_{x \in \Omega} |g(x, \mathcal{Y}_2(x))| \int_0^1 |\Psi(x, t, \mathcal{Y}_1(t)) - \Psi(x, t, \mathcal{Y}_2(t))| dt, \\
&\leq L_2 \lambda_1 \sup_{x \in \Omega} |\mathcal{Y}_1(x) - \mathcal{Y}_2(x)| + L_1 \lambda_2 \sup_{x \in \Omega} |\mathcal{Y}_1(x) - \mathcal{Y}_2(x)|, \\
&\leq (L_2 \lambda_1 + L_1 \lambda_2) \|\mathcal{Y}_1(x) - \mathcal{Y}_2(x)\|_\infty, \quad (2.3)
\end{aligned}$$

F is a contraction mapping on the Banach space $C(\Omega)$ under the condition $(L_2 \lambda_1 + L_1 \lambda_2) < 1$ in **case 1**. Then, there exists a unique solution on $C(\Omega)$ of the Eq. (2.1).

Employing the same technique and similar steps, we establish that F is a contraction mapping on the Banach space $C(\Omega)$ under the condition $(L_2 \lambda_1 + L_1 \lambda_2) < 1$ and the assumptions (i) – (vi) in **case 2**. $\square$

3. BASIC DEFINITIONS

The definitions in this part will be utilized to solve the NQFIEs in the section that follows.

3.1. Legendre-Gauss quadrature rule. An integral function over the domain $[a, b]$ can be approximated using the J -point Legendre-Gauss quadrature rule as follows [12]:

$$\int_a^b g(x) dx = L(g) + E_L(g), \quad (3.1)$$

where the Legendre-Gauss quadrature approximation is

$$L(g) = \sum_{i=1}^J w_i g(\eta_i),$$

and the weights w_i and nodes η_i are given by:

$$w_i = \frac{b-a}{(1-\sigma_i^2)(L'_J(\sigma_i))^2}, \quad \eta_i = \frac{b-a}{2}\sigma_i + \frac{b+a}{2}, \quad i = 1, \dots, J. \quad (3.2)$$

Where σ_i are the roots of the Legendre polynomial of degree J and $i = 1, \dots, J$. If $g \in C^{2J+2}[a, b]$. Then, there exists a constant $c \in]a, b[$, such that the error $E_L(g)$ is estimated as follows [12]:

$$E_L(g) = \frac{2^{2J+3}((J+1)!)^4 g^{(2J+2)}(c)}{(2J+3)((2J+2)!)^3}. \quad (3.3)$$

So that

$$|E_L(g)| \leq \Gamma_J, \quad (3.4)$$

where

$$\Gamma_J = \frac{2^{2J+3}((J+1)!)^4 \max_{c \in]a, b[} |g^{(2J+2)}(c)|}{(2J+3)((2J+2)!)^3}.$$

Example 3.1. Let's approximate the following integral

$$\int_0^\pi \exp(t) dt = e^\pi - 1 \simeq 22.140692632779269006,$$

using the Legendre-Gauss quadrature rule, where $g(t) = \exp(t)$. Then, we have this table

TABLE 1. The estimate $L(g)$ and the errors $E_L(g)$ of the function g

J	2	4	6	8
$L(g)$	21.765157	22.140606	22.14069263003	22.1406926327792451
$E_L(g)$	0.3755	$8.57e^{-05}$	$2.74e^{-09}$	$2.39e^{-14}$

3.2. Vieta-Lucas polynomials. The polynomials of the Vieta-Lucas of degree m ($m \in \mathbb{N}$) are $VL_m(x)$ such that [13]:

$$VL_m(x) = 2 \cos \left(m \arccos \left(\frac{x}{2} \right) \right), \quad x \in [-2, 2]. \quad (3.5)$$

Recurrence relation: The Vieta-Lucas polynomials can be generated using the relation

$$\begin{cases} VL_0(x) = 2, \\ VL_1(x) = x, \\ VL_m(x) = xVL_{m-1}(x) - VL_{m-2}(x), \quad m = 2, 3, 4, \dots \end{cases} \quad (3.6)$$

The first five Vieta-Lucas polynomials are

$$\begin{aligned} VL_0(x) &= 2, \\ VL_1(x) &= x, \\ VL_2(x) &= x^2 - 2, \\ VL_3(x) &= x^3 - 3x, \\ VL_4(x) &= x^4 - 4x^2 + 2. \end{aligned} \quad (3.7)$$

The next figures represent the four Vieta-Lucas polynomials graphs.

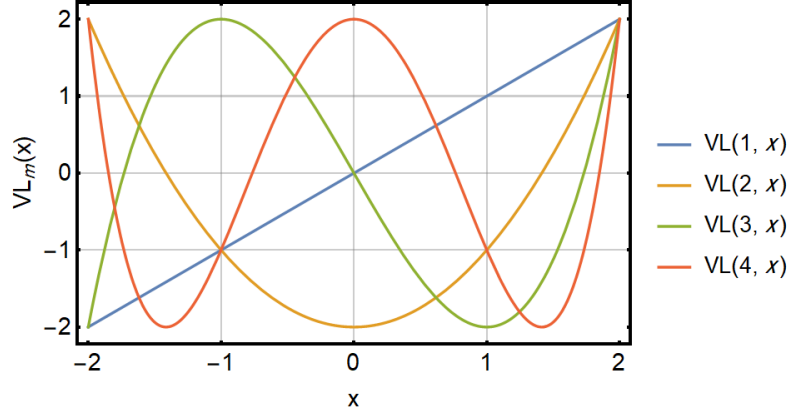


FIGURE 1. Vieta-Lucas polynomials graphs

Express series: The Vieta-Lucas polynomials can be expressed as a power series as

$$VL_m(x) = \begin{cases} 2, & m = 0, \\ \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \frac{m(m-i-1)!}{i!(m-2i)!} x^{m-2i}, & m \neq 0 \dots, \end{cases} \quad (3.8)$$

where $\lfloor \frac{m}{2} \rfloor$ denotes the floor function of $\frac{m}{2}$.

Orthogonality: the Vieta-Lucas polynomials $VL_n(x)$ and $VL_m(x)$ are orthogonal over $[-2, 2]$ with respect to the function $w(x) = \frac{1}{\sqrt{4-x^2}}$. Consequently, the orthogonality condition is given by:

$$\langle VL_n(x), VL_m(x) \rangle_{w(x)} = \int_{-2}^2 VL_n(x) VL_m(x) w(x) dx = \begin{cases} 4\pi, & n = m = 0, \\ 2\pi, & n = m \neq 0, \\ 0, & n \neq m \neq 0. \end{cases} \quad (3.9)$$

In terms of its generating function, the Vieta-Lucas polynomial sequence satisfies:

$$\sum_{m=0}^{\infty} VL_m(x) t^m = \frac{2 - xt}{1 - xt + t^2}. \quad (3.10)$$

3.3. Vieta-Lucas wavelets. The functions $\xi_{s,m}$ defined on area $[0, 1]$ by $\xi_{s,m}(x) = \xi(k, s, m, x)$ are called Vieta-Lucas wavelets if they have the following form [13]:

$$\xi_{s,m}(x) = \begin{cases} \frac{2^{\frac{k+2}{2}}}{\sqrt{\alpha_m}} VL_m(2^{k+2}x - 4s + 2), & \frac{s-1}{2^k} \leq x \leq \frac{s}{2^k}, \\ 0, & \text{otherwise.} \end{cases} \quad (3.11)$$

The parameters are defined as follows: $k \in \mathbb{Z}^+$, $s = 1, 2, \dots, 2^k$, and $m = 0, 1, 2, \dots, M+1$, where $M+1$ represents the maximum polynomial degree. The

normalization value is given by

$$\alpha_m = \begin{cases} 4\pi, & m = 0, \\ 2\pi, & m \geq 1. \end{cases}$$

The next figures represent some of Vieta-Lucas wavelet graphs for $k = 1$ and $s = 1, 2$ where they are defined on the domain $[0, 0.5]$ and $[0.5, 1]$ respectively.

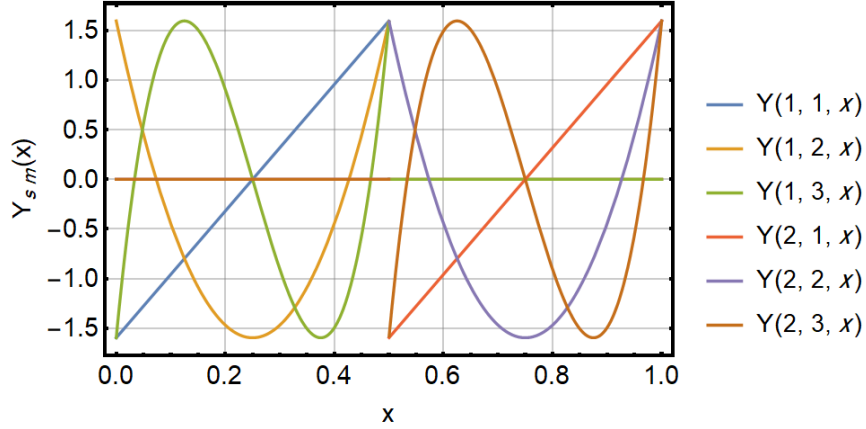


FIGURE 2. Graph of some Vieta-Lucas wavelet for $k = 1$ and $s = 1, 2$.

The Vieta-Lucas wavelets are orthogonal in $L^2_{w_s}[0, 1]$, where the weight function is $w_s(x) = w(2^{k+2}x - 4s + 2)$. That is, they satisfy the condition:

$$\langle \xi_{s,m}(x), \xi_{s',m'}(x) \rangle_{w_s(x)} = \int_0^1 \xi_{s,m}(x) \xi_{s',m'}(x) w_s(x) dt = \begin{cases} 1, & (s, m) = (s', m'), \\ 0, & (s, m) \neq (s', m'). \end{cases} \quad (3.12)$$

Let $\mathcal{Y}(x)$ a function from $L^2_{w_s}[0, 1]$. Then, $\mathcal{Y}(x)$ can be approximated using the Vieta-Lucas wavelets series as

$$\mathcal{Y}(x) = \sum_{s=1}^{\infty} \sum_{m=0}^{\infty} y_{s,m} \xi_{s,m}(x), \quad (3.13)$$

The coefficients $y_{s,m}$ in this expansion are calculated as follows:

$$y_{s,m} = \int_0^1 \mathcal{Y}(x) \xi_{s,m}(x) w_s(x) dx. \quad (3.14)$$

The truncated series expansion in terms of Vieta-Lucas wavelets is given by

$$\mathcal{Y}(x) \simeq \mathcal{Y}^*(x) = \sum_{s=1}^{2^k} \sum_{m=0}^{M+1} y_{s,m} \xi_{s,m}(x) = U^T \xi(x), \quad (3.15)$$

where U and $\xi(x)$ denote vectors of dimension $\mathfrak{M} \times 1$ with $\mathfrak{M} = 2^k(M+2)$. These matrices are structured as

$$U = [y_{1,0}, y_{1,1}, \dots, y_{1,M+1} \mid y_{2,0}, \dots, y_{2,M+1} \mid \dots \mid y_{2^k,0}, \dots, y_{2^k,M+1}]^T, \\ \xi(t) = [\xi_{1,0}(t), \dots, \xi_{1,M+1}(t) \mid \xi_{2,0}(t), \dots, \xi_{2,M+1}(t) \mid \dots \mid \xi_{2^k,0}(t), \dots, \xi_{2^k,M+1}(t)]^T. \quad (3.16)$$

4. PROPOSED METHODS

4.1. Vieta-Lucas wavelets method. This subsection details the numerical solution of the two NQFIE cases through the application of Vieta-Lucas wavelets, integrated with the Legendre-Gauss quadrature rule. Consider NQFIEs of the following form

$$\mathcal{Y}(x) = f(x) + g(x, \mathcal{Y}(x)) \int_0^1 \Psi(x, s, \mathcal{Y}(s)) ds, \quad x \in [0, 1]. \quad (4.1)$$

First, we approximate the function $\mathcal{Y}(x)$ in (4.1) using the Vieta-Lucas wavelets in the next form

$$\mathcal{Y}(x) \simeq \mathcal{Y}^*(x) = U^T \xi(x), \quad (4.2)$$

U denotes a column vector of unknown $y_{s,m}$, that is $\mathfrak{M}$ -dimensional, and $\xi(x)$ is represented as (3.16). Substituting (4.2) into (4.1), we obtain

$$U^T \xi(x) = f(x) + g(x, U^T \xi(x)) \int_0^1 \Psi(x, s, U^T \xi(s)) ds. \quad (4.3)$$

Next, the integral is approximated using the Legendre-Gauss quadrature rule:

$$\int_0^1 \Psi(x, s, U^T \xi(s)) ds = \sum_{i=1}^J w_i \Psi(x, \eta_i, U^T \xi(\eta_i)), \quad (4.4)$$

where the weights and nodes associated with the Legendre-Gauss quadrature rule are represented by w_i and η_i , respectively. Substituting (4.4) into (4.3) yields

$$U^T \xi(x) = f(x) + g(x, U^T \xi(x)) \sum_{i=1}^J w_i \Psi(x, \eta_i, U^T \xi(\eta_i)). \quad (4.5)$$

We now consider two cases for the function $g(x, \mathcal{Y}(x))$:

Case 1: If $g(x, \mathcal{Y}(x))$ is explicitly given, we apply the collocation method with the nodes of Newton-Cotes $x_j = \frac{2j-1}{2\mathfrak{M}}$, for $j = 1, \dots, \mathfrak{M}$ in Eq (4.5), Yielding, a system of nonlinear algebraic equations for the unknown $y_{s,m}$ in the vector U , which is then resolved via Newton's method.

Case 2: If $g(x, \mathcal{Y}(x)) = \int_0^1 v(x, s, \mathcal{Y}(s)) ds$, in Eq.(4.1), we approximate the integral using the same Legendre-Gauss quadrature rule method from (4.4), yielding

$$\int_0^1 v(x, s, U^T \xi(s)) ds = \sum_{l=1}^{J'} w_l v(x, \eta_l, U^T \xi(\eta_l)). \quad (4.6)$$

Substituting Eqs. (4.6) and (4.4) into (4.3), we obtain

$$U^T \xi(x) = f(x) + \sum_{l=1}^{J'} w_l v(x, \eta_l, U^T \xi(\eta_l)) \sum_{i=1}^J w_i \Psi(x, \eta_i, U^T \xi(\eta_i)). \quad (4.7)$$

With J consistently chosen as J' , applying collocation method with the nodes of Newton-Cotes $x_j = \frac{2j-1}{2\mathfrak{M}}$, for $j = 1, \dots, \mathfrak{M}$ at the Eq (4.7). Yielding, a system of nonlinear algebraic equations for the unknown $y_{s,m}$ in the vector U , which is then resolved via Newton's method.

4.2. Convergence analysis.

Theorem 4.1. *Suppose that $\mathcal{Y}(x) \in L_{w_s}^2[0, 1]$ has a bounded second derivative $H > 0$ (i.e., $|\mathcal{Y}''(x)| \leq H$). Then $\mathcal{Y}(x)$ admits an infinite expansion in terms of Vieta-Lucas wavelets, where the series is uniformly convergent as $k, M \rightarrow \infty$. Moreover:*

$$|y_{s,m}| \leq \frac{H\sqrt{\pi}}{2^{\frac{5}{2}} s^{\frac{5}{2}} (m^2 - 1)}, m > 1, s \geq 1 \text{ and } |y_{s,1}| \leq \frac{\sqrt{\pi}}{2^{\frac{5}{2}} s^{\frac{3}{2}}} \max_{0 \leq t \leq 1} |\mathcal{Y}'(x)|, m = 1, s \geq 1.$$

where the coefficients $y_{s,m}$ given in (3.14).

Proof. see [13]. □

Lemma 4.2. *Suppose $f(t)$ is a positive, continuous, decreasing function for all $t \geq s$ and let $f(k) = \Lambda_k$. If the series $\sum \Lambda_s$ converges, then the following inequality holds $\sum_{k=s+1}^{\infty} \Lambda_k \leq \int_s^{\infty} f(t) dt$.*

Theorem 4.3. *Let $\mathcal{Y}(x)$ be a function that satisfies Theorem 4.1. And let $\mathcal{Y}^*(x)$ be the approximation of $\mathcal{Y}(x)$ using Vieta-Lucas wavelets. Then, the error bound estimate is*

$$|\mathcal{Y}(x) - \mathcal{Y}^*(x)| \leq \frac{H(2M+3)}{3 \times 2^{k+1}(M+1)(M+2)} = \Theta_M^k. \quad (4.8)$$

Proof. see [16]. □

Theorem 4.4. *Suppose $\mathcal{Y}(x)$ and $\mathcal{Y}^*(x)$ denote the exact and approximate solutions to equation (2.1), respectively. Provided that conditions (i)–(vi) hold, the inequality $(L_1 \lambda_2 + L_2 \lambda_1) < 1$ is satisfied, and the requirements of Theorem 4.1 are met, then for both **case 1** and **case 2** we obtain the following:*

$$\|\mathcal{Y}(x) - \mathcal{Y}^*(x)\|_{\infty} \leq \max \{ \Theta_M^k, (L_2 \lambda_1 + L_1 \lambda_2) \Theta_M^k + \lambda_2 \Gamma_J \} \text{ and}$$

$$\|\mathcal{Y}(x) - \mathcal{Y}^*(x)\|_{\infty} \leq \max \{ \Theta_M^k, L_1(\lambda_2 + \Gamma'_J) \Theta_M^k + \Gamma_J(\lambda_2 + \Gamma'_J) + \Gamma'_J \lambda_1 \},$$

respectively.

Proof. The Eq. (4.5) in **case 1**:

$$\mathcal{Y}^*(x) = f(x) + g(x, \mathcal{Y}^*(x)) \sum_{i=1}^J w_i \Psi(x, \eta_i, \mathcal{Y}^*(\eta_i)), \quad (4.9)$$

where $\mathcal{Y}^*(x) = U^T \xi(x)$; by subtracting (2.1) and (4.9), we get

$$\begin{aligned}
\mathcal{Y}(x) - \mathcal{Y}^*(x) &= g(x, \mathcal{Y}(x)) \int_0^1 \Psi(x, s, \mathcal{Y}(s)) ds - g(x, \mathcal{Y}^*(x)) \sum_{i=1}^J w_i \Psi(x, \eta_i, \mathcal{Y}^*(\eta_i)), \\
&= g(x, \mathcal{Y}(x)) \int_0^1 \Psi(x, s, \mathcal{Y}(s)) ds - g(x, \mathcal{Y}^*(x)) \int_0^1 \Psi(x, s, \mathcal{Y}(s)) ds \\
&\quad + g(x, \mathcal{Y}^*(x)) \int_0^1 \Psi(x, s, \mathcal{Y}(s)) ds - g(x, \mathcal{Y}^*(x)) \sum_{i=1}^J w_i \Psi(x, \eta_i, \mathcal{Y}^*(\eta_i)), \\
&= (g(x, \mathcal{Y}(x)) - g(x, \mathcal{Y}^*(x))) \int_0^1 \Psi(x, s, \mathcal{Y}(s)) ds \\
&\quad + g(x, \mathcal{Y}^*(x)) \left(\int_0^1 \Psi(x, s, \mathcal{Y}(s)) ds - \sum_{i=1}^J w_i \Psi(x, \eta_i, \mathcal{Y}^*(\eta_i)) \right), \\
&= (g(x, \mathcal{Y}(x)) - g(x, \mathcal{Y}^*(x))) \int_0^1 \Psi(x, s, \mathcal{Y}(s)) ds + g(x, \mathcal{Y}^*(x)) \left(\int_0^1 (\Psi(x, s, \mathcal{Y}(s)) \right. \\
&\quad \left. - \Psi(x, s, \mathcal{Y}^*(s))) ds + \int_0^1 \Psi(x, s, \mathcal{Y}^*(s)) ds - \sum_{i=1}^J w_i \Psi(x, \eta_i, \mathcal{Y}^*(\eta_i)) \right). \tag{4.10}
\end{aligned}$$

Also, we have

$$\begin{aligned}
\|\mathcal{Y}(x) - \mathcal{Y}^*(x)\|_\infty &\leq |g(x, \mathcal{Y}(x)) - g(x, \mathcal{Y}^*(x))| \int_0^1 |\Psi(x, s, \mathcal{Y}(s))| ds + |g(x, \mathcal{Y}^*(x))| \times \\
&\quad \left(\int_0^1 |\Psi(x, s, \mathcal{Y}(s)) - \Psi(x, s, \mathcal{Y}^*(s))| ds + \left| \int_0^1 \Psi(x, s, \mathcal{Y}^*(s)) ds \right. \right. \\
&\quad \left. \left. - \sum_{i=1}^J w_i \Psi(x, \eta_i, \mathcal{Y}^*(\eta_i)) \right| \right). \tag{4.11}
\end{aligned}$$

Assume $\mathcal{Y}(x)$ satisfies conditions (i)–(vi), and from Eq.(3.4), we obtain

$$\|\mathcal{Y}(x) - \mathcal{Y}^*(x)\|_\infty \leq L_2 \|\mathcal{Y}(x) - \mathcal{Y}^*(x)\|_\infty \lambda_1 + \lambda_2 (L_1 \|\mathcal{Y}(x) - \mathcal{Y}^*(x)\|_\infty + \Gamma_J). \tag{4.12}$$

Suppose $\mathcal{Y}(x)$ satisfies the assumptions of Theorem 4.3. Then, we get

$$\|\mathcal{Y}(x) - \mathcal{Y}^*(x)\|_\infty \leq \max \{ \Theta_M^k, (L_2 \lambda_1 + L_1 \lambda_2) \Theta_M^k + \lambda_2 \Gamma_J \}. \tag{4.13}$$

For **case 2**, from (4.7) we have

$$\mathcal{Y}^*(x) = f(x) + \sum_{l=1}^{J'} w_l v(x, \eta_l, \mathcal{Y}^*(\eta_l)) \sum_{i=1}^J w_i \Psi(x, \eta_i, \mathcal{Y}^*(\eta_i)). \tag{4.14}$$

Employing the same technique, we are able to establish the next inequalities

$$\begin{aligned}
\|\mathcal{Y}(x) - \mathcal{Y}^*(x)\|_\infty &\leq \left| \int_0^1 v(x, s, \mathcal{Y}(s)) ds - \sum_{l=1}^{J'} w_l v(x, \eta_l, \mathcal{Y}^*(\eta_l)) \right| \left| \int_0^1 |\Psi(x, s, \mathcal{Y}(s))| ds \right. \\
&\quad \left. + \left| \sum_{l=1}^{J'} w_l v(x, \eta_l, \mathcal{Y}^*(\eta_l)) \right| \left| \int_0^1 \Psi(x, s, \mathcal{Y}(s)) ds - \sum_{i=1}^J w_i \Psi(x, \eta_i, \mathcal{Y}^*(\eta_i)) \right| \right| \\
&\leq \Gamma'_J \lambda_x + (\lambda_2 + \Gamma'_J) (L_1 \|\mathcal{Y}(x) - \mathcal{Y}^*(x)\|_\infty + \Gamma_J), \\
&= L_1 (\lambda_2 + \Gamma'_J) \|\mathcal{Y}(x) - \mathcal{Y}^*(x)\|_\infty + \Gamma_J (\lambda_2 + \Gamma'_J) + \Gamma'_J \lambda_1.
\end{aligned} \tag{4.15}$$

Suppose $\mathcal{Y}(x)$ satisfies the assumptions of Theorem 4.3. Then, we obtain

$$\|\mathcal{Y}(x) - \mathcal{Y}^*(x)\|_\infty \leq \max \{ \Theta_M^k, L_1 (\lambda_2 + \Gamma'_J) \Theta_M^k + \Gamma_J (\lambda_2 + \Gamma'_J) + \Gamma'_J \lambda_1 \}. \tag{4.16}$$

Observe that $\|\mathcal{Y}(x) - \mathcal{Y}^*(x)\|_\infty \rightarrow 0$, as $J, k, m \rightarrow \infty$ since $\Theta_M^k \xrightarrow[k, m \rightarrow \infty]{} 0$, $\Gamma_J \xrightarrow[J \rightarrow \infty]{} 0$, and $\Gamma'_J \xrightarrow[J \rightarrow \infty]{} 0$. □

4.3. Picard iteration method. The description of a straightforward yet effective strategy for dealing with nonlinear equations takes up this part. This technique is sometimes referred to as the Picard iteration method or the method of successive approximation. By beginning with an initial guess, known as the zero approximation, this method finds a sequence that is approximate to the solution of the main problem. . Any chosen real-valued function that will be employed in a recurrence relation to ascertain the other approximations is the zero approximation, as will be observed later. Numerous kinds of nonlinear integral equations can frequently be effectively solved using the successive approximation method. (see, [9, 15]). For NQFIEs in the next form

$$\mathcal{Y}(x) = f(x) + g(x, \mathcal{Y}(x)) \int_0^1 \Psi(x, s, \mathcal{Y}(s)) ds \quad x, s \in \Omega. \tag{4.17}$$

Then, the recurrence relation of Picard iteration method is expressed as

$$\mathcal{Y}_{i+1}(x) = f(x) + g(x, \mathcal{Y}_i(x)) \int_0^1 \Psi(x, s, \mathcal{Y}_i(s)) ds, \quad i = 0, 1, 2, \dots \tag{4.18}$$

Typically, the zero (first) approximation is taken as $\mathcal{Y}_0(x) = 0$ or $f(x)$.

Theorem 4.5. [9] *Let us assume that f, g and Ψ satisfy the conditions (i) – (vi). Additionally, suppose that $(L_2 \lambda_1 + L_1 \lambda_2) < 1$. Then, the iteration approach of (4.18) converges to the unique solution $\mathcal{Y} \in C(\Omega)$ of the Eq (4.17).*

Remark 4.6. We use the Legendre-Gauss quadrature rule in (3.1) to estimate the integral kernel Ψ in Eq. (4.18) to accelerate the calculation and determine the sequence in the Picard iteration method.

Remark 4.7. In **case 2**, where $g(x, \mathcal{Y}_i(x)) = \int_0^1 v(x, s, \mathcal{Y}_i(s)) ds$. Also, we estimate the integral $\int_0^1 v(x, s, \mathcal{Y}_i(s)) ds$ by using the Legendre-Gauss quadrature rule.

5. NUMERICAL EXAMPLES

The accuracy of the proposed methods in section 4 is assessed using the maximum absolute error, defined in this section as:

$$E = \|\mathcal{Y}(x) - \mathcal{Y}^*(x)\|_\infty = \sup_{0 \leq x \leq 1} |\mathcal{Y}(x) - \mathcal{Y}^*(x)|, \quad (5.1)$$

where $\mathcal{Y}(x)$ is the exact solution and $\mathcal{Y}^*(x)$ represents the approximate solution. To demonstrate the method's efficacy, several numerical examples are presented. All computations were performed in Wolfram Mathematica 10.3 software over the interval $0 \leq x \leq 1$ with the parameter $k = 1$ for the Vieta-Lucas wavelets method.

Example 5.1. An examination of the following NQFIE:

$$\mathcal{Y}(x) = \sin(x) - \frac{x \sin(x)}{30(1 + \sin(x))} + g(x, \mathcal{Y}(x)) \int_0^1 \Psi(x, s, \mathcal{Y}(s)) ds, \quad (5.2)$$

where

$$\begin{aligned} g(x, \mathcal{Y}(x)) &= \frac{|\mathcal{Y}(x)|}{3(1 + |\mathcal{Y}(x)|)}, & \Psi(x, s, \mathcal{Y}(s)) &= \frac{x}{5} \arcsin(|\mathcal{Y}(s)|), \\ |g(x, \mathcal{Y}(x))| &\leq \frac{1}{3} = \lambda_2, & |\Psi(x, s, \mathcal{Y}(s))| &\leq \frac{\pi}{10} = \lambda_1, \\ \left| \frac{\partial g(x, \mathcal{Y})}{\partial \mathcal{Y}} \right| &\leq \frac{1}{3} = L_2, & \left| \frac{\partial \Psi(x, s, \mathcal{Y})}{\partial \mathcal{Y}} \right| &\leq \frac{1}{5} = L_1, \end{aligned}$$

The uniqueness of the solution for Eq.(5.2), $\mathcal{Y}(x) = \sin(x)$, is established by the existence Theorem 2.1 where $(L_2\lambda_1 + L_1\lambda_2) < 1$.

The performance of the Vieta-Lucas wavelets method in section 4 is quantified by the maximum absolute error E^1 for $k = 1$ and E^2 for $k = 2$, computed at $M = 2, 6$ and $J = 2, 4, 6$; these results are compiled in Table 2. Figure 3 is the logarithm absolute error graphs of the Vieta-Lucas wavelets method at $M = 2, 6, 10$, $J = 6$.

TABLE 2. Vieta-Lucas wavelets method maximum absolute error for Eq (5.2).

$J = 2$			$J = 4$		$J = 6$	
M	2	6	2	6	2	6
E^1	$4.7851e^{-5}$	$3.2043e^{-11}$	$4.7729e^{-5}$	$3.2024e^{-11}$	$4.7765e^{-5}$	$3.2045e^{-11}$
CPU Ti 1	0.00527	0.02068	0.00732	0.02466	0.00813	0.02966
E^2	$3.2596e^{-6}$	$1.3911e^{-13}$	$3.2592e^{-6}$	$1.3922e^{-13}$	$3.2667e^{-6}$	$1.3944e^{-13}$
CPU Ti 2	0.01151	0.05134	0.01525	0.07432	0.01771	0.08171

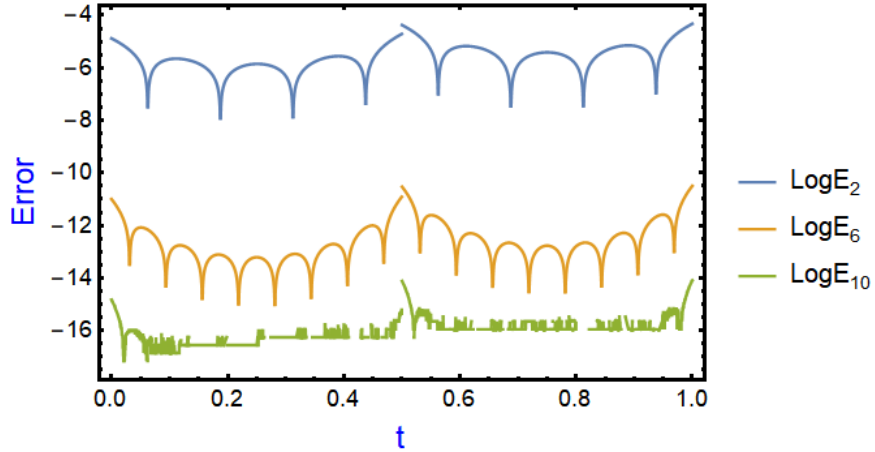


FIGURE 3. Logarithm absolute errors graphs of Vieta-Lucas wavelets method at $M = 2, 6, 10$, $k = 1$, $J = 6$ for Eq (5.2).

Now, the next Table 3 represents the maximum absolute error of Picard iteration method at $n = 2, 4, 6$, $J = 6$ where the zero approximation $\mathcal{Y}_0(x) = 1$ and Figure 4 is the logarithm absolute error graphs of Picard iteration method in the same numbers.

TABLE 3. Picard method maximum absolute error at $J = 6$ for Eq (5.2).

n	2	4	6
E	0.00121358	$8.7738e^{-7}$	$5.9186e^{-10}$
CPU Time	0.0014001	0.0032064	0.0139117

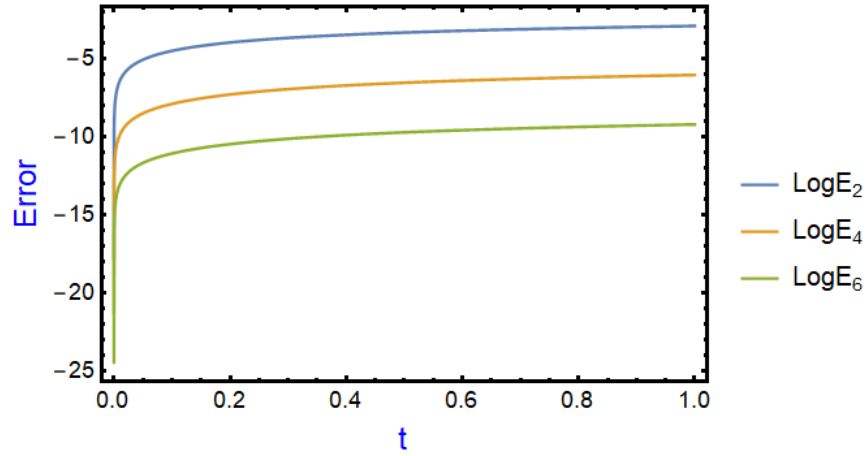


FIGURE 4. Logarithm absolute errors graphs of Picard method at $n = 2, 4, 6$, $J = 6$ for Eq (5.2).

Example 5.2. An examination of the following NQFIE [4]:

$$\mathcal{Y}(x) = \frac{e^x \sin x (2 - x \sinh x)}{2(\sin x + e^x) \cosh x} + \frac{x \mathcal{Y}(x)}{\sin(x) + e^x} \int_0^1 s^2 \tanh(x) \mathcal{Y}(s) ds, \quad x \in [0, 1]. \quad (5.3)$$

The unique solution of Eq.(5.3) is established [4] as $\mathcal{Y}(x) = e^x \sin x$. Applying the Vieta-Lucas wavelets method and Picard iteration method, and comparing these methods and the method in [4], we have the next table.

TABLE 4. Comparing the maximum absolute error for Eq (5.3).

method in[4]		Picard method for $J = 8$		wavelets method for $J = 8$	
$N = 5$	$N = 10$	$N = 5$	$N = 10$	$M = 4$	$M = 10$
$7.342e^{-06}$	$5.516e^{-13}$	$1.93541e^{-04}$	$4.24803e^{-08}$	$9.36292e^{-07}$	$2.57572e^{-14}$

Example 5.3. An examination of the following NQFIE in form **case 2**:

$$\mathcal{Y}(x) = x^2 - \frac{e-1}{450} + \left(\int_0^1 s e^{\mathcal{Y}(s)} ds \right) \left(\int_0^1 \frac{s^2}{25} \mathcal{Y}^3(s) ds \right). \quad (5.4)$$

The exact solution of Eq.(5.4) is $\mathcal{Y}(x) = x^2$. Applying the Picard iteration method. And Vieta-Lucas wavelets method for parameters $J = 6, 8, 10$ and $M = 2, 4$ yields the maximum absolute errors E compiled in the next Tables.

TABLE 5. Vieta-Lucas wavelets method maximum absolute error E for Eq (5.4).

$M = 2$				$M = 4$		
J	6	8	10	6	8	10
E	$1.2505e^{-11}$	$1.8802e^{-15}$	$4.232e^{-16}$	$1.2505e^{-11}$	$1.9082e^{-15}$	$4.1430e^{-16}$
CPU Ti	0.0074718	0.0095706	0.0117185	0.0202526	0.0202526	0.0258019

TABLE 6. Picard method maximum absolute error at $J = 10$ for Eq (5.4).

n	2	4	6	8
E	$2.7302e^{-03}$	$9.3964e^{-08}$	$.2321e^{-11}$	$1.1117e^{-14}$
CPU Time	0.0026911	0.0052776	0.0076317	0.0100275

6. CONCLUSION

In this paper, we establish the existence and uniqueness of NQFIEs under specific conditions in two cases. The last one is a new case (**case 2**). Numerical results and convergence analysis confirm that the Vieta-Lucas wavelet method and the Picard iteration method, combined with the Legendre-Gauss quadrature rule, are accurate and robust solvers for these equations. Furthermore, if the values of M or N and J increase, the maximum absolute error decreases in the two methods.

Generally, the two approach methods converge to the exact solution very rapidly. However, the Vieta-Lucas wavelets method is faster than the Picard iteration method.

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