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CRITICAL POINT THEORY FOR THE p-LAPLACIAN PROBLEMS

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Abstract. In this article we establish the existence of weak solutions for a p-Laplacian differential equation on the half-line under Dirichlet boundary conditions. The proof is based on critical point theory.

Keywords. p-Laplacian operator, differential equations, half-line, mountain pass theorem, minimization principle, critical point theory.
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1. Introduction and preliminaries

We consider the following p-Laplacian boundary value problem posed on the half-line for $p \in [2, +\infty)$,

$$\begin{cases} -\left(|u'(t)|^{p-2}u'(t)\right)' + |u(t)|^{p-2}u(t) &= \lambda f(t, u(t)),\ t \in (0 + \infty), \\ u(0) = u(+\infty) &= 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$, and $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e.

$$\begin{cases} \text{i) the function } t \mapsto f(t, u) \text{ is measurable, for each } u \in \mathbb{R}, \\ \text{ii) the function } u \mapsto f(t, u) \text{ is continuous, for a.e. } t \in (0, +\infty). \end{cases}$$

The study of nonlinear differential equations driven by the p-Laplacian operator has attracted increasing attention due to their mathematical richness and relevance in modeling real-world phenomena. A central focus of current research lies in understanding the existence and multiplicity of solutions. In this context, variational methods and critical point theory have proven to be powerful tools, enabling significant advances in recent years. see [4],[7],[2], [15].
This paper is motivated by the work of Galewski [10], in which the existence of solutions was established for the following problem on the half-line:

$$\begin{cases} -u''(t) + u(t) = \lambda q(t) f(t, u(t)), \quad t \in (0; +\infty), \\ u(0) = u(+\infty) = 0. \end{cases}$$

Here, we generalize the differential operator to the p-Laplacian case and study the existence of solutions for problem (1.1). Using a combination of the mountain

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pass theorem and a minimization principle, we obtain new existence results that extend and refine those available in the literature.

Let us consider the space X defined by

$$X = \Big\{ u \in W_0^{1,p}(0, +\infty),\ u(0) = 0 \Big\},$$

which is equipped with the following norm

$$\|u\| = \Big(\int_0^{+\infty} |u'(t)|^p \mathrm{d}t + \int_0^{+\infty} |u(t)|^p \mathrm{d}t \Big)^{\frac{1}{p}}.$$

Note that if $u \in X$, then $u(+\infty) = 0$ see [3].
And we consider the following spaces :

$$\mathcal{C}_s[0, +\infty) = \{ u \in C([0, +\infty), \mathbb{R}) : \lim_{t \to +\infty} s(t)u(t) \text{ exists } \}$$

endowed with the norm

$$\|u\|_\infty = \sup_{t \in [0, +\infty)} s(t)|u(t)|,$$

where $s\colon [0, +\infty) \to (0, +\infty)$ be continuous and bounded, with $s \in W^{1,q}(0, +\infty)$, and

$$K = \max(\|s\|_{L^q}, \|s'\|_{L^q}),$$

such that $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 1.1. Let X be a Banach space. A functional $J\colon \Omega \to \mathbb{R}$ is called coercive if for every sequence $(x_n)_{n \in \mathbb{N}} \subset X$,

$$\|x_n\| \to +\infty \Longrightarrow J(x_n) \to +\infty.$$

Definition 1.2. Let X be a Banach space. A functional $J\colon X \to (-\infty, +\infty]$ is said to be sequentially weakly lower semi-continuous (*swlsc* for short) if

$$J(x) \le \liminf_{n \to +\infty} J(x_n)$$

as $x_n \rightharpoonup x$ in X, $n \to \infty$.

Lemma 1.3. [1] *Let X be a reflexive Banach space and let $J : X \to \mathbb{R} \cup \{+\infty\}$ be a functional satisfying:*

(1) *Coercivity:* $\lim_{\|x\| \to +\infty} J(x) = +\infty$.
(2) *Sequential weak lower semicontinuity: For every sequence $(x_n) \subset X$ converging weakly to $x \in X$, we have $J(x) \le \liminf_{n \to \infty} J(x_n)$.*

Then J is bounded from below and attains its infimum.

Lemma 1.4. [3] *Let X be a Banach space, $B \subseteq X$ a nonempty, weakly compact set, and $J : B \to \mathbb{R}$ a weakly lower semicontinuous functional. Then J attains its minimum on B; i.e., there exists $x_0 \in B$ such that*

$$J(x_0) = \min_{x \in B} J(x).$$

Definition 1.5. Let X be a real Banach space and let $J : X \to \mathbb{R}$ be a C^1-functional. A sequence $\{x_n\} \subset X$ is called a Palais-Smale sequence for J if:

(1) $(J(x_n))$ is bounded,

(2) $\|J'(x_n)\|_{X^*} \to 0 \quad$ as $n \to +\infty$.

Definition 1.6. The C^1-functional J is said to satisfy the Palais-Smale condition if every Palais-Smale sequence for J possesses a convergent subsequence in the norm topology of X ((PS) condition for brevity).

Lemma 1.7. [12], [14] *Let X be a Banach space, and let $J \in C^1(X, \mathbb{R})$ satisfy $J(0) = 0$. Assume that J satisfies (PS) condition and there exist positive numbers ρ and α such that*
(1) $J(x) \geq \alpha$ if $\|x\| = \rho$,
(2) there exists $x_0 \in X$ such that $\|x_0\| > \rho$ and $J(x_0) < \alpha$.
Then there exists a critical point. It is characterized by

$$J'(x) = 0, \;\; J(x) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \, \gamma(0) = 0, \, \gamma(1) = x_0\}.$$

We need also the following Corduneanu compactness criterion.

Lemma 1.8. [5] *Let D be a bounded set in $C_r[0, +\infty)$. We say that D is relatively compact if the following conditions true:*
(a) D is equicontinuous on any compact sub-interval I in $[0, +\infty)$, i.e.

$$\begin{gathered} \forall I \subset [0, +\infty) \;\; \textit{compact}, \forall \, \varepsilon > 0, \, \exists \, \delta > 0, \, \forall \, t_1, t_2 \in I : \\ |t_1 - t_2| < \delta \Longrightarrow |r(t_1)u(t_1) - r(t_2)u(t_2)| \leq \varepsilon, \forall \, u \in D, \end{gathered}$$

(b) D is equiconvergent at $+\infty$ i.e.,

$$\forall \varepsilon > 0, \exists \, T(\varepsilon) > 0 \;\textit{such that}\; \forall t : t \geq T(\varepsilon) \Longrightarrow |r(t)u(t) - (ru)(+\infty)| \leq \varepsilon, \, \forall \, u \in D,$$

2. Main results

2.1. **Some embedding results.**

Lemma 2.1. *X is continuously embedded in $C_s[0, +\infty)$.*

Proof. For $u \in X$, by using Hölder's inequality, we have

$$
\begin{aligned}
|s(t)u(t)| &= |s(+\infty)u(+\infty) - s(t)u(t)| \\
&= \left| \int_t^{+\infty} (su)'(\theta)d\theta \right| \\
&\leq \left| \int_t^{+\infty} s'(\theta)u(\theta)d\theta \right| + \left| \int_t^{+\infty} s(\theta)u'(\theta)d\theta \right| \\
&\leq \left(\int_0^{+\infty} s'^q(\theta)d\theta \right)^{\frac{1}{q}} \left(\int_0^{+\infty} |u(\theta)|^p d\theta \right)^{\frac{1}{p}} \\
&+ \left(\int_0^{+\infty} s^q(\theta)d\theta \right)^{\frac{1}{q}} \left(\int_0^{+\infty} |u'(\theta)|^p d\theta \right)^{\frac{1}{p}} \\
&\leq \left(\int_0^{+\infty} s'^q(\theta)d\theta \right)^{\frac{1}{q}} \|u\| + \left(\int_0^{+\infty} s^q(\theta)d\theta \right)^{\frac{1}{q}} \|u\| \\
&\leq \max(\|s'\|_{L^q}, \|s\|_{L^q})\|u\|. \\
&\leq K\|u\|,
\end{aligned}
$$

then $\|u\|_\infty \leq K\|u\|$. ☐

The following compact embedding is a fundamental result.

Lemma 2.2. *X embeds compactly into $C_s[0, +\infty)$.*

Proof. Let D be a bounded set in X, then by using Lemma 2.1, D is bounded in $C_s[0, +\infty)$, such that for all $u \in D$, $\|u\| \leq R, (R > 0)$. We will apply Lemma 1.8.
(a) D is equicontinuous on every compact interval I in $[0, +\infty)$.
Let $u \in D$ and $t_1, t_2 \in I$. For $u \in X$, and $t_2 \geq t_1 \geq 0$, from Hölder's inequality, we have

$$
\begin{aligned}
|r(t_2)u(t_2) - r(t_1)u(t_1)| &= \left| \int_{t_1}^{t_2} (ru)'(\tau)d\tau \right| \\
&\leq \left| \int_{t_1}^{t_2} r'(\tau)u(\tau)d\tau \right| + \left| \int_{t_1}^{t_2} r(\tau)u'(\tau)d\tau \right| \\
&\leq \left(\int_{t_1}^{t_2} |r'(\tau)|^q d\tau \right)^{\frac{1}{q}} \left(\int_{t_1}^{t_2} |u(\tau)|^p d\tau \right)^{\frac{1}{p}} \qquad (2.1) \\
&+ \left(\int_{t_1}^{t_2} |r(\tau)|^q d\tau \right)^{\frac{1}{q}} \left(\int_{t_1}^{t_2} |u'(\tau)|^p d\tau \right)^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
|r(t_2)u(t_2)-r(t_1)u(t_1)| &\leq \left(\int_{t_1}^{t_2} |r'(\tau)|^q d\tau\right)^{\frac{1}{q}} \|u\| + \left(\int_{t_1}^{t_2} |r(\tau)|^q d\tau\right)^{\frac{1}{q}} \|u\|, \\
&\leq \max\left[\left(\int_{t_2}^{t_1} r'^q(\tau)d\tau\right)^{\frac{1}{q}}, \left(\int_{t_2}^{t_1} r^q(\tau)d\tau\right)^{\frac{1}{q}}\right]\|u\| \\
&\leq R\max\left[\left(\int_{t_2}^{t_1} r'^q(\tau)d\tau\right)^{\frac{1}{q}}, \left(\int_{t_2}^{t_1} r^q(\tau)d\tau\right)^{\frac{1}{q}}\right] \to 0,
\end{aligned}
\tag{2.2}
$$

as $|t_1 - t_2|$ go to 0.

(b) D is equiconvergent at $+\infty$.

For $t \in [0,+\infty)$ and $u \in D$, using that $(ru)(+\infty) = 0, (ru')(+\infty) = 0$ (note that $u(\infty) = 0, u'(\infty) = 0$ and r is bounded) and by using the Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
|(ru)(t)-(ru)(+\infty)| &= \left|\int_t^{+\infty} (ru)'(\tau)d\tau\right| \\
&= \left|\int_t^{+\infty} r'(\tau)u(\tau)d\tau + \int_t^{+\infty} u'(\tau)r(\tau)d\tau\right| \\
&\leq \max\left[\left(\int_t^{+\infty} r'^q(\tau)d\tau\right)^{\frac{1}{q}}, \left(\int_t^{+\infty} r^q(\tau)d\tau\right)^{\frac{1}{q}}\right]\|u\| \\
&\leq R\max\left[\left(\int_t^{+\infty} r'^q(\tau)d\tau\right)^{\frac{1}{q}}, \left(\int_t^{+\infty} r^q(\tau)d\tau\right)^{\frac{1}{q}}\right] \to 0,
\end{aligned}
\tag{2.3}
$$

as $t \to +\infty$, □

Lemma 2.3. [3] *[Sobolev embedding]*
The space X is continuously embedded into $L^\infty(0,+\infty)$. More precisely, there exists a constant $L > 0$ such that for every $u \in X$,

$$\|u\|_{L^\infty(0,+\infty)} \leq L\,\|u\|.$$

2.2. **Variational Framework.** Let $v \in X$, and multiply the equation in problem (1.1) by v and integrate over $(0,+\infty)$, so we get

$$\int_0^{+\infty} \left[-\left(|u'(t)|^{p-2}u'(t)\right)' + |u(t)|^{p-2}u(t)\right] v(t)\mathrm{d}t = \int_0^{+\infty} \lambda f(t,u(t))v(t)\mathrm{d}t,$$

we have

$$\int_0^{+\infty} \left[|u'(t)|^{p-2}u'(t)v'(t) + |u(t)|^{p-2}u(t)v(t)\right] \mathrm{d}t = \lambda \int_0^{+\infty} f(t,u(t))v(t)\mathrm{d}t.$$

Definition 2.4. We say that a function $u \in X$ is a weak solution of problem (1.1) if

$$\int_0^{+\infty} \left[|u'(t)|^{p-2}u'(t)v'(t) + |u(t)|^{p-2}u(t)v(t)\right] \mathrm{d}t = \lambda \int_0^{+\infty} f(t,u(t))v(t)\mathrm{d}t.$$

for all $v \in X$.

To study problem (1.1), we introduce the functional J defined as follows

$$\begin{cases} J\colon X & \to \quad \mathbb{R} \\ \quad u & \longmapsto \quad J(u) = \frac{1}{p}\|u\|^p - \lambda \displaystyle\int_0^{+\infty} F(t,u(t))\mathrm{d}t, \end{cases}$$

where

$$F(t,u) = \int_0^u f(t,y)dy.$$

If the functional J is Fréchet differentiable at the point $u \in X$ then

$$(J'(u),v) \;=\; \int_0^{+\infty} \left[|u'(t)|^{p-2}u'(t)v'(t) + |u(t)|^{p-2}u(t)v(t)\right] \mathrm{d}t - \lambda \int_0^{+\infty} f(t,u(t))v(t)dt. \tag{2.4}$$

Remark 2.5. If you find a function $u \in X$ such that $J'(u) = 0$, then u is a weak solution of problem (1.1).

2.3. **The sublinear case.**

Theorem 2.6. *Suppose that the following conditions hold:*

- $(\mathbf{H}_1)$ *there exist positive functions $a, b : [0,+\infty) \to (0,+\infty)$ with $a, b \in L^1(0,+\infty) \cap L^{\frac{p}{p-1}}(0,+\infty)$ and $\sigma > 0$ such that*
$$|f(t,u)| \le a(t)|u|^\sigma + b(t), \quad \text{for a.e. } t \in [0,+\infty) \text{ and all } u \in \mathbb{R},$$
- $(\mathbf{H}_2)$ *there exist functions $c_1, c_2 : [0,+\infty) \to (0,+\infty)$ with $c_1,\ c_2 \in L^1(0,+\infty)$, and $\mu > p$ such that*
  - (a) *$F(t,u) \ge c_1(t)|u|^\mu - c_2(t)$, for a.e. $t \ge 0$ and all $u \in \mathbb{R}$,*
  - (b) *$\mu F(t,u) \le uf(t,u)$, for a.e. $t \ge 0$ and all $u \in \mathbb{R}\backslash\{0\}$,*
- $(\mathbf{H}_3)$ $\lim_{u\to 0} \frac{f(t,u)}{|u|^{p-1}} = 0$ *uniformly for a.e. $t \in [0,+\infty)$,*
- $(\mathbf{H}_4)$ *the function $u \longmapsto F(t,u)$ is convex on $\mathbb{R}$ for a.e. $t \in [0,\ +\infty)$,*
- $(\mathbf{H}_5)$ *for any constant $\xi > 0$ there exists a nonnegative function h_ξ for which $\frac{1}{s}h_\xi \in L^1(0,+\infty)$ such that*
$$\sup_{|y|\le\xi} \left| f\left(t, \frac{y}{s(t)}\right)\right| \le h_\xi(t) \quad \text{for a.e. } t \in [0,+\infty).$$

Then problem (1.1) has at least one weak solution nontrivial.

Proof.
Step 1. The functional J is well defined.
Let $u \in X$. Under assumptions (H_1), we have that

$$|F(t,u(t))| \le \frac{a(t)}{\sigma+1}|u(t)|^{\sigma+1} + b(t)|u(t)|.$$

Hence

$$\begin{aligned}\left|\int_0^{+\infty} F(t,u(t))dt\right| &\leq \frac{1}{\sigma+1}\int_0^{+\infty}\frac{a(t)}{s^{\sigma+1}(t)}|s(t)u(t)|^{\sigma+1}dt+\int_0^{+\infty}\frac{b(t)}{s(t)}|s(t)u(t)|dt\\ &\leq \frac{\|u\|_\infty^{\sigma+1}}{\sigma+1}\int_0^{+\infty}\frac{a(t)}{s^{\sigma+1}(t)}dt+\|u\|_\infty\int_0^{+\infty}\frac{b(t)}{s(t)}dt\\ &\leq \frac{K^{\sigma+1}}{\sigma+1}\|u\|^{\sigma+1}\|\frac{a}{s^{\sigma+1}}\|_{L^1}+K\|u\|\|\frac{b}{s}\|_{L^1}\end{aligned}$$

Then

$$\begin{aligned}|J(u)| &\leq \frac{1}{p}\|u\|^p+\lambda\frac{K^{\sigma+1}}{\sigma+1}\|u\|^{\sigma+1}\|\frac{a}{s^{\sigma+1}}\|_{L^1}+\lambda K\|u\|\|\frac{b}{s}\|_{L^1}\\ &< \infty.\end{aligned}$$

step 2 J satisfies the Palais–Smale condition.

Consider a sequence $(u_n)\subset X$ such that $(J(u_n))$ is bounded and $J'(u_n)\to 0$, as $n\to\infty$. We will prove that (u_n) admits a convergent subsequence.
Since $J'(u_n)\to 0$, for any given $\epsilon>0$ there exists n_0 such that $\|J'(u_n)\|\leq\epsilon$ for all $n\geq n_0$. Then, we have

$$|\langle J'(u_n),\ u_n\rangle|\leq\epsilon\|u_n\|,\ \forall n\geq n_0.$$

Moreover, a straightforward calculation yields

$$\langle J'(u_n),u_n\rangle=\int_0^{+\infty}|u_n'(t)|^p\,dt+\int_0^{+\infty}|u_n(t)|^p\,dt-\lambda\int_0^{+\infty}f(t,u_n(t))u_n(t)dt.$$

Now we estimate by $(\mathbf{H}_2)(\mathbf{b})$ that

$$\begin{aligned}-\lambda\int_0^{+\infty}F(t,u_n(t))dt &\geq \frac{-\lambda}{\mu}\left(\int_0^{+\infty}f(t,u_n(t))u_n(t)dt\right)\\ &=\frac{1}{\mu}\langle J'(u_n),u_n\rangle-\frac{1}{\mu}\int_0^{+\infty}\left(u_n'(t)\right)^p dt-\frac{1}{\mu}\int_0^{+\infty}\left(u_n(t)\right)^p dt\\ &\geq\frac{-\epsilon}{\mu}\|u_n\|-\frac{1}{\mu}\|u_n\|^p.\end{aligned}$$

The boundedness of $(J(u_n))$ yields a constant C with $|J(u_n)|\leq C$ for all $n\in\mathbb{N}$, Thus we obtain

$$C-\frac{1}{p}\|u_n\|^p\geq\frac{-\epsilon}{\mu}\|u_n\|-\frac{1}{\mu}\|u_n\|^p$$

which results in

$$C\geq\left(\frac{\mu-p}{p\mu}\right)\|u_n\|^p-\frac{\epsilon}{p}\|u_n\|.$$

Since $\mu>p$, (u_n) is bounded in X i.e., there is some $M_p>0$ such that $\|u_n\|\leq M_p$, for $n\in\mathbb{N}$.

Next, we prove that (u_n) converges strongly to some $u\in X$. Since (u_n) is bounded in X, we can extract a subsequence, still denoted by (u_n), that converges weakly to some $u\in X$ with $\|u\|\leq M_p$. As already noted, by Lemma 2.2.
If (u_n) converges to u in $C_s[0,+\infty)$ and $s(t)>0$ for all $t\in[0,+\infty)$, then it follows that $u_n(t)\to u(t)$ for every $t\in[0,+\infty)$. Since f is a Carathéodory function, we

have $f(t,u_n(t)) \to f(t,u(t))$ as $n \to +\infty$ for almost every $t \in [0,+\infty)$. Then, using (H_1), we obtain

$$\begin{aligned} |f(t,u_n(t))| &\leq a(t)|u_n(t)|^\sigma + b(t) \\ &\leq a(t)\|u_n\|^\sigma_{L^\infty} + b(t) \leq L^\sigma a(t)\|u_n\|^\sigma + b(t) \\ &\leq M_p^\sigma L^\sigma a(t) + b(t), \end{aligned} \tag{2.5}$$

and since $a \in L^1(0,+\infty)$, $\;b \in L^1(0,+\infty)$, we see also that

$$M_p^\sigma L^\sigma a + b \in L^1(0,+\infty).$$

By the Lebesgue dominated convergence theorem we now have

$$\lim_{n\to+\infty} \int_0^\infty f(t,u_n(t))dt = \int_0^\infty f(t,u(t))dt.$$

it follows that

$$\lim_{n\to+\infty} \int_0^{+\infty} \left(f(t,u_n(t)) - f(t,u(t))\right)\left(u_n(t) - u(t)\right) \mathrm{d}t = 0. \tag{2.6}$$

According to (2.4), we get

$$\begin{aligned} \langle J'(u_n) - J'(u), u_n - u\rangle &= \int_0^{+\infty} (|u_n'(t)|^{p-2}u_n'(t) - |u'(t)|^{p-2}u'(t))(u_n'(t) - u'(t))\mathrm{d}t \\ &\quad + \int_0^{+\infty} (|u_n(t)|^{p-2}u_n(t) - |u(t)|^{p-2}u(t))(u_n(t) - u(t))\mathrm{d}t \\ &\quad -\lambda \int_0^{+\infty} \left(f(t,u_n(t)) - f(t,u(t))\right)\left(u_n(t) - u(t)\right)\mathrm{d}t. \end{aligned} \tag{2.7}$$

if $p \geq 2$ there exists $\theta_p > 0$ (see [13]) such that,

$$\left(|x|^{p-2}x - |y|^{p-2}y\right)\left(x - y\right) \geq \theta_p |x-y|^p, (x,y) \in \mathbb{R}.$$

Therefore, we get

$$\begin{aligned} \langle J'(u_n) - J'(u), u_n - u\rangle &\geq \theta_p \|u_n - u\|^p \\ &\quad -\lambda \int_0^{+\infty} \left(f(t,u_n(t)) - f(t,u(t))\right)\left(u_n(t) - u(t)\right)\mathrm{d}t. \end{aligned} \tag{2.8}$$

Since $\lim_{n\to+\infty} J'(u_k) = 0$ and (2.6) , (2.8), we see that $\lim_{n\to+\infty}\|u_n - u\| = 0$.
step 3 J satisfies the first geometric condition of Lemma 1.7.
Let us fix $\lambda > 0$ and let

$$0 < \epsilon \leq \frac{1}{\lambda}.$$

From $(\mathbf{H}_3)$ there exists $\delta > 0$ such that $|f(t,x)| \leq \epsilon |x|^{p-1}$ whenever $|x| \leq \delta$.

$$\begin{aligned} \int_0^\infty |F(t,u(t))|dt &\leq \frac{\epsilon}{p}\int_0^\infty |u(t)|^p dt \\ &\leq \frac{\epsilon}{p}\|u\|^p \end{aligned}$$

Let $0 < \rho \leq \frac{\delta}{K}$. Then for $\|u\| = \rho$, we have

$$\begin{aligned} J(u) &= \frac{1}{p}\|u\|^p - \lambda \int_0^{+\infty} F(t, u(t))\mathrm{d}t \\ &\geq \frac{1}{p}(1-\epsilon\lambda)\|u\|^p \\ &\geq \alpha, \end{aligned}$$

such that $\alpha = \frac{1}{p}(1-\epsilon\lambda)\rho^p$, therefore condition (1) in lemma 1.7 is verified.

step 4 J satisfies the second geometric condition of Lemma 1.7.

By $(\mathbf{H}_2)(\mathbf{a})$, for some $v_0 \in X$, and $\xi \in \mathbb{R}$ we have

$$\begin{aligned} J(\xi v_0) &= \tfrac{1}{p}|\xi|^p\|v_0\|^p - \lambda \int_0^{+\infty} F(t, \xi v_0(t))\mathrm{d}t \\ &\leq \tfrac{1}{p}|\xi|^p\|v_0\|^p - \lambda|\xi|^\mu \int_0^{+\infty} c_1(t)|v_0(t)|^\mu \mathrm{d}t + \int_0^{+\infty} c_2(t)\mathrm{d}t. \end{aligned}$$

Now, since $\mu > p$, for $u_0 = \xi v_0$ we have $J(u_0) \leq 0$ for sufficiently large ξ (i.e., as $\xi \to +\infty$). Then, by Lemma 1.7, J admits a critical point, which is a nontrivial weak solution of problem (1.1). $\square$

Remark 2.7. Assume that $(\mathbf{H}_5)$ holds. The functional J is well-defined and $\mathcal{C}^1$ on X . see [10]

Example 2.8.

$$\begin{cases} -\left(|u'(t)|^{p-2}u'(t)\right)' + |u(t)|^{p-2}u(t) = \lambda u^7(t)\left(\dfrac{5|u(t)|+7|u(t)|\ln(2|u(t)|+9)}{(2|u(t)|+5)^2}\right)(|\cos(3t)|+7) \\ u(0) = u(+\infty) = 0. \end{cases} \tag{2.9}$$

It can be easily checked that all conditions of Theorem 2.6 are satisfied with

$$f(t,u) = u^7(t)\left(\frac{5|u(t)|+7|u(t)|\ln(2|u(t)|+9)}{(2|u(t)|+5)^2}\right)(|\cos(3t)|+7).$$

Therefore problem (2.9) has at least one nontrivial solution for any $\lambda > 0$.

2.4. **A critical point exists on a closed ball.** Let E be a real reflexive Banach space, and let $\Phi, H : E \to \mathbb{R}$ be two convex functionals that are continuously Fréchet differentiable, with derivatives $\varphi, h : E \to E^*$ respectively; i.e., $\Phi' = \varphi$ and $H' = h$. Consider the problem

$$\varphi(u) = h(u), \quad u \in E. \tag{2.10}$$

Define the functional $J : E \to \mathbb{R}$ associated with (2.10) by $J(u) = \Phi(u) - H(u)$, (see [9]).

Theorem 2.9. [9] *Let E be an infinite-dimensional reflexive Banach space.*

(i) *Let $A \subset E$ and suppose there exist $u_0, v \in A$ such that $\varphi(v) = h(u_0)$ and*

$$J(u_0) = \inf_{u \in A} J(u).$$

Then u_0 is a critical point of J, and consequently solves (2.10).

We have the following result

Theorem 2.10. *Suppose that* ($\boldsymbol{H}_1$),($\boldsymbol{H}_4$) *and* ($\boldsymbol{H}_5$) *hold. Under the assumption that* $f(t,0)\neq 0$ *on some subset of* $[0,+\infty)$*, there exists a constant* $\lambda^*>0$ *such that for all* $\lambda\in(0,\lambda^*)$*, problem* (1.1) *possesses at least one nontrivial solution.*

Proof. Set $B:=\{u\in X:\|u\|\leq r\}$, the closed ball of radius r centered at 0. By (2.5) and Lemma 2.3, we obtain the following estimate for any $u\in B$

$$|f(t,u_n(t))|\leq r^\sigma L^\sigma a(t)+b(t),\ \ for\ a.e.\ t\in[0,+\infty).$$

Define $d:=L^\sigma r^\sigma\|a\|_{L^{\frac{p}{p-1}}}+\|b\|_{L^{\frac{p}{p-1}}}$, then we see that for any $v\in X$ by the Hölder's inequality

$$\begin{aligned}\int_0^{+\infty}|f(t,u(t))v(t)|\,dt &\leq L^\sigma r^\sigma\int_0^{+\infty}a\left(t\right)v(t)dt+\int_0^{+\infty}b(t)v(t)dt\\ &\leq \left(L^\sigma r^\sigma\|a\|_{L^{\frac{p}{p-1}}}+\|b\|_{L^{\frac{p}{p-1}}}\right)\|v\|_{L^p}\\ &\leq d\left(\|v\|_{L^p}^p+\|v'\|_{L^p}^p\right)^{\frac{1}{p}}=d\|v\|,\end{aligned}\tag{2.11}$$

we put $\lambda^*=\frac{r^{p-1}}{d}$ and fix $\lambda\in(0,\,\lambda^*)$.
Since the zero function is not a solution to the problem under consideration, any solution we obtain is necessarily nontrivial.
We put

$$J_1\left(u\right)=\frac{1}{p}\int_0^{+\infty}|u'(t)|^pdt+\frac{1}{p}\int_0^{+\infty}|u(t)|^pdt=\frac{1}{p}\,\|u\|^p,$$

and

$$J_2\left(u\right)=\int_0^{+\infty}F(t,u(t))dt,$$

then $J=J_1-\lambda J_2$. Observe that J_1 is weakly lower semicontinuous and J_2 is weakly continuous. Hence J is weakly lower semicontinuous. Moreover, B is weakly compact, so by Lemma 1.4, for every $\lambda>0$ there exists a minimizer $u_0\in B$ of J.
We shall apply Theorem 2.9. Define $\Phi,H:X\to\mathbb{R}$ by the formulas

$$\Phi(u)=\int_0^{+\infty}\frac{1}{p}|u'(t)|^p+\int_0^{+\infty}\frac{1}{p}|u(t)|^p,\qquad H(u)=\lambda\int_0^{+\infty}F(t,u(t))dt,$$

and observe that these are convex C^1 functionals. Now consider the auxiliary Dirichlet problem

$$\begin{cases}-\left(|u'(t)|^{p-2}u'(t)\right)'+|u(t)|^{p-2}u(t) &= \lambda f(t,u_0(t)),\, t\in(0+\infty),\\ u(0)=u(+\infty) &= 0.\end{cases}\tag{2.12}$$

Problem (2.12) admits a unique solution for some $v\in X$. This follows by considering the following associated functional,

$$J_0(u)=\frac{1}{p}\int_0^{+\infty}|(u'(t)|^p+|u(t)|^p\,dt-\lambda\int_0^{+\infty}f(t,u_0(t))u(t)dt,$$

and proving that it is coercive, C^1, weakly lower semicontinuous, and strictly convex. Then the direct method in the calculus of variations (see [11]) yields exactly one solution to (2.12).

We shall prove that $v \in B$. To do so, take equation (2.12) with $u = v$, multiply it by v, and integrate by parts. This yields

$$\int_0^{+\infty} |v'(t)|^p \, dt + \int_0^{+\infty} |v(t)|^p dt = \lambda \int_0^{+\infty} f(t, u_0(t)) v(t) dt.$$

Using (2.11) we have

$$\begin{aligned} \|v\|^p &\leq \lambda d \|v\| \\ &\leq \lambda^* d \|v\|, \end{aligned} \tag{2.13}$$

then

$$\begin{aligned} \|v\|^{p-1} &\leq \lambda^* d \\ &\leq r^{p-1}. \end{aligned} \tag{2.14}$$

From the above we obtain $\|v\| \leq r$, which implies $v \in B$. Hence Theorem 2.11 applies. $\square$

Example 2.11.

$$\begin{cases} -\left(|u'(t)|^{p-2}u'(t)\right)' + |u(t)|^{p-2}u(t) = \lambda u^7(t) \left(\dfrac{5|u(t)| + 7|u(t)| \ln(2|u(t)| + 9)}{(2|u(t)| + 5)^2} \right) (|\cos(3t)| + 7) \\ +\lambda \left(|\sin(2t)| + 2\right), \\ u(0) = u(+\infty) = 0. \end{cases} \tag{2.15}$$

It can be easily verified that for the function,

$$f(t,u) = u^7(t) \left(\frac{5|u(t)| + 7|u(t)| \ln(2|u(t)| + 9)}{(2|u(t)| + 5)^2} \right) (|\cos(3t)| + 7) + \left(|\sin(2t)| + 2\right),$$

all conditions of Theorem 2.10 are satisfied. Consequently, there exists $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$ problem (2.15) admits at least one nontrivial solution.

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