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NORMAL FAMILIES OF MEROMORPHIC FUNCTIONS CONCERNING SHARED VALUES

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Abstract. Let $a(\neq 0), b \in \mathbb{C}$ and $k \in \mathbb{Z}^+$. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D such that each f in $\mathcal{F}$ has only zeros of multiplicity at least $k+2$. For each f in $\mathcal{F}$, if $f + a(f^{(k)})^{k+1} - b$ has at most 1 zero in D, then $\mathcal{F}$ is normal in D. We also give some examples to illustrate that some restricted conditions of our main results are necessary.

1. Introductions and main results

Let D be a domain in $\mathbb{C}$, and $\mathcal{F}$ be a family of meromorphic functions defined in the domain D. $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if for every sequence $\{f_n\}_{n=1}^{\infty}$ contains a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ such that f_{n_j} converges spherically uniformly to a meromorphic function $f(z)$ or ∞.

Let f and g be two meromorphic functions in D, and let $a \in \mathbb{C}$. If f and g have the same a-points(ignoring multiplicity), then we say that f and g share the value a IM.

In 1994 and 2008, Ye[8] and Fang and Zalcman[4] considered a similar problem related the Hayman's conjecture and proved the following result.

Theorem 1.1. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D such that $f \in \mathcal{F}$ has only multiple zeros, let $n \geq 2$ be an integer and $a(\neq 0), b$ be two finite numbers. If for each $f \in \mathcal{F}$,$f + a(f')^n \neq b$, then $\mathcal{F}$ is normal.*

In 2009, Xu et al. extended above theorem, and proved the following result.

Theorem 1.2. [7] *Let f be a transcendental meromorphic function,let n and k be two positive integers such that $n \geq k+1$ and a be a nonzero complex number. Then $f + a(f^{(k)})^n$ assumes every complex value infinitely often.*

Theorem 1.3. [7] *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D such that $f \in \mathcal{F}$ has only zeros of multiplicity at least $k+1$, let n and k be two positive integers such that $n \geq k+1$ and $a(\neq 0), b$ be two finite numbers. If for each $f \in \mathcal{F}$, $f + a(f^{(k)})^n \neq b$, then $\mathcal{F}$ is normal in D.*

In 2013, Chen et al. obtained the following theorem.

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Theorem 1.4. [3] *Let $a(\neq 0), b \in \mathbb{C}$ and n and k be two positive integers such that $n \geq k+2$. Let $\mathcal{F}$ be a family of mreomorphic functions in the plane domain D such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least $k+1$. For each $f, g \in \mathcal{F}$, if $f + a(f^{(k)})^n$ and $g + a(g^{(k)})^n$ share b IM in D, then $\mathcal{F}$ is normal in D.*

Naturally, we pose the following question:
Is the conclusion of Theorem 1.4 also true for $n = k+1$.
First, we give the following counterexample.

Example 1.5. Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_m(z)\}$, where

$$f_m(z) = mz^{k+1},\ z \in D\ , m = 1, 2, \cdots .$$

then

$$f + a(f^{(k)})^{k+1} = [m + a((k+1)!m)^{k+1}]z^{k+1}.$$

Clearly, for each pair $j, l \in \mathbb{Z}^+$, $f_j + a(f_j^{(k)})^{k+1}$ and $f_l + a(f_l^{(k)})^{k+1}$ share the value 0 in D, however, $\mathcal{F}$ is not normal in D since

$$f_m^{\#}(m^{-\frac{1}{k+1}}) = \frac{|f^{(k)}(m^{-\frac{1}{k+1}})|}{1 + |f(m^{-\frac{1}{k+1}})|^{k+1}} = \frac{(k+1)!}{2} m^{\frac{k}{k+1}} \to \infty \ \ as\ m \to \infty,$$

This suggests that some further investigation is necessary for the case $n = k+1$. In the paper, we take up this problem and prove the following results.

Theorem 1.6. *Let $a(\neq 0), b \in \mathbb{C}$ and $k \in \mathbb{Z}^+$. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least $k+2$. If, for each pair of functions f and g in $\mathcal{F}$,$f + a(f^{(k)})^{k+1}$ and $g + a(g^{(k)})^{k+1}$ share b IM in D,then $\mathcal{F}$ is normal in D.*

Theorem 1.7. *Let $a(\neq 0), b \in \mathbb{C}$ and $k \in \mathbb{Z}^+$. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least $k+2$. If, for each f in $\mathcal{F}$,$f + a(f^{(k)})^{k+1} - b$ has at most 1 zero in D, then $\mathcal{F}$ is normal in D.*

Example 1.8. Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_m(z)\}$, where

$$f_m(z) = mz^{k+2},\ z \in D\ , m = 1, 2, \cdots .$$

then

$$f + a(f^{(k)})^{k+1} = [m + a(\frac{(k+2)!m}{2})^{k+1} z^k]z^{k+2}.$$

Clearly, $f_m + a(f_m^{(k)})^{k+2}$ has $k+1(\geq 2)$ distinct zeros in D, however, $\mathcal{F}$ is not normal in D.

Remark 1.9. Example 1.5 shows that the condition that all zeros of f have multiplicity at least $k+2$ is necessary for Theorem 1.6. Example 1.8 illustrates that the condition that $f + a(f^{(k)})^{k+1} - b$ has at most 1 zero is best possible for Theorem 1.7.

2. Some lemmas

In order to prove the Theorem 1.6-1.7, we need the following lemmas.

Lemma 2.1. [6] *Let $\mathcal{F}$ be a family of functions meromorphic in the unit disc, all of whose zeros have multiplicity at least k. Then if $\mathcal{F}$ is not normal, there exist, for each α, $0 \leq \alpha < k$,(i)points z_n, $z_n \to z_0$,$z_0 \in \Delta$;(ii)functions $f_n \in \mathcal{F}$; and (iii)positive numbers $\rho_n \to 0^+$, such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function in $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $\mathrm{g}^\sharp(\xi) \leq \mathrm{g}^\sharp(0) = \mathrm{kA} + 1$. Moreover, g has order at most 2.*

Lemma 2.2. [1] *Let f be an entire function on the complex plane $\mathbb{C}$. If the spherical derivative $f^\sharp$ of f is bounded on $\mathbb{C}$, then f is of exponential type.*

Lemma 2.3. [5] *Let f be a non-constant rational function with $f \neq 0$, and $n \geq 2$,k be two positive integers, and $a \neq 0, b$ be two complex constants, then $f + a(f^{(k)})^n - b$ has at least $nk + 1$ distinct zeros.*

Lemma 2.4. *Let f be a non-constant rational function, all of whose zeros in $\mathbb{C}$ have multiplicity at least $k+2$. let k be a positive integer, and $a \neq 0$ be a complex constant. If f has only one zero in $\mathbb{C}$, then $f + a(f^{(k)})^n$ has at least two distinct zeros.*

Proof. We consider the following cases.

Case 1. If f is a non-constant polynomial. Write

$$f(z) = A(z - z_0)^p, \tag{2.1}$$

where $p \geq k+2$ is integer, $A(\neq 0)$ is a constant.

Since $p \geq k+2$, then $k(p-k-1) \geq k$. So we get

$$\begin{aligned} &f(z) + a(f^{(k)}(z))^{k+1} = \\ &\quad A(z - z_0)^p \left\{ 1 + aA^k[p(p-1)\cdots(p-k+1)]^{k+1}(z - z_0)^{k(p-k-1)} \right\}. \end{aligned} \tag{2.2}$$

Then $f + a(f^{(k)})^{k+1}$ has at least two distinct zeros.

Case 2. If f is a non-polynomial rational function. Write

$$f(z) = B\frac{(z-\alpha)^l}{(z-\beta_1)^{m_1}(z-\beta_2)^{m_2}\cdots(z-\beta_t)^{m_t}}, \tag{2.3}$$

Where $B(\neq 0)$ is a constant and $m_i \geq 1 (i = 1, \cdots, t), l \geq k+2$ are positive integers.

We put

$$M = m_1 + m_2 + \cdots + m_t.$$

From (2.3), we get

$$f^{(k)}(z) = \frac{(z-\alpha)^{l-k}}{(z-\beta_1)^{m_1+k}(z-\beta_2)^{m_2+k}\cdots(z-\beta_t)^{m_t+k}} g(z), \tag{2.4}$$

where $g(z) = B(l-M)(l-M-1)\cdots(l-M-k+1)z^{kt} + \cdots + a_0$ is a polynomial.

If $l < M$ or $l \geq M + k$, then

$$\deg(g) = kt. \tag{2.5}$$

If $M \leq l < M + k$, then

$$\deg(g) \leq l - M - 1 + kt < kt. \tag{2.6}$$

Since $l \geq k + 2$, then $k(l - k - 1) \geq k$.
From (2.3) and (2.4) we get

$$f(z) + a(f^{(k)}(z))^{k+1} = \frac{(z-\alpha)^l \{\varphi(z) + \phi(z)\}}{\prod_{i=1}^{t} (z - \beta_i)^{(m_i+k)(k+1)}}, \tag{2.7}$$

where

$$\varphi(z) = B \prod_{i=1}^{t} (z - \beta_i)^{k(m_i+k+1)}, \;\; \phi(z) = a(z-\alpha)^{k(l-k-1)} g^{k+1}(z). \tag{2.8}$$

Then

$$\deg(\varphi) = kM + k(k+1)t, \tag{2.9}$$

Set $\psi(z) = \varphi(z) + \phi(z)$.

If $l < M$, we obtain $\deg(\phi) = kl + k(k+1)(t-1)$, and so $\deg(\phi) < \deg(\varphi)$. This implies $\psi(\xi)$ has at least one zero.

If $l \geq M + k$, we obtain $\deg(\phi) = kl + k(k+1)(t-1)$, and so $\deg(\phi) \geq \deg(\varphi) - k$.This implies $\psi(\xi)$ has at least one zero.

If $M \leq l < M+k$, we obtain $\deg(\phi) \leq \deg(\varphi) + [M - l - (k+1)^2]$. This implies $\deg(\phi) < \deg(\varphi)$. Therefore $\psi(\xi)$ has at least one zero.

Hence $\psi(\xi)$ has at least one zero $\zeta(\neq \alpha)$, say. We get $f + a(f^{(k)})^n$ has at least two distinct zeros.This proves the lemma 4. $\square$

Lemma 2.5. [2] *Let f be a non-constant rational function with $f \neq 0$ and k a positive integer. Then $f^{(k)} - 1$ has at least $k+1$ distinct zeros.*

Lemma 2.6. *Let f be a non-constant rational function, all of whose zeros in $\mathbb{C}$ have multiplicity at least $k+1$. let k and $n \geq 2$ be two positive integers, and $a, b \in \mathbb{C} \setminus \{0\}$. If f has at least one zero in $\mathbb{C}$, then $a(f^{(k)})^n - b$ has at least two distinct zeros.*

Proof. Since f has multiple zeros at least $k+1$, then $f^{(k)}$ is a non-constant rational function. Noting that a non-constant rational function has only one Picard value, so,for $n \geq 3$, it is easy to obtain that $a(f^{(k)})^n - b$ has at least two distinct zeros.

For $n = 2$, suppose $a(f^{(k)})^2 - b$ has only one zero. Set $c = \sqrt{\frac{b}{a}}$, then $(f^{(k)} - c)(f^{(k)} + c) = 0$ has only one zero. Hence $f^{(k)}(z_0) = -c$, $f^{(k)}(z_0) \neq c$. So we get

$$f^{(k)} - c = \frac{1}{P}, f^{(k)} + c = \frac{A(z-z_0)^l}{P}, \tag{2.10}$$

where $A(\neq 0)$ is a constant,$l \in \mathbb{Z}^+$ and P is a non-constant polynomial. Therefore

$$f^{(k)} - c = \frac{2c}{A(z-z_0)^l - 1}, \tag{2.11}$$

Noting $f^{(k)} - c$ has multiple poles, however, $\frac{2c}{A(z-z_0)^l-1}$ has single pole, which is a contradiction. This proves the lemma 6. □

3. Proof of Theorems

Proof. Theorem 1.6. We consider two cases.

Case 1. $b = 0$. Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. Let $\alpha = k+1$. Then by Lemma 2.1, there exists a sequence of complex numbers $z_j \to z_0$, a sequence of functions $f_j \in \mathcal{F}$ and a sequence of positive numbers $\rho_n \to 0^+$ such that $h_j(\xi) = \rho_j^{-k-1} f_j(z_j + \rho_j \xi)$ converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic functions $h(\xi)$ in $\mathbb{C}$. Also the order of h does not exceed two and by Hurwitz's theorem h has no zero of mulitiplicity less than $k+2$.

On every compact subset of $\mathbb{C}$ which contains no poles of h, we get

$$h_j(\xi) + a(h_j^{(k)}(\xi))^{k+1} =$$

$$\rho_j^{-k-1}\left\{f_j(z_j+\rho_j\xi) + a(f_j^{(k)}(z_j+\rho_j\xi))^{k+1}\right\} \to h(\xi) + a(h^{(k)}(\xi))^{k+1}, \tag{3.1}$$

also locally uniformly with respect to the spherical metric.

If $h(\xi) + a(h^{(k)}(\xi))^{k+1} \equiv 0$, by h has only zeros of multiplicity at least k + 2 then h has no poles,and thus h is entire function. By Lemma 2.2, h is of exponential type since $h^\sharp(\xi) \leq M$. It follows that $-\frac{1}{a}\frac{h(\xi)}{h^{(k)}(\xi)} \equiv (h^{(k)}(\xi))^k$, and note that $h^k = (h^{(k)})^k(\frac{h}{h^{(k)}})^k$,then we have

$$\begin{aligned}
k\ m(r,h) &\leq m(r,(h^{(k)})^k) + km(r,\tfrac{h}{h^{(k)}}) + O(1)\\
&\leq m(r,\tfrac{h}{h^{(k)}}) + \log^+\left|\tfrac{1}{a}\right| + (k)m(r,\tfrac{h}{h^{(k)}}) + O(1)\\
&= (k+1)\left[T(r,\tfrac{h^{(k)}}{h}) - N(r,\tfrac{h}{h^{(k)}})\right] + O(1)\\
&= (k+1)\left[N(r,\tfrac{h^{(k)}}{h}) - N(r,\tfrac{h}{h^{(k)}})\right] + S(r,h)\\
&= (k+1)\left[N(r,\tfrac{1}{h}) + N(r,h^{(k)}) - N(r,h) - N(r,\tfrac{1}{h^{(k)}})\right] + S(r,h)\\
&= (k+1)\left[N(r,\tfrac{1}{h}) - N(r,\tfrac{1}{h^{(k)}})\right] + S(r,h)\\
&\leq (k+1)k\overline{N}(r,\tfrac{1}{h}) + S(r,h) \leq \tfrac{(k+1)k}{k+2}N(r,\tfrac{1}{h}) + S(r,h)\\
&\leq \tfrac{(k+1)k}{k+2}T(r,h) + S(r,h).
\end{aligned}$$

Thus

$$k\ T(r,h) = k\ m(r,h) \leq \frac{(k+1)k}{k+2}T(r,h) + S(r,h).$$

This implies

$$[k-\frac{(k+1)k}{k+2}]T(r,h)=\frac{k}{k+2}T(r,h)\le o(\log r)\ \ (r\to\infty).$$

This is

$$T(r,h)=o(\log r)\ \ (r\to\infty),$$

which is impossible. So $h(\xi)+a(h^{(k)}(\xi))^n\not\equiv 0$.

If $h(\xi)+a(h^{(k)}(\xi))^{k+1}\neq 0$, it follows from the zeros of h have multiplicity at least $k+2$ that $h\neq 0$. Then by Theorem 1.2 and Lemma 2.3, we can get a contradiction.

Hence the function $h(\xi)+a(h^{(k)}(\xi))^{k+1}$ has zeros and $h(\xi)+a(h^{(k)}(\xi))^{k+1}\not\equiv 0$.

We claim $h(\xi)+a(h^{(k)}(\xi))^{k+1}$ has only one zero. If this is not true, then $h(\xi)+a(h^{(k)}(\xi))^{k+1}$ would have at least two distinct zeros. Let ξ_0 and ξ_0^* be two distinct zeros of $h(\xi)+a(h^{(k)}(\xi))^{k+1}$. We choose a small $\delta(>0)$ such that $D_1\bigcap D_2=\emptyset$ and $h(\xi)+a(h^{(k)}(\xi))^{k+1}$ has no other zeros in $D_1\bigcup D_2$ apart from ξ_0 and ξ_0^*, where $D_1=\{\xi:|\xi-\xi_0|<\delta\}$ and $D_2=\{\xi:|\xi-\xi_0^*|<\delta\}$.

From (3.1), by Hurwitz's theorem, there exist points $\xi_j\in D_1$ and $\xi_j^*\in D_2$ such that

$$f_j(z_j+\rho_j\xi_j)+a\left(f_j^{(k)}(z_j+\rho_j\xi_j)\right)^{k+1}=0,$$

$$f_j(z_j+\rho_j\xi_j^*)+a\left(f_j^{(k)}(z_j+\rho_j\xi_j^*)\right)^{k+1}=0.$$

By the assumption of Theorem 1.6, we see that for any integer m and for all j we get

$$f_m(z_j+\rho_j\xi_j)+a\left(f_m^{(k)}(z_j+\rho_j\xi_j)\right)^{k+1}=0,$$

$$f_m(z_j+\rho_j\xi_j^*)+a\left(f_m^{(k)}(z_j+\rho_j\xi_j^*)\right)^{k+1}=0.$$

Fix m and take $j\to\infty$, and note $z_j+\rho_j\xi_j\to z_0$, $z_j+\rho_j\xi_j^*\to z_0$, then

$$f_m(z_0)+a\left(f_m^{(k)}(z_0)\right)^{k+1}=0.$$

Since the zeros of $f_m+a(f_m^{(k)})^{k+1}$ has no accumulation point, so for sufficiently large j we get $z_j+\rho_j\xi_j=z_0,\ z_j+\rho_j\xi_j^*=z_0$.

Hence

$$\xi_j=\xi_j^*=\frac{z_0-z_j}{\rho_j},$$

This contradicts the fact that $\xi_j\in D_1$, $\xi_j^*\in D_2$ and $D_1\bigcap D_2=\emptyset$.

Thus $h(\xi)+a(h^{(k)}(\xi))^{k+1}$ has only one zero. Hence $h(\xi)$ has at most one zero.

We now further consider the following subcases.

Sub-case 1.1. If h is transcendental or h is a rational function with $h\neq 0$, then by Theorem 1.2 and Lemma 2.3, $h(\xi)+a(h^{(k)}(\xi))^{k+1}$ has at least k^2+k+1 distinct zeros, which is a contradiction.

Sub-case 1.2. If h is a rational function with only one zero. By lemma 2.4, we get $h(\xi)+a(h^{(k)}(\xi))^{k+1}$ has at least two distinct zeros, which is a contradiction.

Case 2. $b \neq 0$. Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. Let $\alpha = k$. Then by lemma 2.1, there exists a sequence of complex numbers $z_j \to z_0$, a sequence of functions $f_j \in \mathcal{F}$ and a sequence of positive numbers $\rho_n \to 0^+$ such that $h_j(\xi) = \rho_j^{-k} f_j(z_j + \rho_j \xi)$ converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic functions $h(\xi)$ in $\mathbb{C}$. Also the order of h does not exceed two and by Hurwitz's theorem h has no zero of mulitiplicity less than k+2.

On every compact subset of $\mathbb{C}$ which contains no poles of h, we get

$$\rho_j^k h_j(\xi) + a(h_j^{(k)}(\xi))^{k+1} - b =$$

$$f_j(z_j + \rho_j \xi) + a(f_j^{(k)}(z_j + \rho_j \xi))^{k+1} - b \to a(h^{(k)}(\xi))^{k+1} - b. \qquad (3.2)$$

If $a(h^{(k)}(\xi))^{k+1} - b \equiv 0$, that is $(h^{(k)}(\xi))^{k+1} \equiv \frac{b}{a}$, then h is a polynomial with degree k. But this contradict the fact that all of zeros of h have multiplicity at least k+2.

Thus $a(h^{(k)}(\xi))^{k+1} - b \neq 0$. Let $c_1, c_2, \cdots, c_{k+1}$ are the different roots of $\omega^{k+1} = \frac{b}{a}$. By Nevanlinna's second fundamental theorem we get

$$\begin{aligned} T(r, h^{(k)}) &\leq \overline{N}(r, h^{(k)}) + \sum_{i=1}^{k+1} \overline{N}(r, \tfrac{1}{h^{(k)} - c_i}) + S(r, h^{(k)}) \\ &\leq \overline{N}(r, h^{(k)}) + S(r, h^{(k)}) \\ &\leq \tfrac{1}{k+1} N(r, h^{(k)}) + S(r, h^{(k)}) \\ &\leq \tfrac{1}{2} T(r, h^{(k)}) + S(r, h^{(k)}), \end{aligned}$$

It follows that $T(r, h^{(k)} \leq S(r, h^{(k)})$, a contradiction. Hence $a(h^{(k)}(\xi))^{k+1} - b$ has zeros.

We distinguish two sub-cases.

Sub-case 2.1. If h is transcendental and $a(h^{(k)}(\xi))^{k+1} - b$ has only one zero ξ_0, say. Let $c_1, c_2, \cdots, c_{k+1}$ are the different roots of $\omega^{k+1} = \frac{b}{a}$. Then $h^{(k)}(\xi_0) = c_1$ and $h^{(k)}(\xi) \neq c_2, \cdots, c_{k+1}$. Obviously $\overline{N}(r, \frac{1}{h^{(k)} - c_1}) = O(\log r)$. By Nevanlinna's second fundamental theorem we get

$$\begin{aligned} kT(r, h^{(k)}) &\leq \overline{N}(r, h^{(k)}) + \sum_{i=1}^{k+1} \overline{N}(r, \tfrac{1}{h^{(k)} - c_i}) + S(r, h^{(k)}) \\ &\leq \overline{N}(r, h^{(k)}) + \overline{N}(r, \tfrac{1}{h^{(k)} - c_1}) + S(r, h^{(k)}) \\ &\leq \tfrac{1}{k+1} T(r, h^{(k)}) + S(r, h^{(k)}) \end{aligned}$$

i.e.,

$$(k - \frac{1}{k+1}) T(r, h^{(k)}) \leq S(r, h^{(k)}).$$

which is impossible.

Thus when h is transcendental, $a(h^{(k)}(\xi))^{k+1} - b$ has at least two distinct zeros.

Sub-case 2.2. If h is a rational function with $h \neq 0$. In this case we note that $a(h^{(k)}(\xi))^{k+1} - b = a \prod_{i=1}^{k+1} (h(\xi)^{(k)} - c_i)$. Then by lemma 2.5, we get $a(h^{(k)}(\xi))^{k+1} - b$ has at least $(k+1)^2$ distinct zeros.

Sub-case 2.3. If h is a rational function with at least one zero. By lemma 2.6 we get $a(h^{(k)}(\xi))^{k+1} - b$ has at least two distinct zeros.

We now claim that $a(h^{(k)}(\xi))^{k+1} - b$ has only one zero. By contraries, Let ξ_0 and ξ_0^* be two distinct zeros of $a(h^{(k)}(\xi))^{k+1} - b$. We choose a small $\delta(>0)$ such that $D_1 \bigcap D_2 = \emptyset$ and $a(h^{(k)}(\xi))^{k+1} - b$ has no other zeros in $D_1 \bigcup D_2$ apart from ξ_0 and ξ_0^*, where $D_1 = \{\xi : |\xi - \xi_0| < \delta\}$ and $D_2 = \{\xi : |\xi - \xi_0^*| < \delta\}$.

From (3.2), by Hurwitz's theorem, there exist points $\xi_j \in D_1$ and $\xi_j^* \in D_2$ such that

$$f_j(z_j + \rho_j \xi_j) + a\left(f_j^{(k)}(z_j + \rho_j \xi_j)\right)^{k+1} - b = 0,$$

$$f_j(z_j + \rho_j \xi_j^*) + a\left(f_j^{(k)}(z_j + \rho_j \xi_j^*)\right)^{k+1} - b = 0.$$

By the assumption of Theorem 1.6, we see that for any integer m and for all j we get

$$f_m(z_j + \rho_j \xi_j) + a\left(f_m^{(k)}(z_j + \rho_j \xi_j)\right)^{k+1} - b = 0,$$

$$f_m(z_j + \rho_j \xi_j^*) + a\left(f_m^{(k)}(z_j + \rho_j \xi_j^*)\right)^{k+1} - b = 0.$$

Fix m and take $j \to \infty$, and note $z_j + \rho_j \xi_j \to z_0$, $z_j + \rho_j \xi_j^* \to z_0$, then

$$f_m(z_0) + a\left(f_m^{(k)}(z_0)\right)^{k+1} - b = 0.$$

Since the zeros of $f_m + a(f_m^{(k)})^{k+1} - b$ has no accumulation point, so for sufficiently large j we get $z_j + \rho_j \xi_j = z_0$, $z_j + \rho_j \xi_j^* = z_0$.

Hence

$$\xi_j = \xi_j^* = \frac{z_0 - z_j}{\rho_j},$$

This contradicts the fact that $\xi_j \in D_1$, $\xi_j^* \in D_2$ and $D_1 \bigcap D_2 = \emptyset$.

Therefore $a(h^{(k)}(\xi))^{k+1} - b$ has only one zero, which is a contradiction since $a(h^{(k)}(\xi))^{k+1} - b$ has at least two distinct zeros.

These contradictions shows that $\mathcal{F}$ is normal in D and hence Theorem 1.6 is proved. □

The proof of Theorem 1.7 is almost similar to that of Theorem 1.6, here we omit the proof.

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