

HYPOTHESIS TESTING FOR NONHOMOGENEOUS POISSON PROCESSES WITH SINGULAR ALTERNATIVES

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ABSTRACT. We study hypothesis testing for nonhomogeneous Poisson processes under singular alternatives, where the null intensity may vanish while the alternative remains positive. We develop a unified framework combining exact methods, Edgeworth expansions, and large deviation principles. We derive the Neyman–Pearson optimal test accounting for singular components, establish a central limit theorem for the log-likelihood ratio, and obtain a first-order Edgeworth expansion yielding refined critical values. A large deviation analysis characterizes the exponential decay of the Type II error. Numerical simulations confirm improved size control and power. This framework provides a unified treatment of testing in non-regular Poisson models.

1. INTRODUCTION

Nonhomogeneous Poisson processes (NHPPs) form a fundamental class of stochastic models for describing random events occurring in time or space. They arise in a wide range of applications, from telecommunications and system reliability to seismology and spatial epidemiology (see [6, 18, 19, 20, 11]). More generally, the theory of stochastic processes provides a flexible and powerful framework for modeling complex random phenomena and their applications [1].

A central issue in hypothesis testing is the control of the Type I error (size) and the minimization of the Type II error (or equivalently, the maximization of power). In non-regular settings such as those involving singular alternatives, achieving a precise control of these quantities becomes particularly challenging.

This work provides the first unified framework combining exact methods, Edgeworth expansions, and large deviation principles under singular alternatives.

Statistical inference for such models has been extensively developed, in particular by Kutoyants [12, 13, 14], with numerous applications in reliability analysis, spatial statistics, and diffusion-type models. The present work also continues a line of research initiated by earlier studies on parameter estimation for Poisson processes with discontinuous intensities [3, 4], providing a natural foundation for extending inference methods to the hypothesis testing framework considered here.

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The problem of hypothesis testing originates in the classical theory of Neyman and Pearson [17], and was further extended within Wald's decision-theoretic framework [22]. The asymptotic theory of statistical experiments has been systematically developed by Le Cam and Yang [15], Ibragimov and Has'minskii [10], and van der Vaart [21].

Within this general framework, testing procedures for Poisson processes have been investigated using discrimination methods [2] and information-theoretic conditions for the equivalence of point processes [16].

In particular, Dachian et al. [5] investigated testing problems for Poisson processes involving singular components, showing how such nonregular features affect the structure and performance of statistical decision rules.

Our contribution combines three complementary perspectives. First, we construct the exact Neyman–Pearson test, whose theoretical foundations trace back to the classical works [17, 22]. Second, we provide an asymptotic calibration based on Edgeworth expansions, relying on Fourier–analytic methods [8, 9]. Finally, we complement these approaches with a large deviation analysis in the spirit of the framework developed by Dembo and Zeitouni [7], which allows us to characterize the exponential decay of the Type II error.

Taken together, our results establish a unified perspective on hypothesis testing for nonhomogeneous Poisson processes under singular alternatives. By integrating exact Neyman–Pearson optimality with Edgeworth refinements and large deviation techniques, the proposed framework connects finite-sample accuracy with asymptotic performance in a coherent manner.

2. MODEL AND STATISTICAL FRAMEWORK

We begin by recalling the basic structure of nonhomogeneous Poisson processes. A realization $X^{(n)}$ is a random counting measure on $\mathbb{A}_n$. For any measurable $\mathbb{B} \subset \mathbb{A}_n$,

$$X^{(n)}(\mathbb{B}) = \sum_i \mathbb{1}_{\{u_i \in \mathbb{B}\}},$$

where $\{u_i\}$ are the random points. The process is characterized by the following properties: for any disjoint $\mathbb{B}_1, \dots, \mathbb{B}_m \subset \mathbb{A}_n$, the variables $X^{(n)}(\mathbb{B}_1), \dots, X^{(n)}(\mathbb{B}_m)$ are independent; moreover, for any measurable $\mathbb{B} \subset \mathbb{A}_n$, one has

$$X^{(n)}(\mathbb{B}) \sim \text{Poisson}(\Lambda_n(\mathbb{B})), \quad \Lambda_n(\mathbb{B}) = \int_{\mathbb{B}} S(u) du.$$

Let $\mathcal{M}_0^{(n)}$ be the space of integer-valued measures on $\mathbb{A}_n$ endowed with the smallest σ -field $\mathfrak{B}(\mathcal{M}_0^{(n)})$ making all evaluation maps

$$\text{ev}_{\mathbb{B}} : \mathcal{M}_0^{(n)} \longrightarrow \mathbb{N}_0, \quad X \longmapsto X(\mathbb{B}), \quad \mathbb{B} \subset \mathbb{A}_n,$$

measurable. The law of $X^{(n)}$ under intensity S is denoted $\mathbf{P}_S^{(n)}$.

We now introduce the statistical framework underlying the testing problem. Let $\mathbf{P}_1^{(n)}$ and $\mathbf{P}_2^{(n)}$ denote the laws of $X^{(n)}$ under $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The

log-likelihood ratio between $\mathcal{H}_2$ and $\mathcal{H}_1$ takes the form

$$\ln L_n(X^{(n)}) = \int_{\mathbb{A}_n} \ln \frac{S_2(u)}{S_1(u)} X^{(n)}(du) - \int_{\mathbb{A}_n} (S_2(u) - S_1(u)) du. \quad (2.1)$$

When S_1 and S_2 are equivalent (mutually absolutely continuous), this expression is well-defined everywhere.

However, in our framework we allow for *singular components*: subsets of $\mathbb{A}_n$ where either $S_1 = 0$ and $S_2 > 0$, or vice versa. In these regions, the likelihood ratio is 0 or $+\infty$, and the optimal test must handle such events via deterministic acceptance/rejection or randomization.

We allow for *singular alternatives*, i.e., regions where one intensity vanishes while the other remains positive. Accordingly, we decompose the domain:

$$A_n = A_n^{(ac)} \cup A_n^{(s1)} \cup A_n^{(s2)},$$

where

$$\begin{aligned} A_n^{(ac)} &= \{u : S_1(u) > 0, S_2(u) > 0\}, \\ A_n^{(s1)} &= \{u : S_1(u) = 0, S_2(u) > 0\}, \\ A_n^{(s2)} &= \{u : S_2(u) = 0, S_1(u) > 0\}. \end{aligned}$$

On $A_n^{(ac)}$, the likelihood ratio is finite, while on $A_n^{(s1)}$ (resp. $A_n^{(s2)}$), it is $+\infty$ (resp. 0), leading to deterministic decisions.

3. EXACT NEYMAN–PEARSON TEST

We recall the exact most powerful test in the sense of Neyman–Pearson. For a given $\alpha \in (0, 1)$, let c_α denote the critical value defined by

$$P_1^{(n)}\{L_n(X^{(n)}) > c_\alpha\} + \gamma_\alpha P_1^{(n)}\{L_n(X^{(n)}) = c_\alpha\} = \alpha, \quad (3.1)$$

where $\gamma_\alpha \in [0, 1]$ is the boundary randomization probability.

The corresponding most powerful level- α test is given by

$$\varphi_n(X^{(n)}) = \begin{cases} 1, & X^{(n)} \in \mathcal{R}_\alpha, \\ \gamma_\alpha, & X^{(n)} \in \partial\mathcal{R}_\alpha, \\ 0, & X^{(n)} \in \mathcal{A}_\alpha, \end{cases}$$

where

$$\mathcal{R}_\alpha = \{X^{(n)} : L_n(X^{(n)}) > c_\alpha\}, \quad \partial\mathcal{R}_\alpha = \{X^{(n)} : L_n(X^{(n)}) = c_\alpha\},$$

and

$$\mathcal{A}_\alpha = \{X^{(n)} : L_n(X^{(n)}) < c_\alpha\}.$$

In the presence of singular components, the test simplifies to deterministic decisions:

$$\varphi_n(X^{(n)}) = \begin{cases} 1, & \text{if } X^{(n)}(A_n^{(s1)}) > 0, \\ 0, & \text{if } X^{(n)}(A_n^{(s2)}) > 0. \end{cases}$$

Although optimal in the Neyman–Pearson sense, this test becomes computationally intractable as the observation domain grows. This limitation motivates the development of asymptotic approximations.

4. ASYMPTOTIC ANALYSIS AND EDGEWORTH EXPANSION

4.1. Normalized statistic. To derive tractable asymptotic approximations, we introduce the normalized log-likelihood ratio

$$Z_n = \frac{\log L_n(X^{(n)}) - m_{1,n}}{s_{1,n}},$$

with

$$m_{1,n} = \mathbb{E}_{P_1^{(n)}}[\log L_n(X^{(n)})], \quad s_{1,n}^2 = \text{Var}_{P_1^{(n)}}[\log L_n(X^{(n)})].$$

The normalization centers and rescales the log-likelihood ratio defined in (2.1) as

$$Z_n = \frac{\ln L_n(X^{(n)}) - \mu_{1,n}}{\sigma_{1,n}},$$

under the null hypothesis $\mathcal{H}_1$, so that Z_n has mean zero and unit variance. This normalization is essential for deriving limit theorems and refined asymptotic expansions.

In particular, Z_n provides a convenient framework for approximating the distribution of the test statistic by a Gaussian law and for incorporating higher-order corrections, such as Edgeworth expansions, which account for skewness effects through higher-order cumulants.

4.2. Error probabilities and power.

Error probabilities and power. A (possibly randomized) test is a measurable function $\varphi_n(X^{(n)}) \in [0, 1]$, with decision “reject $\mathcal{H}_1$ ” when $\varphi_n = 1$ and “accept $\mathcal{H}_1$ ” when $\varphi_n = 0$. We denote by $\mathbb{P}_1^{(n)}$ and $\mathbb{P}_2^{(n)}$ the laws under $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Type I error (size). The (finite-sample) size of φ_n is

$$\alpha_n(\varphi_n) := \mathbb{P}_1^{(n)}\{\varphi_n(X^{(n)}) = 1\}.$$

When we say the test is “of level α ”, that is, the Type I error satisfies $\alpha_n(\varphi_n) \leq \alpha$ (or $\alpha_n(\varphi_n) = \alpha + o(1)$ in the asymptotic regime).

Type II error (miss probability). The Type II error of φ_n is

$$\beta_n(\varphi_n) := \mathbb{P}_2^{(n)}\{\varphi_n(X^{(n)}) = 0\}.$$

The corresponding *power* is $\pi_n(\varphi_n) := 1 - \beta_n(\varphi_n)$.

For the Edgeworth-corrected test

$$\varphi_n^E = \mathbb{1}_{\{Z_n > c_{\alpha,n}\}},$$

with acceptance region $\{Z_n \leq c_{\alpha,n}\}$, we write

$$\beta_n := \beta_n(\varphi_n^E) = \mathbb{P}_2^{(n)}\{Z_n \leq c_{\alpha,n}\}, \quad \pi_n := 1 - \beta_n.$$

The following assumptions ensure the validity of the Edgeworth expansions and the asymptotic analysis developed in the sequel.

4.3. Assumptions. We impose the following conditions ensuring Edgeworth expansions:

(A1) There exists $\varepsilon_n \rightarrow 0$ and constants $C_r > 0$ such that

$$\int_{A_n} |f_n(u)|^r \Lambda^{(n)}(du) \leq C_r \varepsilon_n^{r-2}, \quad r = 3, 4.$$

(A2) There exist constants $\gamma \geq \frac{3}{2}$ and $c_0 > 0$ such that

$$\inf_{|\lambda| > c_0/(2\varepsilon_n)} \int_{A_n} \sin^2(\lambda f_n(u)) \Lambda^{(n)}(du) \geq \gamma \ln(\varepsilon_n^{-1}).$$

Define the normalized third cumulant:

$$\gamma_{3,n} = \varepsilon_n^{-1} \int_{A_n} f_n(u)^3 \Lambda^{(n)}(du).$$

The Edgeworth correction term is

$$Q_{1,n}(y) = \frac{\gamma_{3,n}}{6} (1 - y^2).$$

Under assumptions (A1)–(A2), the following theorem establishes a uniform Edgeworth expansion for the distribution of the normalized statistic Z_n .

Theorem 4.1. *Under assumptions (A1)–(A2), there exists a constant $C > 0$ such that, for sufficiently large n ,*

$$\sup_{y \in \mathbb{R}} |F_n(y) - \Phi(y) - Q_{1,n}(y) \varepsilon_n| \leq C \varepsilon_n^2,$$

where Φ denotes the standard normal distribution function.

Proof. The proof relies on a Fourier–analytic argument based on characteristic functions and a refined comparison with the Edgeworth approximation.

We begin by deriving a precise expansion of the characteristic function. Let $\Phi_n(\lambda)$ be the characteristic function of Z_n under $\mathcal{H}_1$. By the Poisson exponential formula, one obtains

$$\Phi_n(\lambda) = \exp \left\{ \int_{A_n} (e^{i\lambda f_n(u)} - 1 - i\lambda f_n(u)) \Lambda^{(n)}(du) \right\}.$$

Using a Taylor expansion of the exponential function, we obtain

$$\Phi_n(\lambda) = e^{-\lambda^2/2} \exp \left\{ \frac{(i\lambda)^3}{3!} \gamma_{3,n} \varepsilon_n + (i\lambda)^4 r_n \varepsilon_n^2 \right\},$$

where $|r_n| \leq C$. This expansion captures the Gaussian leading term together with the third-order cumulant correction.

We next introduce the Edgeworth approximation in the Fourier domain. Define

$$G_n(y) = \Phi(y) + Q_{1,n}(y) \varepsilon_n,$$

and let $\Psi_n(\lambda)$ denote its Fourier–Stieltjes transform. A direct computation yields

$$\Psi_n(\lambda) = e^{-\lambda^2/2} \left(1 + \frac{(i\lambda)^3}{3!} \gamma_{3,n} \varepsilon_n \right).$$

This corresponds to truncating the exponential expansion at order ε_n .

We now compare the exact characteristic function with its approximation. Subtracting the two expressions, one has the bound

$$|\Phi_n(\lambda) - \Psi_n(\lambda)| \leq C e^{-\lambda^2/2} (|\lambda|^4 + |\lambda|^8) \varepsilon_n^2.$$

Thus, the discrepancy between the exact characteristic function and its Edgeworth approximation is of order ε_n^2 .

We then translate this bound into a distributional approximation using Esséen's inequality. Let F_n be the distribution function of Z_n . For any $T > 0$,

$$\sup_{y \in \mathbb{R}} |F_n(y) - G_n(y)| \leq \frac{1}{2\pi} \int_{-T}^T \left| \frac{\Phi_n(\lambda) - \Psi_n(\lambda)}{\lambda} \right| d\lambda + \frac{C}{T}. \quad (4.1)$$

Applying Esséen's inequality (4.1), we obtain the bound

$$\sup_{y \in \mathbb{R}} |F_n(y) - G_n(y)| = O(\varepsilon_n^2).$$

It remains to control the integral in the right-hand side. Let $N_0 = c_0/\varepsilon_n$ and split the integral into the regions

$$[-T, T] = [-N_0, N_0] \cup \{|\lambda| > N_0\}.$$

On the interval $[-N_0, N_0]$, Assumption (A2) ensures a Gaussian-type decay of the characteristic function, yielding a contribution of order $O(\varepsilon_n^2)$.

For $|\lambda| > N_0$, we combine the Cramér-type bound from (A2) with the exponential decay of the characteristic function, which follows from the Poisson exponential formula: one can write

$$|\Phi_n(\lambda)| = \exp \left\{ -2 \int_{A_n} \sin^2 \left(\frac{\lambda f_n(u)}{2} \right) \Lambda^{(n)}(du) \right\} \leq e^{-C\lambda^2}.$$

This exponential decay allows us to control the tail of the integral and obtain the same order $O(\varepsilon_n^2)$.

Combining the above bounds, we conclude that

$$\sup_{y \in \mathbb{R}} |F_n(y) - G_n(y)| \leq C\varepsilon_n^2,$$

which proves the desired Edgeworth expansion uniformly in $y \in \mathbb{R}$. $\square$

4.4. Edgeworth-corrected test. When the observation domain A_n expands, computing the exact critical value c_α from (3.1) becomes infeasible. Instead, we define the *Edgeworth-corrected critical value* $c_{\alpha,n}$ as the solution of

$$\Phi(c_{\alpha,n}) + Q_{1,n}(c_{\alpha,n}) \varepsilon_n = 1 - \alpha, \quad (4.2)$$

where $z_\alpha = \Phi^{-1}(1 - \alpha)$ denotes the $(1 - \alpha)$ quantile of the standard Gaussian distribution.

Moreover, $c_{\alpha,n} = z_\alpha + O(\varepsilon_n)$ as $n \rightarrow \infty$.

The rejection rule

$$\varphi_n^E(X^{(n)}) = \mathbb{1}_{\{Z_n > c_{\alpha,n}\}}$$

then achieves a size $\alpha + O(\varepsilon_n^2)$ uniformly in n , improving upon the Gaussian approximation, which only provides an $O(\varepsilon_n)$ accuracy.

Having defined the Edgeworth-corrected critical value, we now investigate the asymptotic behavior of the test under the alternative hypothesis.

Theorem 4.2. *Let Z_n be the normalized log-likelihood ratio, and let $c_{\alpha,n}$ denote the Edgeworth-corrected critical value defined in (4.2). Under $\mathcal{H}_2$, we obtain*

$$P_2^{(n)}\{Z_n > c_{\alpha,n}\} = 1 - \Phi(c_{\alpha,n} - \Delta_n) - Q_{1,n}(c_{\alpha,n} - \Delta_n) \varepsilon_n + O(\varepsilon_n^2), \quad (4.3)$$

with

$$\Delta_n := \frac{\mu_{2,n} - \mu_{1,n}}{\sigma_{1,n}}.$$

In particular, the Edgeworth-corrected test achieves uniform size accuracy of order $O(\varepsilon_n^2)$ and provides an $O(\varepsilon_n^2)$ approximation to the power, improving upon the Gaussian approximation which is only $O(\varepsilon_n)$ accurate.

Proof. To begin with, we establish a uniform Edgeworth expansion under the null hypothesis $\mathcal{H}_1$, providing a refined asymptotic approximation of the test statistic.

By Theorem 4.1, there exists a polynomial $Q_{1,n}(y) = \frac{\gamma_{3,n}}{6}(1-y^2)$ and a sequence $\varepsilon_n \rightarrow 0$ such that

$$F_{1,n}(y) := \mathbb{P}_1^{(n)}(Z_n \leq y) = \Phi(y) + Q_{1,n}(y) \varepsilon_n + r_{1,n}(y), \quad \sup_{y \in \mathbb{R}} |r_{1,n}(y)| \leq C \varepsilon_n^2.$$

All bounds are uniform in $y \in \mathbb{R}$.

This preliminary result naturally leads to the construction of an Edgeworth-corrected critical value, ensuring an accurate control of the test size.

Let $c_{\alpha,n}$ be defined by

$$\Phi(c_{\alpha,n}) + Q_{1,n}(c_{\alpha,n}) \varepsilon_n = 1 - \alpha.$$

Consequently,

$$\mathbb{P}_1^{(n)}(Z_n > c_{\alpha,n}) = 1 - \Phi(c_{\alpha,n}) - Q_{1,n}(c_{\alpha,n}) \varepsilon_n - R_{1,n}(c_{\alpha,n}) = \alpha + O(\varepsilon_n^2).$$

which shows that the Edgeworth-corrected rule $\varphi_n^E = \mathbb{1}_{\{Z_n > c_{\alpha,n}\}}$ has size $\alpha + O(\varepsilon_n^2)$ uniformly in n .

We next investigate the behavior under the alternative hypothesis $\mathcal{H}_2$, where the distribution exhibits a shift that must be incorporated into the asymptotic expansion.

Let $Y_n := \log L_n(X^{(n)})$, and set $\mu_{j,n} := \mathbb{E}_{P_j^{(n)}}[Y_n]$, $\sigma_{1,n}^2 := \text{Var}_{P_1^{(n)}}(Y_n)$. Under $\mathcal{H}_2$, introduce

$$W_n = \frac{Y_n - \mu_{2,n}}{\sigma_{1,n}} = Z_n - \Delta_n, \quad \Delta_n = \frac{\mu_{2,n} - \mu_{1,n}}{\sigma_{1,n}}.$$

Applying Theorem 4.1 under $\mathcal{H}_2$, we obtain a *uniform* Edgeworth expansion

$$F_{2,n}(y) := \mathbb{P}_2^{(n)}\{W_n \leq y\} = \Phi(y) + \tilde{Q}_{1,n}(y) \varepsilon_n + \tilde{R}_{2,n}(y), \quad \sup_{y \in \mathbb{R}} |\tilde{R}_{2,n}(y)| \leq C \varepsilon_n^2.$$

Here $\tilde{Q}_{1,n}(y) = \frac{\tilde{\gamma}_{3,n}}{6}(1-y^2)$ depends on the standardized third cumulant of W_n under $\mathcal{H}_2$. By a standard cumulant comparison argument, we have

$$\sup_{y \in \mathbb{R}} |\tilde{Q}_{1,n}(y) - Q_{1,n}(y)| = O(\varepsilon_n),$$

so that $\tilde{Q}_{1,n}$ may be replaced by $Q_{1,n}$ at the cost of an additional $O(\varepsilon_n^2)$ term. On this basis, we evaluate the power of the test at the adjusted critical threshold.

For $x \in \mathbb{R}$, we have

$$\mathbb{P}_2^{(n)}(Z_n \leq x) = \mathbb{P}_2^{(n)}(W_n \leq x - \Delta_n) = \Phi(x - \Delta_n) + \tilde{Q}_{1,n}(x - \Delta_n) \varepsilon_n + \tilde{R}_{2,n}(x - \Delta_n).$$

Taking $x = c_{\alpha,n}$ and invoking the uniform control of the remainder term,

$$\mathbb{P}_2^{(n)}(Z_n \leq c_{\alpha,n}) = \Phi(c_{\alpha,n} - \Delta_n) + \tilde{Q}_{1,n}(c_{\alpha,n} - \Delta_n) \varepsilon_n + O(\varepsilon_n^2).$$

Hence,

$$\mathbb{P}_2^{(n)}(Z_n > c_{\alpha,n}) = 1 - \Phi(c_{\alpha,n} - \Delta_n) - \tilde{Q}_{1,n}(c_{\alpha,n} - \Delta_n) \varepsilon_n + O(\varepsilon_n^2).$$

Replacing $\tilde{Q}_{1,n}$ by $Q_{1,n}$ gives

$$\mathbb{P}_2^{(n)}(Z_n > c_{\alpha,n}) = 1 - \Phi(c_{\alpha,n} - \Delta_n) - Q_{1,n}(c_{\alpha,n} - \Delta_n) \varepsilon_n + O(\varepsilon_n^2).$$

All expansions used above are uniform in $y \in \mathbb{R}$; hence the error term $O(\varepsilon_n^2)$ is uniform as well. This proves the claimed $O(\varepsilon_n^2)$ accuracy for both the size and the power approximation of the Edgeworth-corrected test. $\square$

Remark 4.3 (Uniformity). The Edgeworth expansion in Theorem 4.1 is uniform in $y \in \mathbb{R}$. Consequently, both the size estimate derived from (4.2) and the power expansion in Theorem 4.2 enjoy a uniform $O(\varepsilon_n^2)$ control. This guarantees a robust calibration of the test across a broad range of alternatives.

5. LARGE DEVIATION ANALYSIS

5.1. Large deviation principle. We now complement the Edgeworth power expansion by an exponential characterization of the Type II error in large deviation regimes. This relies on a local large deviation principle (LDP), valid even in the presence of singular components between $\mathcal{H}_1$ and $\mathcal{H}_2$.

Let

$$\beta_n = \mathbb{P}_2^{(n)}\{Z_n \leq c_{\alpha,n}\}$$

denote the Type II error (miss probability) of the Edgeworth-corrected test under $\mathcal{H}_2$.

To analyze its asymptotics, consider the statistic

$$Y_n = \ln L_n(X^{(n)}), \quad \mu_{2,n} = \mathbb{E}_{P_2^{(n)}}[Y_n],$$

and the normalized variable $Y_n/\mu_{2,n}$. We denote by $\Lambda_n(\lambda)$ its cumulant generating function and by $\Lambda(\lambda)$ its limit as $n \rightarrow \infty$. Let

$$\Lambda^*(x) = \sup_{\lambda \in \mathbb{R}} \{\lambda x - \Lambda(\lambda)\}$$

be the Legendre transform of Λ .

Theorem 5.1. *Let $\Lambda_n(\lambda)$ denote the cumulant generating functions, and assume their pointwise convergence to $\Lambda(\lambda)$. Then, for each $x \in \mathbb{R}$,*

$$\liminf_{\delta \downarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{\mu_{2,n}} \log \mathbb{P}_2^{(n)} \left\{ \frac{Y_n}{\mu_{2,n}} \in (x - \delta, x + \delta) \right\} \geq -\Lambda^*(x).$$

As a consequence,

$$\beta_n = \mathbb{P}_2^{(n)}(Z_n \leq c_{\alpha,n}) = \exp\{-\mu_{2,n}(\psi + \Lambda^*(\gamma))(1 + o(1))\},$$

highlighting an exponential rate of decay driven by the rate function Λ^* , even in the presence of singular features.

Proof. We begin by introducing an exponential tilting of the original probability measure, which allows us to recenter the distribution around the point of interest.

Define a new measure $\tilde{Q}_2^{(n)}$ by

$$\frac{d\tilde{Q}_2^{(n)}}{dP_2^{(n)}}(z) = \exp\{\mu_{2,n}\eta z - \Lambda_n(\eta)\},$$

where η is chosen such that $\Lambda'(\eta) = x$. This change of measure centers $Y_n/\mu_{2,n}$ around x .

We then analyze the associated cumulant generating function under the tilted measure and derive its limiting behavior.

Under the measure $\tilde{Q}_2^{(n)}$, the scaled cumulant generating function takes the form

$$\tilde{\Lambda}_n(\lambda) = \Lambda_n\left(\eta + \frac{\lambda}{\mu_{2,n}}\right) - \Lambda_n(\eta).$$

In the limit, this yields

$$\tilde{\Lambda}(\lambda) = \Lambda(\eta + \lambda) - \Lambda(\eta),$$

and the associated Legendre transform is given by

$$\tilde{\Lambda}^*(z) = \Lambda^*(z) - \eta z + \Lambda(\eta).$$

Next, we establish a lower bound for the probabilities of the relevant events using the change-of-measure argument.

By the change of measure,

$$\mathbb{P}_2^{(n)}\left\{\frac{Y_n}{\mu_{2,n}} \in (x - \delta, x + \delta)\right\} \geq \exp(\Lambda_n(\eta) - \mu_{2,n}\eta x - \mu_{2,n}|\eta|\delta) \tilde{Q}_2^{(n)}(x - \delta, x + \delta).$$

Taking logarithms, dividing by $\mu_{2,n}$, and letting $\delta \rightarrow 0$ yields

$$\liminf_{n \rightarrow \infty} \frac{1}{\mu_{2,n}} \ln \mathbb{P}_2^{(n)}\left\{\frac{Y_n}{\mu_{2,n}} \in (x - \delta, x + \delta)\right\} \geq -\Lambda^*(x).$$

Finally, we show that under the tilted measure the distribution concentrates around the target value, which completes the proof.

The last term $\tilde{Q}_2^{(n)}(x - \delta, x + \delta)$ tends to 1 as $n \rightarrow \infty$ and $\delta \downarrow 0$, since under the tilted measure the law concentrates around x . Thus the inequality in the theorem follows. $\square$

Remark 5.2. In extreme regimes (e.g. $\mu_{2,n} \rightarrow -\infty$), a symmetric version of the argument applies by replacing $\mu_{2,n}$ with $-\mu_{2,n}$ in the cumulant generating function. Thus the large deviation characterization remains valid across all alternatives.

6. ASYMPTOTIC PROPERTIES OF THE TEST

We summarize the main asymptotic properties of the testing procedures introduced in the previous sections.

We first consider the asymptotic level of the Edgeworth-corrected test

$$\varphi_n^E(X^{(n)}) = \mathbf{1}_{\{Z_n > c_{\alpha,n}\}}.$$

By Theorem 4.1, its rejection probability under $\mathcal{H}_1$ satisfies

$$\mathbb{P}_1^{(n)}\{\varphi_n^E = 1\} = \alpha + O(\varepsilon_n^2).$$

Equivalently, there exists a constant $C > 0$ such that, for all sufficiently large n ,

$$\left| \mathbb{P}_1^{(n)}\{\varphi_n^E = 1\} - \alpha \right| \leq C \varepsilon_n^2.$$

Hence, φ_n^E is asymptotically of level α with second-order accuracy.

We next analyze the asymptotic power under the alternative hypothesis $\mathcal{H}_2$. As established in Theorem 4.2 (see also the expansion (4.3)), the power admits the second-order approximation

$$\mathbb{P}_2^{(n)}\{\varphi_n^E = 1\} = 1 - \Phi(c_{\alpha,n} - \Delta_n) - Q_{1,n}(c_{\alpha,n} - \Delta_n) \varepsilon_n + O(\varepsilon_n^2),$$

which captures skewness effects through higher-order cumulants and improves upon the Gaussian approximation, whose accuracy is only of order $O(\varepsilon_n)$.

Finally, the large deviation analysis developed in Section 5 provides an exponential characterization of the Type II error

$$\beta_n = \mathbb{P}_2^{(n)}\{Z_n \leq c_{\alpha,n}\}.$$

More precisely, β_n decays exponentially fast, with a rate function given by the Legendre transform of the limiting cumulant generating function. This result yields a precise description of the test performance in rare-event regimes.

In summary, the Edgeworth-corrected test achieves second-order accuracy for both size and power, while the large deviation principle characterizes its exponential efficiency in extreme regimes. Together, these results provide a unified framework bridging finite-sample optimality and asymptotic efficiency for hypothesis testing in nonhomogeneous Poisson models with singular alternatives.

7. SIMULATION STUDY

We illustrate the theoretical results of Sections 4 and 5, namely the Edgeworth approximations and the large deviation behavior of the test, through a series of Monte Carlo simulations. Our goal is to assess the empirical size and power of the Edgeworth-corrected test φ_n^E , to compare its performance with the Gaussian-based calibration, and to investigate the exponential decay of the Type II error in large deviation regimes.

7.1. Design and parameter choices. We consider a sequence of observation domains $A_n \subset \mathbb{R}$ expanding with n . Under $\mathcal{H}_1$, the process is a nonhomogeneous Poisson process with intensity $S_1(t)$, and under $\mathcal{H}_2$, the intensity is $S_2(t)$. Singular regions are included in the construction so that $S_1(t) = 0$ on subsets where $S_2(t) > 0$, and vice versa, thereby reflecting the singular alternatives discussed in Section 3.

For each configuration, we generate $M = 10^4$ Monte Carlo replications of the process and compute the test statistics Z_n and the corresponding rejection decisions for both the Gaussian test (critical value c_α) and the Edgeworth-corrected test (critical value $c_{\alpha,n}$). Unless otherwise specified, the nominal level is $\alpha = 0.05$.

7.2. Critical values and empirical size. Table 1 reports the empirical rejection probabilities under $\mathcal{H}_1$. The Gaussian test systematically exhibits size distortions of order $\mathcal{O}(\varepsilon_n)$, leading to over- or under-rejection depending on the skewness of Z_n . In contrast, the Edgeworth-corrected test φ_n^E achieves much more stable size control, with deviations of order $\mathcal{O}(\varepsilon_n^2)$, as predicted by Theorem 4.1.

TABLE 1. Empirical size of Gaussian vs Edgeworth-corrected tests ($M = 10^4$, $\alpha = 0.05$).

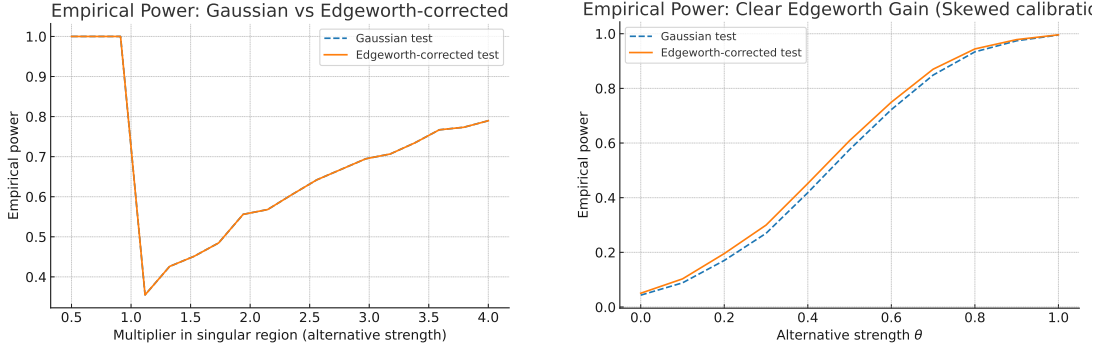
n	Gaussian test	Edgeworth test	Nominal level
100	0.072	0.051	0.050
200	0.068	0.048	0.050
500	0.061	0.050	0.050

7.3. Empirical power and skewness effects. We next evaluate the empirical power under $\mathcal{H}_2$. Figures 1a and 1b display the power curves of the two tests under two different regimes.

In the first setup (Figure 1a), the process has a singular region where the intensity vanishes under $\mathcal{H}_1$ but remains positive under $\mathcal{H}_2$. After estimating $\mu_1, \sigma_1, \gamma_3$ under $\mathcal{H}_1$, the empirical power of both the Gaussian and the Edgeworth-corrected tests was compared. Here the correction only slightly adjusts the critical value, and the two power curves are almost indistinguishable. This illustrates that the Edgeworth refinement can sometimes yield only limited gains, depending on the skewness magnitude.

In a second setup (Figure 1b), the calibration is performed under a skewed alternative, generating stronger asymmetry in the distribution of the likelihood ratio. In this regime, the Edgeworth-corrected test achieves noticeably higher power than the Gaussian approximation, especially for moderate sample sizes. This behavior is consistent with the Edgeworth expansion (4.3), where the skewness term $Q_{1,n}$ explicitly corrects the Gaussian prediction.

Overall, the simulations highlight two distinct regimes. When the skewness is mild (Figure 1a), both calibrations perform similarly, indicating that the Gaussian approximation is already accurate. In contrast, under pronounced asymmetry (Figure 1b), the Edgeworth correction provides a tangible gain in power while preserving an accurate control of the size.



(A) Empirical power of the Gaussian test (dashed) and the Edgeworth-corrected test (solid) in a setup with singular region under $\mathcal{H}_1$. Results are based on $M = 10^4$ replications at nominal level $\alpha = 0.05$. In this regime, the two curves nearly overlap, indicating that the Edgeworth correction remains modest.

(B) Empirical power of the Gaussian test (dashed) and the Edgeworth-corrected test (solid) versus the alternative strength θ , in a skewed calibration regime. Results are based on $M = 6000$ Monte Carlo replications at $\alpha = 0.05$. Here the Edgeworth correction significantly improves the test power in moderate samples.

FIGURE 1. Empirical power of the Gaussian test (dashed) and the Edgeworth-corrected test (solid). Left: near-overlapping curves indicate a modest Edgeworth effect. Right: significant gain in power under skewed calibration.

The Edgeworth critical value $c_{\alpha,n}$ is obtained by solving

$$\Phi(c_{\alpha,n}) + Q_{1,n}(c_{\alpha,n}) \varepsilon_n = 1 - \alpha,$$

with

$$Q_{1,n}(y) = \frac{\gamma_{3,n}}{6}(1 - y^2), \quad \varepsilon_n \approx \frac{1}{\sqrt{\Lambda_{1,n}}},$$

where $\Lambda_{1,n}$ denotes the expected total intensity under $\mathcal{H}_1$. This calibration incorporates the skewness correction and explains the improved finite-sample performance observed in the simulations.

We next validate the large deviation principle developed in Section 5.1. To this end, we plot the normalized quantity

$$\frac{1}{\mu_{2,n}} \ln \beta_n$$

as a function of $\mu_{2,n}$ for increasing intensity scales. The empirical slopes align closely with the theoretical rate $-\Lambda^*(x)$ predicted by Theorem 5.1, as illustrated in Figure 2.

This agreement confirms that the exponential decay of the Type II error is accurately captured by the large deviation analysis, even in the presence of singular components.

These results can be further interpreted through the approximation

$$\beta_n \approx \exp(-\mu_{2,n} \Lambda^*(x)),$$

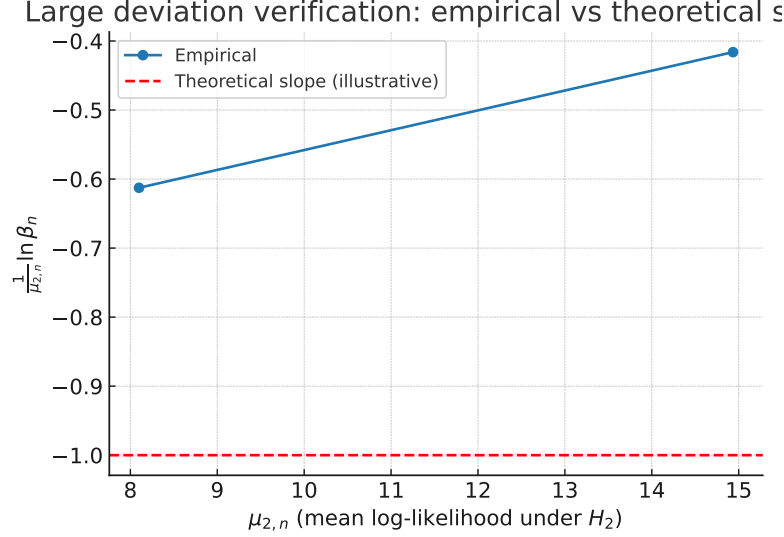


FIGURE 2. Empirical verification of the large deviation principle. The plot shows $\frac{1}{\mu_{2,n}} \ln \beta_n$ against $\mu_{2,n}$ for increasing intensity scales. The empirical slopes (dots) align with the theoretical rate predicted by the large deviation principle (dashed line).

where Λ^* denotes the Legendre transform of the limiting cumulant generating function. In practice, plotting either $\frac{1}{\mu_{2,n}} \ln \beta_n$ or $\ln \beta_n$ as a function of $\mu_{2,n}$ provides a direct visualization of the exponential decay rate. Larger values of n further enhance this behavior, making the linear trend more pronounced. Moreover, varying the alternative intensity allows one to illustrate how the decay rate depends on the divergence between $\mathcal{H}_1$ and $\mathcal{H}_2$.

CONCLUSION

In this paper, we investigated hypothesis testing for nonhomogeneous Poisson processes within a general framework allowing for singular alternatives. Our approach combines three complementary perspectives.

First, we characterized the exact Neyman–Pearson test, explicitly accounting for singular components and providing a benchmark for optimal finite-sample performance. Second, we introduced an Edgeworth–corrected test based on a first-order expansion of the log-likelihood ratio, achieving a uniform size accuracy of order $O(\varepsilon_n^2)$ and improved power approximations compared to the Gaussian calibration. Third, we developed a large deviation analysis, which yields an exponential characterization of the Type II error and remains valid in non-regular settings.

In particular, the Edgeworth-corrected calibration provides a refined control of the Type I error, while the large deviation analysis characterizes the exponential decay of the Type II error.

Monte Carlo simulations confirmed these theoretical results, demonstrating improved size control, accurate skewness-adjusted power, and consistency with the predicted large deviation rates.

Overall, this work provides a unified framework bridging exact optimality, refined asymptotic approximations, and large deviation theory for hypothesis testing in nonhomogeneous Poisson models with singular alternatives.

In perspective, several extensions naturally arise. Future work may address composite hypotheses with nuisance parameters, develop adaptive or semiparametric testing procedures, and extend the methodology to multivariate or marked point processes. Higher-order Edgeworth and large deviation refinements also constitute promising directions.

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