Gulf Journal of Mathematics
Vol 23, Issue 1 (2026) 1-16

ISSN: 2309-4966
https://doi.org/10.56947/d4kq7d88

# BLOW-UP OF SOLUTIONS TO $p(x)$-LAPLACIAN PARABOLIC EQUATIONS WITH GENERAL NONLINEAR SOURCE

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Abstract. In this paper, we study a $p(x)$-Laplacian parabolic equation with a general nonlinear source term. We show that the solution may blow up in finite time at positive initial energy. Moreover, under some suitable assumptions about the nonlinear source term, we prove that solutions blow up in finite time at arbitrarily high initial energy.

*Keywords.* $p(x)$-Laplacian, Parabolic equation, Weak solution, Blow-up.
*2020 Mathematics Subject Classification.* Primary 35K92; Secondary 35B44

## 1. Introduction and preliminaries

In this paper, we consider the following initial boundary value problem of the $p(x)$-Laplacian parabolic equation:

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = f(u), & x \in \Omega, \quad t > 0, \\ u = 0, & x \in \partial\Omega, \quad t \geq 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is open bounded domain, with smooth boundary $\partial\Omega$, $p(\cdot)$ is continous and bounded function on $\overline{\Omega}$ such that

$$2 < p_- \leq p(x) \leq p_+ < +\infty, \tag{1.2}$$

where $p_- = \operatorname{ess\,inf}_{x\in\Omega} p\,(x)\,,\ p_+ = \operatorname{ess\,sup}_{x\in\Omega} p\,(x),\ u_0 \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ is non-negative and non-trivial. The function $f$ is locally Lipschitz continuous and satisfies $f(0) = 0$, $f(s) > 0$ for all $s > 0$. Considering the following assumption

$$\alpha(x)F(s) \leq sf(s) + \beta s^{p(x)} + \gamma, \quad (s > 0) \tag{H1}$$

where

$$\alpha_- = \operatorname*{ess\,inf}_{x\in\Omega} \alpha\,(x) \leq \alpha(x) \leq \alpha_+ = \operatorname*{ess\,sup}_{x\in\Omega} \alpha\,(x)\,, \tag{1.3}$$

$$0 < \beta < (\alpha_- - p_+)\lambda_1/p_-, \tag{1.4}$$

*Date*: Received: Apr 12, 2026; Revised: May 3, 2026; Accepted: May 14, 2026.
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and $\lambda_1$ is the first eigenvalue of the $p(x)$-Laplacian operator, which is given by

$$\lambda_1 = \inf_{u \in W_0^{1,p(x)}(\Omega)\backslash\{0\}} \frac{\int_\Omega |\nabla u|^{p(x)} dx}{\int_\Omega |u|^{p(x)} dx} > 0, \tag{1.5}$$

$\gamma > 0$, will be detemined later, and $F$ is defined by

$$F(s) = \int_0^s f(\tau) d\tau.$$

The functions $p(\cdot)$ and $\alpha(\cdot)$ also satisfy the following assumption

$$p_+ < \alpha_- \leq \alpha(x) \leq \alpha_+ < p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)}, & \text{if} \quad p_+ < n, \\ +\infty, & \text{if} \quad p_+ \geq n. \end{cases} \tag{H2}$$

In what follows, we denote by $(\cdot, \cdot)$ the inner product of $L^2(\Omega)$ and by $\|\cdot\|_{r(x)}$ the norm of $L^{r(x)}(\Omega)$, $(1 \leq r(x) \leq +\infty)$.

The $p(x)$-Laplacian parabolic equation is frequently encountered in the theory of nonlinear elasticity, electrorheological fluids, image processing, calculus of variations, and so on; see [1, 4, 17, 23, 24]. Consequently, model (1.1) when $p(x) \equiv p$ (constant exponent) has been extensively investigated by numerous researchers [3, 5, 8, 13, 18, 19, 25, 27], particularly it appears in the non-Newtonian fluid theory see [9].

Recently, the problem (1.1) has been examined in numerous research papers, considering the general source term, which can be expressed in various forms see for instance, [8], when $p(x)$ is constant $(p(x) \equiv p)$ and [2, 10], when $p(x)$ is variable.

H. Ding and J. Zhou in [8] studied the problem

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = f(u), & x \in \Omega, \quad t > 0, \\ u = 0, & x \in \partial\Omega, \quad t \geq 0, \\ u(x, 0) = u_0(x) \geq 0, & x \in \bar{\Omega}, \end{cases}$$

they showed that the solution may blow up in finite time at positive initial energy. Moreover, under some suitable assumptions about the nonlinear source term, they demonstrated the blow up solution in finite time at arbitrarily high initial energy

In [2] The authors investigated the problem

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{m(x)-2} \nabla u\right) = |u|^{p(x)-2} u + f, & x \in \Omega \times (0, T), \\ u(x, t) = 0, & x \in \partial\Omega \times [0, T), \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases}$$

They proved that when $f \equiv 0$, any solution with a nontrivial initial datum blows up in finite time.

A. Lalmi et al. [16] considered the problem (1.1) with the nonlinear logarithm source term as follows:

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right) = |u|^{p(x)-2} u \log|u|, & x \in \Omega, \quad t > 0, \\ u(x, t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases}$$

with $2 < p_- \leq p(x) \leq p_+ < +\infty$. By applying the logarithmic Sobolev inequality and the potential well method, the researchers obtained results regarding the global existence and blow-up of weak solutions under certain conditions.

The main goal of this work is to analyse the aforementioned blow-up phenomenon of nonnegative solutions to equation (1.1) that occur in finite time when the initial energy is positive. We also look at what happens to the non-negative solutions (1.1) in the presence of arbitrarily high initial energy. Do they blow up in finite time? Here, we will examine nonstandard growth evolution equations, specifically $p(x)$-Laplacian parabolic equations within spaces of variable exponents, and we will develop proofs that are consistent with these spaces. We mention that local existence is established in [14, Theorem 2.4, Theorem 2.5].

The rest of the paper is adjusted as follows: In Section 2, we present preliminary results that are essential for the subsequent parts of the paper. Section 3 focuses on auxiliary results that are necessary for proving the main findings. In Section 4, we discuss the blow-up of weak solutions results, which are the primary objective of this paper. The method used here is concavity in variable exponent spaces. For a more profound understanding of the fundamental properties of these spaces, we suggest that the reader refer to the relevant references [7, 15, 21], which highlight the key features important for subsequent discussions.

**Definition 1.1.** A function $u$ is said to be a weak solution of problem (1.1) on $\Omega \times [0, T)$ if it satisfies the initial condition and

$$\int_\Omega u_t w dx + \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla w dx = \int_\Omega f(u) w dx.$$

for all $w \in W_0^{1,p(x)}(\Omega)$, and for almost every $t \in (0, T)$.

## 2. Auxiliary results

Let us start this section by defining the following energy functional on the space $W_0^{1,p(x)}(\Omega)$:

$$J(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \int_\Omega F(u) dx + \gamma |\Omega|, \tag{2.1}$$

and elaborate certain auxiliary lemmas, which are needed later in the proof of the main results.

A key for proving Theorem 3.1 below is the following lemma:

**Lemma 2.1.** *Assume that the function $f$ satisfies the following condition*

$$\eta_1 s^{\alpha(x)-1} \leq f(s) \leq \eta_2 \left(s^{\alpha(x)-1} + 1\right), \quad \textit{for all } s > 0, \tag{2.2}$$

*where $\eta_1, \eta_2$ are two positive constants with $\eta_1 \alpha_- > \eta_2$. Then there exists a positive initial data function $u_0 \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ such that*

$$0 < J(u_0) < M(u_0), \tag{2.3}$$

*where*

$$M(u_0) = \frac{\lambda_1(\alpha_+ - p_-) - \beta p_-}{2\alpha_+} \left( \|u_0\|_2^2 - \frac{(p_- - 2)|\Omega|}{p_-} \right) + \frac{\gamma(\alpha_+ - 1)|\Omega|}{\alpha_+}, \tag{2.4}$$

*Proof.* We first prove that for some positive $u_0 \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ such that the norm $\|u_0\|_2$ is large enough, we have

$$\begin{aligned}&\frac{(\lambda_1(\alpha_+ - p_-) - \beta p_-)(p_- - 2)|\Omega|}{2p_-} + \eta_2 \int_\Omega u_0 dx - \frac{\eta_1\alpha_- - \eta_2}{\alpha_-}\int_\Omega u_0^{\alpha(x)} dx \\ &\quad < \frac{\lambda_1(\alpha_+ - p_-) - \beta p_-}{2}\|u_0\|_2^2 - \left(\frac{\alpha_+}{p_-} - \frac{1}{p_+}\right)\int_\Omega |\nabla u_0|^{p(x)} dx. \end{aligned} \tag{2.5}$$

since $p_- - 2 > 0$ by (1.2), $\lambda_1(\alpha_+ - p_-) - \beta p_- > 0$ and $\eta_1\alpha_- - \eta_2 > 0$.
Indeed, from (1.2) and (H2), we have $2 < p(x) < \alpha(x) < p^*(x)$ then by the continuous embeddings $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^1(\Omega)$ [11, Theorem 1.11], [6, Theorem 6.29] and [7, Theorem 8.3.1], using [11, Theorem 1.3] (see also [12]) for $u_0$ and $\nabla u_0$, with the assumptions that

$$\|\nabla u\|_{p(x)},\ \|u\|_{\alpha(x)},\ \|u\|_{p(x)} > 1, \tag{2.6}$$

due to the previous embeds, since the norm $\|u_0\|_2$ is assumed large enough, we obtain

$$\begin{aligned}&\frac{(\lambda_1(\alpha_+ - p_-) - \beta p_-)(p_- - 2)|\Omega|}{2p_-} < \|u_0\|_2\Bigg[\|u_0\|_2\Bigg[\frac{\lambda_1(\alpha_+ - p_-) - \beta p_-}{2} \\ &+ \|u_0\|_2^{p_- - 2}\left(\frac{\eta_1\alpha_- - \eta_2}{\alpha_- C_1^{\alpha_-}}\|u_0\|_2^{\alpha_- - p_-} - \left(\frac{\alpha_+}{p_-} - \frac{1}{p_+}\right)\frac{1}{S^{p_-}}\right)\Bigg] - \eta_2 C_2\Bigg]. \end{aligned} \tag{2.7}$$

where $C_1, C_2$ are the constants of the Lebesgue embeddings and $S$ is the best constant of the Sobolev embedding. That is (2.7) true by the choices

$$\|u_0\|_2 > \max\left\{1, \left(\frac{(p_- - 2)|\Omega|}{p_-}\right)^{\frac{1}{2}}, \left(\left(\frac{\alpha_+}{p_-} - \frac{1}{p_+}\right)\frac{\alpha_- C_1^{\alpha_-}}{(\eta_1\alpha_- - \eta_2)S^{p_-}}\right)^{\frac{1}{\alpha_- - p_-}}\right\},$$

and

$$\begin{aligned} C_2 &< \frac{\|u_0\|_2^{p_- - 1}}{\eta_2}\left(\frac{\eta_1\alpha_- - \eta_2}{\alpha_- C_1^{\alpha_-}}\|u_0\|_2^{\alpha_- - p_-} - \frac{\alpha_+ p_+ - p_-}{p_- p_+ S^{p_-}}\right) \\ &+ \frac{\lambda_1(\alpha_+ - p_-) - \beta p_-}{2\eta_2}\|u_0\|_2. \end{aligned}$$

We get that (2.5) follows.

Now, from (1.2), (2.1), (2.2) we have

$$\begin{aligned}&\frac{1}{p_+}\int_\Omega |\nabla u_0|^{p(x)} dx - \frac{\eta_2}{\alpha_-}\int_\Omega u_0^\alpha dx - \eta_2\int_\Omega u_0 dx + \gamma|\Omega| \le J(u_0) \\ &\le \frac{1}{p_-}\int_\Omega |\nabla u_0|^{p(x)} dx - \frac{\eta_1}{\alpha_+}\int_\Omega u_0^\alpha dx + \gamma|\Omega|,\end{aligned}$$

choose $\gamma$ such that

$$\begin{aligned}&\frac{\eta_2}{\alpha_-|\Omega|}\int_\Omega u_0^{\alpha(x)} dx + \frac{\eta_2}{|\Omega|}\int_\Omega u_0 dx - \frac{1}{p_+|\Omega|}\int_\Omega |\nabla u_0|^{p(x)} dx < \gamma < \frac{\eta_1}{|\Omega|}\int_\Omega u_0^{\alpha(x)} dx \\ &- \frac{\alpha_+}{p_-|\Omega|}\int_\Omega |\nabla u_0|^{p(x)} dx + \frac{(\lambda_1(\alpha_+ - p_-) - \beta p_-)}{2|\Omega|}\left(\|u_0\|_2^2 - \frac{(p_- - 2)|\Omega|}{2p_-}\right), \end{aligned} \tag{2.8}$$

then, we can find that

$$\begin{aligned} 0 &< \frac{1}{p_+}\int_\Omega |\nabla u_0|^{p(x)}dx - \frac{\eta_2}{\alpha_-}\int_\Omega u_0^{\alpha(x)}dx - \eta_2\int_\Omega u_0 dx + \gamma|\Omega| \\ &\le \frac{1}{p_-}\int_\Omega |\nabla u_0|^{p(x)}dx - \frac{\eta_1}{\alpha_+}\int_\Omega u_0^{\alpha(x)}dx + \gamma|\Omega| < M(u_0). \end{aligned}$$

The given inequality (2.8) is also true from (2.5), and the proof is complete. □

*Remark* 2.2. Since Lemma 2.1 shows that $J(u_0) < M(u_0)$, that is to say

$$M(u_0) - J(u_0) > 0,$$

regardless of whether $M(u_0)$ is positive or negative. We can assume, without loss of generality, that $M(u_0) > 0$ throughout the paper.

Now, we suppose that the nonlinear term $f$ possesses a concrete expression; we use the fountain theorem (see [26]) for finding the finite time blow-up solution at arbitrarily high initial energy (Theorem 3.3 below). Let

$$f(s) = \gamma s^{\alpha(x)-1} + \frac{\beta p(x)}{\alpha(x)-p(x)}s^{p(x)-1}, \quad s > 0, \tag{2.9}$$

obviously, we have

$$F(s) = \frac{\gamma}{\alpha(x)}s^{\alpha(x)} + \frac{\beta}{\alpha(x)-p(x)}s^{p(x)}. \tag{2.10}$$

The nonlinear term $f$ in (2.9) obviously fulfils (H1). Additionally, if $f$ take the form (2.9), we can deduce from (2.1) and (2.10) that

$$J(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \gamma\int_\Omega \frac{1}{\alpha(x)}u^{\alpha(x)}dx - \beta\int_\Omega \frac{1}{\alpha(x)-p(x)}u^{p(x)}dx + \gamma|\Omega| \tag{2.11}$$

Here are some auxiliary lemmas needed to prove Theorem 3.3 below.

**Lemma 2.3.** *Let* (H2) *and* (2.9) *hold. There exist functions* $\{\xi_k\}_{k=1}^\infty \subset W_0^{1,p(x)}(\Omega)$ *satisfying*

$$\begin{aligned} \widetilde{J}(\xi_k) &= \int_\Omega \frac{1}{p(x)}|\nabla \xi_k|^{p(x)}dx - \gamma\int_\Omega \frac{1}{\alpha(x)}|\xi_k|^{\alpha(x)}dx \\ &\quad - \beta\int_\Omega \frac{1}{\alpha(x)-p(x)}|\xi_k|^{p(x)}dx + \gamma|\Omega| \to \infty \text{ as } k\to\infty \end{aligned} \tag{2.12}$$

*Proof.* It suffices to demonstrate that $\widetilde{J}$ meets the assumptions of [26, Theorem 3.6, page 58]; this proves the Lemma. Given the separability of $W_0^{1,p(x)}(\Omega)$ [7, Theorem 8.1.13], one can select a base $\{e_j\}_{j=1}^\infty$ of $W_0^{1,p(x)}(\Omega)$ and a base $\{l_j\}_{j=1}^\infty$ of $W^{-1,p'(x)}(\Omega)$ such that $\|\nabla e_j\|_{p(x)} = 1$ and $\|l_j\|_{W^{-1,p'(x)}(\Omega)} = 1$, with $l_j(e_i) = 1$ if $i = j$ and $l_j(e_i) = 0$ if $i \neq j$, where $W^{-1,p'(x)}(\Omega)$ is the dual space of $W_0^{1,p(x)}(\Omega)$.

For $j = 1, 2, \cdots$, we set

$$\mathcal{H}_j = \operatorname{span}\{e_j\} = \{\kappa e_j : \kappa\in\mathbb{R}\}.$$

Thus $\mathcal{H}_j \perp \mathcal{H}_i$ for $i \neq j$, i.e., $l_i(\kappa e_j) = 0$ and $l_j(\kappa e_i) = 0$ for any $\kappa \in \mathbb{R}$. In the following sense, for $k = 1, 2, \cdots$, we set

$$\mathcal{W}_k = \bigoplus_{j=1}^{k} \mathcal{H}_j, \quad \mathcal{V}_k = \overline{\bigoplus_{j=k}^{\infty} \mathcal{H}_j},$$

then

$$\mathcal{V}_{k+1} = \mathcal{W}_k^{\perp}, \quad W_0^{1,p(x)}(\Omega) = \mathcal{W}_k \bigoplus \mathcal{V}_{k+1}$$

and $\mathcal{W}_k \subset W_0^{1,p(x)}(\Omega)$ with $\dim \mathcal{W}_k < \infty$.

At First, we have $\widetilde{J} \in C^1\left(W_0^{1,p(x)}(\Omega), \mathbb{R}\right)$; this is clear from (2.10) and [16, Proposition 2.10], and it is also clear that $\widetilde{J}$ is an even functional. Next, we follow three steps:

**Step 1.** We will demonstrate that $\widetilde{J}$ satisfies the statement (1) of [26, Theorem 3.6]. Indeed, let

$$\varrho_k = \sup_{\xi \in \mathcal{V}_k, \|\nabla \xi\|_{p(x)} = 1} \|\xi\|_{\alpha(x)} \tag{2.13}$$

then it follows that $0 < \varrho_{k+1} \leq \varrho_k$. So, there is a $\varrho \geq 0$ such that

$$\varrho_k \to \varrho \text{ as } k \to \infty.$$

For every $k$, there is $\xi_k \in \mathcal{V}_k$ with $\|\nabla \xi_k\|_p = 1$ and

$$\|\xi_k\|_{\alpha(x)} > \frac{\varrho}{2} \geq 0.$$

By the definition of $\mathcal{V}_k$, we arrive at

$$\xi_k \rightharpoonup 0 \text{ weakly in } W_0^{1,p(x)}(\Omega) \text{ as } k \to \infty.$$

The compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$ since by [20, Proposition 3.3] in view of (H2), leads to

$$\xi_k \to 0 \text{ strongly in } L^{\alpha(x)}(\Omega) \text{ as } k \to \infty,$$

which means that

$$\lim_{k \to \infty} \varrho_k = 0. \tag{2.14}$$

Now, by (1.5), (2.13), [11, Theorem 1.3], assuming that $\|\nabla \xi\|_{p(x)} \geq 1$, and in view of (2.14), for $\xi \in \mathcal{V}_k$, we have

$$\begin{aligned}
\widetilde{J}(\xi) &\geq \frac{1}{p_+} \int_\Omega |\nabla \xi|^{p(x)} dx - \frac{\gamma}{\alpha_-} \int_\Omega |\xi|^{\alpha(x)} dx - \frac{\beta}{\alpha_- - p_+} \int_\Omega |\xi|^{p(x)} dx \\
&\geq \left( \frac{1}{p_+} - \frac{\beta}{\lambda_1 (\alpha_- - p_+)} \right) \|\nabla \xi\|_{p(x)}^{p_-} - \frac{\gamma \varrho_k^{\alpha_-}}{\alpha_-} \|\nabla \xi\|_{p(x)}^{\alpha_+}.
\end{aligned}$$

Choose

$$r_k = \left( \frac{\lambda_1 (\alpha_- - p_+) - \beta p_+}{\lambda_1 \gamma \varrho_k^{\alpha_-} (\alpha_- - p_+)} \right)^{\frac{1}{\alpha_+ - p_-}}.$$

So, if $\xi \in \mathcal{V}_k$ and $\|\nabla \xi\|_{p(x)} = r_k$, we get

$$\widetilde{J}(\xi) \geq \frac{\lambda_1 (\alpha_- - p_+) - \beta p_+}{\alpha_- p_+ \lambda_1} \left( \frac{\lambda_1 (\alpha_- - p_+) - \beta p_+}{\lambda_1 \gamma \varrho_k^{\alpha_-} (\alpha_- - p_+)} \right)^{\frac{p_-}{\alpha_+ - p_-}}$$

and $\widetilde{J}$ satisfies [26, Theorem 3.6] $(A_3)$.

**Step 2.** we prove that $\widetilde{J}$ fulfils assertion $(A_2)$ of [26, Theorem 3.6]. For any $\rho_k > 0$ large enough and $\widehat{\xi} \in \mathcal{W}_k$ with $\|\nabla \widehat{\xi}\|_{p(x)} = 1$, we have

$$\begin{aligned}
\widetilde{J}(\rho_k \widehat{\xi}) &= \int_\Omega \frac{1}{p(x)} |\nabla (\rho_k \widehat{\xi})|^{p(x)} dx - \gamma \int_\Omega \frac{1}{\alpha(x)} |\rho_k \widehat{\xi}|^{\alpha(x)} dx \\
&\quad - \beta \int_\Omega \frac{1}{\alpha(x) - p(x)} |\rho_k \widehat{\xi}|^{p(x)} dx + \gamma |\Omega| \\
&\leq \rho_k^{p_+} \left( \frac{1}{p_-} \int_\Omega |\nabla \widehat{\xi}|^{p(x)} dx - \frac{\gamma}{\alpha_+} \rho_k^{\alpha_- - p_+} \int_\Omega |\widehat{\xi}|^{\alpha(x)} dx \right. \\
&\quad \left. - \frac{\beta}{\alpha_+ - p_-} \int_\Omega |\widehat{\xi}|^{p(x)} dx \right) + \gamma |\Omega|. \qquad (2.15)
\end{aligned}$$

Moreover, given that $\dim \mathcal{W}_k < \infty$ and $\|\nabla \widehat{\xi}\|_{p(x)} = 1$, we can refer to [11, Theorem 1.3] to state that for certain $\iota_1, \iota_2 > 0$, we may write

$$\begin{aligned}
&\int_\Omega |\nabla \widehat{\xi}|^{p(x)} dx = \|\nabla \widehat{\xi}\|_{p(x)}^{p_\pm} = 1, \\
&\iota_1^{\alpha_\pm} \leq \min \left\{ \|\widehat{\xi}\|_{\alpha(x)}^{\alpha_-}, \|\widehat{\xi}\|_{\alpha(x)}^{\alpha_+} \right\} \leq \int_\Omega |\widehat{\xi}|^{\alpha(x)} dx \leq \iota_2^{\alpha_\pm}, \\
&\iota_1^{p_\pm} \leq \min \left\{ \|\widehat{\xi}\|_{p(x)}^{p_-}, \|\widehat{\xi}\|_{p(x)}^{p_+} \right\} \leq \int_\Omega |\widehat{\xi}|^{p(x)} dx \leq \iota_2^{p_\pm},
\end{aligned}$$

which, along with (2.15), yields

$$\widetilde{J}(\rho_k \widehat{\xi}) \leq \rho_k^{p_+} \left( \frac{1}{p_-} - \frac{\gamma \iota_1^{\alpha_\pm}}{\alpha_+} \rho_k^{\alpha_- - p_+} - \frac{\beta \iota_1^{p_\pm}}{\alpha_+ - p_-} \right) + \gamma |\Omega| \to -\infty \text{ as } \rho_k \to \infty.$$

Set $\xi = \rho_k \widehat{\xi}$, then

$$\|\nabla \xi\|_{p(x)} = \rho_k \|\nabla \widehat{\xi}\|_{p(x)} = \rho_k \text{ and } \xi \in \mathcal{W}_k.$$

Thus, for sufficiently large $\rho_k > r_k$, we get that $\widetilde{J}$ satisfies [26, Theorem 3.6] $(A_2)$.

**Step 3.** we prove that $\widetilde{J}$ satisfies [26, Theorem 3.6] $(A_4)$. Let $\{\xi_j\}_{j=1}^\infty \subset W_0^{1,p(x)}(\Omega)$ be a sequence satisfying, for any $c > 0$,

$$\widetilde{J}(\xi_j) \to c \quad \text{and} \quad \left\| \widetilde{J}'(\xi_j) \right\|_{W^{-1,p'(x)}(\Omega)} \to 0 \quad \text{as} \quad j \to \infty.$$

Then there exists $\mathcal{C}_1 > c$ and $\mathcal{C}_2 > 0$ independent of $j$ such that

$$\widetilde{J}(\xi_j) \leq \mathcal{C}_1 \quad \text{and} \quad \left\| \widetilde{J}'(\xi_j) \right\|_{W^{-1,p'(\Omega)}} \leq \mathcal{C}_2 \quad \text{for} \quad j = 1, 2, \cdots$$

Formula (1.5), in view of [11, Theorem 1.3] and [16, Proposition 2.10] give

$$\begin{aligned}
&\mathcal{C}_1+\frac{\mathcal{C}_2}{\alpha_+}\max\left\{\|\nabla\xi_j\|_{p(x)}^{p-},\|\nabla\xi_j\|_{p(x)}^{p+}\right\}\\
&\geq \mathcal{C}_1+\frac{\mathcal{C}_2}{\alpha_+}\int_\Omega|\nabla\xi_j|^{p(x)}dx\\
&\geq \widetilde{J}\left(\xi_j\right)-\frac{1}{\alpha_+}\left\langle\widetilde{J}'\left(\xi_j\right),\xi_j\right\rangle\\
&\geq \frac{\alpha_+-p_-}{\alpha_+p_-}\int_\Omega|\nabla\xi_j|^{p(x)}dx-\frac{\beta}{\alpha_+}\int_\Omega|\xi_j|^{p(x)}dx+\gamma|\Omega|\\
&\geq \frac{\lambda_1\left(\alpha_+-p_-\right)-\beta p_-}{\alpha_+p_-\lambda_1}\int_\Omega|\nabla\xi_j|^{p(x)}dx+\gamma|\Omega|\\
&\geq \frac{\lambda_1\left(\alpha_+-p_-\right)-\beta p_-}{\alpha_+p_-\lambda_1}C\max\left\{\|\nabla\xi_j\|_{p(x)}^{p-},\|\nabla\xi_j\|_{p(x)}^{p+}\right\}+\gamma|\Omega|,\quad j=1,2,\cdots,
\end{aligned}\tag{2.16}$$

where $C=\frac{\min\left\{\|\nabla\xi_j\|_{p(x)}^{p-},\|\nabla\xi_j\|_{p(x)}^{p+}\right\}}{\max\left\{\|\nabla\xi_j\|_{p(x)}^{p-},\|\nabla\xi_j\|_{p(x)}^{p+}\right\}}$ and $\langle\cdot,\cdot\rangle$ denotes the duality bracket between $W^{-1,p'(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, then there exists a $\mathcal{C}_3>0$ independent of $j$ such that

$$\|\nabla\xi_j\|_{p(x)}\leq\mathcal{C}_3,\quad j=1,2,\cdots\tag{2.17}$$

Thus we can find $\xi\in W_0^{1,p(x)}(\Omega)$ and a subsequence of $\{\xi_j\}_{j=1}^\infty$ (still denoted by $\{\xi_j\}_{j=1}^\infty$ ) such that

$$\xi_j\rightharpoonup\xi\quad\text{weakly in } W_0^{1,p(x)}(\Omega)\quad\text{as}\quad j\to\infty.\tag{2.18}$$

So, from (2.17) and (2.18), using the weak lower semicontinuity of the norm (see [7, Theorem 3.2.9 p. 77]) we obtain

$$\|\nabla\xi\|_{p(x)}\leq\liminf_{j\to\infty}\|\nabla\xi_j\|_{p(x)}\leq\mathcal{C}_3.\tag{2.19}$$

For any $\zeta\in W_0^{1,p(x)}(\Omega)$, it yields

$$\begin{aligned}
\left\langle\widetilde{J}'(\xi),\zeta\right\rangle&=\int_\Omega|\nabla\xi|^{p(x)-2}\nabla\xi\nabla\zeta dx-\gamma\int_\Omega|\xi|^{\alpha(x)-2}\xi\zeta dx\\
&\quad-\beta\int_\Omega\frac{p(x)}{\alpha(x)-p(x)}|\xi|^{p(x)-2}\xi\zeta dx,
\end{aligned}\tag{2.20}$$

consequently, replacing $\zeta$ by $\xi_j-\xi$, it holds

$$\begin{aligned}
\left\langle\widetilde{J}'(\xi_j)-\widetilde{J}'(\xi),\xi_j-\xi\right\rangle&=\int_\Omega\left(|\nabla\xi_j|^{p(x)-2}\nabla\xi_j-\nabla\xi|^{p(x)-2}\nabla\xi\right)\nabla(\xi_j-\xi)dx\\
&\quad-\gamma\int_\Omega\left(|\xi_j|^{\alpha(x)-2}\xi_j-|\xi|^{\alpha(x)-2}\xi\right)(\xi_j-\xi)dx\\
&\quad-\beta\int_\Omega\frac{p(x)}{\alpha(x)-p(x)}\left(|\xi_j|^{p(x)-2}\xi_j-|\xi|^{p(x)-2}\xi\right)(\xi_j-\xi)dx
\end{aligned}\tag{2.21}$$

Looking to (2.18), the compact embeddings given in [20, Proposition 3.3], $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ give that $\{\xi_j\}_{j=1}^{\infty}$ is bounded in $L^{\alpha(x)}(\Omega) \cap L^{p(x)}(\Omega)$, and

$$\xi_j \to \xi \quad \text{strongly in } L^{\alpha(x)}(\Omega) \cap L^{p(x)}(\Omega) \quad \text{as} \quad j \to \infty. \tag{2.22}$$

Thus (2.20) (with $\zeta$ is replaced by $\xi_j - \xi$), using the generalized Hölder inequality (see [15, 21]), (2.18) and (2.22), becomes

$$\begin{aligned} \left|\left\langle \widetilde{J}'(\xi), \xi_j - \xi \right\rangle\right| &\le \left|\int_\Omega |\nabla \xi|^{p(x)-2} \nabla \xi \nabla (\xi_j - \xi) dx\right| + \gamma \int_\Omega \left||\xi|^{\alpha(x)-2}\xi(\xi_j - \xi)\right| dx \\ &\quad + \frac{\beta p_+}{\alpha_- - p_+} \int_\Omega \left||\xi|^{p(x)-2}\xi(\xi_j - \xi)\right| dx \\ &\le \left|\int_\Omega |\nabla \xi|^{p(x)-2} \nabla \xi \nabla (\xi_j - \xi) dx\right| + \gamma \||\xi|^{\alpha(x)-1}\|_{\alpha'(x)} \|\xi_j - \xi\|_{\alpha(x)} \\ &\quad + \frac{\beta p_+}{\alpha_- - p_+} \||\xi|^{p(x)-1}\|_{p'(x)} \|\xi_j - \xi\|_{p(x)} \to 0 \text{ as } j \to \infty. \end{aligned} \tag{2.23}$$

On the other hand, it follows from (2.17) and (2.19) that

$$\left|\left\langle \widetilde{J}'(\xi_j), \xi_j - \xi \right\rangle\right| \le 2\mathcal{C}_3 \left\| \widetilde{J}'(\xi_j) \right\|_{W^{-1,p'(x)}(\Omega)} \to 0 \quad \text{as} \quad j \to \infty$$

which, when combined with (2.21) and (2.23), using the $p(x)$-Laplacian strong monotonicity inequality (see [22]), since by (1.2), yields

$$\begin{aligned} &\mathcal{C}_4 \left\| \nabla \xi_j - \nabla \xi \right\|_{p(x)}^{p_-} \\ &\le \int_\Omega \left( |\nabla \xi_j|^{p(x)-2} \nabla \xi_j - |\nabla \xi|^{p(x)-2} \nabla \xi \right) \nabla \left( \xi_j - \xi \right) dx \\ &\le \left\langle \widetilde{J}'(\xi_j) - \widetilde{J}'(\xi), \xi_j - \xi \right\rangle + \gamma \left\| |\xi_j|^{\alpha(x)-2}\xi_j - |\xi|^{\alpha(x)-2}\xi \right\|_{\alpha'(x)} \left\| \xi_j - \xi \right\|_{\alpha(x)} \\ &+ \frac{\beta p_+}{\alpha_- - p_+} \left\| |\xi_j|^{p(x)-2}\xi_j - |\xi|^{p(x)-2}\xi \right\|_{p'(x)} \left\| \xi_j - \xi \right\|_{p(x)} \to 0 \quad \text{as} \quad j \to \infty \end{aligned}$$

This shows that, $\widetilde{J}$ satisfies [26, Theorem 3.6] $(A_4)$.

Finally, based on the previous steps, we conclude that (2.12) is valid. □

**Lemma 2.4.** *Assume that* (H2) *and* (2.9) *hold. Then for any constant* $\kappa \ge 0$, *there exists a nonnegative function* $\psi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ *such that* $J(\psi) = \kappa$.

*Proof.* Let

$$v_k = |\xi_k| \in W_0^{1,p(x)}(\Omega),$$

where $\{\xi_k\}_{k=1}^{\infty}$ is the sequence given in Lemma 2.3. So, from this Lemma, we have

$$J(v_k) \to \infty \quad \text{as} \quad k \to +\infty.$$

Thus, for any constant $\kappa \ge 0$, there exists $v_k$ satisfying $J(v_k) \ge 2\kappa$.

Now let $\{\zeta_j\}_{j=1}^{\infty} \subset C_0^\infty(\Omega)$ be a sequence of nonnegative functions, then, in view of the log-Hölder continuity condition for the function $p(\cdot)$ (see [7]) it can be seen that

$$|J(\zeta_j) - J(v_k)| \to 0 \quad \text{as} \quad j \to \infty \quad \text{if} \quad \zeta_j \to v_k \quad \text{in} \quad W_0^{1,p(x)}(\Omega) \quad \text{as} \quad j \to \infty.$$

Thus, there exists a function $\zeta_j \in C_0^\infty(\Omega)$ such that $J(\zeta_j) \geq \kappa$. Let

$$\begin{aligned} g(\rho) = J(\rho\zeta_j) = & \int_\Omega \frac{1}{p(x)} |\nabla \rho\zeta_j|^{p(x)} dx - \gamma \int_\Omega \frac{1}{\alpha(x)} |\rho\zeta_j|^{\alpha(x)} dx \\ & - \beta \int_\Omega \frac{1}{\alpha(x) - p(x)} |\rho\zeta_j|^{p(x)} dx + \gamma|\Omega| \\ \leq & \frac{\rho^{p\pm}}{p_-} \int_\Omega |\nabla \zeta_j|^{p(x)} dx - \frac{\gamma\rho^{\alpha\pm}}{\alpha_+} \int_\Omega |\zeta_j|^{\alpha(x)} dx \\ & - \frac{\beta\rho^{p\pm}}{\alpha_+ - p_-} \int_\Omega |\zeta_j|^{p(x)} dx + \gamma|\Omega|, \quad \forall \zeta \geq 1 \end{aligned}$$

Obviously, the function $g(\rho)$ is continuous and

$$\lim_{\rho \to \infty} g(\rho) = -\infty,$$

since from (H2). Let $R(g(\rho))$ denote the range of $g(\rho)$, then we know $R(g(\rho)) \supset (-\infty, g(1)]$. Owing to $g(1) = J(\zeta_j)$, we obtain from $J(\zeta_j) \geq \kappa$ that there exists $\widehat{\rho} \geq 1$ such that $\psi = \widehat{\rho}\zeta_j \in C_0^\infty(\Omega) \subset W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ satisfies $J(\psi) = \kappa$. $\square$

## 3. Blow-up results

This section aims to prove that finite-time blow-up occurs for weak solutions to problem (1.1), given that the initial data $u_0$ belongs to $W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$. The following present the blow-up results of the paper.

**Theorem 3.1.** *Suppose that $u_0 \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ is a positive function. If it hold* (H1) *and*

$$J(u_0) < \max\{0, M(u_0)\}, \tag{3.1}$$

*then any positive weak solution $u$ to* (1.1) *blows up at the finite time $T_* < +\infty$ in the sense of*

$$\lim_{t \to T_*^-} \|u\|_2^2 = +\infty. \tag{3.2}$$

*Proof.* Let $u = u(t)$ for $t \in [0, T)$ be the positive weak solution to equation (1.1) as presented in [5]. This solution is contingent on the initial data $u_0$ that satisfies condition (3.1), where $T$ represents the maximum existence time.

Consider the functional

$$\Gamma(t) = \int_0^t \int_\Omega u^2 dx ds + \Theta, \quad t \geq 0, \tag{3.3}$$

where $\Theta$ is a constant that will be determined later, we have

$$\begin{aligned} \Gamma'(t) &= \int_\Omega u^2 dx \\ &= \int_0^t \int_\Omega 2uu_s dx ds + \int_\Omega u_0^2 dx, \end{aligned} \tag{3.4}$$

then, we get

$$\frac{1}{2}\Gamma''(t) = \int_{\Omega} u u_t dx$$
$$= -\int_{\Omega} |\nabla u|^{p(x)} dx + \int_{\Omega} u f(u)\, dx. \tag{3.5}$$

On the other hand we may write

$$\begin{aligned}\int_0^t \int_{\Omega} u_s^2 dx ds &= -\left[\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \int_{\Omega} \frac{1}{p(x)} |\nabla u_0|^{p(x)} dx\right] \\ &+ \left[\int_{\Omega} F(u) dx - \int_{\Omega} F(u_0) dx\right] \\ &\leq -\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\Omega} F(u) dx - \gamma|\Omega| \\ &+ \int_{\Omega} \frac{1}{p(x)} |\nabla u_0|^{p(x)} dx - \int_{\Omega} F(u_0) dx + \gamma|\Omega| \\ &= -J(u) + J(u_0)\end{aligned} \tag{3.6}$$

Therefore, from (H1), (2.1) and (3.6) we obtain

$$\begin{aligned}\frac{1}{2}\Gamma''(t) &\geq -\int_{\Omega} |\nabla u|^{p(x)} dx + \alpha_+ \int_{\Omega} F(u)\, dx - \beta \int_{\Omega} u^{p(x)} dx - \gamma|\Omega| \\ &= -\alpha_+ J(u) + \frac{\alpha_+ - p_-}{p_-} \int_{\Omega} |\nabla u|^{p(x)} dx - \beta \int_{\Omega} u^{p(x)} dx + \gamma(\alpha_+ - 1)|\Omega| \\ &\geq -\alpha_+ J(u_0) + \frac{\lambda_1(\alpha_+ - p_-) - \beta p_-}{p_-} \int_{\Omega} u^{p(x)} dx + \gamma(\alpha_+ - 1)|\Omega| \\ &+ \alpha_+ \int_0^t \int_{\Omega} u_s^2 dx ds\end{aligned} \tag{3.7}$$

We discuss two cases

**Case 1.** $J(u_0) \leq 0$

Here we use the embedding $L^{p(x)}(\Omega) \hookrightarrow L^2(\Omega)$, [17, Proposition 2.4] since by (1.2). Then [11, Theorem 1.3] in view of (2.6), (3.4), and (3.7) lead to

$$\begin{aligned}\frac{1}{2}\Gamma''(t) &\geq \frac{\lambda_1(\alpha_+ - p_-) - \beta p_-}{p_-} \|u\|_{p(x)}^{p_-} \\ &\geq \frac{\lambda_1(\alpha_+ - p_-) - \beta p_-}{p_- C_{2,p(x)}^{p_-}} \left(\|u\|_2^2\right)^{p_-/2} \\ &= \frac{\lambda_1(\alpha_+ - p_-) - \beta p_-}{p_- C_{2,p(x)}^{p_-}} \left(\Gamma'(t)\right)^{p_-/2},\end{aligned} \tag{3.8}$$

where $C_{2,p(x)}$ is the embedding constant. Integrating (3.8) over $[0,t]$ we find

$$\|u(t)\|_2^2 \geq \left(\frac{p_- C_{2,p(x)}^{p_-} \|u_0\|_2^{2-p_-}}{p_- C_{2,p(x)}^{p_-} \|u_0\|_2^{2-p_-} - (\lambda_1(\alpha_+ - p_-) - \beta p_-)(p_- - 2)\, t}\right)^{2/(p_- - 2)}.$$

This shows the blow up of weak solution to (1.1) in the sense of (3.2) with finite time

$$T_* = \frac{p_- C_{2,p(x)}^{p_-} \|u_0\|_2^{2-p_-}}{(\lambda_1(\alpha_+ - p_-) - \beta p_-)(p_- - 2)}.$$

**Case 2.** $0 < J(u_0) < M(u_0)$

In this case (3.7) may be derived using the generalised Hölder inequality and Young's inequality

$$\begin{aligned}\frac{1}{2}\Gamma''(t) \geq & -\alpha_+ J(u_0) + \frac{\lambda_1(\alpha_+ - p_-) - \beta p_-}{2}\int_\Omega u^2 dx + \gamma(\alpha_+ - 1)|\Omega| \\ & + \alpha_+ \int_0^t \int_\Omega u_s^2 dx ds - \frac{(\lambda_1(\alpha_+ - p_-) - \beta p_-)(p_- - 2)|\Omega|}{2p_-}, \end{aligned} \tag{3.9}$$

we add and subtract the term $[(\lambda_1(\alpha_+ - p_-) - \beta p_-)/2]\|u_0\|_2^2$ on the right-hand side of (3.9),in view of (3.4), resulting in the following

$$\begin{aligned}\frac{1}{2}\Gamma''(t) \geq & \alpha_+ (M(u_0) - J(u_0)) + \alpha_+ \int_0^t \int_\Omega u_s^2 dx ds \\ & + \frac{\lambda_1(\alpha_+ - p_-) - \beta p_-}{2}\int_0^t \int_\Omega 2uu_s dx ds. \end{aligned} \tag{3.10}$$

Moreover, using Hölder's inequality, we can conclude from (3.4) that

$$\begin{aligned}\left[\Gamma'(t) - \|u_0\|_2^2\right]^2 &= \left(\int_0^t \int_\Omega 2uu_s dx ds\right)^2 \\ &\leq 4\int_0^t \|u\|_2^2 ds \int_0^t \|u_s\|_2^2 ds. \end{aligned} \tag{3.11}$$

Combining (3.3), (3.10) and (3.11) we get

$$\begin{aligned}\Gamma(t)\Gamma''(t) \geq & 2\alpha_+ (M(u_0) - J(u_0)) + \frac{\alpha_+}{2}\left[\Gamma'(t) - \|u_0\|_2^2\right]^2 \\ & + (\lambda_1(\alpha_+ - p_-) - \beta p_-)\left[\Gamma'(t) - \|u_0\|_2^2\right] \\ & = 2\alpha_+ (M(u_0) - J(u_0)) + \frac{\alpha_+}{2}\left[\Gamma'(t) - \|u_0\|_2^2 + \frac{\lambda_1(\alpha_+ - p_-) - \beta p_-}{\alpha_+}\right]^2 \\ & - \left(\frac{\lambda_1(\alpha_+ - p_-) - \beta p_-}{\alpha_+}\right)^2, \end{aligned} \tag{3.12}$$

choose $\|u_0\|_2 = \sqrt{(\lambda_1(\alpha_+ - p_-) - \beta p_-)/\alpha_+}$ and $\alpha_+$ to be sufficiently large; we find

$$\Gamma(t)\Gamma''(t) - \frac{\alpha_+}{2}(\Gamma'(t))^2 \geq 2\alpha_+ (M(u_0) - J(u_0)) - \left(\frac{\lambda_1(\alpha_+ - p_-) - \beta p_-}{\alpha_+}\right)^2 > 0.$$

This inevitably leads to the blow up in finite time $T_*$ (see [16, Lemma 4.2]) of the weak solutions to (1.1) in the sense of (3.2). The proof of the theorem is achieved. □

*Remark* 3.2. The blow-up result in case 2, $(0 < J(u_0) < M(u_0))$, may be obtained in another way as given in [8, Proof of Theorem 1], by assuming that the solution $u$ exists globally and then we get a contradiction. Indeed, (3.9) can be written

$$\begin{aligned}\frac{1}{2}\Gamma''(t) &= \left(\frac{1}{2}\|u\|_2^2 - \Theta\right)' \\ &\geq \left(\lambda_1(\alpha_+ - p_-) - \beta p_-\right)\left(\frac{1}{2}\|u\|_2^2 - \Theta\right), \end{aligned} \tag{3.13}$$

where

$$\Theta = \frac{\left(\lambda_1(\alpha_+ - p_-) - \beta p_-\right)(p_- - 2)|\Omega| - 2p_-\gamma(\alpha_+ - 1) + 2p_-\alpha_+ J(u_0)}{2p_-\left(\lambda_1(\alpha_+ - p_-) - \beta p_-\right)}.$$

An integration of (3.13) on $[0,t]$ yields

$$\|u\|_2^2 \geq \left(\|u_0\|_2^2 - 2\Theta\right)e^{(\lambda_1(\alpha_+ - p_-) - \beta p_-)t} + 2\Theta, \tag{3.14}$$

and (3.14) shows that $\|u\|_2^2 \geq 0$ since by $(\lambda_1(\alpha_+ - p_-) - \beta p_-) \geq 0$ and Remark 2.2. Using [8, (18)] we arrive at

$$\left(\|u_0\|_2^2 - 2\Theta\right)e^{(\lambda_1(\alpha_+ - p_-) - \beta p_-)t} + 2\Theta \leq \left((J(u_0))^{1/2} t^{1/2} + \|u_0\|_2\right)^2, \quad t \geq 0. \tag{3.15}$$

But (3.15) cannot hold true when $t$ is large enough, leading to a contradiction. Consequently, this conclusion achieves the result.

**Theorem 3.3.** *Let $u_0 \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, suppose that (H1) holds. If (H2) and (2.9) hold, then for any constant $\mathcal{B} \geq 0$, there exists an initial positive function $u_{\mathcal{B}} \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ that satisfies (3.1) with $J(u_{\mathcal{B}}) = \mathcal{B}$, and the positive weak solution $u$ to (1.1) with $u_{\mathcal{B}}$ blows up in finite time in the sense of (3.2).*

*Proof.* Assume $\Omega_1$ and $\Omega_2$ are two arbitrary disjoint open subsets of $\Omega$. Let

$$v \in \left(W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)\right) \setminus \{0\}$$

be an arbitrary nonnegative function satisfying

$$\operatorname{supp}(v) = \overline{\{x \in \Omega : v(x) \neq 0\}} \subset \Omega_1.$$

Then for any constant $\mathcal{B} \geq 0$, one can choose $\epsilon > 0$ large enough such that

$$J(\epsilon v) \leq 0, \tag{3.16}$$

and

$$\frac{\lambda_1(\alpha_+ - p_-) - \beta p_-}{2\alpha_+}\left(\|\epsilon v\|_2^2 - \frac{(p_- - 2)|\Omega|}{p_-}\right) + \frac{\gamma(\alpha_+ - 1)|\Omega|}{\alpha_+} > \mathcal{B}. \tag{3.17}$$

From Lemma 2.4, one can select a nonnegative function

$$\varphi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$$

such that

$$\operatorname{supp}(\varphi) \subset \Omega_2 \quad \text{and} \quad J(\varphi) = \gamma|\Omega| + \mathcal{B} - J(\epsilon v).$$

Then by the choice $u_{\mathcal{B}} = \epsilon v + \varphi$, we infer from (3.16) and (3.17) that

$$J\left(u_{\mathcal{B}}\right) = J(\epsilon v) + J(\varphi) - \gamma|\Omega| = \mathcal{B},$$

and

$$\begin{aligned}
&\frac{\lambda_1(\alpha_+ - p_-) - \beta p_-}{2\alpha_+}\left(\|u_{\mathcal{B}}\|_2^2 - \frac{(p_- - 2)|\Omega|}{p_-}\right) + \frac{\gamma\left(\alpha_+ - 1\right)|\Omega|}{\alpha_+} \\
&\geq \frac{\lambda_1(\alpha_+ - p_-) - \beta p_-}{2\alpha_+}\left(\|\epsilon v\|_2^2 - \frac{(p_- - 2)|\Omega|}{p_-}\right) + \frac{\gamma\left(\alpha_+ - 1\right)|\Omega|}{\alpha_+} \\
&> \mathcal{B} = J\left(u_{\mathcal{B}}\right).
\end{aligned}$$

Taking $u_{\mathcal{B}}$ as the initial value, the blow-up result follows from Theorem 3.1. $\square$

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