

GLOBAL STABILITY OF A FRACTIONAL TWO-STRAIN SEIR MODEL

MOHSINE AARIR^{1*}, SANAA HARROUDI^{1,2}, KARAM ALLALI¹ and JAOUAD DANANE³

ABSTRACT. In this work, we study a fractional-order Susceptible-Exposed-Infected-Recovered epidemic model with general incidence rates. The model is formulated as a system of fractional differential equations. We determine the equilibrium points and compute the basic reproduction numbers using the next-generation matrix method. The global stability of equilibria is established using Lyapunov functions and LaSalle's invariance principle. Numerical simulations are performed to illustrate the theoretical results. The analysis shows that the stability of equilibria depends on the reproduction numbers, while the fractional-order parameter affects the convergence rate.

Keywords. Caputo fractional-order derivative; two-strain epidemic model; general incidence function; local and global stability; Lyapunov function method.

1. INTRODUCTION

The study of infectious disease dynamics is generally formulated through mathematical models, most often expressed as systems of differential equations [1]. During the 18th century, Daniel Bernoulli proposed a mathematical model describing smallpox transmission, marking one of the earliest foundational contributions to the field of mathematical epidemiology [2]. Mathematical modeling has become an essential tool in epidemiology for analyzing the dynamics of infectious diseases affecting human populations, including HIV [3, 4], coronavirus infections [5], hepatitis B (HBV) and other pathogens [6, 7]. A common approach to modeling infectious diseases is based on the classical SIR compartmental framework, where the population is divided into susceptible ($\mathcal{S}$), infected ($\mathcal{I}$), and recovered ($\mathcal{R}$) classes [8, 9]. This model is particularly appropriate for diseases that do not involve a latent phase, meaning that newly infected individuals become immediately infectious. In many realistic situations, infectious diseases involve a latency period during which infected individuals are not yet infectious. To account for this feature, the $SEIR$ compartmental model is introduced, incorporating an additional exposed class $\mathcal{E}$ that represents individuals who are infected but not

Date: Received: Apr 13, 2026; Revised: May 6, 2026; Accepted: May 14, 2026.

* Corresponding author

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yet capable of transmitting the disease, and who become infectious after an incubation period [10, 11, 12]. The classical formulation of $\mathcal{SEIR}$ models relies on systems of ordinary differential equations (ODEs) [13, 14, 15]. Nowadays, fractional differential equations have emerged as an effective tool for modeling the dynamics of infectious diseases, since the integer-order derivatives used in ODEs can be generalized to fractional derivatives, allowing the incorporation of memory effects. Consequently, many studies have employed systems of fractional differential equations (FDEs) to better describe biological systems [16, 17, 18]. For example, Danane et al. [19] proposed a fractional-order model for the dynamics of HBV infection that accounts for memory effects and consists of five fractional differential equations describing the interactions between healthy phagocytes, infected phagocytes, capsids, viruses and antibodies; the authors established the positivity, boundedness of solutions and global stability of the equilibrium points.

The spread of infectious diseases caused by pathogens with multiple variants, such as COVID-19 and HIV, is often modeled using multi-strain extensions of the classical SIR or $\mathcal{SEIR}$ frameworks. In many studies, disease transmission is represented through bilinear incidence functions of the form βSI , where β denotes the transmission rate. A significant portion of the literature focuses on the case of two interacting strains [20, 21], in which the infected population is divided into two distinct classes, typically denoted by $\mathcal{I}_1$ and $\mathcal{I}_2$, corresponding to individuals infected by each strain. Similarly, the exposed class is partitioned into $\mathcal{E}_1$ and $\mathcal{E}_2$, representing individuals in the latent stage for each strain. These models typically involve two bilinear incidence rates of the form $\beta_1 SI_1$ and $\beta_2 SI_2$. Recent studies have explored the dynamics of two-strain epidemic models under various assumptions. For instance, Baba et al. [22] developed a two-strain model and conducted a qualitative analysis by proving the existence, positivity, and boundedness of solutions, along with the global stability of the equilibria, complemented by numerical simulations. In a related work, Yaagoub et al. [23] proposed a two-strain epidemic model that incorporates a quarantine mechanism, where the authors established similar analytical properties and investigated the global stability of equilibria, supported by numerical experiments to highlight the role of quarantine. More recent studies have investigated fractional epidemic models with multiple strains and their stability properties [24, 25].

However, the previously proposed two-strain epidemic models use only ODEs and the general incidence functions. This classical approach is well suited for analyzing short-term epidemic dynamics and facilitates the computation of key epidemiological quantities such as the basic reproduction number R_0 and the stability of equilibrium points; however, it does not account for memory effects or long-term dependencies often observed in real epidemics. To overcome these limitations, we extend this framework by incorporating Caputo fractional derivatives and general incidence functions, where the fractional-order formulation captures memory and hereditary effects, while the generalized incidence provides a more flexible representation of non-linear transmission dynamics, leading to a richer and more realistic description of disease propagation that better reproduces real-world epidemic patterns.

The following equation system will describe our model:

$$\begin{cases} {}^C D_t^\alpha \mathcal{S}(t) = \lambda - f_1(\mathcal{S}, \mathcal{I}_1) \mathcal{I}_1 - f_2(\mathcal{S}, \mathcal{I}_2) \mathcal{I}_2 - \mu \mathcal{S}, \\ {}^C D_t^\alpha \mathcal{E}_1(t) = f_1(\mathcal{S}, \mathcal{I}_1) \mathcal{I}_1 - (\gamma_1 + \mu) \mathcal{E}_1, \\ {}^C D_t^\alpha \mathcal{E}_2(t) = f_2(\mathcal{S}, \mathcal{I}_2) \mathcal{I}_2 - (\gamma_2 + \mu) \mathcal{E}_2, \\ {}^C D_t^\alpha \mathcal{I}_1(t) = \gamma_1 \mathcal{E}_1 - (\sigma_1 + \mu) \mathcal{I}_1, \\ {}^C D_t^\alpha \mathcal{I}_2(t) = \gamma_2 \mathcal{E}_2 - (\sigma_2 + \mu) \mathcal{I}_2, \\ {}^C D_t^\alpha \mathcal{R}(t) = \sigma_1 \mathcal{I}_1 + \sigma_2 \mathcal{I}_2 - \mu \mathcal{R}. \end{cases} \quad (1.1)$$

with initial conditions: $\mathcal{S}_0 \geq 0$, $\mathcal{E}_{1,0} \geq 0$, $\mathcal{E}_{2,0} \geq 0$, $\mathcal{I}_{1,0} \geq 0$, $\mathcal{I}_{2,0} \geq 0$ and $\mathcal{R}_0 \geq 0$.

Here, $\mathcal{S}(t)$ denotes the susceptible population, $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$ the latent (exposed) classes for strains 1 and 2, $\mathcal{I}_1(t)$ and $\mathcal{I}_2(t)$ the infectious classes and $\mathcal{R}(t)$ the recovered class. The parameter λ represents the recruitment rate, μ the natural mortality, γ_1 and γ_2 , the progression rates from exposed to infectious, σ_1 and σ_2 the recovery rates. The general incidence functions $f_1(\mathcal{S}, \mathcal{I}_1)$ and $f_2(\mathcal{S}, \mathcal{I}_2)$ describe the transmission dynamics of each strain and satisfy standard biological conditions:

$$\begin{aligned} (H_1): & f_1, f_2 \in C^1(\mathbb{R}_+^2, \mathbb{R}_+), \text{ with } f_1(0, \mathcal{I}_1) = f_2(0, \mathcal{I}_2) = 0, \quad \forall \mathcal{I}_1, \mathcal{I}_2 \geq 0; \\ (H_2): & \frac{\partial f_1}{\partial \mathcal{S}}(\mathcal{S}, \mathcal{I}_1) \geq 0, \quad \frac{\partial f_2}{\partial \mathcal{S}}(\mathcal{S}, \mathcal{I}_2) \geq 0, \quad \forall \mathcal{S} > 0, \forall \mathcal{I}_1, \mathcal{I}_2 \geq 0; \\ (H_3): & \frac{\partial f_1}{\partial \mathcal{I}_1}(\mathcal{S}, \mathcal{I}_1) \leq 0, \quad \frac{\partial f_2}{\partial \mathcal{I}_2}(\mathcal{S}, \mathcal{I}_2) \leq 0, \quad \forall \mathcal{S} \geq 0, \mathcal{I}_1, \mathcal{I}_2 \geq 0. \end{aligned}$$

The three above properties (H₁), (H₂) and (H₃), these conditions, for both functions f_1 and f_2 , are satisfied by many classical biological incidence functions, such as the bilinear incidence function $\alpha \mathcal{S}$ [26, 27], the saturated incidence functions $\frac{\alpha \mathcal{S}}{1 + \sigma_1 \mathcal{S}}$ or $\frac{\alpha \mathcal{S}}{1 + \sigma_2 \mathcal{I}}$ [28, 29], the Beddington–DeAngelis incidence function $\frac{\alpha \mathcal{S}}{1 + \sigma_1 \mathcal{S} + \sigma_2 \mathcal{I}}$ [30, 31], the Crowley–Martin incidence function $\frac{\alpha \mathcal{S}}{1 + \sigma_1 \mathcal{S} + \sigma_2 \mathcal{I} + \sigma_1 \sigma_2 \mathcal{S} \mathcal{I}}$ [32, 33] and the non-monotone incidence function $\frac{\alpha \mathcal{S}}{1 + \sigma \mathcal{I}^2}$ [34, 35]. Moreover, we analyze the qualitative proprieties of the the proposed model, including positivity, boundeness, equilibrium states and their stability. A schematic representation of the proposed two-strain epidemiological model (1.1) is presented in Fig. 1.

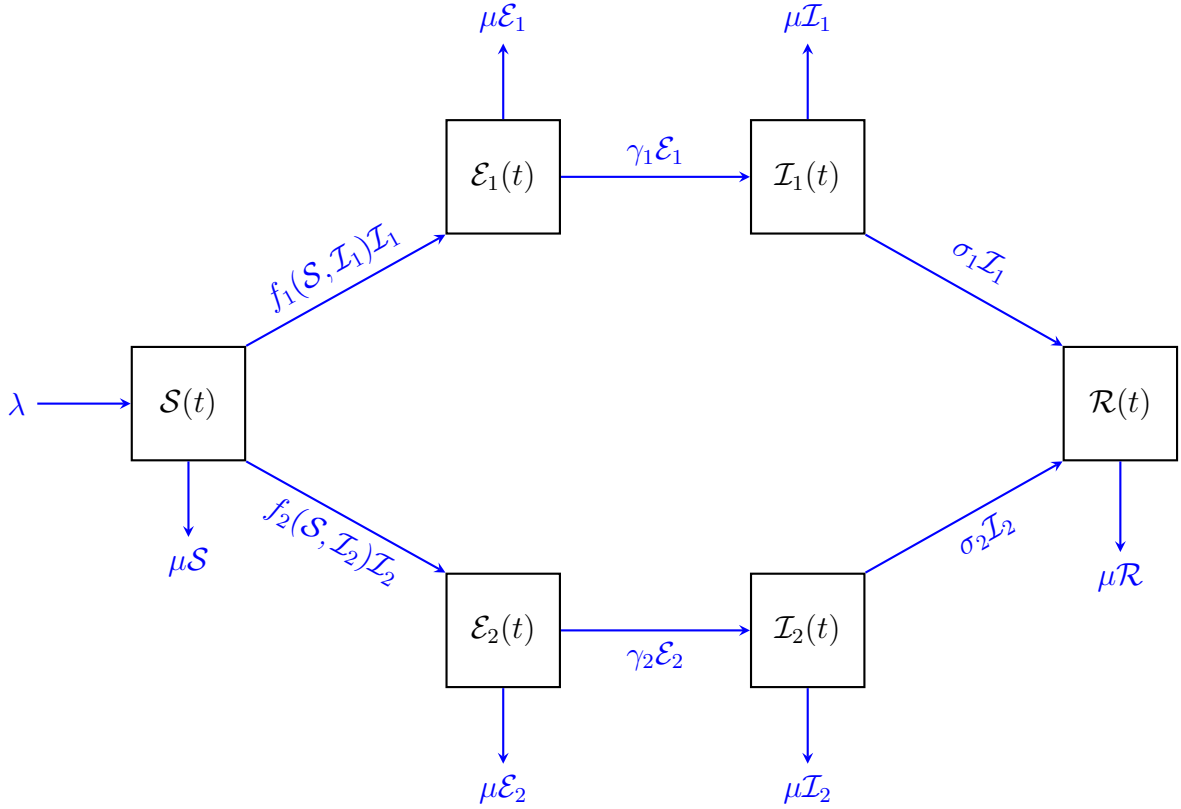


FIGURE 1. Flow diagram of the two-strain SEIR model.

The remainder of this paper is organized as follows. Section 2 introduces the basic definitions related to fractional derivatives. In Section 3, we investigate the well-posedness of the model by establishing the positivity and boundedness of solutions. Section 4 is devoted to the characterization of equilibrium points in relation to the basic reproduction numbers of each strain. The global stability of these equilibria is analyzed in Section 5 using appropriate Lyapunov functionals. Numerical simulations are presented in Section 6 to support the theoretical results, and the paper concludes with a final section summarizing the main findings.

2. PRELIMINARY RESULTS

Definition 2.1. [36] Let $\psi : [0, \infty) \rightarrow \mathbb{R}$ be a differentiable function and $0 < \alpha \leq 1$. The Caputo derivative of fractional order α is defined as

$${}^C D_t^\alpha \psi(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\psi'(s)}{(t-s)^\alpha} ds, \quad t > 0.$$

Definition 2.2. [37] The Mittag-Leffler function E_α of the parameters $\alpha > 0$ defined by: $E_\alpha(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(\alpha i + 1)}$.

Let $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ where $n > 0$ and let the following fractional order system

$${}^C D_t^\alpha Y(t) = g(Y), \quad (2.1)$$

With $Y(0) = Y_0$, $Y_0 \in \mathbb{R}^n$ and $0 < \alpha < 1$. To establish the existence and uniqueness of solutions to system (2.1), we present the following result

Proposition 2.3. [38] *If the function g satisfies the two following conditions:*

- (1) $g(Y)$ and $\frac{\partial g}{\partial Y}(Y)$ are continuous on $\mathbb{R}^n$.
- (2) $\|g(Y)\| < K_1 + K_2\|Y\|$ for all $Y \in \mathbb{R}^n$, with K_1 and K_2 are two positive constants. Then, the system (2.1) has a unique solution defined on $\mathbb{R}_+^n$.

Lemma 2.4. ([39] *LaSalle's invariance principale*) *Let M be a compact set of $\mathbb{R}_+^n$. All solutions of the system (2.1), starting from M , remain in M for all t , if there exists a class C^1 function $L(x)$ defined on M and satisfying the following condition:*

$$D^\alpha L \leq 0. \quad (2.4)$$

Let E be the set of the points in M where $D^\alpha V = 0$, and D be the largest invariant set in E . Then, every solution $y(t)$ starting in M approaches D as $t \rightarrow \infty$.

3. EXISTENCE, POSITIVITY AND BOUNDEDNESS OF SOLUTIONS

Here, we establish that the solutions of model (1.1) are strictly positive and bounded. For biological consistency, we consider initial conditions such that, $\mathcal{S}_0$, $\mathcal{E}_{1,0}$, $\mathcal{E}_{2,0}$, $\mathcal{I}_{1,0}$, $\mathcal{I}_{2,0}$ and $\mathcal{R}_0$, are all non-negative. The result is formulated in the following proposition:

Proposition 3.1. *For any positive initial conditions $\mathcal{S}_0$, $\mathcal{E}_{1,0}$, $\mathcal{E}_{2,0}$, $\mathcal{I}_{1,0}$, $\mathcal{I}_{2,0}$ and $\mathcal{R}_0$, the system (1.1) admits a unique positive and bounded solution defined on $[0, +\infty)$ for all $t > 0$.*

Proof. We can rewrite our model (1.1) in a compact form as

$${}^C D_t^\alpha \mathcal{U}(t) = \mathcal{F}(\mathcal{U}), \quad (3.1)$$

where $\mathcal{F}(\mathcal{U}) = (\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2, \mathcal{R})$ and,

$$\mathcal{F}(\mathcal{U}) = \begin{pmatrix} \lambda - f_1(\mathcal{S}, \mathcal{I}_1)\mathcal{I}_1 - f_2(\mathcal{S}, \mathcal{I}_2)\mathcal{I}_2 - \mu\mathcal{S} \\ f_1(\mathcal{S}, \mathcal{I}_1)\mathcal{I}_1 - (\gamma_1 + \mu)\mathcal{E}_1 \\ f_2(\mathcal{S}, \mathcal{I}_2)\mathcal{I}_2 - (\gamma_2 + \mu)\mathcal{E}_2 \\ \gamma_1\mathcal{E}_1 - (\sigma_1 + \mu)\mathcal{I}_1 \\ \gamma_2\mathcal{E}_2 - (\sigma_2 + \mu)\mathcal{I}_2 \\ \sigma_1\mathcal{I}_1 + \sigma_2\mathcal{I}_2 - \mu\mathcal{R} \end{pmatrix}.$$

For the fractional case, we can decompose:

$$\mathcal{F}(\mathcal{U}) = \mathcal{A}_1 + \mathcal{A}_2\mathcal{U} + \mathcal{A}_3\mathcal{U},$$

with,

$$\mathcal{A}_1 = \begin{pmatrix} \lambda \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} -\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\gamma_1 + \mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\gamma_2 + \mu) & 0 & 0 & 0 \\ 0 & \gamma_1 & 0 & -(\sigma_1 + \mu) & 0 & 0 \\ 0 & 0 & \gamma_2 & 0 & -(\sigma_2 + \mu) & 0 \\ 0 & 0 & 0 & \sigma_1 & \sigma_2 & -\mu \end{pmatrix},$$

$$\mathcal{A}_3 = \begin{pmatrix} 0 & 0 & 0 & -f_1(\mathcal{S}, \mathcal{I}_1) & -f_2(\mathcal{S}, \mathcal{I}_2) & 0 \\ 0 & 0 & 0 & f_1(\mathcal{S}, \mathcal{I}_1) & 0 & 0 \\ 0 & 0 & 0 & 0 & f_2(\mathcal{S}, \mathcal{I}_2) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, we obtain

$$\|\mathcal{F}(\mathcal{U}(t))\| \leq \|\mathcal{A}_1\| + (\|\mathcal{A}_2\| + \|\mathcal{A}_3\|) \|U(t)\| = \omega + \theta \|U(t)\|,$$

where $\omega = \|\mathcal{A}_1\|$ and $\theta = \|\mathcal{A}_2 + \mathcal{A}_3\|$. Hence, the conditions of proposition (2.3) are satisfied, guaranteeing the uniqueness of the solution on $\mathbb{R}^+$. To ensure positivity, we observe that

$$\begin{cases} {}^c D_t^\alpha \mathcal{S}|_{\mathcal{S}=0} = \lambda > 0, \\ {}^c D_t^\alpha \mathcal{E}_1|_{\mathcal{E}_1=0} = f_1(\mathcal{S}, \mathcal{I}_1) \mathcal{I}_1 \geq 0, \\ {}^c D_t^\alpha \mathcal{E}_2|_{\mathcal{E}_2=0} = f_2(\mathcal{S}, \mathcal{I}_2) \mathcal{I}_2 \geq 0, \\ {}^c D_t^\alpha \mathcal{I}_1|_{\mathcal{I}_1=0} = \gamma_1 \mathcal{E}_1 > 0, \\ {}^c D_t^\alpha \mathcal{I}_2|_{\mathcal{I}_2=0} = \gamma_2 \mathcal{E}_2 > 0, \\ {}^c D_t^\alpha \mathcal{R}|_{\mathcal{R}=0} = \sigma_1 \mathcal{I}_1 + \sigma_2 \mathcal{I}_2 > 0. \end{cases}$$

Thus, all solutions remain non-negative.

Finally, for boundedness, let $\mathcal{N} = \mathcal{S} + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{R}$.

Summing all equations gives ${}^c D_t^\alpha \mathcal{N}(t) = \lambda - \mu \mathcal{N}(t)$ and solving this fractional linear

equation yields $\mathcal{N}(t) < \frac{\lambda}{\mu} + \left(\mathcal{N}(0) - \frac{\lambda}{\mu} \right) E_\alpha(-\mu t^\alpha)$, where $E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}$

is the Mittag-Leffler function. Since, $0 < E_\alpha(-\mu t^\alpha) < 1$, we get $\mathcal{N}(t) < \mathcal{N}(0) + \frac{\lambda}{\mu}$,

completing the proof of positivity and boundedness. $\square$

4. THE STEADY STATES

This section establishes the existence of three endemic equilibria along with a disease-free equilibrium. Since the first five equations of system (1.1) are independent of $\mathcal{R}$ and the sixth equation depends solely on the total population, it

can be omitted. Thus, the system reduces to

$$\begin{cases} {}^C D_t^\alpha \mathcal{S}(t) = \lambda - f_1(\mathcal{S}, \mathcal{I}_1) \mathcal{I}_1 - f_2(\mathcal{S}, \mathcal{I}_2) \mathcal{I}_2 - \mu \mathcal{S}, \\ {}^C D_t^\alpha \mathcal{E}_1(t) = f_1(\mathcal{S}, \mathcal{I}_1) \mathcal{I}_1 - (\gamma_1 + \mu) \mathcal{E}_1, \\ {}^C D_t^\alpha \mathcal{E}_2(t) = f_2(\mathcal{S}, \mathcal{I}_2) \mathcal{I}_2 - (\gamma_2 + \mu) \mathcal{E}_2, \\ {}^C D_t^\alpha \mathcal{I}_1(t) = \gamma_1 \mathcal{E}_1 - (\sigma_1 + \mu) \mathcal{I}_1, \\ {}^C D_t^\alpha \mathcal{I}_2(t) = \gamma_2 \mathcal{E}_2 - (\sigma_2 + \mu) \mathcal{I}_2, \end{cases} \quad (4.1)$$

with, $\mathcal{R} = \mathcal{N} - \mathcal{S} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{I}_1 - \mathcal{I}_2$.

The basic reproduction number is defined as the average number of secondary infections produced by a typical infected individual. The basic reproduction number is obtained via the next-generation matrix method, where $R_0 = \rho(FV^{-1})$ denotes the spectral radius of the matrix FV^{-1} . Here, ρ denotes the spectral radius, F represents the nonnegative matrix associated with new infections, and V corresponds to the transition matrix describing the transfer of individuals between infected compartments. In this model, the infectious classes are $\mathcal{I}_1$ and $\mathcal{I}_2$. Accordingly, the matrices F and V are given by:

$$F = \begin{pmatrix} 0 & 0 & f_1(\frac{\lambda}{\mu}, 0) & 0 \\ 0 & 0 & 0 & f_2(\frac{\lambda}{\mu}, 0) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \gamma_1 + \mu & 0 & 0 & 0 \\ 0 & \gamma_2 + \mu & 0 & 0 \\ -\gamma_1 & 0 & \sigma_1 + \mu & 0 \\ 0 & -\gamma_2 & 0 & \sigma_2 + \mu \end{pmatrix}.$$

So

$$FV^{-1} = \begin{pmatrix} \frac{f_1(\frac{\lambda}{\mu}, 0)\gamma_1}{(\gamma_1 + \mu)(\sigma_1 + \mu)} & 0 & \frac{f_1(\frac{\lambda}{\mu}, 0)}{(\sigma_1 + \mu)}, 0 & 0 \\ 0 & \frac{f_2(\frac{\lambda}{\mu}, 0)\gamma_2}{(\gamma_2 + \mu)(\sigma_2 + \mu)} & 0 & \frac{f_2(\frac{\lambda}{\mu}, 0)}{(\sigma_1 + \mu)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The basic reproduction number is the spectral radius of the matrix FV^{-1} . This fact implies that $R_0 = \max \{R_0^1, R_0^2\}$ with,

$$R_0^1 = \frac{f_1(\frac{\lambda}{\mu}, 0)\gamma_1}{(\gamma_1 + \mu)(\sigma_1 + \mu)}, \quad R_0^2 = \frac{f_2(\frac{\lambda}{\mu}, 0)\gamma_2}{(\gamma_2 + \mu)(\sigma_2 + \mu)}.$$

We denote $a = \gamma_1 + \mu$, $b = \gamma_2 + \mu$, $c = \sigma_1 + \mu$, $d = \sigma_2 + \mu$.

Then

$$R_0^1 = \frac{f_1(\frac{\lambda}{\mu}, 0)\gamma_1}{ac}, \quad R_0^2 = \frac{f_2(\frac{\lambda}{\mu}, 0)\gamma_2}{bd}.$$

The strain 1 and strain 2 reproduction numbers are denoted by R_0^1 and R_0^2 , respectively.

Theorem 4.1. *The system (4.1) has a unique disease-free equilibrium $\mathcal{E}_f$ and three endemic equilibria under the following conditions:*

- If $R_0^1 > 1$, then the strain 1 endemic equilibrium $\mathcal{E}_{S_1}$ exists.
- If $R_0^2 > 1$, then the strain 2 endemic equilibrium $\mathcal{E}_{S_2}$ exists.
- If $\min\{R_0^1, R_0^2\} > 1$, then the both strains endemic equilibrium $\mathcal{E}_{S_t}$ exists.

Proof. To determine the equilibrium points of system (4.1), we consider the following system of equations:

$$\begin{cases} 0 = \lambda - f_1(\mathcal{S}, \mathcal{I}_1)\mathcal{I}_1 - f_2(\mathcal{S}, \mathcal{I}_2)\mathcal{I}_2 - \mu\mathcal{S}, \\ 0 = f_1(\mathcal{S}, \mathcal{I}_1)\mathcal{I}_1 - (\gamma_1 + \mu)\mathcal{E}_1, \\ 0 = f_2(\mathcal{S}, \mathcal{I}_2)\mathcal{I}_2 - (\gamma_2 + \mu)\mathcal{E}_2, \\ 0 = \gamma_1\mathcal{E}_1 - (\sigma_1 + \mu)\mathcal{I}_1, \\ 0 = \gamma_2\mathcal{E}_2 - (\sigma_2 + \mu)\mathcal{I}_2. \end{cases} \quad (4.2)$$

From the equilibrium system (4.2), we obtain the following results:

- (1) Disease-free equilibrium: When $\mathcal{I}_1 = 0$ and $\mathcal{I}_2 = 0$, the system admits the disease-free equilibrium

$$\mathcal{E}_f = \left(\frac{\lambda}{\mu}, 0, 0, 0, 0 \right).$$

- (2) Strain 1 endemic equilibrium: When $\mathcal{I}_1 \neq 0$ and $\mathcal{I}_2 = 0$, the strain 1 endemic equilibrium is given by: $\mathcal{E}_{s_1} = \left(\mathcal{S}_1^*, \frac{\lambda - \mu\mathcal{S}_1^*}{a}, 0, \frac{\gamma_1(\lambda - \mu\mathcal{S}_1^*)}{ac}, 0 \right)$, where,

$$\mathcal{S}_1^* \in \left[0, \frac{\lambda}{\mu} \right].$$

Define now a function Ψ on $[0, +\infty[$ as follows: $\Psi(\mathcal{S}) = f_1\left(\mathcal{S}, \frac{\gamma_1(\lambda - \mu\mathcal{S})}{ac}\right) - \frac{ac}{\gamma_1}$.

$$\text{We have } \frac{\partial \Psi(\mathcal{S})}{\partial \mathcal{S}} = \frac{\partial f_1(\mathcal{S}, \mathcal{I}_1)}{\partial \mathcal{S}} + \frac{\partial f_1(\mathcal{S}, \mathcal{I}_1)}{\partial \mathcal{I}_1} \left(\frac{-\mu\gamma_1}{ac} \right),$$

From the hypothesis (H_2) and (H_3) , we have $\frac{\partial \Psi(\mathcal{S})}{\partial \mathcal{S}} \geq 0$.

$$\text{Moreover, } \Psi(0) = f_1(0, \mathcal{I}_{1,s_1}^*) - \frac{ac}{\gamma_1} = -\frac{ac}{\gamma_1} < 0.$$

Therefore, for $R_0^1 > 1$, we have $\Psi\left(\frac{\lambda}{\mu}\right) = f_1\left(\frac{\lambda}{\mu}, 0\right) - \frac{ac}{\gamma_1} = \frac{ac}{\gamma_1}(R_0^1 - 1) > 0$.

Therefore, a unique strain 1 endemic equilibrium exists: $\mathcal{E}_{s_1} = (\mathcal{S}_1^*, \mathcal{E}_{1,s_1}^*, \mathcal{E}_{2,s_1}^*, \mathcal{I}_{1,s_1}^*, \mathcal{I}_{2,s_1}^*)$, with $\mathcal{S}_1^* \in \left] 0, \frac{\lambda}{\mu} \right[$, $\mathcal{E}_{1,s_1}^* > 0$, $\mathcal{I}_{1,s_1}^* > 0$ and $\mathcal{E}_{2,s_1}^* = \mathcal{I}_{2,s_1}^* = 0$.

- (3) Strain 2 endemic equilibrium: When $\mathcal{I}_2 \neq 0$ and $\mathcal{I}_1 = 0$, the strain 2 endemic equilibrium is $\mathcal{E}_{s_2} = \left(\mathcal{S}_2^*, 0, \frac{\lambda - \mu \mathcal{S}_2^*}{b}, 0, \frac{\gamma_2(\lambda - \mu \mathcal{S}_2^*)}{bd} \right)$, where $\mathcal{S}_2^* \in \left[0, \frac{\lambda}{\mu} \right]$.

Define also a function Φ on $[0, +\infty[$ as follows: $\Phi(\mathcal{S}) = f_2 \left(\mathcal{S}, \frac{\gamma_2(\lambda - \mu \mathcal{S})}{bd} \right) - \frac{bd}{\gamma_2}$.

Then $\frac{\partial \Phi(\mathcal{S})}{\partial \mathcal{S}} = \frac{\partial f_2(\mathcal{S}, \mathcal{I}_2)}{\partial \mathcal{S}} + \frac{\partial f_2(\mathcal{S}, \mathcal{I}_2)}{\partial \mathcal{I}_2} \left(\frac{-\mu \gamma_2}{bd} \right)$. Using the hypothesis

(H_2) and (H_3) , we have $\frac{\partial \Phi(\mathcal{S})}{\partial \mathcal{S}} \geq 0$.

Moreover, $\Phi(0) = f_2(0, \mathcal{I}_{2,s_2}^*) - \frac{bd}{\gamma_2} = -\frac{bd}{\gamma_2} < 0$

and for $R_0^2 > 1$, $\Phi \left(\frac{\lambda}{\mu} \right) = f_2 \left(\frac{\lambda}{\mu}, 0 \right) - \frac{bd}{\gamma_2} = \frac{bd}{\gamma_2} (R_0^2 - 1) > 0$.

Hence, a unique strain 2 endemic equilibrium exists: $\mathcal{E}_{s_2} = (\mathcal{S}_2^*, \mathcal{E}_{1,s_2}^*, \mathcal{E}_{2,s_2}^*, \mathcal{I}_{1,s_2}^*, \mathcal{I}_{2,s_2}^*)$, with $\mathcal{S}_2^* \in \left] 0, \frac{\lambda}{\mu} \right[$, $\mathcal{E}_{2,s_2}^* > 0$, $\mathcal{I}_{2,s_2}^* > 0$ and $\mathcal{E}_{1,s_2}^* = \mathcal{I}_{1,s_2}^* = 0$.

- (4) Both endemic equilibrium: When $\mathcal{I}_1 \neq 0$ and $\mathcal{I}_2 \neq 0$, the global endemic equilibrium is $\mathcal{E}_t = (\mathcal{S}_t^*, \mathcal{E}_{1,t}^*, \mathcal{E}_{2,t}^*, \mathcal{I}_{1,t}^*, \mathcal{I}_{2,t}^*)$, where, $\mathcal{E}_{1,t}^* = \frac{c}{\gamma_1} \mathcal{I}_{1,t}^*$, $\mathcal{E}_{2,t}^* =$

$$\frac{d}{\gamma_2} \mathcal{I}_{2,t}^*, \quad \mathcal{S}_t^* = \frac{1}{\mu} \left[\lambda - \frac{f_1\left(\frac{\lambda}{\mu}, 0\right)}{R_0^1} \mathcal{I}_{1,t}^* - \frac{f_2\left(\frac{\lambda}{\mu}, 0\right)}{R_0^2} \mathcal{I}_{2,t}^* \right].$$

With, $\lambda \geq \frac{f_1\left(\frac{\lambda}{\mu}, 0\right)}{R_0^1} \mathcal{I}_{1,t}^* + \frac{f_2\left(\frac{\lambda}{\mu}, 0\right)}{R_0^2} \mathcal{I}_{2,t}^*$, $R_0^1 > 1$ and $R_0^2 > 1$.

□

5. GLOBAL STABILITY OF EQUILIBRIA

5.1. Global stability analysis of disease-free equilibrium state.

Theorem 5.1. *If $R_0 < 1$, then the disease-free equilibrium $\mathcal{E}_f$ is globally asymptotically stable.*

Proof. In $\mathbb{R}_+$, we first consider the Lyapunov function as follows:

$$\mathcal{P}_0(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) = \mathcal{S} - \mathcal{S}_0^* - \int_{\mathcal{S}_0^*}^{\mathcal{S}} \frac{f_1(\mathcal{S}_0^*, 0)}{f_1(X, 0)} dX + \mathcal{E}_1 + \mathcal{E}_2 + \frac{a}{\gamma_1} \mathcal{I}_1 + \frac{b}{\gamma_2} \mathcal{I}_2.$$

Computing the time derivative yields

$$\begin{aligned}
{}^C D_t^\alpha \mathcal{P}_0(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) &\leq {}^C D_t^\alpha \mathcal{S} - \frac{f_1(\mathcal{S}_0^*, 0)}{f_1(\mathcal{S}, 0)} {}^C D_t^\alpha \mathcal{S} + {}^C D_t^\alpha \mathcal{E}_1 + {}^C D_t^\alpha \mathcal{E}_2 + \frac{a}{\gamma_1} {}^C D_t^\alpha \mathcal{I}_1 + \frac{b}{\gamma_2} {}^C D_t^\alpha \mathcal{I}_2 \\
&\leq \mu \mathcal{S}_0^* \left(1 - \frac{\mathcal{S}}{\mathcal{S}_0^*}\right) \left(1 - \frac{f_1(\mathcal{S}_0^*, 0)}{f_1(\mathcal{S}, 0)}\right) + \frac{ac}{\gamma_1} \mathcal{I}_1 \left(\frac{f_1(\mathcal{S}, \mathcal{I}_1)}{f_1(\mathcal{S}, 0)} R_0^1 - 1\right) \\
&\quad + \frac{bd}{\gamma_2} \mathcal{I}_2 \left(\frac{f_1(\mathcal{S}_0^*, 0)}{f_1(\mathcal{S}, 0)} \frac{f_2(\mathcal{S}, \mathcal{I}_2)}{f_2(\mathcal{S}_0^*, 0)} R_0^2 - 1\right) \\
&\leq \mu \mathcal{S}_0^* \left(1 - \frac{\mathcal{S}}{\mathcal{S}_0^*}\right) \left(1 - \frac{f_1(\mathcal{S}_0^*, 0)}{f_1(\mathcal{S}, 0)}\right) + \frac{ac}{\gamma_1} \mathcal{I}_1 (R_0^1 - 1) \\
&\quad + \frac{bd}{\gamma_2} \mathcal{I}_2 \left(\frac{f_1(\mathcal{S}_0^*, 0)}{f_1(\mathcal{S}, 0)} \frac{f_2(\mathcal{S}, \mathcal{I}_2)}{f_2(\mathcal{S}_0^*, 0)} R_0^2 - 1\right).
\end{aligned}$$

We analyze the following two cases:

- If $\mathcal{S} \leq \mathcal{S}_0^*$, using (H_2) we have: $\frac{f_2(\mathcal{S}, \mathcal{I}_2)}{f_2(\mathcal{S}_0^*, 0)} \leq 1$,
then,

$$\begin{aligned}
{}^C D_t^\alpha \mathcal{P}_0(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) &\leq \mu \mathcal{S}_0^* \left(1 - \frac{\mathcal{S}}{\mathcal{S}_0^*}\right) \left(1 - \frac{f_1(\mathcal{S}_0^*, 0)}{f_1(\mathcal{S}, 0)}\right) + \frac{ac}{\gamma_1} \mathcal{I}_1 (R_0^1 - 1) \\
&\quad + \frac{bd}{\gamma_2} \mathcal{I}_2 \left(\frac{f_1(\mathcal{S}_0^*, 0)}{f_1(\mathcal{S}, 0)} R_0^2 - 1\right).
\end{aligned}$$

Since, $R_0^2 \leq \frac{f_1(\mathcal{S}, 0)}{f_1(\mathcal{S}_0^*, 0)} \leq 1$, we obtain $\frac{f_1(\mathcal{S}_0^*, 0)}{f_1(\mathcal{S}, 0)} R_0^2 - 1 \leq 0$.

Otherwise, $1 - \frac{f_1(\mathcal{S}_0^*, 0)}{f_1(\mathcal{S}, 0)} \leq 0$, thus $\mu \mathcal{S}_0^* \left(1 - \frac{\mathcal{S}}{\mathcal{S}_0^*}\right) \left(1 - \frac{f_1(\mathcal{S}_0^*, 0)}{f_1(\mathcal{S}, 0)}\right) \leq 0$.

- If $\mathcal{S} > \mathcal{S}_0^*$, using (H_2) we have, $\frac{f_2(\mathcal{S}, \mathcal{I}_2)}{f_2(\mathcal{S}_0^*, 0)} > 1$ and $\frac{f_1(\mathcal{S}_0^*, 0)}{f_1(\mathcal{S}, 0)} < 1$,
then,

$$\begin{aligned}
{}^C D_t^\alpha \mathcal{P}_0(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) &\leq \mu \mathcal{S}_0^* \left(1 - \frac{\mathcal{S}}{\mathcal{S}_0^*}\right) \left(1 - \frac{f_1(\mathcal{S}_0^*, 0)}{f_1(\mathcal{S}, 0)}\right) + \frac{ac}{\gamma_1} \mathcal{I}_1 (R_0^1 - 1) \\
&\quad + \frac{bd}{\gamma_2} \mathcal{I}_2 \left(\frac{f_2(\mathcal{S}, \mathcal{I}_2)}{f_2(\mathcal{S}_0^*, 0)} R_0^2 - 1\right).
\end{aligned}$$

Since, $R_0^2 < \frac{f_2(\mathcal{S}_0^*, 0)}{f_2(\mathcal{S}, \mathcal{I}_2)} < 1$, we obtain: $\frac{f_2(\mathcal{S}, \mathcal{I}_2)}{f_2(\mathcal{S}_0^*, 0)} R_0^2 - 1 \leq 0$.

Also, $\mu \mathcal{S}_0^* \left(1 - \frac{\mathcal{S}}{\mathcal{S}_0^*}\right) \left(1 - \frac{f_1(\mathcal{S}_0^*, 0)}{f_1(\mathcal{S}, 0)}\right) \leq 0$.

The discussion above shows that, if $R_0^2 \leq 1$ and $R_0^1 \leq 1$. Then, ${}^C D_t^\alpha \mathcal{P}_0(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) \leq 0$.

Let $V_0 = \{(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) / {}^C D_t^\alpha \mathcal{P}_0(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) = 0\}$, then the largest positively invariant subset of V_0 reduces to the singleton $\mathcal{E}_f$. Consequently, by Lemma (2.4), the disease-free equilibrium is globally asymptotically stable.. $\square$

5.2. Global stability analysis of the strain-1 endemic state. We assume that the function f_1 satisfies the following condition for the global stability $\mathcal{E}_{s_1}$:

$$\left(1 - \frac{f_1(\mathcal{S}, \mathcal{I}_1)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)}\right) \left(\frac{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_1)} - \frac{\mathcal{I}_1}{\mathcal{I}_{1,s_1}^*}\right) \leq 0, \quad \forall \mathcal{S}, \mathcal{I}_1 > 0. \quad (5.1)$$

Theorem 5.2. *If $R_0^2 \leq 1 < R_0^1$, then The strain 1 endemic equilibrium $\mathcal{E}_{s_1}$ is globally asymptotically stable*

Proof. First, let us define the Lyapunov function $\mathcal{P}_1$ by:

$$\begin{aligned} \mathcal{P}_1(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) &= \mathcal{S} - \mathcal{S}_1^* - \int_{\mathcal{S}_1^*}^{\mathcal{S}} \frac{f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*)}{f_1(X, \mathcal{I}_{1,s_1}^*)} dX \\ &\quad + \mathcal{E}_{1,s_1}^* \left(\frac{\mathcal{E}_1}{\mathcal{E}_{1,s_1}^*} - \ln \left(\frac{\mathcal{E}_1}{\mathcal{E}_{1,s_1}^*} \right) - 1 \right) + \mathcal{E}_2 \\ &\quad + \frac{a}{\alpha_1} \mathcal{I}_{1,s_1}^* \left(\frac{\mathcal{I}_1}{\mathcal{I}_{1,s_1}^*} - \ln \left(\frac{\mathcal{I}_1}{\mathcal{I}_{1,s_1}^*} \right) - 1 \right) + \frac{b}{\alpha_2} \mathcal{I}_2. \end{aligned}$$

Computing the time derivative yields

$$\begin{aligned} {}^C D_t^\alpha \mathcal{P}_1(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) &\leq (\lambda - f_1(\mathcal{S}, \mathcal{I}_1) \mathcal{I}_1 - f_2(\mathcal{S}, \mathcal{I}_2) \mathcal{I}_2 - \mu \mathcal{S}) \left(1 - \frac{f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)} \right) \\ &\quad + (f_1(\mathcal{S}, \mathcal{I}_1) \mathcal{I}_1 - a \mathcal{E}_1) \left(1 - \frac{\mathcal{E}_{1,s_1}^*}{\mathcal{E}_1} \right) \\ &\quad + (f_2(\mathcal{S}, \mathcal{I}_2) \mathcal{I}_2 - b \mathcal{E}_2) \\ &\quad + \frac{a}{\gamma_1} (\gamma_1 \mathcal{E}_1 - c \mathcal{I}_1) \left(1 - \frac{\mathcal{I}_{1,s_1}^*}{\mathcal{I}_1} \right) \\ &\quad + \frac{b}{\gamma_2} (\gamma_2 \mathcal{E}_2 - d \mathcal{I}_2). \end{aligned}$$

We have

$$\begin{cases} \lambda = \mu \mathcal{S}_1^* + f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*) \mathcal{I}_{1,s_1}^*, \\ f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*) \mathcal{I}_{1,s_1}^* = \frac{ac}{\gamma_1} \mathcal{I}_{1,s_1}^* = a \mathcal{E}_{1,s_1}^*, \\ \frac{\mathcal{E}_{1,s_1}^*}{\mathcal{I}_{1,s_1}^*} = \frac{c}{\gamma_1}. \end{cases}$$

Therefore,

$$\begin{aligned}
{}^C D_t^\alpha \mathcal{P}_1(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) &\leq \mu \mathcal{S}_1^* \left(1 - \frac{f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)} \right) - \mu \mathcal{S} \left(1 - \frac{f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)} \right) \\
&\quad + f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*) \mathcal{I}_{1,s_1}^* - f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*) \mathcal{I}_1^* \frac{f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)} \\
&\quad + \frac{f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)} f_1(\mathcal{S}, \mathcal{I}_1) \mathcal{I}_1 - f_1(\mathcal{S}, \mathcal{I}_1) \frac{\mathcal{I}_1 \mathcal{E}_{1,s_1}^*}{\mathcal{E}_1} \\
&\quad + a \mathcal{E}_{1,s_1}^* - \frac{ac}{\gamma_1} \mathcal{I}_1 - a \mathcal{I}_{1,s_1}^* \frac{\mathcal{E}_1}{\mathcal{I}_1} + \frac{ac}{\gamma_1} \mathcal{I}_{1,s_1}^* \\
&\quad + \frac{f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)} f_2(\mathcal{S}, \mathcal{I}_2) \mathcal{I}_2 - \frac{bd}{\gamma_2} \mathcal{I}_2.
\end{aligned}$$

Then,

$$\begin{aligned}
{}^C D_t^\alpha \mathcal{P}_1(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) &\leq a \mathcal{E}_{1,s_1}^* \left(4 - \frac{a \mathcal{E}_{1,s_1}^*}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*) \mathcal{I}_{1,s_1}^*} - \frac{f_1(\mathcal{S}, \mathcal{I}_1) \mathcal{I}_1}{a \mathcal{E}_1} - \frac{\mathcal{I}_{1,s_1}^* \mathcal{E}_1}{\mathcal{I}_1 \mathcal{E}_{1,s_1}^*} - \frac{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_1)} \right) \\
&\quad + a \mathcal{E}_{1,s_1}^* \left(\frac{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_1)} + \frac{f_1(\mathcal{S}, \mathcal{I}_1)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)} \frac{\mathcal{I}_1}{\mathcal{I}_{1,s_1}^*} - \frac{\mathcal{I}_1}{\mathcal{I}_{1,s_1}^*} - 1 \right) \\
&\quad - \frac{bd}{\gamma_2} \mathcal{I}_2 + \frac{f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)} f_2(\mathcal{S}, \mathcal{I}_2) \mathcal{I}_2. \\
{}^C D_t^\alpha \mathcal{P}_1(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) &\leq a \mathcal{E}_{1,s_1}^* \left(4 - \frac{a \mathcal{E}_{1,s_1}^*}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*) \mathcal{I}_{1,s_1}^*} - \frac{f_1(\mathcal{S}, \mathcal{I}_1) \mathcal{I}_1}{a \mathcal{E}_1} - \frac{\mathcal{I}_{1,s_1}^* \mathcal{E}_1}{\mathcal{I}_1 \mathcal{E}_{1,s_1}^*} - \frac{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_1)} \right) \\
&\quad + a \mathcal{E}_{1,s_1}^* \left(\frac{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_1)} + \frac{f_1(\mathcal{S}, \mathcal{I}_1)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)} \frac{\mathcal{I}_1}{\mathcal{I}_{1,s_1}^*} - \frac{\mathcal{I}_1}{\mathcal{I}_{1,s_1}^*} - 1 \right) \\
&\quad + \frac{bd}{\gamma_2} \mathcal{I}_2 \left(\frac{f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)} \frac{f_2(\mathcal{S}, \mathcal{I}_2)}{f_2(\mathcal{S}_0^*, 0)} R_0^2 - 1 \right).
\end{aligned}$$

Using (H_2) and (H_3) , we have, $\frac{f_2(\mathcal{S}, \mathcal{I}_2)}{f_2(\mathcal{S}_0^*, 0)} \leq 1$, then

$$\begin{aligned}
{}^C D_t^\alpha \mathcal{P}_1(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) &\leq a \mathcal{E}_{1,s_1}^* \left(4 - \frac{a \mathcal{E}_{1,s_1}^*}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*) \mathcal{I}_{1,s_1}^*} - \frac{f_1(\mathcal{S}, \mathcal{I}_1) \mathcal{I}_1}{a \mathcal{E}_1} - \frac{\mathcal{I}_{1,s_1}^* \mathcal{E}_1}{\mathcal{I}_1 \mathcal{E}_{1,s_1}^*} - \frac{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_1)} \right) \\
&\quad + a \mathcal{E}_{1,s_1}^* \left(\frac{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_1)} + \frac{f_1(\mathcal{S}, \mathcal{I}_1)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)} \frac{\mathcal{I}_1}{\mathcal{I}_{1,s_1}^*} - \frac{\mathcal{I}_1}{\mathcal{I}_{1,s_1}^*} - 1 \right) \\
&\quad + \frac{bd}{\gamma_2} \mathcal{I}_2 \left(\frac{f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)} R_0^2 - 1 \right).
\end{aligned}$$

From (5.1), we have

$$\frac{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_1)} + \frac{f_1(\mathcal{S}, \mathcal{I}_1)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)} \frac{\mathcal{I}_1}{\mathcal{I}_{1,s_1}^*} - \frac{\mathcal{I}_1}{\mathcal{I}_{1,s_1}^*} - 1 = \left(1 - \frac{f_1(\mathcal{S}, \mathcal{I}_1)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)}\right) \left(\frac{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_1)} - \frac{\mathcal{I}_1}{\mathcal{I}_{1,s_1}^*}\right) \leq 0.$$

Making use of the inequality relating the arithmetic and geometric means yields:

$$\left(4 - \frac{a\mathcal{E}_{1,s_1}^*}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*) \mathcal{I}_{1,s_1}^*} - \frac{f_1(\mathcal{S}, \mathcal{I}_1) \mathcal{I}_1}{a\mathcal{E}_1} - \frac{\mathcal{I}_{1,s_1}^* \mathcal{E}_1}{\mathcal{I}_1 \mathcal{E}_{1,s_1}^*} - \frac{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_1)}\right) \leq 0$$

We consider two cases:

– If $\mathcal{S}_1^* \leq \mathcal{S}$, using (H_2) , we have: $\frac{f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)} \leq 1$, since $\left(\frac{f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)} R_0^2 - 1\right) \leq$

0,

we obtain, for $R_0^2 \leq 1$, the following ${}^C D_t^\alpha \mathcal{P}_1 \leq 0$.

– If $\mathcal{S} \leq \mathcal{S}_1^*$, using (H_2) , it follows that: $\frac{f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)} \geq 1$,

since $R_0^2 \leq \frac{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*)} \leq 1$, we obtain $\frac{f_1(\mathcal{S}_1^*, \mathcal{I}_{1,s_1}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,s_1}^*)} R_0^2 - 1 \leq 0$, which implies,

${}^C D_t^\alpha \mathcal{P}_1 \leq 0$.

By the above discussion, we deduce that if $R_0^2 \leq 1$, we will have ${}^C D_t^\alpha \mathcal{P}_1 \leq 0$.

Let $V_1 = \{(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) / {}^C D_t^\alpha \mathcal{P}_1(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) = 0\}$, then the largest positively invariant subset of V_1 reduces to the singleton $\mathcal{E}_1$. Consequently, by Lemma (2.4), the strain-1 endemic equilibrium $\mathcal{E}_1$ is globally asymptotically stable..

□

5.3. Global stability analysis of the strain-2 endemic state. We assume that the function f_2 satisfies the following condition for the global stability $\mathcal{E}_{s_2}$:

$$\left(1 - \frac{f_2(\mathcal{S}, \mathcal{I}_2)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)}\right) \left(\frac{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_2)} - \frac{\mathcal{I}_2}{\mathcal{I}_{2,s_2}^*}\right) \leq 0, \quad \forall \mathcal{S}, \mathcal{I}_2 > 0 \quad (5.2)$$

Theorem 5.3. *If $R_0^1 \leq 1 < R_0^2$, then The strain 2 endemic equilibrium $\mathcal{E}_{s_2}$ is globally asymptotically stable*

Proof. We first consider the following Lyapunov function $\mathcal{P}_2$ as follows:

$$\begin{aligned} \mathcal{P}_2(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) &= \mathcal{S} - \mathcal{S}_2^* - \int_{\mathcal{S}_2^*}^{\mathcal{S}} \frac{f_2(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)}{f_2(X, \mathcal{I}_{2,s_2}^*)} dX + \mathcal{E}_{2,s_2}^* \left(\frac{\mathcal{E}_2}{\mathcal{E}_{2,s_2}^*} - \ln \left(\frac{\mathcal{E}_2}{\mathcal{E}_{2,s_2}^*} \right) - 1 \right) \\ &\quad + \mathcal{E}_1 + \frac{b}{\alpha_2} \mathcal{I}_{2,s_2}^* \left(\frac{\mathcal{I}_2}{\mathcal{I}_{2,s_2}^*} - \ln \left(\frac{\mathcal{I}_2}{\mathcal{I}_{2,s_2}^*} \right) - 1 \right) + \frac{a}{\alpha_1} \mathcal{I}_1. \end{aligned}$$

Computing the time derivative yields

$$\begin{aligned}
{}^C D_t^\alpha \mathcal{P}_2(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) \leq & (\lambda - f_1(\mathcal{S}, \mathcal{I}_1)\mathcal{I}_1 - f_2(\mathcal{S}, \mathcal{I}_2)\mathcal{I}_2 - \mu\mathcal{S}) \left(1 - \frac{f_2(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)}\right) \\
& + (f_2(\mathcal{S}, \mathcal{I}_2)\mathcal{I}_2 - b\mathcal{E}_2) \left(1 - \frac{\mathcal{E}_{2,s_2}^*}{\mathcal{E}_2}\right) \\
& + (f_1(\mathcal{S}, \mathcal{I}_1)\mathcal{I}_1 - a\mathcal{E}_1) \\
& + \frac{b}{\gamma_2}(\gamma_2\mathcal{E}_2 - d\mathcal{I}_2) \left(1 - \frac{\mathcal{I}_{2,s_2}^*}{\mathcal{I}_2}\right) \\
& + \frac{a}{\gamma_1}(\gamma_1\mathcal{E}_1 - c\mathcal{I}_1).
\end{aligned}$$

We have

$$\begin{cases} \lambda = \mu\mathcal{S}_2^* + f_2(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)\mathcal{I}_{2,s_2}^*, \\ f_2(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)\mathcal{I}_{2,s_2}^* = \frac{bd}{\gamma_2}\mathcal{I}_{2,s_2}^* = b\mathcal{E}_{2,s_2}^*, \\ \frac{\mathcal{E}_{2,s_2}^*}{\mathcal{I}_{2,s_2}^*} = \frac{d}{\gamma_2}. \end{cases}$$

Therefore,

$$\begin{aligned}
{}^C D_t^\alpha \mathcal{P}_2(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) \leq & \mu\mathcal{S}_2^* \left(1 - \frac{f_2(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)}\right) - \mu\mathcal{S} \left(1 - \frac{f_2(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)}\right) \\
& + f_2(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)\mathcal{I}_{2,s_2}^* - f_2(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)\mathcal{I}_2^* \frac{f_2(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)} \\
& + \frac{f_2(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)} f_2(\mathcal{S}, \mathcal{I}_2)\mathcal{I}_2 - f_2(\mathcal{S}, \mathcal{I}_2) \frac{\mathcal{I}_2 \mathcal{E}_{2,s_2}^*}{\mathcal{E}_2} \\
& + b\mathcal{E}_{2,s_2}^* - \frac{bd}{\gamma_2}\mathcal{I}_2 - b\mathcal{I}_{2,s_2}^* \frac{\mathcal{E}_2}{\mathcal{I}_2} + \frac{bd}{\gamma_2}\mathcal{I}_{2,s_2}^* \\
& + \frac{f_2(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)} f_1(\mathcal{S}, \mathcal{I}_1)\mathcal{I}_1 - \frac{ac}{\gamma_1}\mathcal{I}_1.
\end{aligned}$$

Then,

$$\begin{aligned}
{}^C D_t^\alpha \mathcal{P}_2(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) \leq & b\mathcal{E}_{2,s_2}^* \left(4 - \frac{b\mathcal{E}_{2,s_2}^*}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)\mathcal{I}_{2,s_2}^*} - \frac{f_2(\mathcal{S}, \mathcal{I}_2)\mathcal{I}_2}{b\mathcal{E}_2} - \frac{\mathcal{I}_{2,s_2}^*\mathcal{E}_2}{\mathcal{I}_2\mathcal{E}_{2,s_2}^*} - \frac{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_2)}\right) \\
& + b\mathcal{E}_2^* \left(\frac{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_2)} + \frac{f_2(\mathcal{S}, \mathcal{I}_2)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)} \frac{\mathcal{I}_2}{\mathcal{I}_{2,s_2}^*} - \frac{\mathcal{I}_2}{\mathcal{I}_{2,s_2}^*} - 1\right) \\
& - \frac{ac}{\gamma_1}\mathcal{I}_1 + \frac{f_2(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)} f_1(\mathcal{S}, \mathcal{I}_1)\mathcal{I}_1.
\end{aligned}$$

$$\begin{aligned}
{}^C D_t^\alpha \mathcal{P}_2(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) \leq & b\mathcal{E}_{2,s_2}^* \left(4 - \frac{b\mathcal{E}_{2,s_2}^*}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*) \mathcal{I}_{2,s_2}^*} - \frac{f_2(\mathcal{S}, \mathcal{I}_2) \mathcal{I}_2}{b\mathcal{E}_2} - \frac{\mathcal{I}_{2,s_2}^* \mathcal{E}_2}{\mathcal{I}_2 \mathcal{E}_{2,s_2}^*} - \frac{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_2)} \right) \\
& + b\mathcal{E}_2^* \left(\frac{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_2)} + \frac{f_2(\mathcal{S}, \mathcal{I}_2)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)} \frac{\mathcal{I}_2}{\mathcal{I}_{2,s_2}^*} - \frac{\mathcal{I}_2}{\mathcal{I}_{2,s_2}^*} - 1 \right) \\
& + \frac{ac}{\gamma_1} \mathcal{I}_1 \left(\frac{f_2(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)} \frac{f_1(\mathcal{S}, \mathcal{I}_1)}{f_1(\mathcal{S}_0^*, 0)} R_0^1 - 1 \right).
\end{aligned}$$

From (H_2) and (H_3) , we have, $\frac{f_1(\mathcal{S}, \mathcal{I}_1)}{f_1(\mathcal{S}_0^*, 0)} \leq 1$,
then,

$$\begin{aligned}
{}^C D_t^\alpha \mathcal{P}_2(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) \leq & b\mathcal{E}_{2,s_2}^* \left(4 - \frac{b\mathcal{E}_{2,s_2}^*}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*) \mathcal{I}_{2,s_2}^*} - \frac{f_2(\mathcal{S}, \mathcal{I}_2) \mathcal{I}_2}{b\mathcal{E}_2} - \frac{\mathcal{I}_{2,s_2}^* \mathcal{E}_2}{\mathcal{I}_2 \mathcal{E}_{2,s_2}^*} - \frac{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_2)} \right) \\
& + b\mathcal{E}_2^* \left(\frac{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_2)} + \frac{f_2(\mathcal{S}, \mathcal{I}_2)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)} \frac{\mathcal{I}_2}{\mathcal{I}_{2,s_2}^*} - \frac{\mathcal{I}_2}{\mathcal{I}_{2,s_2}^*} - 1 \right) \\
& + \frac{ac}{\gamma_1} \mathcal{I}_1 \left(\frac{f_2(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)} R_0^1 - 1 \right).
\end{aligned}$$

From (5.2), we have

$$\left(\frac{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_2)} + \frac{f_2(\mathcal{S}, \mathcal{I}_2)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)} \frac{\mathcal{I}_2}{\mathcal{I}_{2,s_2}^*} - \frac{\mathcal{I}_2}{\mathcal{I}_{2,s_2}^*} - 1 \right) = \left(1 - \frac{f_2(\mathcal{S}, \mathcal{I}_2)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)} \right) \left(\frac{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_2)} - \frac{\mathcal{I}_2}{\mathcal{I}_{2,s_2}^*} \right) \leq 0.$$

Making use of the inequality relating the arithmetic and geometric means yields:

$$\left(4 - \frac{b\mathcal{E}_{2,s_2}^*}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*) \mathcal{I}_{2,s_2}^*} - \frac{f_2(\mathcal{S}, \mathcal{I}_2) \mathcal{I}_2}{b\mathcal{E}_2} - \frac{\mathcal{I}_{2,s_2}^* \mathcal{E}_2}{\mathcal{I}_2 \mathcal{E}_{2,s_2}^*} - \frac{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_2)} \right) \leq 0.$$

We discuss two cases:

- If $\mathcal{S}_2^* \leq \mathcal{S}$, from (H_2) we will have: $\frac{f_2(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)} \leq 1$, since $\left(\frac{f_2(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)} R_0^1 - 1 \right) \leq$

0, we obtain, from $R_0^1 \leq 1$, We will have ${}^C D_t^\alpha \mathcal{P}_2 \leq 0$.

- If $\mathcal{S} \leq \mathcal{S}_2^*$, from (H_2) , we will have: $\frac{f_2(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)} \geq 1$, from $R_0^1 \leq \frac{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)} \leq$

1, we will get

$$\frac{f_1(\mathcal{S}_2^*, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_{2,s_2}^*)} R_0^1 - 1 \leq 0, \text{ which implies, } {}^C D_t^\alpha \mathcal{P}_2 \leq 0.$$

According to the discussion above, if $R_0^1 \leq 1$, we will have ${}^C D_t^\alpha \mathcal{P}_2 \leq 0$.

Let $V_2 = \{(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) / {}^C D_t^\alpha \mathcal{P}_2(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) = 0\}$, then the largest positively invariant subset of V_2 reduces to the singleton $\mathcal{E}_2$. Therefore, by Lemma (2.4), the strain-2 endemic equilibrium $\mathcal{E}_2$ is globally asymptotically stable.. $\square$

5.4. Global stability of the both strains endemic state. To ensure the global stability of $\mathcal{E}_{st}$, we assume that the functions f_1 and f_2 satisfy the following condition:

$$\left(1 - \frac{f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)} \frac{f_2(\mathcal{S}, \mathcal{I}_2)}{f_2(\mathcal{S}_t^*, \mathcal{I}_{2,t}^*)}\right) \left(\frac{f_2(\mathcal{S}_t^*, \mathcal{I}_{2,s_2}^*)}{f_2(\mathcal{S}, \mathcal{I}_2)} \left(\frac{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}^*, \mathcal{I}_{1,t}^*)} - \frac{\mathcal{I}_2}{\mathcal{I}_{2,s_2}^*}\right)\right) \leq 0, \quad \forall \mathcal{S}, \mathcal{I}_1, \mathcal{I}_2 > 0 \quad (5.3)$$

Theorem 5.4. *If $R_0^1 > 1$ and $R_0^2 > 1$, then the endemic equilibrium $\mathcal{E}_{st}$ is globally asymptotically stable.*

Proof. The Lyapunov function $\mathcal{P}_3$, which is defined by:

$$\begin{aligned} \mathcal{P}_3(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) &= \mathcal{S} - \mathcal{S}_t^* - \int_{\mathcal{S}_t^*}^{\mathcal{S}} \frac{f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*)}{f_1(X, \mathcal{I}_{1,t}^*)} dX \\ &\quad + \mathcal{E}_{2,t}^* \left(\frac{\mathcal{E}_2}{\mathcal{E}_{2,t}^*} - \ln \left(\frac{\mathcal{E}_2}{\mathcal{E}_{2,t}^*} \right) - 1 \right) \\ &\quad + \mathcal{E}_{1,t}^* \left(\frac{\mathcal{E}_1}{\mathcal{E}_{1,t}^*} - \ln \left(\frac{\mathcal{E}_1}{\mathcal{E}_{1,t}^*} \right) - 1 \right) \\ &\quad + \frac{b}{\alpha_2} \mathcal{I}_{2,t}^* \left(\frac{\mathcal{I}_2}{\mathcal{I}_{2,t}^*} - \ln \left(\frac{\mathcal{I}_2}{\mathcal{I}_{2,t}^*} \right) - 1 \right) \\ &\quad + \frac{a}{\alpha_1} \mathcal{I}_{1,t}^* \left(\frac{\mathcal{I}_1}{\mathcal{I}_{1,t}^*} - \ln \left(\frac{\mathcal{I}_1}{\mathcal{I}_{1,t}^*} \right) - 1 \right). \end{aligned}$$

Computing the time derivative yields

$$\begin{aligned} {}^C D_t^\alpha \mathcal{P}_3(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) &\leq \left(1 - \frac{f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)}\right) {}^C D_t^\alpha \mathcal{S} \\ &\quad + \left(1 - \frac{\mathcal{E}_{2,t}^*}{\mathcal{E}_2}\right) {}^C D_t^\alpha \mathcal{E}_2 \\ &\quad + \left(1 - \frac{\mathcal{E}_{1,t}^*}{\mathcal{E}_1}\right) {}^C D_t^\alpha \mathcal{E}_1 \\ &\quad + \frac{b}{\gamma_2} \left(1 - \frac{\mathcal{I}_{2,t}^*}{\mathcal{I}_2}\right) {}^C D_t^\alpha \mathcal{I}_2 \\ &\quad + \frac{a}{\gamma_1} \left(1 - \frac{\mathcal{I}_{1,t}^*}{\mathcal{I}_1}\right) {}^C D_t^\alpha \mathcal{I}_1. \end{aligned}$$

We have

$$\begin{cases} \lambda = \mu \mathcal{S}_t^* + f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*) \mathcal{I}_{1,t}^* + f_2(\mathcal{S}_t^*, \mathcal{I}_{2,t}^*) \mathcal{I}_{2,t}^*, \\ f_2(\mathcal{S}_t^*, \mathcal{I}_{2,t}^*) \mathcal{I}_{2,t}^* = b \mathcal{E}_{2,t}^*, \quad f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*) \mathcal{I}_{1,t}^* = a \mathcal{E}_{1,t}^*, \\ \frac{\mathcal{E}_{2,t}^*}{\mathcal{I}_{2,t}^*} = \frac{d}{\gamma_2}, \quad \frac{\mathcal{E}_{1,t}^*}{\mathcal{I}_{1,t}^*} = \frac{c}{\gamma_1}. \end{cases}$$

As a result,

$$\begin{aligned} {}^C D_t^\alpha \mathcal{P}_3(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) \leq & a \mathcal{E}_{1,t}^* \left(4 - \frac{a \mathcal{E}_{1,t}^*}{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*) \mathcal{I}_{1,t}^*} - \frac{f_1(\mathcal{S}, \mathcal{I}_1) \mathcal{I}_1}{a \mathcal{E}_1} - \frac{\mathcal{I}_{1,t}^* \mathcal{E}_1}{\mathcal{I}_1 \mathcal{E}_{1,t}^*} - \frac{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}, \mathcal{I}_1)} \right) \\ & + a \mathcal{E}_{1,t}^* \left(\frac{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}, \mathcal{I}_1)} + \frac{f_1(\mathcal{S}, \mathcal{I}_1) \mathcal{I}_1}{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*) \mathcal{I}_{1,t}^*} - \frac{\mathcal{I}_1}{\mathcal{I}_{1,t}^*} - 1 \right) \\ & + b \mathcal{E}_{2,t}^* \left(4 - \frac{f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)} - \frac{f_2(\mathcal{S}, \mathcal{I}_2) \mathcal{I}_2}{b \mathcal{E}_2} - \frac{\mathcal{I}_{2,t}^* \mathcal{E}_2}{\mathcal{I}_2 \mathcal{E}_{2,t}^*} - \frac{b \mathcal{E}_{2,t}^* f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)}{f_2(\mathcal{S}, \mathcal{I}_2) f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*) \mathcal{I}_{2,t}^*} \right) \\ & + b \mathcal{E}_{2,t}^* \left(\frac{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*)} \frac{f_2(\mathcal{S}_t^*, \mathcal{I}_{2,t}^*)}{f_2(\mathcal{S}, \mathcal{I}_2)} + \frac{f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*)} \frac{f_2(\mathcal{S}, \mathcal{I}_2) \mathcal{I}_2}{f_2(\mathcal{S}_t^*, \mathcal{I}_{2,t}^*) \mathcal{I}_2^*} - \frac{\mathcal{I}_2}{\mathcal{I}_{2,t}^*} - 1 \right) \\ & + \mu \mathcal{S}_t^* \left(1 - \frac{f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)} \right) \left(1 - \frac{\mathcal{S}}{\mathcal{S}_t^*} \right). \end{aligned}$$

The following inequalities are used:

$$\begin{aligned} 1 - \frac{f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)} &\geq 0, \quad \text{for } \mathcal{S} \geq \mathcal{S}_t^*, \\ 1 - \frac{f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)} &< 0, \quad \text{for } \mathcal{S} < \mathcal{S}_t^*. \end{aligned}$$

Then, $\left(1 - \frac{f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)} \right) \left(1 - \frac{\mathcal{S}}{\mathcal{S}_t^*} \right) \leq 0$.

Making use of the inequality relating the arithmetic and geometric means yields:

$$\left(4 - \frac{a \mathcal{E}_{1,t}^*}{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*) \mathcal{I}_{1,t}^*} - \frac{f_1(\mathcal{S}, \mathcal{I}_1) \mathcal{I}_1}{a \mathcal{E}_1} - \frac{\mathcal{I}_{1,t}^* \mathcal{E}_1}{\mathcal{I}_1 \mathcal{E}_{1,t}^*} - \frac{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}, \mathcal{I}_1)} \right) \leq 0$$

$$\left(4 - \frac{f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)} - \frac{f_2(\mathcal{S}, \mathcal{I}_2) \mathcal{I}_2}{b \mathcal{E}_2} - \frac{\mathcal{I}_{2,t}^* \mathcal{E}_2}{\mathcal{I}_2 \mathcal{E}_{2,t}^*} - \frac{b \mathcal{E}_{2,t}^* f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)}{f_2(\mathcal{S}, \mathcal{I}_2) f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*) \mathcal{I}_{2,t}^*} \right) \leq 0$$

From (5.1) we have:

$$\left(\frac{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}, \mathcal{I}_1)} + \frac{f_1(\mathcal{S}, \mathcal{I}_1)}{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)} \frac{\mathcal{I}_1}{\mathcal{I}_{1,t}^*} - \frac{\mathcal{I}_1}{\mathcal{I}_{1,t}^*} - 1 \right) = \left(1 - \frac{f_1(\mathcal{S}, \mathcal{I}_1)}{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)} \right) \left(\frac{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}, \mathcal{I}_1)} - \frac{\mathcal{I}_1}{\mathcal{I}_{1,t}^*} \right) \leq 0$$

Also, from (5.3) we have

$$\left(\frac{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*)} \frac{f_2(\mathcal{S}_t^*, \mathcal{I}_{2,t}^*)}{f_2(\mathcal{S}, \mathcal{I}_2)} + \frac{f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)} \frac{f_2(\mathcal{S}, \mathcal{I}_2)}{f_2(\mathcal{S}_t^*, \mathcal{I}_{2,t}^*)} \frac{\mathcal{I}_2}{\mathcal{I}_{2,t}^*} - \frac{\mathcal{I}_2}{\mathcal{I}_{2,t}^*} - 1 \right) = (1 - \Theta) \left(\frac{1}{\Theta} - \frac{\mathcal{I}_2}{\mathcal{I}_{2,t}^*} \right) \leq 0$$

$$\text{with, } \Theta = \frac{f_1(\mathcal{S}_t^*, \mathcal{I}_{1,t}^*)}{f_1(\mathcal{S}, \mathcal{I}_{1,t}^*)} \frac{f_2(\mathcal{S}, \mathcal{I}_2)}{f_2(\mathcal{S}_t^*, \mathcal{I}_{2,t}^*)}$$

We conclude that if $R_0^1 > 1$ and $R_0^2 > 1$, then ${}^C D_t^\alpha \mathcal{P}_3 \leq 0$

Let $V_3 = \{(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) / {}^C D_t^\alpha \mathcal{P}_3(\mathcal{S}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{I}_1, \mathcal{I}_2) = 0\}$, then the largest invariant positive subset of V_2 is the singleton $\mathcal{E}_2$. Therefore from Lemma (2.4), we conclude that the strain-2 endemic equilibrium point $\mathcal{E}_2$ is globally asymptotically stable. $\square$

6. NUMERICAL SIMULATIONS

In this section, we present numerical simulations to support the theoretical results established in the previous section. For this purpose, we employ the numerical scheme proposed by Odibat et al. [40]. Based on the formulation given in (3.1), the model (1.1) can be rewritten in the following form:

$$\begin{cases} {}^C D_t^\alpha \mathcal{U}(t) = \mathcal{F}(\mathcal{U}(t)), \\ \mathcal{U}(0) = \mathcal{U}_0, \quad \mathcal{U}_0 \in \mathbb{R}_+^6. \end{cases} \quad (6.1)$$

Where, $\mathcal{U}_0 = (\mathcal{S}_0, \mathcal{E}_{1,0}, \mathcal{E}_{2,0}, \mathcal{I}_{1,0}, \mathcal{I}_{2,0}, \mathcal{R}_0)$, is the initial condition. So the problem (6.1) can be resolved by applying the numerical technique that follows:

$$\begin{aligned} \mathcal{U}(t_j) &= \frac{h^\alpha}{\Gamma(\alpha + 2)} \left((j-1)^{\alpha+1} - (j-\alpha-1)j^\alpha \right) \mathcal{F}(\mathcal{U}(t_0)) + \mathcal{U}(t_0) \\ &+ \frac{h^\alpha}{\Gamma(\alpha + 2)} \sum_{i=1}^{j-1} \left((j-i+1)^{\alpha+1} - 2(j-i)^{\alpha+1} + (j-i-1)^{\alpha+1} \right) \mathcal{F}(\mathcal{U}(t_i)) \\ &+ \frac{h^\alpha}{\Gamma(\alpha + 2)} \mathcal{F}(\mathcal{U}(t_{j-1})) + \frac{h^\alpha}{\Gamma(\alpha + 1)} \mathcal{F}(\mathcal{U}(t_{j-1})). \end{aligned}$$

Where $t_j = t(j-1) + h$, for $j = 0, 1, \dots, N-1$ and $\mathcal{U}_0 = (10, 5, 4.5, 3, 2.8, 2)$.

Numerical simulations will be performed in this section in order to check numerically the stability of all the equilibria and to confirm our theoretical results. To this end, different incidence functions will be taken into consideration to have a wide view on the infection dynamics. More precisely, numerical simulations will

be performed for two non-monotonic incidence rates $f_1(\mathcal{S}, \mathcal{I}_1) = \frac{\beta_1 \mathcal{S}}{1 + m_1 \mathcal{I}_1^2}$ and $f_2(\mathcal{S}, \mathcal{I}_2) = \frac{\beta_2 \mathcal{S}}{1 + m_2 \mathcal{I}_2^2}$. To assert the theoretical results obtained in the previous sections, we use the parameter values listed in table (1).

Parameters	Figure 2	Figure 3	Figure 4	Figure 5
λ	1	1	1	1
β_1	0.18	0.51	0.2	0.6
β_2	0.16	0.13	0.4	0.06
m_1	2.55	2	2	0.14
m_2	3.5	1.5	1.5	0.14
σ_1	0.66	0.4	1	0.15
σ_2	0.76	0.76	0.5	0.15
γ_1	0.4	0.4	0.4	0.5
γ_2	0.5	0.31	0.3	0.5
μ	0.21	0.2	0.2	0.2

TABLE 1. Table of parameters

Parameters	$\mathcal{E}_f$	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_{st}$
R_0^1	0.6	2.7778	0.5556	6.1224
R_0^2	0.5263	0.3789	1.7143	6.1224

TABLE 2. Values of reproduction numbers

Figure (2) shows the dynamics of the model components for different values of the fractional-order parameter α . The results indicate that all solution trajectories approach the disease-free equilibrium $\mathcal{E}_f = (5, 0, 0, 0, 0, 0)$, regardless of the value of α . The basic reproduction numbers are reported in column 1 of Table (2); These values meet the conditions ensuring the global stability of this equilibrium point, as established in Theorem (5.1).

Figure (3) illustrates the infection dynamics for various values of the fractional-order parameter α . It is observed that all solution trajectories converge to the equilibrium point $\mathcal{E}_{s_1} = (1.812, 1.207, 0, 0.270, 0, 1.410)$. This behavior indicates the persistence of strain 1, while the components associated with strain 2 tend to zero. The corresponding reproduction numbers and inhibition parameters, reported in column 2 of Table (2), satisfy the conditions ensuring the global stability of this equilibrium, as established in Theorem (5.2).

A similar behavior is observed in Figure (4), where all trajectories converge to the equilibrium point $\mathcal{E}_{s_2} = (2.476, 0, 0.857, 0, 0.367, 1.212)$. In this case, strain 2 persists while strain 1 dies out. The reproduction numbers and inhibition parameters reported in column 3 of Table (2) fulfill the conditions guaranteeing the global stability of this equilibrium, as established in Theorem (5.3).

Finally, Figure (5) shows that all trajectories converge to the coexistence equilibrium point $\mathcal{E}_{st} = (0.875, 0.295, 0.179, 1.492, 1.114, 1.601)$. Here, both strains

persist. Furthermore, both basic reproduction numbers exceed unity, whereas the corresponding inhibitory reproduction numbers remain below one (see column 4 of Table (2)). These results are in agreement with the theoretical conclusions established in Theorem (5.4).

Overall, these numerical results are consistent with the theoretical analysis, confirming the global stability of the equilibrium points. With regard to the influence of the fractional-order parameter, it is observed that smaller values of α lead to a faster convergence of the model variables toward equilibrium, whereas larger values result in a slower convergence rate, highlighting the memory effect inherent to fractional-order systems. More specifically, while this parameter does not alter the global stability of model (1.1), it has a significant impact on the speed at which solutions approach equilibrium.

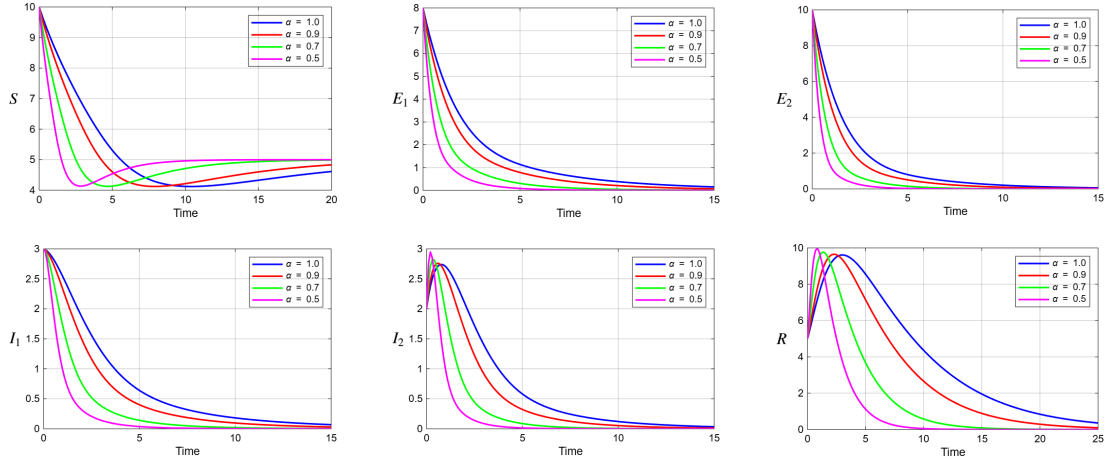


FIGURE 2. Stability of the free equilibrium point with different value of α .

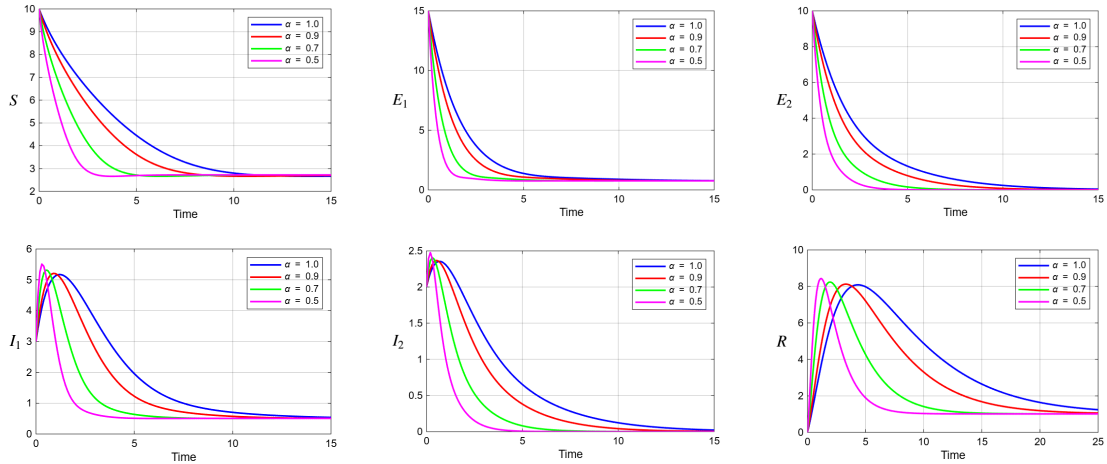


FIGURE 3. Stability of the strain-1 endemic equilibrium point with different value of α .

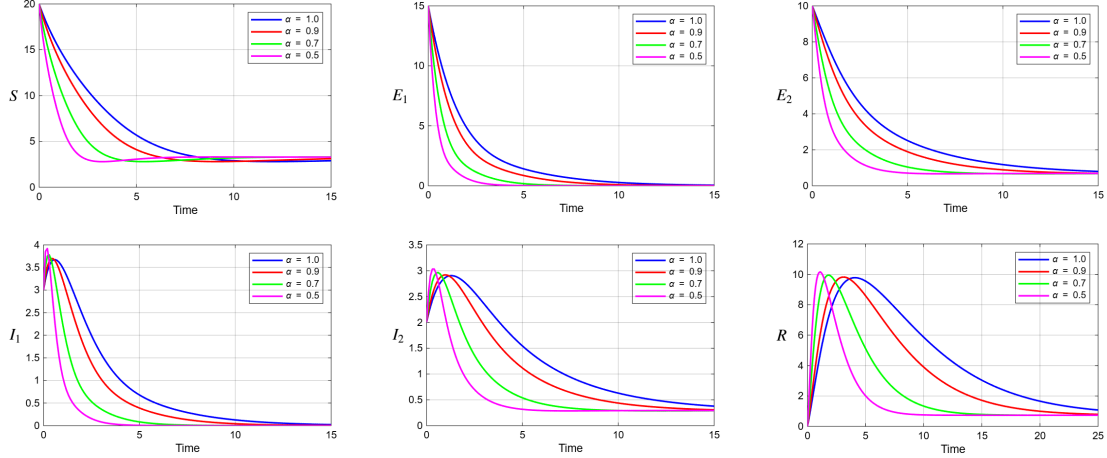


FIGURE 4. Stability of the strain-2 endemic equilibrium point with different value of α .

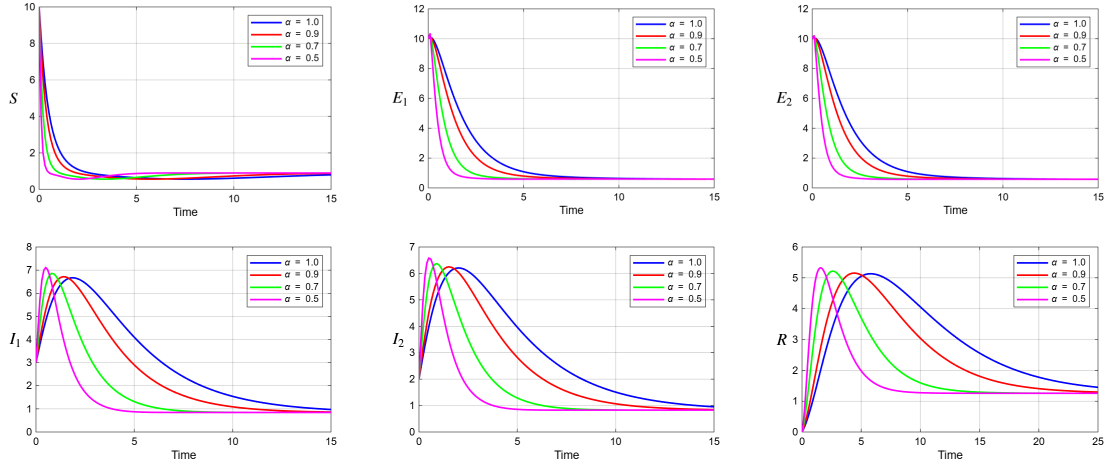


FIGURE 5. Stability of the total endemic equilibrium point with different value of α .

7. CONCLUSION

In this work, we analyze the global stability of an epidemic model involving two general incidence functions. The proposed framework consists of six epidemiological compartments: susceptible individuals, two exposed classes, two infectious classes, and recovered individuals, forming an extended $\mathcal{SEIR}$ structure. We establish the existence, positivity and boundedness of solutions, ensuring the well-posedness of the model. The following formula is used to determine the disease-free equilibrium, the strain-1 endemic equilibrium, the strain-2 endemic equilibrium, and the both endemic equilibrium involving both strains. We determine the global stability of the equilibria based on the value of the fundamental reproduction number R_0 by constructing an appropriate Lyapunov functional. Numerical simulations are carried out to validate the theoretical results.

and to examine the influence of the fractional-order parameter α on the system dynamics. The results show that, for all considered incidence functions, the value of α does not affect the stability of the equilibrium points, but significantly influences the convergence rate of the solutions toward steady states. In particular, larger values of α , corresponding to stronger memory effects, lead to slower convergence, whereas smaller values result in a faster approach to equilibrium.

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^{1*} LABORATORY OF MATHEMATICS, COMPUTER SCIENCE AND APPLICATIONS, FACULTY OF SCIENCES AND TECHNOLOGIES, UNIVERSITY HASSAN II OF CASABLANCA, PO Box 146, MOHAMMEDIA, 146, MOROCCO.

Email address: aarir.mohsine91@gmail

Email address: allali@hotmail.com

² ENCG OF CASABLANCA, UNIVERSITY HASSAN II, CASABLANCA, MOROCCO.

Email address: sanaa.harroudi@gmail.com

³ LABORATORY OF SYSTEMS MODELIZATION AND ANALYSIS FOR DECISION SUPPORT, NATIONAL SCHOOL OF APPLIED SCIENCES, HASSAN FIRST UNIVERSITY, BERRECHID, 26100, MOROCCO.

Email address: jaouaddanane@gmail.com