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TIKHONOV REGULARIZATION FOR KINETIC INVERSE SOURCE PROBLEMS

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ABSTRACT. Reconstructing a spatial source in stationary kinetic transport equations is a complex inverse problem complicated by grazing boundary singularities. We introduce a zero-extension framework for boundary-vanishing sources that bypasses these singularities without artificial compact-support constraints. This yields a robust forward mapping with exact gradient trace control. To stably invert noisy boundary measurements, we formulate a variational Tikhonov regularization strategy governed by Morozov's discrepancy principle. Relying solely on the bounded linearity and injectivity of the forward operator, we analyze the regularization scheme and show that the approximations converge strongly to the exact source.

Keywords. Inverse source problem, stationary kinetic transport, Tikhonov regularization.
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1. Introduction

Particle dynamics in phase space are typically modeled via kinetic transport equations, which serve as the mathematical backbone of imaging techniques such as SPECT. The associated inverse source problem—recovering an unknown internal spatial source distribution $f(x)$ from boundary measurements—is intrinsically ill-posed because the transport operator $\mathcal{L} = v \cdot \nabla_x + \mu$ lacks elliptic smoothing.

A major analytical hurdle in this context is the phase-space behavior at the grazing boundary $\{v \cdot \nu = 0\}$. The inverse operator $\mathcal{K} = \mathcal{L}^{-1}$ does not naturally commute with spatial gradients: differentiating along characteristics introduces terms proportional to $(v \cdot \nu)^{-1}$, leading to singular blow-ups. Traditionally, resolving these grazing singularities requires either physically unrealistic boundary gradient data or severely restrictive artificial support margins, though recent computational advances have begun exploring gradient-type conditions and interior measurements to mitigate these issues [11, 9]. In the broader context of mathematical analysis, recent studies have extensively explored various boundary value

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techniques, numerical approximations, and regularization methods for complex partial differential inverse problems (see, e.g., [12], [13]).

To overcome this analytical bottleneck, we leverage the natural zero-trace property. By extending the source by zero across the boundary, we fix the integration domain and completely bypass the need to differentiate the backward exit time $\tau_-(x, v)$. This zero-extension strategy yields exact gradient trace control, definitively ensuring that the inflow trace of $\nabla_x u$ vanishes within $W_v(\Omega)$.

Our analysis relies centrally on the *Characteristic Co-area Formula* (Lemma 2.10). While this identity is a classical cornerstone of transport theory [3, 4, 5], its application here necessitates a self-contained and systematic development. Specifically, we establish the *Uniqueness of Characteristic Parametrization* (Lemma 2.7) and show that the Jacobian of the affine flow $(x_0, s) \mapsto x_0 + sv$ is precisely $|v \cdot \nu(x_0)|$. This enables the fiberwise L^2 slicing and convergence arguments that form the bedrock of our characteristic trace limits.

Finally, we address the stable inversion of this forward mapping using Tikhonov regularization. The strong norm convergence of our scheme fundamentally relies on the injectivity of the forward mapping—a strict uniqueness property that has been extensively documented for inverse problems in stationary Boltzmann-type equations [10]. Although achieving strong convergence without compactness or source conditions is a standard result in abstract Hilbert spaces [6], embedding this theory within the kinetic transport framework is highly non-trivial. It requires the precise gradient trace control provided by our zero-extension approach. Building on this robust analytical foundation, future research can naturally extend to distributed regularization and joint discretization strategies.

2. Phase-Space Geometry and Forward Mapping

Consider a strictly convex domain $D \subset \mathbb{R}^n$ ($n \geq 2$) with a twice continuously differentiable boundary and let $G \subset \mathbb{R}^n$ be a bounded, Lebesgue-measurable velocity set satisfying

$$0 < c_0 \leq |v| \leq c_1 \quad \forall\, v \in G, \qquad \text{and} \qquad v \in G \;\Rightarrow\; -v \in G. \tag{2.1}$$

The symmetry condition $-v \in G$ is physically natural (particles travel in all directions) and is needed in Lemma 2.6, Remark 2.12, and Proposition 3.3. Define $\Omega = D \times G$ and the inflow/outflow boundaries

$$\Gamma_\pm := \{(x, v) \in \partial D \times G : \pm v \cdot \nu(x) > 0\}.$$

Transport is formulated on space

$$W_v(\Omega) := \{u \in L^2(\Omega) : v \cdot \nabla_x u \in L^2(\Omega)\}, \qquad \|u\|_{W_v}^2 := \|u\|_{L^2(\Omega)}^2 + \|v \cdot \nabla_x u\|_{L^2(\Omega)}^2.$$

By Cessenat's trace theorem [3, 4] (see also [5, Ch. XXI, §1]), there exists a continuous trace operator $\gamma_\pm : W_v(\Omega) \to L^2(\Gamma_\pm, |v \cdot \nu|\, dS\, dv)$ and consider

$$v \cdot \nabla_x u + \mu(x, v) u = f(x) \quad \text{in } \Omega, \qquad \gamma_- u = 0. \tag{2.2}$$

Assumption 1. $\mu \in W^{1,\infty}(\Omega)$, $\mu \geq \mu_0 > 0$, and $f \in H_0^1(D)$.

Remark 2.1. $H_0^1(D)$ is the closure of $C_c^\infty(D)$ in $H^1(D)$. By [2, Proposition 9.18], any $f \in H_0^1(D)$ admits a canonical zero extension $\tilde{f} \in H^1(\mathbb{R}^n)$ satisfying $\tilde{f}|_D = f$ and $\|\tilde{f}\|_{H^1(\mathbb{R}^n)} = \|f\|_{H^1(D)}$ (isometry). If $f_m \in C_c^\infty(D) \to f$ in $H^1(D)$, the zero extensions $\tilde{f}_m \in C_c^\infty(\mathbb{R}^n)$ converge in $H^1(\mathbb{R}^n)$ by the same isometry. Similarly, μ admits an extension $\tilde{\mu} \in W^{1,\infty}(\mathbb{R}^n \times G)$.

Remark 2.2. The assumption $f \in H_0^1(D)$ is essential: the isometric zero extension of Remark 2.1 characterizes $H_0^1(D)$, and it is this property that prevents any boundary contribution when computing weak spatial derivatives via the characteristic formula (2.26), thereby bypassing grazing singularities without artificial support margins. Extending the framework to $f \in H^1(D)$ with non-vanishing trace would require additional boundary correction terms in the derivative formula and remains an open problem.

Lemma 2.3. *Let $D \subset \mathbb{R}^n$ be a convex set, $(x, v) \in \Omega$. We introduce the backward exit time*

$$\tau_-(x, v) := \inf\{s > 0 : x - sv \notin D\}.$$

Due to convexity, the characteristic line leaves the domain permanently after this time. Therefore, $x - sv \notin D$ for all $s > \tau_-(x, v)$, consequently

$$\tilde{f}(x - sv) = 0 \quad \textit{for all } s > \tau_-(x, v).$$

Proof. If $x - s_0 v \in D$ for some $s_0 > \tau_-(x, v)$, then convexity gives $x - tv \in D$ for $t \in [0, s_0]$, so $\inf\{t > 0 : x - tv \notin D\} \geq s_0 > \tau_-(x, v)$, a contradiction. The conclusion $\tilde{f}(x - sv) = 0$ follows since $\tilde{f} \equiv 0$ outside D. ☐

Remark 2.4. For $x \in D$, $\tau_-(x, v) > 0$ and $\tau_-(x, v) \leq T := \operatorname{diam}(D)/c_0 < \infty$. We extend τ_- to all of $\overline{\Omega}$ by setting

$$\tau_-(x, v) := 0 \quad \text{for } (x, v) \in \partial D \times G \text{ with } v \cdot \nu(x) \leq 0. \tag{2.3}$$

For inflow and grazing boundary points ($v \cdot \nu(x) \leq 0$), the ray $x - sv$ does not enter D for $s > 0$, so $\tau_- = 0$ is natural. For outflow points $(x, v) \in \Gamma_+$, $\tau_-(x, v) := \inf\{s > 0 : x - sv \notin D\} > 0$ is the backward chord length: since $(-v) \cdot \nu(x) < 0$ and $-v \in G$ by (2.1), Lemma 2.5(i) applied at $(x, -v) \in \Gamma_-$ gives $x - sv \in D$ for small $s > 0$. For every $(x, v) \in \Omega$, the backward exit point $x - \tau_-(x, v)v \in \partial D$ (by continuity and Lemma 2.3), giving

$$\tau_-(x, v) \geq \frac{\operatorname{dist}(x, \partial D)}{c_1} \quad \forall\, (x, v) \in \Omega. \tag{2.4}$$

Lemma 2.5. *Let $(x_0, v_0) \in \Gamma_-$. Then there exists $\varepsilon_0 = \varepsilon_0(x_0, v_0) > 0$ such that $x_0 + sv_0 \in D$ and $\tau_-(x_0 + sv_0, v_0) \leq s$ for all $s \in (0, \varepsilon_0)$.*

Proof. Let $F \in C^2(\mathbb{R}^n)$ satisfy $D = \{F < 0\}$, $\partial D = \{F = 0\}$ and $\nabla F(x) = |\nabla F(x)|\nu(x) \neq 0$ on ∂D [7, Appendix C]. Since $(x_0, v_0) \in \Gamma_-$ implies $v_0 \cdot \nu(x_0) < 0$ and $|\nabla F(x_0)| > 0$, the Taylor expansion $F(x_0 + sv_0) = s|\nabla F(x_0)|(v_0 \cdot \nu(x_0)) + O(s^2)$ is strictly negative for all sufficiently small $s > 0$, yielding $x_0 + sv_0 \in D$. Setting $x_s := x_0 + sv_0$, we observe that $x_s - sv_0 = x_0 \in \partial D \setminus D$, so s belongs to the set $\{t > 0 : x_s - tv_0 \notin D\}$, from which $\tau_-(x_s, v_0) \leq s$ follows immediately. ☐

Lemma 2.6. *The backward exit time τ_-, extended as in (2.3), is lower semicontinuous on $\overline{\Omega}$. Moreover, τ_- is C^2 near every non-grazing point $(x_0, v_0) \in \overline{\Omega}$ with $v_0 \cdot \nu(y_0) \neq 0$, where $y_0 := x_0 - \tau_-(x_0, v_0)v_0$.*

Proof. Let F be the global C^2 defining function as in Lemma 2.5 [7, Appendix C].

Lower semicontinuity. Let $(x_j, v_j) \to (x_0, v_0)$ in $\overline{\Omega}$ and set $\ell := \liminf_j \tau_-(x_j, v_j)$. We extract a subsequence achieving the limit inferior, still indexed by j, and treat three cases according to the position of x_j relative to ∂D.

When $x_j \in D$, the exit point $y_j := x_j - \tau_-(x_j, v_j)v_j$ lies in ∂D: since $x_j - sv_j \in D$ for $s \in [0, \tau_-)$ and D is open, by Lemma 2.3 and continuity, $x_j - \tau_-(x_j, v_j)v_j \in \overline{D} \setminus D = \partial D$. Boundedness of $\{y_j\}$ and compactness of ∂D yield a subsequence $y_j \to y_0 \in \partial D$, so $y_0 = x_0 - \ell v_0 \in \partial D$. Since D is open, $x_0 - \ell v_0 \notin D$, giving $\tau_-(x_0, v_0) \leq \ell$.

When $x_j \in \partial D$ with $v_j \cdot \nu(x_j) > 0$, since $(-v_j) \cdot \nu(x_j) < 0$ and $-v_j \in G$ by (2.1), the point $(x_j, -v_j) \in \Gamma_-$. Applying Lemma 2.5 at this inflow point yields $\varepsilon_j > 0$ with $x_j - sv_j \in D$ for $s \in (0, \varepsilon_j)$, so $\tau_-(x_j, v_j) \geq \varepsilon_j > 0$. The same compactness argument gives $y_j := x_j - \tau_-(x_j, v_j)v_j \to y_0 = x_0 - \ell v_0 \in \partial D$. If $v_0 \cdot \nu(x_0) \leq 0$, then $\tau_-(x_0, v_0) = 0 \leq \ell$; if $v_0 \cdot \nu(x_0) > 0$, then $x_0 - \ell v_0 \notin D$ gives $\tau_-(x_0, v_0) \leq \ell$.

When $x_j \in \partial D$ with $v_j \cdot \nu(x_j) \leq 0$, convention (2.3) gives $\tau_-(x_j, v_j) = 0$, so $\ell = 0$. Continuity of $(x, v) \mapsto v \cdot \nu(x)$ then forces $v_0 \cdot \nu(x_0) \leq 0$, hence $\tau_-(x_0, v_0) = 0 = \ell$.

In all cases $\tau_-(x_0, v_0) \leq \ell$, establishing lower semicontinuity.

C^2 smoothness. The exit time solves $\mathcal{G}(x, v, \tau) := F(x - \tau v) = 0$ near (x_0, v_0, τ_0) with $\tau_0 := \tau_-(x_0, v_0)$. Since $\partial_\tau \mathcal{G} = -v_0 \cdot \nabla F(y_0) = -|\nabla F(y_0)|(v_0 \cdot \nu(y_0)) \neq 0$, the C^2 implicit function theorem yields a local C^2 representation $\tau_- = \sigma(x, v)$ near (x_0, v_0). □

Lemma 2.7. *Let $D \subset \mathbb{R}^n$ be strictly convex with C^2 boundary, and fix $v \in G$. For $x \in D$, there exists a unique pair $(x_0, s) \in \partial D_v^- \times (0, \tau_+(x_0, v))$ with $x = x_0 + sv$, where $\partial D_v^- := \{x_0 \in \partial D : v \cdot \nu(x_0) < 0\}$ and $\tau_+(x_0, v) := \inf\{t > 0 : x_0 + tv \notin D\}$.*

Proof. Let $F \in C^2(\mathbb{R}^n)$ satisfy $D = \{F < 0\}$ and $\nabla F(x) = |\nabla F(x)|\nu(x) \neq 0$ on ∂D [7, Appendix C]. We first record two consequences of strict convexity that will be invoked throughout.

Supporting hyperplane property. For every $y \in \partial D$, the supporting hyperplane theorem (see [7, Appendix B]) gives $\nu(y) \cdot (z - y) \leq 0$ for all $z \in \overline{D}$. Since any supporting hyperplane meets a convex body in a convex subset of ∂D, and a convex subset of ∂D consisting of more than one point would contain an interior segment contradicting strict convexity, the tangent hyperplane $H_y := \{z : \nu(y) \cdot (z - y) = 0\}$ satisfies $H_y \cap \overline{D} = \{y\}$. In particular, if $w \cdot \nu(y) = 0$, then $\nu(y) \cdot (y + tw - y) = 0$ for all t, so every point $y + tw$ $(t \neq 0)$ lies in $H_y \setminus \{y\}$ and hence outside $\overline{D}$.

Existence. Set $s := \tau_-(x, v) > 0$ (positive by (2.4)) and $x_0 := x - sv \in \partial D$. We claim $v \cdot \nu(x_0) < 0$. For every $\varepsilon \in (0, s)$, the point $x_0 + \varepsilon v = x - (s - \varepsilon)v$ lies in D (since $s - \varepsilon < \tau_-(x, v)$), so $F(x_0 + \varepsilon v) < 0$. The Taylor expansion $F(x_0 + \varepsilon v) = \varepsilon|\nabla F(x_0)|(v \cdot \nu(x_0)) + O(\varepsilon^2)$ combined with dividing by $\varepsilon > 0$ and letting $\varepsilon \to 0^+$ yields $v \cdot \nu(x_0) \leq 0$. The case $v \cdot \nu(x_0) = 0$ is excluded by the supporting hyperplane property: it would force $x_0 + \varepsilon v \notin \overline{D}$ for all $\varepsilon \neq 0$, contradicting $x_0 + \varepsilon v \in D$ for $\varepsilon \in (0, s)$. Hence $v \cdot \nu(x_0) < 0$, i.e. $x_0 \in \partial D_v^-$.

It remains to verify $s \in (0, \tau_+(x_0, v))$. Since $v \cdot \nu(x_0) < 0$, the first-order Taylor expansion shows $x_0 + tv \in D$ for small $t > 0$, so $\mathcal{I} := \{t \geq 0 : x_0 + tv \in \overline{D}\}$ is a non-empty convex closed subset of $[0, \infty)$, hence an interval $[0, \tau_+(x_0, v)]$ with $\tau_+(x_0, v) < \infty$ (since D is bounded). Because $x_0 + sv = x \in D$, we have $s \in (0, \tau_+(x_0, v))$, so (x_0, s) is a valid pair.

Uniqueness. Suppose $x = x_0 + sv = x_0' + s'v$ with both pairs in $\partial D_v^- \times (0, \infty)$ and $s > s'$. Then $x_0' = x_0 + (s - s')v$ lies on the ray $\{x_0 + tv : t > 0\}$ at parameter $t = s - s' > 0$, and $x_0' \in \partial D$. Since $x_0 + tv \in D$ for $t \in (0, \tau_+(x_0, v))$, only boundary point on this ray is the exit point $x_0 + \tau_+(x_0, v)v$, forcing $s - s' = \tau_+(x_0, v)$. At that exit point, expanding $F(x_0 + (\tau_+(x_0, v) + \varepsilon)v) > 0$ and dividing by $\varepsilon \to 0^+$ gives $v \cdot \nu(x_0') \geq 0$. The case $v \cdot \nu(x_0') = 0$ is excluded by the supporting hyperplane property: it would force $x_0' - \varepsilon v \notin \overline{D}$ for $\varepsilon > 0$, contradicting $x_0' - \varepsilon v = x_0 + (s - s' - \varepsilon)v \in D$ for small ε. Therefore $v \cdot \nu(x_0') > 0$, contradicting $x_0' \in \partial D_v^-$. Hence $s = s'$ and $x_0 = x_0'$. □

Remark 2.8. Strict convexity is essential for the supporting hyperplane argument in Lemma 2.7: for merely convex domains, the tangent hyperplane may meet $\overline{D}$ along an entire face, and the forward ray could travel along ∂D, destroying the bijectivity of the characteristic parametrization.

Lemma 2.9. *Let D be strictly convex with C^2 boundary, $v \in G$, $x_0 \in \partial D_v^-$, and $s \in (0, \tau_+(x_0, v))$. Then $\tau_-(x_0 + sv, v) = s$.*

Proof. Set $x_s := x_0 + sv$. Since $v \cdot \nu(x_0) < 0$, the proof of Lemma 2.7 establishes that $\{t \geq 0 : x_0 + tv \in \overline{D}\} = [0, \tau_+(x_0, v)]$, an interval.

For $t \in (0, s)$, we have $x_s - tv = x_0 + (s - t)v$ with $s - t \in (0, \tau_+(x_0, v))$, so $x_s - tv \in D$. At $t = s$, $x_s - sv = x_0 \in \partial D \setminus D$, so $x_s - sv \notin D$. For $t > s$, set $r := s - t < 0$, giving $x_s - tv = x_0 + rv$. Since the tangent hyperplane H_{x_0} supports $\overline{D}$, we have $\nu(x_0) \cdot (z - x_0) \leq 0$ for all $z \in \overline{D}$. Evaluating at $z = x_0 + rv$ gives $\nu(x_0) \cdot (rv) = r(v \cdot \nu(x_0)) > 0$ (as $r < 0$ and $v \cdot \nu(x_0) < 0$), so $x_0 + rv \notin \overline{D}$, i.e. $x_s - tv \notin \overline{D}$.

Thus $\{t > 0 : x_s - tv \notin D\}$ contains s and all $t > s$, but contains no element of $(0, s)$. Hence $\tau_-(x_s, v) = s$. □

Lemma 2.10. *Let D, G be as above. For $v \in G$, define ∂D_v^- and τ_+ as in Lemma 2.7.*

(I) ***Exact Jacobian.*** *For any C^1 local chart $\phi : U \to \partial D$ ($U \subset \mathbb{R}^{n-1}$ open), the map $\Psi_v(u, s) := \phi(u) + sv$ satisfies*

$$|\det D\Psi_v(u, s)| = |v \cdot \nu(\phi(u))| \cdot \sqrt{g(u)} \quad \forall\, (u, s) \in U \times (0, \tau_+(\phi(u), v)), \tag{2.5}$$

where $\sqrt{g(u)} := |\det[\partial_1 \phi(u), \ldots, \partial_{n-1} \phi(u), \nu(\phi(u))]|$. The Jacobian is independent of s.

(II) ***Exact co-area formula.*** *For measurable $h : \Omega \to [0, \infty]$:*

$$\int_D h(x, v)\, dx = \int_{\partial D_v^-} \int_0^{\tau_+(x_0, v)} h(x_0 + sv, v)\, |v \cdot \nu(x_0)|\, ds\, dS(x_0). \tag{2.6}$$

Integrating over $v \in G$ and applying to $|h|^2$:

$$\|h\|^2_{L^2(\Omega)} = \int_G \int_{\partial D_v^-} \int_0^{\tau_+(x_0,v)} |h(x_0+sv,v)|^2 |v\cdot\nu(x_0)|\, ds\, dS(x_0)\, dv. \tag{2.7}$$

(III) ***Fiberwise L^2 slicing.*** *For every $h \in L^2(\Omega)$, for $|v\cdot\nu|\, dS\, dv$-a.e. $(x_0, v_0) \in \Gamma_-$:*

$$s \mapsto h(x_0+sv_0, v_0) \in L^2(0, \tau_+(x_0, v_0)). \tag{2.8}$$

(IV) ***Fiberwise L^2 convergence.*** *If $h_m \to h$ in $L^2(\Omega)$, there exists a subsequence $\{m_j\}$ such that for $|v\cdot\nu|\, dS\, dv$-a.e. $(x_0, v_0) \in \Gamma_-$:*

$$h_{m_j}(x_0 + \cdot v_0, v_0) \to h(x_0 + \cdot v_0, v_0) \quad \text{in } L^2(0, \tau_+(x_0, v_0)). \tag{2.9}$$

Proof. Part (I). Fix a C^1 local chart $\phi : U \to \partial D$. The columns of $D\Psi_v(u,s)$ are $[\partial_1\phi(u) \,|\, \cdots \,|\, \partial_{n-1}\phi(u) \,|\, v]$, independent of s since Ψ_v is affine in s. Decomposing $v = (v\cdot\nu(\phi(u)))\nu(\phi(u)) + v^\perp$ with $v^\perp$ in the tangent space of ∂D at $\phi(u)$, multilinearity of the determinant and column operations gives

$$\det[\partial_1\phi, \ldots, \partial_{n-1}\phi, v] = (v\cdot\nu(\phi(u)))\, \det[\partial_1\phi, \ldots, \partial_{n-1}\phi, \nu(\phi(u))],$$

and taking absolute values yields (2.5).

Part (II). We establish (2.6) via a partition of unity. Since $\partial D_v^- = \{v\cdot\nu < 0\}$ is relatively open in the compact C^2 manifold ∂D, we cover ∂D by finitely many C^2 charts $\{(U_k, \phi_k)\}_{k=1}^N$ and fix a subordinate C^∞ partition of unity $\{\psi_k\}$ with $\mathrm{supp}(\psi_k) \subset \phi_k(U_k)$ and $\sum_k \psi_k \equiv 1$ on ∂D [7, Appendix C].

The weighted surface measure $|v\cdot\nu|\, dS$ is intrinsic: on any chart overlap, the transition map $\Phi_{jk} = \phi_k^{-1} \circ \phi_j$ satisfies $|\det D\Phi_{jk}|\sqrt{g_k \circ \Phi_{jk}} = \sqrt{g_j}$, so $|v\cdot\nu|\sqrt{g_k}\, dw = |v\cdot\nu|\sqrt{g_j}\, du$ under the change of variables $w = \Phi_{jk}(u)$; the Jacobian factor is atlas-independent.

Locally, for each k, Lemma 2.7 ensures that $\Psi_v(u,s) = \phi_k(u) + sv$ is a bijection from $\mathcal{D}_k := \{(u,s) : u \in U_k,\ 0 < s < \tau_+(\phi_k(u), v)\}$ onto its image $D_k \subset D$: injectivity follows from uniqueness of the parametrization, and surjectivity holds because any $x \in D_k$ traces back to its unique entry point $x_0 = x - \tau_-(x,v)v \in \phi_k(U_k)$. The classical change-of-variables theorem with Jacobian (2.5) gives

$$\int_{D_k} h(x,v)\, dx = \int_{\phi_k(U_k)\cap\partial D_v^-} \int_0^{\tau_+(x_0,v)} h(x_0+sv, v)\, |v\cdot\nu(x_0)|\, ds\, dS(x_0).$$

Decomposing $h = \sum_k h_k$ with $h_k(x,v) := \psi_k(x - \tau_-(x,v)v)\, h(x,v)$ (measurable since τ_- is Borel measurable by Lemma 2.6), summing over k, and using $\sum_k \psi_k(x_0) = 1$ on ∂D_v^-, we obtain (2.6). The global identity (2.7) follows by applying (2.6) to $|h|^2$ and integrating over G via Tonelli's theorem.

Part (III). Since $\|h\|^2_{L^2(\Omega)} < \infty$, Fubini applied to the non-negative integrand in (2.7) shows that $\int_0^{\tau_+} |h(x_0+sv_0, v_0)|^2 |v_0\cdot\nu(x_0)|\, ds < \infty$ for $dS\, dv$-almost every $(x_0, v_0) \in \Gamma_-$. At any such point with $|v_0\cdot\nu(x_0)| > 0$, the fiber map belongs to $L^2(0, \tau_+(x_0, v_0))$.

Part (IV). Applying (2.7) to $|h_m - h|^2$, the function $F_m(x_0, v) := \int_0^{\tau_+} |h_m - h|^2\, ds$ satisfies $\int_{\Gamma_-} F_m |v\cdot\nu|\, dS\, dv = \|h_m - h\|^2_{L^2(\Omega)} \to 0$, so $F_m \to 0$ in $L^1(\Gamma_-)$. Any

L^1-convergent sequence admits an a.e.-convergent subsequence, giving $\{m_j\}$ with $F_{m_j}(x_0, v_0) \to 0$ a.e., which is (2.9). $\square$

Lemma 2.11. *Let $w \in W_v(\Omega)$. Then for $|v \cdot \nu|\, dS\, dv$-a.e. $(x_0, v_0) \in \Gamma_-$, the map $s \mapsto w(x_0 + sv_0, v_0)$ has an absolutely continuous representative on $[0, \varepsilon]$ for every small $\varepsilon > 0$, satisfying $\frac{d}{ds} w(x_0 + sv_0, v_0) = (v_0 \cdot \nabla_x w)(x_0 + sv_0, v_0)$ a.e., and*

$$\gamma_- w(x_0, v_0) = \lim_{s \to 0^+} w(x_0 + sv_0, v_0) \quad |v \cdot \nu|\, dS\, dv\text{-a.e.} \tag{2.10}$$

Proof. Fiberwise regularity. Applying Lemma 2.10(III) to $h = w$ and $h = v_0 \cdot \nabla_x w$ yields a full-$|v \cdot \nu|\, dS\, dv$-measure set $\Sigma_0 \subset \Gamma_-$ such that

$$w(x_0 + \cdot v_0, v_0) \in L^2(0, \varepsilon_0) \quad \text{and} \quad (v_0 \cdot \nabla_x w)(x_0 + \cdot v_0, v_0) \in L^2(0, \varepsilon_0) \tag{2.11}$$

for every $(x_0, v_0) \in \Sigma_0$ and every $\varepsilon_0 < \tau_+(x_0, v_0)$.

Distributional derivative on fibers. Let $\rho_\delta \in C_c^\infty(B(0, \delta))$ be a standard mollifier with unit mass and set $w_\delta := w(\cdot, v) * \rho_\delta$. Standard mollification gives

$$w_\delta \to w \;\text{ and }\; v_0 \cdot \nabla_x w_\delta \to v_0 \cdot \nabla_x w \quad \text{in } L^2(\Omega). \tag{2.12}$$

Applying Lemma 2.10(IV) to $\{w_\delta\}$ extracts a subsequence $\delta_j \to 0$ and full-measure $\Sigma_1 \subset \Gamma_-$ with

$$w_{\delta_j}(x_0 + \cdot v_0, v_0) \to w(x_0 + \cdot v_0, v_0) \quad \text{in } L^2(0, \varepsilon_0), \tag{2.13}$$

and applying Lemma 2.10(IV) once more to $\{v_0 \cdot \nabla_x w_{\delta_j}\}$, passing to a further subsequence (still labelled $\{\delta_j\}$) to preserve (2.13), yields full-measure $\Sigma_2 \subset \Gamma_-$ with

$$(v_0 \cdot \nabla_x w_{\delta_j})(x_0 + \cdot v_0, v_0) \to (v_0 \cdot \nabla_x w)(x_0 + \cdot v_0, v_0) \quad \text{in } L^2(0, \varepsilon_0). \tag{2.14}$$

Fix $(x_0, v_0) \in \Sigma_0 \cap \Sigma_1 \cap \Sigma_2$. Setting $\Phi(s) := w(x_0 + sv_0, v_0)$ and $\psi(s) := (v_0 \cdot \nabla_x w)(x_0 + sv_0, v_0)$, for any $\phi \in C_c^\infty(0, \varepsilon_0)$ and δ_j smaller than the distance from the trajectory to ∂D, classical calculus and integration by parts (no boundary terms since ϕ is compactly supported) give $\int w_{\delta_j} \phi' = -\int (v_0 \cdot \nabla_x w_{\delta_j}) \phi$. Passing to the limit using (2.13)–(2.14) shows that

$$\int_0^{\varepsilon_0} \Phi(s) \phi'(s)\, ds = -\int_0^{\varepsilon_0} \psi(s) \phi(s)\, ds \quad \forall\, \phi \in C_c^\infty(0, \varepsilon_0), \tag{2.15}$$

so $\Phi' = \psi$ in $\mathcal{D}'$. Since $\psi \in L^1$, the embedding $W^{1,1} \hookrightarrow C$ gives $\Phi \in C([0, \varepsilon_0])$ with $\Phi(0^+) := \lim_{s \to 0^+} \Phi(s)$ well-defined.

Identification with the Cessenat trace. The trace $\gamma_- w$ is the unique element of $L^2(\Gamma_-)$ satisfying Green's formula

$$\int_{\Gamma_+} \gamma_+ w\, \varphi\, |v \cdot \nu| - \int_{\Gamma_-} \gamma_- w\, \varphi\, |v \cdot \nu| = \int_\Omega \big((v \cdot \nabla_x w) \varphi + w (v \cdot \nabla_x \varphi) \big)\, dx\, dv \tag{2.16}$$

for all $\varphi \in C^1(\overline{\Omega})$ [3, 4, 5], with uniqueness guaranteed by density of $C^1(\overline{\Omega})|_{\Gamma_-}$ in $L^2(\Gamma_-)$.

For the mollified functions, classical traces coincide with characteristic limits: $\gamma_- w_{\delta_j}(x_0, v_0) = \Phi_{\delta_j}(0^+)$. The continuity of the Cessenat trace operator combined with $\|w_{\delta_j} - w\|_{W_v} \to 0$ gives $\gamma_- w_{\delta_j} \to \gamma_- w$ in $L^2(\Gamma_-)$, and passing to a subsequence yields

$$\Phi_{\delta_j}(0^+) \to \gamma_- w(x_0, v_0) \quad \text{a.e. on } \Gamma_-. \tag{2.17}$$

To bridge $\Phi_{\delta_j}(0^+)$ and $\Phi(0^+)$, we use the absolutely continuous representations

$$\Phi_{\delta_j}(0^+) = \Phi_{\delta_j}(s) - \int_0^s (v_0 \cdot \nabla_x w_{\delta_j})(x_0 + tv_0, v_0)\, dt, \quad \Phi(0^+) = \Phi(s) - \int_0^s \psi(t)\, dt.$$

Subtracting, integrating over $s \in (0, \varepsilon_0)$, and dividing by ε_0:

$$|\Phi_{\delta_j}(0^+) - \Phi(0^+)| \le \frac{\|\Phi_{\delta_j} - \Phi\|_{L^1(0,\varepsilon_0)}}{\varepsilon_0} + \|(v_0 \cdot \nabla_x w_{\delta_j})(x_0 + \cdot v_0, v_0) - \psi\|_{L^1(0,\varepsilon_0)}. \tag{2.18}$$

Both terms vanish by the embedding $L^2 \hookrightarrow L^1$ and the fiberwise limits (2.13)–(2.14). Hence $\Phi_{\delta_j}(0^+) \to \Phi(0^+)$ for every $(x_0, v_0) \in \Sigma_1 \cap \Sigma_2$. Intersecting with (2.17) on the a.e.-full set $\Sigma_1 \cap \Sigma_2$ gives $\Phi(0^+) = \gamma_- w(x_0, v_0)$ a.e., establishing (2.10). □

Remark 2.12. By (2.1), $-v_0 \in G$ whenever $v_0 \in G$. For $w \in W_v(\Omega)$,

$$\gamma_+ w(x_0, v_0) = \lim_{s \to 0^+} w(x_0 - sv_0, v_0) \quad |v \cdot \nu|\, dS\, dv\text{-a.e. } (x_0, v_0) \in \Gamma_+. \tag{2.19}$$

The proof is identical to Lemma 2.11 with v_0 replaced by $-v_0 \in G$: since $v_0 \cdot \nu(x_0) > 0$ gives $(-v_0) \cdot \nu(x_0) < 0$, the point $(x_0, -v_0) \in \Gamma_-$ and the lemma applies in the direction $-v_0$.

Theorem 2.13. *Under Assumption 1, (2.2) has a unique solution $u \in W_v(\Omega)$. Moreover, $u \in L^2(G; H^1(D))$, $\nabla_x u \in W_v(\Omega)^n$, and $\gamma_-(\nabla_x u) = 0$ in $L^2(\Gamma_-)^n$. The operator $\mathcal{K} : H_0^1(D) \to W_v(\Omega)$, $\mathcal{K} f := u$, is bounded and linear.*

Proof. Throughout, set $C_\mu := T \|\nabla_x \tilde{\mu}\|_{L^\infty}$.

Characteristic representation and gradient formula. Standard L^2 transport theory [5] guarantees a unique $u \in W_v(\Omega)$. The method of characteristics, combined with Lemma 2.3, yields the representation over the fixed domain $[0, T]$, where $T := \mathrm{diam}(D)/c_0$:

$$u(x, v) = \int_0^T a(x, v, s)\, \tilde{f}(x - sv)\, ds, \quad a(x, v, s) := e^{-\int_0^s \tilde{\mu}(x - tv, v)\, dt} \in [e^{-\|\mu\|_\infty T}, 1]. \tag{2.20}$$

Extending to $[0, T]$ introduces no error by Lemma 2.3. Direct differentiation of the exponential factor yields the pointwise bound

$$|(\partial_{x_i} a)(x, v, s)| = \left| a \int_0^s \partial_{x_i} \tilde{\mu}(x - tv, v)\, dt \right| \le s \|\nabla_x \tilde{\mu}\|_{L^\infty} \le C_\mu. \tag{2.21}$$

To compute the weak spatial derivative, we verify absolute integrability: for fixed $v \in G$ and $\varphi \in C_c^\infty(D)$ supported on a compact set K,

$$\int_0^T \int_{\mathbb{R}^n} |a\, \tilde{f}(x - sv)\, \partial_{x_i} \varphi|\, dx\, ds \le T \|f\|_{L^2(D)}\, |K|^{1/2} \|\partial_{x_i} \varphi\|_{L^\infty} < \infty, \tag{2.22}$$

so Fubini's theorem applies. Substituting (2.20) into $-\langle u(\cdot,v),\partial_{x_i}\varphi\rangle$, applying Fubini, performing the affine substitution $y = x - sv$ (identity Jacobian), and using the chain-rule identity $\partial_{x_i}\varphi(x)|_{x=y+sv} = \partial_{y_i}[\varphi(y+sv)]$, we obtain

$$-\langle u(\cdot,v),\partial_{x_i}\varphi\rangle = -\int_0^T\int_{\mathbb{R}^n} a(y+sv,v,s)\,\tilde f(y)\,\partial_{y_i}[\varphi(y+sv)]\,dy\,ds. \tag{2.23}$$

To shift the derivative onto the source, we approximate $\tilde f$ by a sequence $\tilde f_m \in C_c^\infty(\mathbb{R}^n)$ with $\tilde f_m \to \tilde f$ in $H^1(\mathbb{R}^n)$. Since $H^1(\mathbb{R}^n)$ is a Banach space, we may pass to a subsequence satisfying $\|\tilde f_m - \tilde f\|_{H^1} \le 2^{-m}$. Integration by parts in y_i and the chain rule applied to a give

$$-\int_{\mathbb{R}^n} a\,\tilde f_m\,\partial_{y_i}[\varphi(y+sv)]\,dy = \int_{\mathbb{R}^n}\big[(\partial_{x_i}a)\,\tilde f_m + a\,\partial_{y_i}\tilde f_m\big]\varphi(y+sv)\,dy. \tag{2.24}$$

Defining the L^2 dominating functions

$$g := |\tilde f| + \textstyle\sum_m |\tilde f_m - \tilde f|, \quad h := |\partial_{y_i}\tilde f| + \textstyle\sum_m |\partial_{y_i}\tilde f_m - \partial_{y_i}\tilde f|, \tag{2.25}$$

Minkowski's inequality gives $\|g\|_{L^2} \le \|\tilde f\|_{L^2} + 1 < \infty$ and $\|h\|_{L^2} < \infty$. Since $\varphi(y+sv) \ne 0$ only for y in the bounded set $K' := \bigcup_{s\in[0,T]}(K - sv)$, Cauchy–Schwarz ensures local L^1 integrability, and the integrand in (2.24) is dominated by the m-independent bound $\|\varphi\|_{L^\infty}(C_\mu g + h)$. The Dominated Convergence Theorem yields

$$\partial_{x_i}u(x,v) = \int_0^T\big[(\partial_{x_i}a)\,\tilde f(x-sv) + a\,(\partial_{x_i}\tilde f)(x-sv)\big]\,ds \quad \text{a.e.} \tag{2.26}$$

H^1, W_v, and kinetic regularity bounds. Minkowski's integral inequality applied to (2.20) and (2.26) gives

$$\|u(\cdot,v)\|_{L^2(D)} \le T\|f\|_{L^2(D)}, \qquad \|\partial_{x_i}u(\cdot,v)\|_{L^2(D)} \le T(C_\mu+1)\|f\|_{H^1(D)}. \tag{2.27}$$

Since $|G| < \infty$, integrating over G gives $u \in L^2(G;H^1(D))$. Rearranging the governing equation as $v\cdot\nabla_x u = f - \mu u \in L^2(\Omega)$ confirms $u \in W_v(\Omega)$ and the boundedness of $\mathcal{K}$.

To confirm that $\partial_{x_i}u$ possesses kinetic regularity, we test $\partial_{x_i}(v\cdot\nabla_x u)$ against $\phi \in C_c^\infty(\Omega)$. Because v is x-independent, $v\cdot\nabla_x(\partial_{x_i}\varphi) = \partial_{x_i}(v\cdot\nabla_x\varphi)$ for any test function. Using this and successive integrations by parts:

$$\begin{aligned}\langle\partial_{x_i}(v\cdot\nabla_x u),\phi\rangle &\underset{(1)}{=} -\langle v\cdot\nabla_x u,\partial_{x_i}\phi\rangle \underset{(2)}{=} \langle u, v\cdot\nabla_x(\partial_{x_i}\phi)\rangle \\ &\underset{(3)}{=} \langle u,\partial_{x_i}(v\cdot\nabla_x\phi)\rangle \underset{(4)}{=} -\langle\partial_{x_i}u, v\cdot\nabla_x\phi\rangle \underset{(5)}{=} \langle v\cdot\nabla_x(\partial_{x_i}u),\phi\rangle,\end{aligned} \tag{2.28}$$

where (1) is distributional differentiation; (2) integration by parts for $v\cdot\nabla_x$; (3) x-independence of v; (4)–(5) successive integrations by parts. Hence $[\partial_{x_i}, v\cdot\nabla_x] = 0$ in $\mathcal{D}'(\Omega)$, and differentiating the transport equation yields

$$v\cdot\nabla_x(\partial_{x_i}u) = \partial_{x_i}f - (\partial_{x_i}\mu)u - \mu(\partial_{x_i}u) \in L^2(\Omega), \tag{2.29}$$

so $\partial_{x_i}u \in W_v(\Omega)$ and $\|\nabla_x u\|_{W_v(\Omega)^n} \le C_{\mathrm{grad}}\|f\|_{H^1(D)}$.

Vanishing inflow trace. We proceed by density, selecting $\{f_k\} \subset C_c^\infty(D)$ with $f_k \to f$ in $H_0^1(D)$ and defining $u_k := \mathcal{K}f_k$, $w_k := \nabla_x u_k$. Set $d_k :=$

$\operatorname{dist}(\operatorname{supp}(f_k), \partial D) > 0$ and $V_k := \{(x,v) \in \overline{\Omega} : \tau_-(x,v) < d_k/c_1\}$. By lower semicontinuity of τ_- (Lemma 2.6), V_k is open; by convention (2.3), $\Gamma_- \subset V_k$.

For any $(x,v) \in V_k \cap \Omega$ and $s \in [0, \tau_-(x,v))$, the distance $\operatorname{dist}(x - sv, \partial D) \leq c_1\tau_-(x,v) < d_k$, so the characteristic never intersects $\operatorname{supp}(f_k)$. By (2.20) and (2.26):

$$u_k = 0 \quad \text{and} \quad (w_k)_i = 0 \quad \text{a.e. on } V_k \cap \Omega, \tag{2.30}$$

and (2.29) then gives $v \cdot \nabla_x (w_k)_i = 0$ a.e. on $V_k \cap \Omega$ as well.

Define $F_k := |(w_k)_i|^2 + |v \cdot \nabla_x (w_k)_i|^2 = 0$ a.e. on $V_k \cap \Omega$. The upper integration limit d_k/c_1 in the following co-area integral is justified by Lemma 2.9: for any $(x_0, v_0) \in \Gamma_-$ and $s \in (0, d_k/c_1)$, that lemma gives $\tau_-(x_0 + sv_0, v_0) = s < d_k/c_1$, so $(x_0 + sv_0, v_0) \in V_k \cap \Omega$. Applying the co-area formula (2.6):

$$0 = \int_{V_k \cap \Omega} F_k \, dx \, dv = \int_G \int_{\partial D_v^-} \int_0^{\min(\tau_+, d_k/c_1)} F_k(x_0 + sv, v) \, |v \cdot \nu(x_0)| \, ds \, dS \, dv. \tag{2.31}$$

Since the integrand is non-negative, the Fubini-Tonelli theorem extracts a full-measure $\Lambda_k \subset \Gamma_-$ with $\int_0^{\varepsilon_k} F_k(x_0 + sv_0, v_0) \, ds = 0$ for $(x_0, v_0) \in \Lambda_k$, where $\varepsilon_k := \min\{\varepsilon_0, d_k/c_1\}$ and ε_0 is from Lemma 2.5, hence

$$(w_k)_i = 0 \ \text{ and } \ v_0 \cdot \nabla_x (w_k)_i = 0 \quad \text{a.e. } s \in (0, \varepsilon_k). \tag{2.32}$$

By Lemma 2.11, there is a full-measure $\Sigma_k \subset \Gamma_-$ on which the fiber profile $\Phi(s) := (w_k)_i(x_0 + sv_0, v_0)$ is absolutely continuous with $\Phi'(s) = v_0 \cdot \nabla_x (w_k)_i (x_0 + sv_0, v_0)$ a.e. For any $(x_0, v_0) \in \Sigma_k \cap \Lambda_k$, equations (2.32) give $\Phi' = 0$ a.e. on $(0, \varepsilon_k)$, forcing Φ constant; since $\Phi = 0$ a.e., the constant is zero. The trace characterization (2.10) then yields $\gamma_-(w_k)_i(x_0, v_0) = \lim_{s \to 0^+} \Phi(s) = 0$, so $\gamma_-(w_k)_i = 0$ in $L^2(\Gamma_-)$.

Finally, $\|\nabla_x u_k - \nabla_x u\|_{W_v^n} \leq C_{\mathrm{grad}} \|f_k - f\|_{H^1} \to 0$, and the strong continuity of γ_- (Cessenat's theorem) gives $\gamma_-(\nabla_x u) = \lim_k \gamma_-(w_k) = 0$. $\square$

3. Variational Regularization

The observation operator $\mathcal{T} := \gamma_+ \circ \mathcal{K} : H_0^1(D) \to L^2(\Gamma_+, |v \cdot \nu|)$ is bounded and linear. For injectivity we take $\mu = \mu(x)$. Throughout, $H_0^1(D)$ carries the inner product

$$\langle f, g \rangle_{H_0^1} := \int_D \nabla f \cdot \nabla g \, dx, \qquad \|f\|_{H_0^1} := \|\nabla f\|_{L^2(D)}, \tag{3.1}$$

equivalent to the full $H^1(D)$ norm by Poincaré's inequality. For each $g \in L^2(\Gamma_+, |v \cdot \nu|)$, the functional $f \mapsto \langle \mathcal{T} f, g \rangle_{L^2(\Gamma_+)}$ is bounded and linear on $H_0^1(D)$, so the Riesz representation theorem on $(H_0^1(D), \langle \cdot, \cdot \rangle_{H_0^1})$ yields a unique adjoint $\mathcal{T}^* : L^2(\Gamma_+, |v \cdot \nu|) \to H_0^1(D)$ satisfying $\langle \mathcal{T} f, g \rangle_{L^2(\Gamma_+)} = \langle f, \mathcal{T}^* g \rangle_{H_0^1}$ for all $f \in H_0^1(D)$. The operator $\mathcal{T}^*\mathcal{T} + \alpha I$ is then self-adjoint, coercive with constant α, and boundedly invertible by the Lax–Milgram theorem.

Remark 3.1. Since $\mu \geq \mu_0 > 0$, the injectivity conditions below are satisfied. For $\mu \geq 0$, existence and uniqueness of (2.2) hold by [5].

Definition 3.2. For $\mu = \mu(x) \in C^1(\overline{D})$, $\mu \geq 0$, define $\mathcal{A}_\mu : L^2(D) \to L^2(\Gamma_+, |v \cdot \nu|\, dS\, dv)$ by

$$(\mathcal{A}_\mu f)(x, v) := \int_0^T \exp\Big(- \int_0^s \mu(x - tv)\, dt \Big)\, \tilde{f}(x - sv)\, ds, \quad (x, v) \in \Gamma_+, \quad (3.2)$$

where $T := \operatorname{diam}(D)/c_0$. Since $\tilde{f} = 0$ outside D, only the chord $s \in [0, \tau_-(x, v)]$ contributes.

Proposition 3.3. *Assume $\mu \in C^1(\overline{D})$, $\mu \geq 0$, and $\{v/|v| : v \in G\}$ contains an open subset of S^{n-1}. Then $\mathcal{T}$ is injective under any one of:*

(i) $n = 2$ *(Novikov [14]);*
(ii) $n \geq 3$, *μ spatially constant > 0 (Finch [8]);*
(iii) $n \geq 3$, $\|\mu\|_{C^1} < \kappa(D)$ ([1, Theorem 2.1]).

Proof. We show $\mathcal{T} f = \mathcal{A}_\mu f$ in $L^2(\Gamma_+)$. For a.e. $(x, v) \in \Gamma_+$, Remark 2.12 gives $\gamma_+ u(x, v) = \lim_{\varepsilon \to 0^+} u(x - \varepsilon v, v)$, and $x - \varepsilon v \in D$ for small $\varepsilon > 0$ by Remark 2.4. From (2.20), substituting $\sigma = s + \varepsilon$ and $r = \varepsilon + t$ to transform the inner integral:

$$u(x - \varepsilon v, v) = \int_\varepsilon^{T+\varepsilon} e^{-\int_\varepsilon^\sigma \tilde{\mu}(x - rv, v)\, dr}\, \tilde{f}(x - \sigma v)\, d\sigma. \quad (3.3)$$

As $\varepsilon \to 0^+$, the integrand converges pointwise to $e^{-\int_0^\sigma \mu(x - rv, v)\, dr}\, \tilde{f}(x - \sigma v)$ (using $\tilde{\mu} = \mu$ on D and the support constraint $\tilde{f} = 0$ outside D from Lemma 2.3). The L^2 dominator $\sigma \mapsto |\tilde{f}(x - \sigma v)|$ on $[0, \tau_+(x, -v)]$, provided by Lemma 2.10(III) and hence L^1, governs all integrands in (3.3) uniformly in $\varepsilon \in (0, 1]$. The Dominated Convergence Theorem gives $\gamma_+ u(x, v) = (\mathcal{A}_\mu f)(x, v)$.

Under conditions (i)–(iii), injectivity of $\mathcal{A}_\mu$ on $L^2(D)$ follows; since $H_0^1(D) \hookrightarrow L^2(D)$, we have $\mathcal{T} f = 0 \Rightarrow f = 0$. $\square$

Remark 3.4. $|v| \geq c_0 > 0$ and D bounded: every characteristic exits in time $\leq T < \infty$.

3.1. **Morozov Discrepancy Principle and Strong Convergence.** Given $\|u_\delta - \mathcal{T} f_{\text{exact}}\|_{L^2(\Gamma_+)} \leq \delta$, define

$$J_\alpha(f) := \tfrac{1}{2} \|\mathcal{T} f - u_\delta\|^2_{L^2(\Gamma_+)} + \tfrac{\alpha}{2} \|f\|^2_{H_0^1}, \quad \alpha > 0. \quad (3.4)$$

Remark 3.5. Strong convergence without compactness and without source conditions is classical abstractly [6]; the contribution is its integration with the nonstandard transport forward mapping of Theorem 2.13, whose gradient trace control requires the zero-extension framework.

Proposition 3.6. *For any $\alpha > 0$, J_α has a unique minimizer $f_\alpha \in H_0^1(D)$.*

Proof. Existence. Since $J_\alpha(0) = \frac{1}{2}\|u_\delta\|^2 < \infty$, the infimum is finite. A minimizing sequence $\{f_k\}$ satisfies $\|f_k\|^2_{H_0^1} \leq (2/\alpha)(\inf J_\alpha + 1) < \infty$, so reflexivity and Banach–Alaoglu yield a weakly convergent subsequence $f_{k_j} \rightharpoonup f$. Bounded linearity of $\mathcal{T}$ gives $\mathcal{T} f_{k_j} \rightharpoonup \mathcal{T} f$,

$$f_{k_j} \rightharpoonup f \;\Rightarrow\; \mathcal{T} f_{k_j} \rightharpoonup \mathcal{T} f \quad \text{in } L^2(\Gamma_+), \quad (3.5)$$

and weak lower semicontinuity of the convex continuous functionals gives $J_\alpha(f) \le \inf J_\alpha$.

Uniqueness. The strict convexity of the squared Hilbert norm $\|f\|^2_{H^1_0}$ makes J_α strictly convex, so the minimizer is unique. □

Lemma 3.7. *Let $\phi(\alpha) := \|\mathcal{T} f_\alpha - u_\delta\|_{L^2(\Gamma_+)}$ for $\alpha > 0$. Then:*

(i) $\phi(\alpha) = \alpha\|(\mathcal{T}\mathcal{T}^* + \alpha I)^{-1} u_\delta\|$;

(ii) *ϕ is continuous on $(0,\infty)$, non-decreasing, and strictly increasing on $\{\alpha : \phi(\alpha) < \|u_\delta\|\}$;*

(iii) *$\phi(\alpha) \to \operatorname{dist}(u_\delta, \overline{\operatorname{ran}(\mathcal{T})})$ as $\alpha \to 0^+$;*

(iv) *$\phi(\alpha) \to \|u_\delta\|$ as $\alpha \to +\infty$.*

Proof. The minimizer satisfies

$$(\mathcal{T}^*\mathcal{T} + \alpha I) f_\alpha = \mathcal{T}^* u_\delta. \tag{3.6}$$

The resolvent identity $\mathcal{T}(\mathcal{T}^*\mathcal{T}+\alpha I)^{-1}\mathcal{T}^* = (\mathcal{T}\mathcal{T}^*+\alpha I)^{-1}\mathcal{T}\mathcal{T}^*$ (see [6, Lemma 2.8]) yields $\mathcal{T} f_\alpha - u_\delta = -\alpha(\mathcal{T}\mathcal{T}^* + \alpha I)^{-1} u_\delta$, which is (i).

Continuity. Spectral theorem for the non-negative operator $\mathcal{T}\mathcal{T}^*$ gives $\alpha \mapsto (\mathcal{T}\mathcal{T}^* + \alpha I)^{-1}$ continuous in strong operator topology, since $\lambda \mapsto (\lambda + \alpha)^{-1}$ is continuous in α uniformly on the spectrum and bounded by α^{-1}. Hence $\alpha \mapsto (\mathcal{T}\mathcal{T}^* + \alpha I)^{-1} u_\delta$ is norm-continuous, and ϕ is continuous.

Monotonicity. Differentiating (3.6) in α gives $\frac{d}{d\alpha} f_\alpha = -(\mathcal{T}^*\mathcal{T}+\alpha I)^{-1} f_\alpha$. Using $\mathcal{T}^*(\mathcal{T} f_\alpha - u_\delta) = -\alpha f_\alpha$:

$$\begin{aligned} \tfrac{d}{d\alpha}\phi(\alpha)^2 &= 2\langle \mathcal{T} f_\alpha - u_\delta, \mathcal{T}\tfrac{d}{d\alpha} f_\alpha\rangle \\ &= 2\alpha\langle f_\alpha, (\mathcal{T}^*\mathcal{T} + \alpha I)^{-1} f_\alpha\rangle \\ &= 2\alpha\|(\mathcal{T}^*\mathcal{T} + \alpha I)^{-1/2} f_\alpha\|^2 \ge 0. \end{aligned} \tag{3.7}$$

On $\{\phi(\alpha) < \|u_\delta\|\}$, if $f_\alpha = 0$ then (3.6) forces $\mathcal{T}^* u_\delta = 0$, giving $\phi(\alpha) = \|u_\delta\|$ by (i), a contradiction. Hence $f_\alpha \ne 0$ and the derivative is strictly positive, establishing (ii).

Limits. The spectral theorem gives $\alpha(\mathcal{T}\mathcal{T}^* + \alpha I)^{-1} \to P_{\ker(\mathcal{T}^*)}$ strongly as $\alpha \to 0^+$, and $\to I$ strongly as $\alpha \to +\infty$. Applying (i) yields (iii) and (iv). □

Remark 3.8. Fix $\tau > 1$ and assume

$$\operatorname{dist}(u_\delta, \overline{\operatorname{ran}(\mathcal{T})}) < \tau\delta < \|u_\delta\|. \tag{3.8}$$

The first inequality holds since $\|\mathcal{T} f_{\mathrm{exact}} - u_\delta\| \le \delta < \tau\delta$. The second requires a non-trivial signal-to-noise ratio, satisfied for small δ when $\mathcal{T} f_{\mathrm{exact}} \ne 0$. By Lemma 3.7(ii)–(iv) and IVT, there exists a unique $\alpha(\delta) > 0$ with

$$\|\mathcal{T} f_{\alpha(\delta)} - u_\delta\| = \tau\delta. \tag{3.9}$$

As $\delta \to 0$: $\phi(\alpha(\delta)) = \tau\delta \to 0$ and monotonicity give $\alpha(\delta) \to 0$, and $\delta^2/\alpha(\delta) \to 0$ by standard theory [6, Theorem 4.17].

Theorem 3.9. *Let $f_{\mathrm{exact}} \in H^1_0(D)$, $\|u_\delta - \mathcal{T} f_{\mathrm{exact}}\|_{L^2(\Gamma_+)} \le \delta$ with (3.8), and $\alpha(\delta)$ satisfy (3.9). For any $\delta_k \to 0$, $f_{\alpha(\delta_k)} \to f_{\mathrm{exact}}$ strongly in $H^1_0(D)$.*

Proof. Write $f_k := f_{\alpha(\delta_k)}$ and $\alpha_k := \alpha(\delta_k)$.

Uniform bound. The minimality condition $J_{\alpha_k}(f_k) \le J_{\alpha_k}(f_{\text{exact}})$ gives $\frac{\tau^2\delta_k^2}{2} + \frac{\alpha_k}{2}\|f_k\|^2 \le \frac{\delta_k^2}{2} + \frac{\alpha_k}{2}\|f_{\text{exact}}\|^2$. Rearranging with $1-\tau^2 < 0$ and $\delta_k^2/\alpha_k \to 0$ (Remark 3.8):

$$\|f_k\|_{H_0^1}^2 \le \|f_{\text{exact}}\|_{H_0^1}^2 + \frac{(1-\tau^2)\delta_k^2}{\alpha_k} \le \|f_{\text{exact}}\|_{H_0^1}^2, \tag{3.10}$$

and hence

$$\limsup_k \|f_k\|_{H_0^1}^2 \le \|f_{\text{exact}}\|_{H_0^1}^2. \tag{3.11}$$

Weak limit identification. By (3.10) and reflexivity, every subsequence of $\{f_k\}$ admits a weakly convergent further subsequence; let $f_{k_j} \rightharpoonup f_*$. From (3.9), $\|\mathcal{T}f_{k_j} - \mathcal{T}f_{\text{exact}}\| \le (\tau+1)\delta_{k_j} \to 0$, so $\mathcal{T}f_{k_j} \to \mathcal{T}f_{\text{exact}}$ strongly. Since a sequence in a Hilbert space cannot converge strongly to one limit and weakly to another (due to the uniqueness of the weak limit [2, Proposition 3.5]), the weak convergence $\mathcal{T}f_{k_j} \rightharpoonup \mathcal{T}f_*$ established in (3.5) forces $\mathcal{T}f_* = \mathcal{T}f_{\text{exact}}$. Injectivity (Proposition 3.3) yields $f_* = f_{\text{exact}}$. Since every subsequence converges weakly to the same limit:

$$f_k \rightharpoonup f_{\text{exact}} \quad \text{weakly in } H_0^1(D). \tag{3.12}$$

Strong convergence via Radon–Riesz. Weak lower semicontinuity and (3.11) give $\|f_{\text{exact}}\|^2 \le \liminf_k \|f_k\|^2 \le \limsup_k \|f_k\|^2 \le \|f_{\text{exact}}\|^2$, hence $\|f_k\|_{H_0^1} \to \|f_{\text{exact}}\|_{H_0^1}$. Combined with (3.12) and $\langle f_k, f_{\text{exact}}\rangle_{H_0^1} \to \|f_{\text{exact}}\|_{H_0^1}^2$:

$$\|f_k - f_{\text{exact}}\|_{H_0^1}^2 = \|f_k\|_{H_0^1}^2 - 2\langle f_k, f_{\text{exact}}\rangle_{H_0^1} + \|f_{\text{exact}}\|_{H_0^1}^2 \to 0. \qquad \square$$

Remark 3.10. Under $f_{\text{exact}} \in \mathcal{R}((\mathcal{T}^*\mathcal{T})^\nu)$ for $\nu > 0$, standard results [6, Chapter 4] yield $\|f_k - f_{\text{exact}}\|_{H_0^1} = \mathcal{O}(\delta^{2\nu/(2\nu+1)})$. Verifying this source condition within the transport geometry is a direction for future work.

4. Conclusion

We established a fully rigorous zero-extension framework for the inverse source problem in kinetic transport, achieving exact gradient trace control without artificial support margins. While the assumption $f \in H_0^1(D)$ introduces a physical limitation by requiring the source to vanish entirely at the domain boundary, this mathematically strategic restriction enables us to rigorously bypass boundary correction terms and grazing singularities without relying on artificial support conditions.

The logical structure is as follows. The symmetry condition $-v \in G$ (eq. (2.1)) enables the outflow boundary argument in Lemma 2.6 (Sub-case $x_j \in \partial D$ with $v_j \cdot \nu > 0$), the outflow trace characterization (Remark 2.12), and the ART identity in Proposition 3.3. The exit-time monotonicity (Lemma 2.5) is established first and is invoked within Lemma 2.6 without any forward reference. The Uniqueness of Characteristic Parametrization (Lemma 2.7) establishes bijectivity of the co-area map; the key geometric ingredient is the supporting hyperplane property of strictly convex domains, which forces $H_y \cap \overline{D} = \{y\}$ and excludes the grazing case

$v \cdot \nu(x_0) = 0$ both in the existence and uniqueness parts of the proof. The companion Lemma 2.9 yields the exact identity $\tau_-(x_0 + sv, v) = s$: the case $t \in (0, s)$ uses the interval structure $\{t \geq 0 : x_0 + tv \in \overline{D}\} = [0, \tau_+(x_0, v)]$ established in Lemma 2.7; the case $t = s$ is immediate from $x_0 \in \partial D$; and the case $t > s$ applies the supporting hyperplane inequality to show $\nu(x_0) \cdot (rv) = r(v \cdot \nu(x_0)) > 0$ for $r = s - t < 0$. This exact identity justifies the upper limit d_k/c_1 in the vanishing trace argument (2.31). The Jacobian exactness (Lemma 2.10(I)) follows from the affine structure of straight-line flow; the co-area formula (Lemma 2.10(II)) is established via a complete partition-of-unity argument that renders the atlas-independence of the weighted surface measure $|v \cdot \nu|\, dS$ explicit. The trace identification in Lemma 2.11 uses an averaging argument (eq. (2.18)), giving a direct L^1-norm bound without extracting further subsequences. The commutator identity (2.28) rigorously establishes $[\partial_{x_i}, v \cdot \nabla_x] = 0$ using x-independence of v.

For the Tikhonov framework, the inner product (3.1) makes the normal equation unambiguous, and the adjoint $\mathcal{T}^*$ is defined rigorously via the Riesz representation theorem. Lemma 3.7 establishes continuity of $\phi(\alpha)$ via the spectral theorem. The ART derivation (3.3) uses the correct transformed inner integral $\int_\varepsilon^\sigma \tilde{\mu}(x - rv, v)\, dr$, and the dominator is identified as an L^2 function via Lemma 2.10(III). These results provide mathematical foundations for recent computational methods [11, 9].

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