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A NOVEL NUMERICAL METHOD FOR A CLASS OF HATTAF FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Fractional operators of differentiation and integration have become essential tools for modeling a wide range of phenomena encountered in science, industry and engineering. This paper presents a direct numerical scheme for fractional differential equations involving the generalized Hattaf fractional (GHF) derivative, avoiding the common approach of converting these equations into integral forms. In addition, a comprehensive theoretical analysis of the scheme's consistency, stability and convergence is provided. Furthermore, the inclusion of both purely mathematical examples and an application to an economic model highlights the practical relevance of the proposed method.

Keywords. Generalized Hattaf fractional derivative, numerical scheme, convergence, stability, consistence
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1. Introduction

By allowing derivatives of non-integer order, fractional differential equations (FDEs) broaden the scope of classical ordinary differential equations (ODEs). Owing to their ability to capture memory effects and hereditary properties inherent in many real-world processes, FDEs have become a powerful modeling framework in various scientific and engineering domains, including epidemiology [1], cancer modeling [2], cardiology [3], economic systems [4] and thermo-viscoplastic contact phenomena [5].

Over the past few years, significant attention has been devoted to the development of new formulations of fractional derivatives. In 2020, Hattaf proposed a generalized fractional derivative characterized by a non-local and non-singular kernel, encompassing both Riemann-Liouville and Caputo frameworks [6]. This formulation was introduced to better capture memory effects in the analysis of dynamical systems arising in various application areas [7, 9, 8, 10]. Moreover, the

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generalized Hattaf fractional (GHF) operator unifies numerous well-known fractional operators of differentiation that have been extensively studied in previous works, notably that presented in [11, 12, 13].

A large class of FDEs cannot be handled effectively through analytical approaches, which has motivated the development of various numerical techniques aimed at obtaining accurate approximations of their solutions. For instance, a numerical scheme designed for equations involving the Atangana-Baleanu fractional derivative was presented in [14]. This approach does not rely on a predictor-corrector strategy and yields numerical results that are in strong agreement with exact solutions, providing efficient convergence and reliable accuracy. In addition, Shiri et al. [15, 16, 17] proposed methods based on collocation strategies as well as predictor-corrector algorithms to treat FDEs governed by Caputo [18] and Atangana-Baleanu [12] fractional operators. Furthermore, Hattaf and collaborators proposed a novel numerical technique based on Lagrange polynomial interpolation to approximate the solution of FDEs characterized by nonlinear and non-autonomous behavior within the GHF derivative [19]. In a subsequent study [20], the same author introduced an additional numerical approach specifically designed for nonlinear FDEs expressed through the GHF operator. More recently, a numerical method has been introduced in [21] for the solution of FDEs based on the generalized Hattaf mixed fractional derivative [22], which integrates various forms of fractional operators with singular and non-singular kernels.

The aforementioned methods typically rely on converting FDEs into integral forms. In contrast, the present work introduces a novel numerical framework that allows for the direct treatment of nonlinear and non-autonomous FDEs, without recasting them into integral forms. For this purpose, Section 2 introduces the proposed direct method. Section 3 presents a comprehensive theoretical analysis of the scheme's consistency, stability and convergence. Section 4 provides various numerical examples to assess the performance of the proposed method. Section 5 applies the method to an economic model for numerical simulations. Lastly, Section 6 ends the present study.

2. Description of the novel numerical method

This section introduces a novel finite difference numerical scheme for solving directly the following FDE involving the GHF derivative:

$$\mathcal{D}_{0,\phi}^{\alpha,\beta}\psi(\zeta) = f(\zeta,\psi(\zeta)), \tag{2.1}$$

with $\psi(0)=\psi_0$. Let $\zeta_k = kh$, where $k\in\mathbb{N}$ and h denotes the time step size. In order to construct a discrete approximation of the Hattaf fractional derivative, which will lead to the numerical scheme, we begin with the following preliminary lemma.

Lemma 2.1. *Let $(\phi\psi)\in C^2([\zeta_k,\zeta_{k+1}])$. Then for any $\mu\in[\zeta_k,\zeta_{k+1}]$, we have*

$$(\phi\psi)'(\mu) = \frac{(\phi\psi)(\zeta_{k+1})-(\phi\psi)(\zeta_k)}{h} - (m-\mu)(\phi\psi)''(\zeta_k) + O(h^2), \tag{2.2}$$

where $m=\frac{\zeta_{k+1}+\zeta_k}{2}$ denotes the midpoint of (ζ_k,ζ_{k+1}).

Proof. Let $(\phi\psi) \in C^2([\zeta_k, \zeta_{k+1}])$. For any $\mu \in (\zeta_k, \zeta_{k+1})$, it follows from Taylor's formula that there exist $\xi_1, \xi_2 \in (\zeta_k, \zeta_{k+1})$ such that

$$(\phi\psi)(\zeta_{k+1}) = (\phi\psi)(\mu) + (\zeta_{k+1} - \mu)(\phi\psi)'(\mu) + \frac{(\zeta_{k+1} - \mu)^2}{2}(\phi\psi)''(\xi_1),$$

and

$$(\phi\psi)(\zeta_k) = (\phi\psi)(\mu) + (\zeta_k - \mu)(\phi\psi)'(\mu) + \frac{(\zeta_k - \mu)^2}{2}(\phi\psi)''(\xi_2).$$

Subtracting the second expansion from the first yields

$$(\phi\psi)(\zeta_{k+1}) - (\phi\psi)(\zeta_k) = h(\phi\psi)'(\mu) + \frac{(\zeta_{k+1} - \mu)^2}{2}(\phi\psi)''(\xi_1) - \frac{(\mu - \zeta_k)^2}{2}(\phi\psi)''(\xi_2).$$

Dividing by h and rearranging the terms, we obtain

$$\begin{aligned}
(\phi\psi)'(\mu) &= \frac{(\phi\psi)(\zeta_{k+1}) - (\phi\psi)(\zeta_k)}{h} - \frac{(\zeta_{k+1} - \mu)^2}{2h}(\phi\psi)''(\xi_1) + \frac{(\mu - \zeta_k)^2}{2h}(\phi\psi)''(\xi_2), \\
&= \frac{(\phi\psi)(\zeta_{k+1}) - (\phi\psi)(\zeta_k)}{h} - \frac{(\zeta_{k+1} - \mu)^2}{2h}\big((\phi\psi)''(\xi_1) - (\phi\psi)''(\xi_2)\big) \\
&\quad - (\frac{\zeta_{k+1} + \zeta_k}{2} - \mu)(\phi\psi)''(\xi_2).
\end{aligned}$$

Since $\zeta_{k+1} - \mu < h$ and using mean value theorem, we get

$$(\phi\psi)'(\mu) = \frac{(\phi\psi)(\zeta_{k+1}) - (\phi\psi)(\zeta_k)}{h} - (\frac{\zeta_{k+1} + \zeta_k}{2} - \mu)(\phi\psi)''(\xi_2) + O(h^2).$$

Let $m = \frac{\zeta_{k+1}+\zeta_k}{2}$ denote the midpoint of (ζ_k, ζ_{k+1}). Then

$$(\phi\psi)'(\mu) = \frac{(\phi\psi)(\zeta_{k+1}) - (\phi\psi)(\zeta_k)}{h} - (m - \mu)(\phi\psi)''(\zeta_k) + O(h^2).$$

□

Now, we derive a discrete approximation of the Hattaf fractional derivative using the estimate established in the previous Lemma 2.1.

For $\zeta = \zeta_{n+1}$, we have:

$$\mathcal{D}^{\alpha,\beta}_{0,\phi}\psi(\zeta_{n+1}) = \frac{\mathcal{N}(\alpha)}{(1-\alpha)\phi(\zeta_{n+1})}\int_0^{\zeta_{n+1}} (\phi\psi)'(\tau)\,\mathrm{E}_\beta\left[-\mu_\alpha\left(\zeta_{n+1} - \tau\right)^\beta\right] d\tau.$$

Employing the progressive difference approximation (2.2) for $(\phi\psi)'(\tau)$ on the interval $[\zeta_k, \zeta_{k+1}]$ leads to

$$\begin{aligned}
\mathcal{D}^{\alpha,\beta}_{0,\phi}\psi(\zeta_{n+1}) \simeq{}& \frac{\mathcal{N}(\alpha)}{(1-\alpha)\phi(\zeta_{n+1})}\sum_{k=0}^{n}\int_{\zeta_k}^{\zeta_{k+1}} \frac{\phi(\zeta_{k+1})\psi(\zeta_{k+1}) - \phi(\zeta_k)\psi(\zeta_k)}{h} \\
&\times \mathrm{E}_\beta\left[-\mu_\alpha\left(\zeta_{n+1} - \tau\right)^\beta\right] d\tau, \\
={}& \frac{\mathcal{N}(\alpha)}{(1-\alpha)\phi(\zeta_{n+1})}\sum_{k=0}^{n}\left[\phi(\zeta_{n-k+1})\psi(\zeta_{n-k+1}) - \phi(\zeta_{n-k})\psi(\zeta_{n-k})\right] \\
&\times\Big((k+1)\mathrm{E}_{\beta,2}\left[-\mu_\alpha(k+1)h^\beta\right] - k\mathrm{E}_{\beta,2}\left[-\mu_\alpha(kh)^\beta\right]\Big).
\end{aligned}$$

Let $\mathrm{E}_n = \mathrm{E}_{\beta,2}\left[-\mu_\alpha (nh)^\beta\right]$ and $A_n = (n+1)\mathrm{E}_{n+1} - n\mathrm{E}_n$. We get

$$\mathcal{D}_{0,\phi}^{\alpha,\beta}\psi(\zeta_{n+1}) = \frac{\mathcal{N}(\alpha)}{(1-\alpha)\phi(\zeta_{n+1})}\sum_{k=0}^{n}\left[\phi(\zeta_{n-k+1})\psi(\zeta_{n-k+1}) - \phi(\zeta_{n-k})\psi(\zeta_{n-k})\right] \times\Big((k+1)\mathrm{E}_{k+1} - k\mathrm{E}_k\Big).$$

The GHF derivative can be written in discrete form as follows:

$$\mathcal{D}_{0,\phi}^{\alpha,\beta}\psi(\zeta_{n+1}) = \frac{\mathcal{N}(\alpha)}{(1-\alpha)\phi(\zeta_{n+1})}\sum_{k=0}^{n} A_k\left[\phi(\zeta_{n-k+1})\psi(\zeta_{n-k+1}) - \phi(\zeta_{n-k})\psi(\zeta_{n-k})\right]. \tag{2.3}$$

By substituting (2.3) into (2.1) and applying the ρ-weighted scheme, we obtain

$$\frac{\mathcal{N}(\alpha)}{(1-\alpha)\phi(\zeta_{n+1})}\sum_{k=0}^{n}\left[\phi(\zeta_{n-k+1})\psi(\zeta_{n-k+1}) - \phi(\zeta_{n-k})\psi(\zeta_{n-k})\right] A_k = \rho f(\zeta_{n+1}, \psi(\zeta_{n+1})) + (1-\rho) f(\zeta_n, \psi(\zeta_n)), \quad \textit{where} \quad k = 0,1,2,\ldots,n-1 \quad \textit{and} \quad \rho\in[0,1].$$

Consequently, we get the direct numerical method as follows:

$$\psi_{n+1} = \frac{\phi(\zeta_n)}{\phi(\zeta_{n+1})}\,\psi_n - \frac{1}{\phi(\zeta_{n+1})\,\mathrm{E}_1}\sum_{k=1}^{n} A_k\left[\phi(\zeta_{n-k+1})\,\psi_{n-k+1} - \phi(\zeta_{n-k})\,\psi_{n-k}\right] + \frac{1-\alpha}{\mathcal{N}(\alpha)\,\mathrm{E}_1}\left[\rho\, f(\zeta_{n+1}, \psi(\zeta_{n+1})) + (1-\rho)\, f(\zeta_n, \psi(\zeta_n))\right]. \tag{2.4}$$

Remark 2.2. Depending on the choice of ρ, various numerical methods can be derived, including the implicit scheme for $\rho = 1$ and the explicit scheme for $\rho = 0$.

3. Consistency, stability and convergence analysis

This section is devoted to the analysis of the numerical error of the approximation scheme given in (2.4).

In the analysis of the numerical scheme based on the GHF derivative, the discrete coefficients

$$A_k = (k+1)E_{k+1} - kE_k,$$

play a fundamental role in stability analysis. We have the following lemma.

Lemma 3.1. *If $0 < \beta < 1$, then $(A_n)_n$ is a decreasing sequence and $|A_n| \le 1$.*

Proof. Let $\zeta_k = kh$ and define

$$g(\zeta) = \zeta E_{\beta,2}(-\mu_\alpha \zeta^\beta).$$

Then

$$g(\zeta_k) = khE_k.$$

Therefore,

$$A_k = \frac{g(\zeta_{k+1}) - g(\zeta_k)}{h}.$$

It suffices to prove that g is concave. Using the classical derivative identity for Mittag–Leffler functions (see, [24]),

$$\frac{d}{d\zeta}\Big(\zeta\, E_{\beta,2}(-\mu_\alpha \zeta^\beta)\Big) = E_{\beta,1}(-\mu_\alpha \zeta^\beta),$$

we obtain

$$g'(t) = E_{\beta,1}(-\mu_\alpha \zeta^\beta).$$

For $0 < \beta < 1$, $E_{\beta,1}(-\lambda\zeta^\beta)$ is completely monotone for $\lambda > 0$ [25], which leads that $g''(\zeta) \leq 0$, and g is concave. Thus,

$$\frac{g(\zeta_{k+2}) - g(\zeta_{k+1})}{h} \leq \frac{g(\zeta_{k+1}) - g(\zeta_k)}{h},$$

which implies

$$A_{k+1} \leq A_k.$$

Since g is differentiable on $[\zeta_k, \zeta_{k+1}]$, the mean value theorem ensures the existence of $\xi_k \in (\zeta_k, \zeta_{k+1})$ such that

$$g(\zeta_{k+1}) - g(\zeta_k) = g'(\xi_k)(\zeta_{k+1} - \zeta_k) = g'(\xi_k)\, h.$$

Hence,

$$A_k = g'(\xi_k) = E_{\beta,1}\Big(-\mu_\alpha \xi_k^\beta\Big).$$

For $0 < \beta \leq 1$ and $\mu_\alpha > 0$, the function $\zeta \mapsto E_{\beta,1}(-\mu_\alpha\zeta^\beta)$ is completely monotone on $(0, \infty)$. In particular, it is positive and decreasing, and $E_{\beta,1}(0) = 1$. Hence, for all $\zeta > 0$,

$$0 < E_{\beta,1}(-\mu_\alpha \zeta^\beta) \leq 1.$$

Applying this with $\zeta = \xi_k$ yields $|A_k| \leq 1$. □

Theorem 3.2. *Assume that $\phi\psi \in C^2([0,T])$ and $0 < \beta < 1$. The numerical approximation of $\psi(\zeta)$ is expressed by*

$$\begin{aligned}\psi(\zeta_{n+1}) = \frac{\phi(\zeta_n)}{\phi(\zeta_{n+1})}\,\psi(\zeta_n) &+ \frac{1}{\phi(\zeta_{n+1})\,\mathrm{E}_1}\sum_{k=1}^{n} A_k\left[\phi(\zeta_{n-k+1})\,x(\zeta_{n-k+1}) - \phi(\zeta_{n-k})\,x(\zeta_{n-k})\right] \\ &- \frac{1-\alpha}{\mathcal{N}(\alpha)\,\mathrm{E}_1}\left[\rho\, f(\zeta_{n+1}, \psi(\zeta_{n+1})) + (1-\rho)\, f(\zeta_n, \psi(\zeta_n))\right] + R_n^{\alpha,\beta}, \qquad (3.1)\end{aligned}$$

where $\left|R_n^{\alpha,\beta}\right| \leq Ch^2$, with C being a positive constant independent of h.

Proof. From the discretization of the GHF derivative evaluated at ζ_{n+1}, it follows that

$$\begin{aligned}\mathcal{D}_{0,w}^{\alpha,\beta}\psi(\zeta_{n+1}) &= \frac{\mathcal{N}(\alpha)}{(1-\alpha)\phi(\zeta_{n+1})}\int_0^{\zeta_{n+1}} (\phi\psi)'(\tau)\mathrm{E}_\beta\left[-\mu_\alpha\left(\zeta_{n+1}-\tau\right)^\beta\right]d\tau, \\ &= \frac{\mathcal{N}(\alpha)}{(1-\alpha)\phi(\zeta_{n+1})}\,\frac{\phi(\zeta_{k+1})\psi(\zeta_{k+1}) - \phi(\zeta_k)\psi(\zeta_k)}{h} \\ &\quad \times \int_0^{\zeta_{n+1}} \mathrm{E}_\beta\left[-\mu_\alpha\left(\zeta_{n+1}-\tau\right)^\beta\right]d\tau + R_n^{\alpha,\beta},\end{aligned}$$

where the truncation error $R_n^{\alpha,\beta}$ is expressed by:

$$R_n^{\alpha,\beta} = -\frac{h\mathcal{N}(\alpha)}{2(1-\alpha)\phi(\zeta_{n+1})}\sum_{k=0}^{n}\int_{\zeta_k}^{\zeta_{k+1}}(m-\tau)(\phi\psi)''(\zeta_k)\mathrm{E}_\beta\left[-\mu_\alpha\left(\zeta_{n+1}-\tau\right)^\beta\right]d\tau.$$

Hence,

$$\begin{aligned}|R_n^{\alpha,\beta}| &\leq \frac{\mathcal{N}(\alpha)h^2}{2(1-\alpha)\phi(\zeta_{n+1})}\sum_{k=0}^{n}|(m-\mu)|\max_{0<\xi<\xi_k}|(\phi\psi)''(\xi)||A_k|.\\ &\leq \frac{\mathcal{N}(\alpha)h^3}{4(1-\alpha)\phi(\zeta_{n+1})}\sum_{k=0}^{n}\max_{0<\xi<\xi_k}|(\phi\psi)''(\xi)|.\end{aligned}$$

Consequently,

$$\left|R_n^{\alpha,\beta}\right| \leq Ch^2,$$

where C is a constant. □

Remark 3.3. The estimate $\left|R_n^{\alpha,\beta}\right| \leq C\,h^2$, implies that the truncation error tends to zero as $h \to 0$ and is of order $O(h^2)$. Consequently, the numerical scheme is consistent with second-order accuracy in time.

Now, we consider the stability of the fractional numerical scheme (2.4). Stability in this context means that small perturbations in the initial conditions do not result in large errors in the numerical solution.

Theorem 3.4. *Let ψ_n and $\tilde{\psi}_n$ be two numerical solutions generated by the proposed scheme corresponding to two different initial conditions. Define the error sequence*

$$e_n = \psi_n - \tilde{\psi}_n.$$

Assume that $\phi \in C^1([0,T])$, f is Lipschitz continuous with respect to its second argument with Lipschitz constant $L > 0$, and $\phi(\zeta) > 0$ for all $t \in [0,T]$. Then the numerical scheme is stable.

Proof. Subtracting the two numerical schemes satisfied by ψ_n and $\tilde{\psi}_n$, we obtain

$$\begin{aligned}|e_{n+1}| \leq{}& \frac{|\phi(\zeta_n)|}{|\phi(\zeta_{n+1})|}\,|e_n| + \frac{1}{\phi(\zeta_{n+1})E_1}\sum_{k=1}^{n}|A_k|\,|g_{n-k+1}-g_{n-k}|\\ &+\frac{1-\alpha}{\mathcal{N}(\alpha)E_1}\Big[\rho\,|f(\zeta_{n+1},\psi_{n+1})-f(\zeta_{n+1},\tilde{\psi}_{n+1})|\\ &+(1-\rho)\,|f(\zeta_n,\psi_n)-f(\zeta_n,\tilde{\psi}_n)|\Big],\end{aligned}\tag{3.2}$$

where $g_{n-k+1} = \phi(\zeta_{n-k+1})e_{n-k+1}$. Since $\phi \in C^1([0,T])$, it is continuous on the compact interval $[0,T]$. By the extreme value theorem, ϕ attains its minimum and maximum on $[0,T]$. Therefore, there exist two positive constants ϕ_* and ϕ^* such that

$$0 < \phi_* \leq \phi(\zeta) \leq \phi^*, \quad \text{for all}\,\zeta \in [0,T].$$

According to the Lipschitz continuity of f and the monotonicity of (A_k), we have

$$|e_{n+1}| \leq \frac{\phi^*}{\phi_*}|e_n| + \frac{1}{\phi_* E_1}\Big[A_1|g_n| + A_n|g_0| + \sum_{k=1}^{n-1}|A_{k+1}-A_k|\,|g_{n-k}|\Big] + \frac{(1-\alpha)L}{\mathcal{N}(\alpha)E_1}\big[\rho|e_{n+1}| + (1-\rho)|e_n|\big].$$

Thus,

$$|e_{n+1}| \leq \frac{\phi^*}{\phi_*}|e_n| + \frac{2A_1}{\phi_* E_1}\max_{1\leq j\leq n}|g_j| + \frac{(1-\alpha)L}{\mathcal{N}(\alpha)E_1}\big[\rho|e_{n+1}| + (1-\rho)|e_n|\big].$$

Hence,

$$\Big(1 - \frac{L(1-\alpha)\rho}{\mathcal{N}(\alpha)E_1}\Big)|e_{n+1}| \leq \frac{\phi^*}{\phi_*}|e_n| + \frac{L(1-\alpha)(1-\rho)}{\mathcal{N}(\alpha)E_1}|e_n| + \frac{2A_1}{\phi_* E_1}\max_{1\leq j\leq n}|g_j|.$$

Assume that $1 - \frac{L(1-\alpha)\rho}{\mathcal{N}(\alpha)E_1} > 0$ and let $M_n = \max_{0\leq j\leq n}|e_j|$. Then

$$M_{n+1} \leq \kappa\, M_n,$$

with

$$\kappa = \frac{\frac{\phi^*}{\phi_*} + \frac{L(1-\alpha)(1+\rho)}{\mathcal{N}(\alpha)E_1} + \frac{2A^*}{\phi_* E_1}}{1 - \frac{L(1-\alpha)\rho}{\mathcal{N}(\alpha)E_1}}.$$

If $\kappa < 1$, then (M_n) is non-increasing and

$$|e_n| \leq M_0, \quad \text{for all } n \in \mathbb{N}.$$

Therefore, the numerical scheme is stable. □

Next, we investigate the convergence of the numerical scheme (2.4).

Theorem 3.5. *Assume that ψ_n is an approximation of the exact solution $\psi(\zeta_n)$ and that $f(\zeta,\psi)$ satisfies the Lipschitz condition with respect to ψ with Lipschitz constant $L > 0$. Then*

$$\max_{0\leq n\leq N}|\psi(\zeta_n) - \psi_n| = O(h).$$

Proof. Denote by

$$\tilde{e_n} = \psi(\zeta_n) - \psi_n, \qquad \lambda = \frac{1-\alpha}{\mathcal{N}(\alpha)E_1}.$$

Then, we obtain

$$\tilde{e}_{n+1} = \frac{\phi(\zeta_n)}{\phi(\zeta_{n+1})}\tilde{e}_n - \frac{1}{\phi(\zeta_{n+1})E_1}\sum_{k=1}^{n}A_k\big[\phi(\zeta_{n-k+1})\tilde{e}_{n-k+1} - \phi(\zeta_{n-k})\tilde{e}_{n-k}\big] + \lambda\Big[\rho\big(f(\zeta_{n+1},\psi(\zeta_{n+1})) - f(\zeta_{n+1},\psi_{n+1})\big) + (1-\rho)\big(f(\zeta_n,\psi(\zeta_n)) - f(\zeta_n,\psi_n)\big)\Big] + R^{\alpha,\beta}_{n+1}.$$

Taking the absolute value, we get

$$|\tilde{e}_{n+1}| \leq \frac{\phi^*}{\phi_*}|\tilde{e}_n| + \frac{1}{\phi_* E_1}\left|\sum_{k=1}^{n} A_k\big[\phi(\zeta_{n-k+1})\tilde{e}_{n-k+1} - \phi(\zeta_{n-k})\tilde{e}_{n-k}\big]\right| + \lambda L\big(\rho|\tilde{e}_{n+1}| + (1-\rho)|\tilde{e}_n|\big) + |R_{n+1}^{\alpha,\beta}|.$$

Hence,

$$(1-\lambda L\rho)|\tilde{e}_{n+1}| \leq \left(\frac{\phi^*}{\phi_*} + \lambda L(1-\rho)\right)|\tilde{e}_n| + \frac{1}{\phi_* E_1}S_n + |R_{n+1}^{\alpha,\beta}|,$$

where

$$S_n = \left|\sum_{k=1}^{n} A_k\big[\phi(\zeta_{n-k+1})\tilde{e}_{n-k+1} - \phi(\zeta_{n-k})\tilde{e}_{n-k}\big]\right|.$$

Using the monotonicity of (A_k), we obtain

$$S_n \leq 2A_1\phi^* \max_{0\leq j\leq n}|\tilde{e}_j|.$$

Thus,

$$(1-\lambda L\rho)|\tilde{e}_{n+1}| \leq \left(\frac{\phi^*}{\phi_*} + \frac{2A_1\phi^*}{\phi_* E_1} + \lambda L(1-\rho)\right)\max_{0\leq j\leq n}|\tilde{e}_j| + |R_{n+1}^{\alpha,\beta}|.$$

Assume that $1-\lambda L\rho > 0$ and let

$$C = \frac{\phi^*}{\phi_*} + \frac{2A_1\phi^*}{\phi_* E_1} + \lambda L(1-\rho), \qquad q = \frac{C}{1-\lambda L\rho}.$$

Then

$$|\tilde{e}_{n+1}| \leq q\max_{0\leq j\leq n}|\tilde{e}_j| + \frac{1}{1-\lambda L\rho}|R_{n+1}^{\alpha,\beta}|.$$

Define

$$\tilde{M}_n = \max_{0\leq j\leq n}|\tilde{e}_j|.$$

We obtain the recurrence relation

$$\tilde{M}_{n+1} \leq q\tilde{M}_n + \frac{1}{1-\lambda L\rho}|R_{n+1}^{\alpha,\beta}|.$$

By iteration, we get

$$\tilde{M}_n \leq q^n\tilde{M}_0 + \frac{1}{1-\lambda L\rho}\sum_{k=0}^{n-1} q^k \max_{0\leq j\leq n}|R_j^{\alpha,\beta}|.$$

Assume that $q<1$. Since $\tilde{M}_0 = 0$, it follows that

$$\tilde{M}_n \leq \frac{C}{(1-\lambda L\rho)(1-q)}h.$$

Therefore,

$$\max_{0\leq j\leq n}|\tilde{e}_j| \leq \delta h.$$

□

4. Illustrative examples

This section aims to evaluate the performance of the proposed approach by applying it to fractional differential equations for which exact solutions are known.
Example 1. We consider the following fractional differential equation:

$$\begin{cases} D^{\alpha,\beta}_{0,\phi}\psi(\zeta) = \zeta^2, \\ \psi(0) = 0. \end{cases} \tag{4.1}$$

Setting $\phi(\zeta) = 1$ and applying the Hattaf fractional integral operator [23] to both sides of (4.1), one obtains the exact solution in the following form:

$$\psi(\zeta) = \frac{1-\alpha}{\mathcal{N}(\alpha)}\zeta^2 + \frac{2\alpha\zeta^{\beta+2}}{\Gamma(\beta+3)\mathcal{N}(\alpha)}.$$

We next apply the developed method to compute an approximation of the solution of (4.1). Throughout the numerical simulations the normalization function is defined as follows:

$$\mathcal{N}(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}.$$

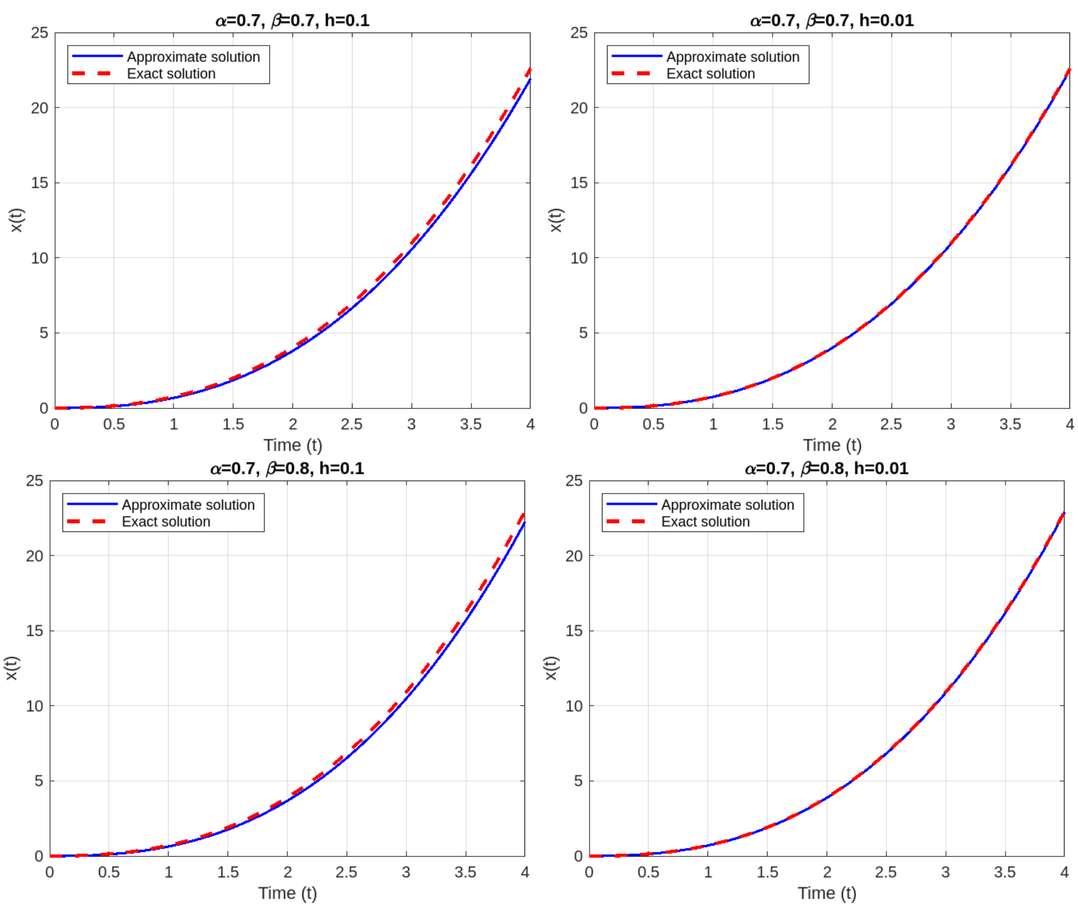

Figure 1. Comparison of analytical and numerical solutions of (4.1) for various values of α, β and h.

TABLE 1. Computed errors for different step sizes h in Example (4.1), with $\alpha = 0.7$ and $\beta = 0.8$.

Step of discretization h	Error
0.1	5.26904×10^{-4}
0.01	5.58229×10^{-7}
0.001	5.63514×10^{-10}

Figure 1 shows a comparison between the exact solution and its numerical approximation for equation (4.1), for different values of h, α and β. We also present in Table 1 the corresponding error values obtained for α=0.7 and β=0.8, computed for various discretization step sizes h.

Example 4.1. Consider the following fractional differential equation:

$$\begin{cases} D^{\alpha,\beta}_{0,\phi}\psi(\zeta) = \sin(\zeta), \\ \psi(0) = 0. \end{cases} \tag{4.2}$$

Setting $\phi(\zeta) = 1$ and applying the Hattaf fractional integral operator [23] to both sides of (4.2), one obtains the exact solution in the following form:

$$\psi(\zeta) = \frac{1-\alpha}{\mathcal{N}(\alpha)}\sin(\zeta) + \frac{\alpha}{\Gamma(\beta)\mathcal{N}(\alpha)}\int_0^\zeta (\zeta-\tau)^{\beta-1}\phi(\zeta)\sin(\zeta)d\tau.$$

Subsequently, the proposed method is applied to approximate the solution of (4.2).

TABLE 2. Computed errors for different step sizes h in Example (4.2), with $\alpha = 0.7$ and $\beta = 0.8$.

Step of discretization h	Error
0.1	-3.64754×10^{-5}
0.01	-4.23649×10^{-8}
0.001	-4.29936×10^{-11}

Table 2 presents the difference between the numerical approximation produced by the numerical scheme and the analytical solution of (4.2), computed for several time step sizes h, with $\alpha = 0.7$ and $\beta = 0.8$. We observe that the error remains negative across all tested values of h, which indicates a slight underestimation of the exact solution by the numerical scheme. Despite this, the magnitude of the error is very small and decreases significantly as the time step is reduced. In particular, it decreases from the order of 10^{-5} for $h = 0.1$ to the order of 10^{-11} for $h = 0.001$, confirming both the accuracy and convergence of the method. Overall, the results validate the reliability and performance of the proposed method, even in the context of generalized fractional derivatives.

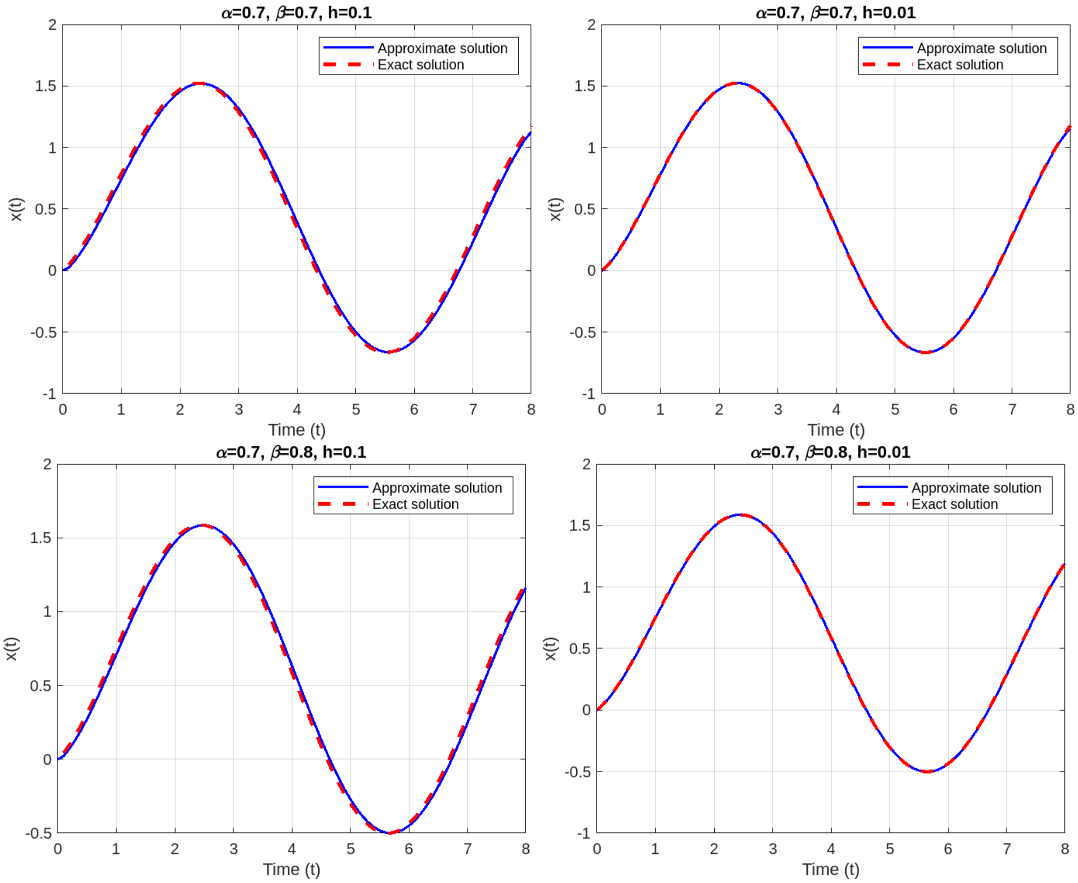

FIGURE 2. Comparison between analytical and numerical solutions of (4.2) for different choices of α, β, and step size h.

5. Application to economics

This section presents and illustrates the dynamics of an economic model by applying our novel numerical method.

Firstly, we propose an economic model that describes the interaction between two essential factors that are capital, which represents the stock of productive assets such as machinery, equipment and infrastructure, as well as labour, which corresponds to the workforce or economically active population participating in productive activities. This model provides a mathematical representation of how these two factors jointly determine output and influence the evolution of an economy over time. By linking capital accumulation with changes in the labour force, it allows for the analysis of production levels, employment, productivity and long-term economic adjustment. Therefore, the proposed capital-labour model is formulated by the following nonlinear system of FDEs:

$$\begin{cases} D^{\alpha,\beta}_{0,\phi}x(\zeta) = \gamma\, x(\zeta)\left(1-\dfrac{x(\zeta)}{M}\right) - \dfrac{\eta\, x(\zeta)\, y(\zeta)}{c_0+c_1x(\zeta)+c_2y(\zeta)+c_3x(\zeta)y(\zeta)}, \\ D^{\alpha,\beta}_{0,\phi}y(\zeta) = \dfrac{\eta\, x(\zeta)\, y(\zeta)}{c_0+c_1x(\zeta)+c_2y(\zeta)+c_3x(\zeta)y(\zeta)} - \delta\, y(\zeta), \end{cases} \tag{5.1}$$

which is a generalization of the mathematical models presented in [27, 28]. The variables $x(\zeta)$ and $y(\zeta)$ represent the number of free jobs and the total labour force at time ζ, respectively. The parameter γ corresponds to the intrinsic per capita growth rate of job opportunities and M defines the upper bound associated with the maximum capacity of available jobs, which is linked to the level of invested capital. The parameter δ characterizes the rate at which individuals leave the workforce. In the proposed fractional model, the interaction between the labour force and available employment opportunities is captured through the Hattaf-Yousfi functional response [29] defined by $\dfrac{\eta x y}{c_0 + c_1 x + c_2 y + c_3 xy}$, where c_0, c_1, c_2 and c_3 are nonnegative constants and the positive factor η is the maximum growth of labour force. In addition, the following result provides a sufficient condition for the local asymptotic stability of the capital-labour equilibrium $E(\bar{x}, \bar{y})$ of (5.1).

Theorem 5.1. *The capital-labour equilibrium $E(\bar{x}, \bar{y})$ is locally asymptotically stable if $\beta_1 = \frac{a_1}{a_0} > 0$ and $\beta_2 = \frac{a_2}{a_0} > 0$, where*

$$\begin{aligned}
a_0 &= \mathcal{N}(\alpha) - \mathcal{N}(\alpha)(1-\alpha)(a_{22} + a_{11}) + (1-\alpha)^2(a_{11}a_{22} - a_{12}a_{21}),\\
a_1 &= -\mathcal{N}(\alpha)(1-\alpha)\mu_\alpha(a_{22} + a_{11}) + 2(1-\alpha)^2\mu_\alpha(a_{11}a_{22} - a_{12}a_{21}),\\
a_2 &= \mu_\alpha^2(1-\alpha)^2(a_{11}a_{22} - a_{12}a_{21}),
\end{aligned}$$

with $a_{11} = \left(1 - \frac{1}{\bar{x}}\frac{c_0 + c_2\bar{y}}{c_0 + c_1\bar{x} + c_2\bar{y} + c_3\bar{x}\bar{y}}\right)\delta\bar{y}$, $a_{12} = -\frac{\delta(c_0 + c_1\bar{x})}{c_0 + c_1\bar{x} + c_2\bar{y} + c_3\bar{x}\bar{y}}$, $a_{21} = \frac{\delta\bar{y}}{\bar{x}}\frac{(\alpha_0 + \alpha_2\bar{y})}{c_0 + c_1\bar{x} + c_2\bar{y} + c_3\bar{x}\bar{y}}$, $a_{22} = \delta\left(\frac{c_0 + c_1\bar{x}}{(c_0 + c_1\bar{x} + c_2\bar{y} + c_3\bar{x}\bar{y})^2} - 1\right)$.

Proof. By introducing the change of variables $U = x - \bar{x}$ and $V = y - \bar{y}$, we obtain the linearized system of (5.1) given by

$$\begin{cases} D_{0,\phi}^{\alpha,\beta}U = a_{11}U + a_{12}V,\\ D_{0,\phi}^{\alpha,\beta}V = a_{21}U + a_{22}V, \end{cases} \tag{5.2}$$

It follows from the first equation of (5.2) that

$$\phi(\zeta)D_{0,\phi}^{\alpha,\beta}U(\zeta) = a_{11}\phi(\zeta)U(\zeta) + a_{12}\phi(\zeta)V(\zeta). \tag{5.3}$$

Let

$$\tilde{U}(p) = \mathcal{L}\{\phi(\zeta)U(\zeta)\}(p), \tag{5.4}$$

$$\tilde{V}(p) = \mathcal{L}\{\phi(\zeta)V(\zeta)\}(p). \tag{5.5}$$

Using the Laplace transform on Eq. (5.3), it follows that

$$\mathcal{L}\{\phi(\zeta)D_{0,\phi}^{\alpha,\beta}U(\zeta)\}(p) = \mathcal{L}\{a_{11}\phi(\zeta)U(\zeta) + a_{12}\phi(\zeta)V(\zeta)\}(p).$$

Hence,

$$\mathcal{N}(\alpha)(p^\beta\tilde{U}(p) - p^{\beta-1}w(\zeta_0)U(\zeta_0)) = (p^\beta + \mu_\alpha)(1-\alpha)a_{11}\tilde{U}(p) + (1-\alpha)(p^\beta + \mu_\alpha)a_{12}\tilde{V}(p).$$

Consequently,

$$(\mathcal{N}(\alpha)p^\beta - (p^\beta + \mu_\alpha)(1-\alpha)a_{11})\tilde{U}(p) - (p^\beta + \mu_\alpha)(1-\alpha)a_{12}\tilde{V}(p) = \mathcal{N}(\alpha)\phi(\zeta_0)U(\zeta_0)p^{\beta-1}.$$

In a similar manner, the second equation of system (5.2) yields

$$\mathcal{N}(\alpha)(p^{\beta}\tilde{V}(p) - p^{\beta-1}w(0)Y(0)) = (p^{\beta} + \mu_{\alpha})(1-\alpha)(C\tilde{U}(p) + D\tilde{V}(p)).$$

Thus,

$$-(1-\alpha)(p^{\beta}+\mu_{\alpha})a_{21}\tilde{U}(p)+(p^{\beta}\mathcal{N}(\alpha)-a_{22}(p^{\beta}+\mu_{\alpha})(1-\alpha))\tilde{V}(p) = \mathcal{N}(\alpha)p^{\beta-1}\phi(\zeta_0)Y(\zeta_0).$$

Hence, system (5.2) can be written as

$$\begin{cases} (\mathcal{N}(\alpha)p^{\beta} - (1-\alpha)(p^{\beta}+\mu_{\alpha})a_{11})\tilde{U}(p) - (1-\alpha)(p^{\beta}+\mu_{\alpha})a_{12}\tilde{V}(p) = \mathcal{N}(\alpha)\phi(\zeta_0)U(\zeta_0)p^{\beta-1}, \\ -(1-\alpha)(p^{\beta}+\mu_{\alpha})a_{21}\tilde{U}(p) + (\mathcal{N}(\alpha)p^{\beta} - a_{22}(1-\alpha)(p^{\beta}+\mu_{\alpha}))\tilde{V}(p) = \mathcal{N}(\alpha)p^{\beta-1}\phi(\zeta_0)V(\zeta_0). \end{cases} \tag{5.6}$$

This implies that

$$\mathcal{D}(p)\tilde{W}(p) = q,$$

where

$$\mathcal{D}(p) = \begin{pmatrix} \mathcal{N}(\alpha)p^{\beta} - (1-\alpha)(p^{\beta}+\mu_{\alpha})a_{11} & -(1-\alpha)(p^{\beta}+\mu_{\alpha})a_{12} \\ -(1-\alpha)(p^{\beta}+\mu_{\alpha})a_{21} & \mathcal{N}(\alpha)p^{\beta} - a_{22}(1-\alpha)(p^{\beta}+\mu_{\alpha}) \end{pmatrix}, \tag{5.7}$$

$$\tilde{W}(p) = \begin{pmatrix} \tilde{U}(p) \\ \tilde{V}(p) \end{pmatrix},$$

$$q = \mathcal{N}(\alpha)p^{\beta-1}\phi(\zeta_0)\begin{pmatrix} U(\zeta_0) \\ V(\zeta_0) \end{pmatrix}.$$

Let $\lambda = p^{\beta}$. Then the characteristic equation associated with the equilibrium $E(\bar{x}, \bar{y})$ is given by:

$$a_0\lambda^2 + a_1\lambda + a_2 = 0, \tag{5.8}$$

If $a_0 \neq 0$, then Eq. (5.8) reduces to

$$\lambda^2 + \beta_1\lambda + \beta_2 = 0. \tag{5.9}$$

Since $\beta_1 > 0$ and $\beta_2 > 0$, we deduce that all the roots of Eq. (5.9) have negative real parts, which implies that the capital-labour equilibrium $E(\bar{x}, \bar{y})$ is locally asymptotically stable. This completes the proof. □

We now apply our proposed numerical scheme to investigate the dynamical behavior of our fractional capital-labour model (5.1). For the purpose of numerical simulations, the model's parameters are selected as follows: $M = 100$, $\eta = 1$, $\delta = 0.2$, and $\gamma = 1$, together with $c_0 = 1$, $c_1 = 1$, $c_2 = 0.1$ and $c_3 = 0.001$. Under these parameters, $E(20, 80)$ with $a_{11} = -0.85$, $a_{12} = -0.18$, $a_{21} = 0.58$ and $a_{22} = -0.01$. It follows from Table 3 and Theorem 5.1 that $E(20, 80)$ is locally asymptotically stable. Figure 3 demonstrates this result.

TABLE 3. Numerical verification of the asymptotic stability conditions

α	β_1	β_2
0.2	$0.134 > 0$	$0.003 > 0$
0.4	$0.31 > 0$	$0.018 > 0$
0.6	$0.5 > 0$	$0.045 > 0$
0.8	$0.69 > 0$	$0.08 > 0$
1	$0.87 > 0$	$0.119 > 0$

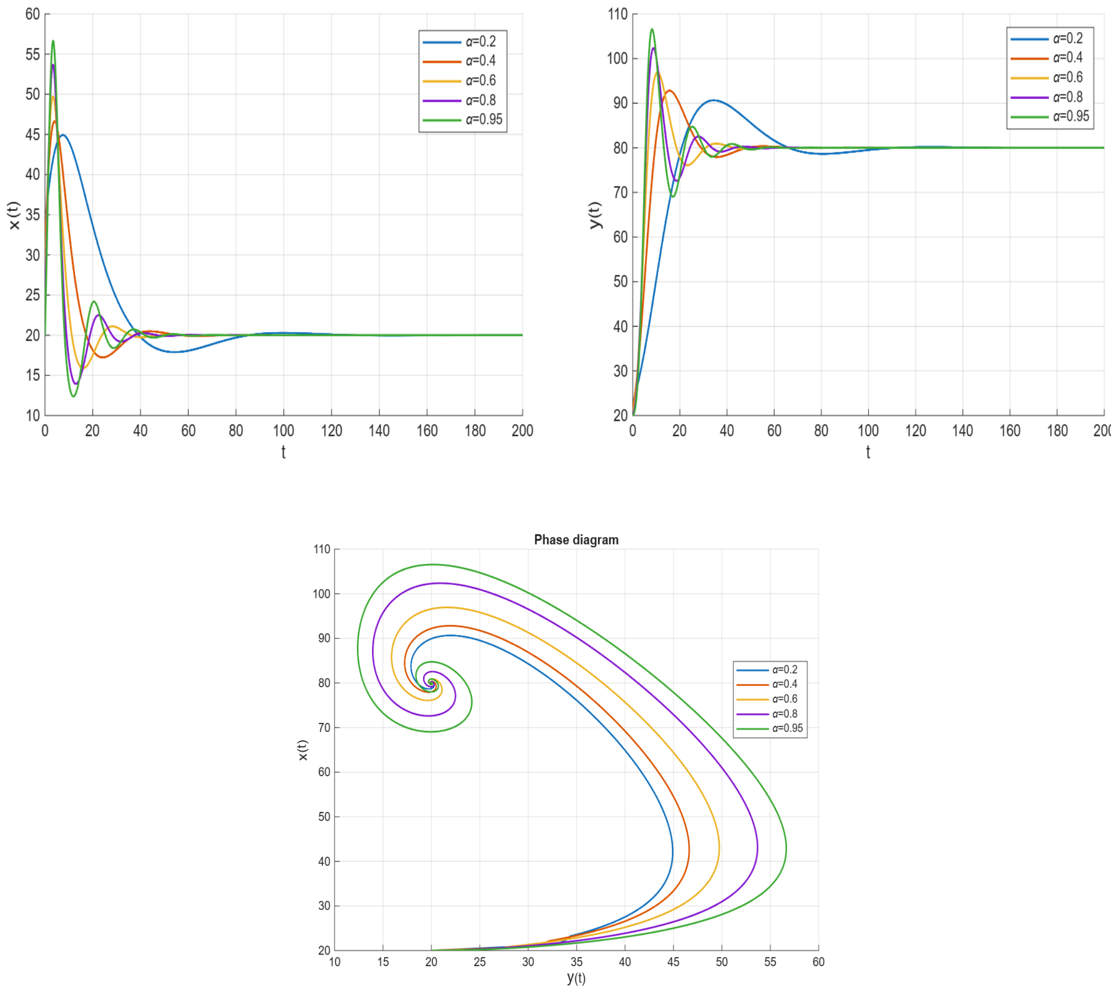

FIGURE 3. Stability of the capital-labour equilibrium E.

6. CONCLUSION

This study introduced a novel direct numerical approach for solving FDEs with GHF differential operator, which encompasses several well-known non-singular operators, including the weighted Atangana-Baleanu, the Atangana-Baleanu and the Caputo-Fabrizio fractional derivatives. The proposed scheme was validated using two example problems with known analytical solutions, in addition to nonlinear fractional equations modeling the dynamics of a capital-labour model. Both theoretical and numerical investigations demonstrated that the method is simple to implement, accurate and exhibits rapid convergence.

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