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EXISTENCE RESULT FOR SOME NONLINEAR L_φ-ELLIPTIC PROBLEMS.

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Abstract. The present paper is dedicated to study the existence of renormalized solutions for the following elliptic problem

$$\begin{cases} -\operatorname{div}\left(a(x,u,\nabla u)\right)+H(x,u,\nabla u)=\mu-\operatorname{div}\left(\phi(x,u)\right)-\operatorname{div}\left(\Phi(u)\right) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

in the setting of Musielak-Orlicz Sobolev spaces, where the Carathéodory function $\phi(x,u)$ satisfies certain growth conditions, $H(x,s,\xi)$ satisfies the sign and the growth condition and $\Phi\in C^0(\mathbb{R},\mathbb{R}^{\mathbb{N}})$, the datum $\mu=f-\operatorname{div}(F)$ is a measure with $f\in L^1(\Omega)$ and $F\in(E_{\bar\varphi}(\Omega))^N$.

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1. Introduction

Let Ω be an open bounded subset included in $\mathbb{R}^N(N\geq 2)$, with a Lipschitz boundary $\partial\Omega$. In this manuscript we focus on proving the existence of renormalized solutions for a class of nonlinear elliptic problems of the form

$$\begin{cases} -\operatorname{div}\left(a(x,u,\nabla u)\right)+H(x,u,\nabla u)=\mu-\operatorname{div}\left(\phi(x,u)\right)-\operatorname{div}\left(\Phi(u)\right) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

here, the nonlinear term $\phi(x,u)$ is a Carathéodory function that satisfying the growth condition

$$|\phi(x,s)| \;\leq\; \gamma(x)+\bar{\varphi}_x^{-1}\big(Q(x,|s|)\big). \tag{1.2}$$

such that $Q\prec\prec\varphi$. $H(x,s,\xi)$ is a Carathéodory functiony-that satisfying the sign and the growth conditions, with $\Phi\in C^0(\mathbb{R},\mathbb{R}^{\mathbb{N}})$, and $\mu=f-\operatorname{div}(F)$ where the datum $f(x)$ belongs to $L^1(\Omega)$ and $F(x)\in(E_{\bar\varphi}(\Omega))^N$.

Due to the fact that $f(x)\in L^1(\Omega)$, the equation of (1.1) does not belongs to the dual space. To overcome this challenge, we adopt the concept of renormalized

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solutions. It is worth mentioning that Lions and DiPerna [21] introduced this notion of solutions for the study of Boltzmann equations. Later, it was adapted to nonlinear elliptic equations by Boccardo et al. in [17].

The nonlinear elliptic problem (1.1) has been widely studied, in the classical Sobolev spaces. Boccardo and Orsina have proved in [16] the existence of T-solutions for the problems (1.1) when $\phi = 0$ by using L^1-version of Minty's lemma. For a general measure data and when $a(x,s,\xi) = a(x,\xi)$ the existence of renormalized solutions for (1.1) was established in [20] by Dal-Maso et al. (see also [32] for the weak solutions). In [15], Boccardo has showed that there exists at least one entropy solution for the problem (1.1) in the case of $\phi(x,s) = \phi(s) \in C^0(\mathbb{R})$ and $F = 0$. In the case where the right-hand side belongs to $W^{-1,p'}(\Omega)$ and $H(x,s,\xi) = H(x,s)$ the authors have addressed in [17] the existence of renormalized solutions for the problem (1.1). For the variable exponent Sobolev spaces, we refer the reader to ([8],[9],[10] and [11]).

Concerning the framework of Musielak–Orlicz Sobolev spaces, Sabiki et al. have studied in [22] the existence of solutions for the elliptic problem (1.1), such that $F = 0$ and ϕ satisfies the following condition

$$|\phi(x,s)| \;\leq\; \gamma(x)\bar{P}_x^{-1}\big(P(x,|s|)\big). \tag{1.3}$$

where $\gamma(\cdot) \in L^\infty(\Omega)$ and $P \prec\prec \varphi$.The problem (1.1) has been addressed in the parabolic case by Azraibi et. al in [7], where the authors have investigated the existence of entropy solutions. In framework of Orlicz-Sobolev spaces we refer the reader to [29].

Moreover, numerous research works deal with the existence of solutions to elliptic and parabolic problems under various hypotheses and in different frameworks (see [23, 3] for further details).

The principal novelty of the present work lies in establishing the existence of renormalized solutions without assuming the Δ_2-condition on the Musielak–Orlicz function φ, we only assume that the conjugate function $\bar{\varphi}$ satisfies this condition. This weaker assumption creates some difficulties in the analysis and calculus. Indeed, in the case where the Δ_2-condition is not imposed on φ, some important properties of Musielak-Orlicz spaces are lost : the space $L_\varphi(\Omega)$ is no longer reflexive, and the equality $E_\varphi(\Omega) = L_\varphi(\Omega)$ no longer holds. A further difficulty comes from the lack of coercivity of the equation in (1.1). To face these obstacles, we use an approximation method, then we show some a priori estimates in order to pass to the limit.

This paper is structured as follows : Section 2 presents the main definitions and properties of our functional spaces. Section 3 contains the fundamental lemmas necessary for proving our main result. In Section 4, we introduce the essential assumptions that ensure the existence of at least one renormalized solution for elliptic problem (1.1). The existence theorem and its proof are provided in Section 5.

2. Preliminaries

Let Ω be an open bounded subset of $\mathbb{R}^N (N \geq 2)$, with a Lipschitz boundary $\partial\Omega$.
We consider the real-valued function $\varphi(x,t) : \Omega \times \mathbb{R}^+ \longmapsto \mathbb{R}^+$, and $\varphi(x,t)$ is called a Musielak-Orlicz function if the following conditions are verified :

(i) $\varphi(x,\cdot)$ is an N-function, i.e. convex, continuous, strictly increasing, with $\varphi(x,0) = 0$ and $\operatorname{ess\,inf}_{x\in\Omega} \varphi(x,t) > 0$ for all $t > 0$, and

$$\lim_{t\to 0} \sup_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = +\infty. \tag{2.1}$$

(ii) The function $\varphi(\cdot,t)$ is measurable for any $t \geq 0$.

We denote $\varphi_x(t) = \varphi(x,t)$ and φ_x^{-1} its reciprocal function with respect to t, it means that

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\varphi_x^{-1}(t)) = t.$$

We say that $\varphi(x,t)$ satisfies the Δ_2-condition, if for some $l > 0$ there exists a nonnegative function $M(\cdot) \in L^1(\Omega)$ such that

$$\varphi(x,2t) \leq l\varphi(x,t) + M(x) \quad \text{a.e. } x \in \Omega \text{ and for all } t \geq 0. \tag{2.2}$$

In the case where (2.2) is satisfied for any $t \geq t_0 > 0$, then φ is said verified Δ_2-condition near infinity.
Let φ and γ be two Musielak-Orlicz functions, φ is said dominate γ near infinity (resp. globally) and we set $\gamma \prec \varphi$ if there exist two positive constants c and t_0 such that

$$\gamma(x,t) \leq \varphi(x,ct) \text{ a.e } x \in \Omega \text{ and for all } t \geq t_0 \text{ (resp. } t \geq 0).$$

The function $\gamma(x,t)$ grows essentially less rapidly than $\varphi(x,t)$ at 0 (resp. near infinity) and we write $\gamma \prec\prec \varphi$, if for every positive constant c we have

$$\lim_{t\to 0} \sup_{x\in\Omega} \left(\frac{\gamma(x,t)}{\varphi(x,ct)} \right) = 0 \quad \left(\text{resp. } \lim_{t\to\infty} \sup_{x\in\Omega} \left(\frac{\gamma(x,t)}{\varphi(x,ct)} \right) = 0 \right).$$

If $\gamma \prec\prec \varphi$, then for any $\varepsilon > 0$ there exists a positive function $h \in L^1(\Omega)$ such that $\gamma(x,t) \leq \varphi(x,\varepsilon t) + h(x)$ for all $t \geq 0$ and a.e. $x \in \Omega$ (see [24]).

For a Musielak-Orlicz function $\varphi(x,t)$, we define the convex modular

$$\varrho_\varphi(u) = \int_\Omega \varphi(x,|u(x)|)\, dx.$$

The Musielak-Orlicz Lebesgue's space $L_\varphi(\Omega)$ is defined as follows

$$L_\varphi(\Omega) = \Big\{ u : \Omega \longmapsto \mathbb{R} \text{ measurable} : \ \varrho_\varphi(u/\lambda) < \infty \text{ for some } \lambda > 0 \Big\},$$

equipped with the Luxemburg norm

$$\|u\|_\varphi = \inf\{\lambda > 0 : \ \varrho_\varphi(u/\lambda) \leq 1\}.$$

We denote by $\bar{\varphi}(x,t)$ the Musielak-Orlicz function complementary to (or conjugate of) $\varphi(x,t)$ defined by

$$\bar{\varphi}(x,t) = \sup_{s\geq 0}\{st - \varphi(x,s)\} \quad \text{a.e. in } \Omega, \tag{2.3}$$

Musielak-Orlicz Lebesgue's spaces share the same properties as Orlicz Lebesgue's spaces (Fenchel-Young's inequality see (see [30])) and (generalized Hölder's inequality (see [30])).
The closure of bounded measurable functions with compact support is denoted by $E_\varphi(\Omega)$; we have $(E_\varphi(\Omega))^* = L_{\bar{\varphi}}(\Omega)$. For more properties of these spaces we refer to [4, 13, 30].

We define the Musielak-Orlicz-Sobolev spaces $W^1L_\varphi(\Omega)$ and $W^1E_\varphi(\Omega)$ as follows

$$W^1L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) \quad \text{with} \quad |\nabla u| \in L_\varphi(\Omega)\},$$

and

$$W^1E_\varphi(\Omega) = \{u \in E_\varphi(\Omega) \quad \text{with} \quad |\nabla u| \in E_\varphi(\Omega)\}.$$

The Sobolev space $W^1L_\varphi(\Omega)$ is equipped with the following norm

$$\|u\|_{1,\varphi} = \|u\|_\varphi + \|\nabla u\|_\varphi. \tag{2.4}$$

The space $(W^1L_\varphi(\Omega), \|\cdot\|_{1,\varphi})$ is a Banach space (see [6]).
The spaces $W^1L_\varphi(\Omega)$ (resp. $W^1E_\varphi(\Omega)$) are identified to a subspace of the product of $N+1$ copies of $L_\varphi(\Omega)$ (resp. $E_\varphi(\Omega)$), this product is denoted by $\Pi L_\varphi(\Omega)$ (resp. $\Pi E_\varphi(\Omega)$). Throughout this paper we use the following weak topology $\sigma(\Pi L_\varphi(\Omega), \Pi E_{\bar{\varphi}}(\Omega))$ and $\sigma(\Pi L_\varphi(\Omega), \Pi L_{\bar{\varphi}}(\Omega))$.

The closure of the Schwartz space $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,\varphi}$ in $W^1L_\varphi(\Omega)$ is denoted by $W_0^1E_\varphi(\Omega)$, and we define $W_0^1L_\varphi(\Omega)$ as the weak $\sigma(\Pi L_\varphi(\Omega), \Pi E_{\bar{\varphi}}(\Omega))$ closure of $C_0^\infty(\Omega)$ in $W^1L_\varphi(\Omega)$.
The dual spaces of $W_0^1L_\varphi(\Omega)$ (resp. $W_0^1E_\varphi(\Omega)$) are isomorphic to $W^{-1}L_{\bar{\varphi}}(\Omega)$ (resp. $W^{-1}E_{\bar{\varphi}}(\Omega)$).
We say that $u_n \to u$ *modularly* in $W^1L_\varphi(\Omega)$ if there exists $\lambda > 0$ such that

$$\varrho_\varphi\Big(\frac{u_n - u}{\lambda}\Big) + \varrho_\varphi\Big(\frac{|\nabla u_n - \nabla u|}{\lambda}\Big) \to 0.$$

3. Some fundamental lemmas

This section recalls some standard and basic Lemmas in Musielak-Orlicz-Sobolev spaces.

Lemma 3.1 (see [4]). *If φ satisfies the log-Hölder continuity condition and $\bar{\varphi}(x,1) \leq c_1$ a.e. in Ω, then $C_0^\infty(\Omega)$ is dense in $L_\varphi(\Omega)$ and in $W_0^1L_\varphi(\Omega)$ for the modular convergence.*

Lemma 3.2 (see [2],[4]). *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$, and let $\varphi(x,t)$ be a Musielak-Orlicz function that satisfies the assumptions of lemma 3.1.*

Then, there exist two constants $d_1, d_2 > 0$ and $\lambda > 0$ depending only on Ω and φ such that

$$\int_\Omega \varphi(x,|u|)dx \le d_1 + d_2 \int_\Omega \varphi(x, \lambda|\nabla u|)\, dx \qquad \text{for any } u \in W_0^1 L_\varphi(\Omega).$$

Moreover, there exists a constant $C > 0$ depends only on Ω and φ such that

$$\|u\|_\varphi \le C \|\nabla u\|_\varphi \qquad \forall u \in W_0^1 L_\varphi(\Omega).$$

Lemma 3.3 (see [13]). *If Ω has the segment property, then for every $u \in W_0^1 L_\varphi(\Omega)$ there exists $\{(u_n)_n\} \subset C_0^\infty(\Omega)$ converging modularly to u in $W_0^1 L_\varphi(\Omega)$. If additionally $u \in L^\infty(\Omega)$, then $\|u_n\|_\infty \le (N+1)\|u\|_\infty$.*

Lemma 3.4. *([31]) Let $F : \mathbb{R} \mapsto \mathbb{R}$ be an uniformly Lipschitz function, with $F(0) = 0$. If $u \in W_0^1 L_\varphi(\Omega)$, then $F(u) \in W_0^1 L_\varphi(\Omega)$. Moreover, if the set D of discontinuity points of $F'(\cdot)$ is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \dfrac{\partial u}{\partial x_i} & a.e \text{ in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & a.e \text{ in } \{x \in \Omega : u(x) \in D\}. \end{cases} \tag{3.1}$$

Lemma 3.5. *Let $k > 0$, we define the truncation function $T_k(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ by*

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k, \\ s \dfrac{k}{|k|} & \text{if } |s| > k. \end{cases} \tag{3.2}$$

Lemma 3.6. *(see [14]) Let $(h_n)_n$ be a sequence in $L_\varphi(\Omega)$ that converges in measure to some measurable function h, and if h_n remains uniformly bounded in $L_\varphi(\Omega)$, then $h \in L_\varphi(\Omega)$ and $h_n \rightharpoonup h$ weakly in $L_\varphi(\Omega)$ for $\sigma\Big(L_\varphi, E_{\bar\varphi}\Big)$.*

Lemma 3.7. *(see [14]) Let $v_n, v \in L_\varphi(\Omega)$. Assuming that v_n converges modularly to v then $v_n \rightharpoonup v$ weakly in $L_\varphi(\Omega)$ for $\sigma\Big(L_\varphi, L_{\bar\varphi}\Big)$.*

Lemma 3.8. *(see [28]) Assuming that (4.2) – (4.4) hold true, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W_0^1 L_\varphi(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^1 L_\varphi(\Omega)$ for $\sigma\Big(\prod L_\varphi, \prod L_{\bar\varphi}\Big)$, and let $(a(x, u_n, \nabla u_n))_n$ be an uniformly bounded sequence in $(L_{\bar\varphi}(\Omega))^N$ with*

$$\lim_{\tau\to\infty} \lim_{n\to\infty} \int_\Omega (a(x, u_n, \nabla u_n) - a(x, u, \nabla u \chi_{H_\tau})))(\nabla u_n - \nabla u \chi_{H_\tau})\, dx = 0, \tag{3.3}$$

with $H_\tau = \{x \in \Omega : \ |\nabla u| \le \tau\}$. Then, $u_n \longrightarrow u$ modularly in $W_0^1 L_\varphi(\Omega)$ and

$$a(x, u_n, \nabla u_n) \cdot \nabla u_n \longrightarrow a(x, u, \nabla u) \cdot \nabla u \quad \text{strongly in} \quad L^1(\Omega). \tag{3.4}$$

4. Essential assumptions

We recall our nonlinear elliptic problem

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + H(x,u,\nabla u) = \mu - \operatorname{div}(\phi(x,u)) - \operatorname{div}(\Phi(u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

We consider the operator $Au : \mathrm{D}(A) \subset W_0^1 L_\varphi(\Omega) \mapsto W^{-1} L_{\bar{\varphi}}(\Omega)$ given by

$$Au = -\operatorname{div}\left(a(x, u, \nabla u)\right),$$

where $a(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}^{\mathbb{N}}$ is a Carathéodory function verifying the conditions below

$$|a(x, s, \xi)| \leq \beta\Big(d(x) + \bar{\varphi_x}^{-1}\big(P(x, k_1|s|)\big) + \bar{\varphi_x}^{-1}\big(\varphi(x, k_2|\xi|)\big)\Big), \tag{4.2}$$

where $\beta, k_1, k_2 > 0$ and $d(x) \in E_{\bar{\varphi}}(\Omega)$ such that $P \prec\prec \varphi$.

$$\left(a(x, s, \xi) - a\left(x, s, \zeta\right)\right)\left(\xi - \zeta\right) > 0 \text{ for any } \xi \neq \zeta, \tag{4.3}$$

$$a(x, s, \xi)\xi \geq \alpha\varphi\big(x, |\xi|\big) + \varphi\big(x, |s|\big), \tag{4.4}$$

with α is a positive real constant.
The Carathéodory function $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ checks the conditions

$$H(x, s, \xi)s \geq 0 \quad \text{for any} \quad s \in \mathbb{R}, \tag{4.5}$$

and

$$|H(x, s, \xi)| \leq h_0(x) + b(s)\varphi(x, |\xi|), \tag{4.6}$$

such that $h_0 \in L^1(\Omega)$ and $b(\cdot)$ is a continuous nonnegative increasing function lying in $L^\infty(\Omega)$.
The measure data can be decomposed as follows $\mu = f - \operatorname{div} F$ with :

$$f \in L^1(\Omega) \quad \text{and} \quad F \in (E_{\bar{\varphi}}(\Omega))^N. \tag{4.7}$$

The nonlinear term $\phi(x, s) : \Omega \times \mathbb{R} \mapsto \mathbb{R}^N$ is a Carathéodory function, verifying

$$|\phi(x, s)| \ \leq \ \gamma(x) + \bar{\varphi_x}^{-1}\big(Q(x, |s|)\big), \tag{4.8}$$

and

$$\Phi(s) \in C^0(\mathbb{R}, \mathbb{R}^{\mathbb{N}}). \tag{4.9}$$

5. Main result

This section is entirely devoted to state the existence result with its associated proof.
First of all, we present the definition of renormalized solutions for the nonlinear elliptic problem (4.1).

Definition 5.1. A renormalized solution of the nonlinear problem (4.1) is a measurable function u defined on Ω, satisfying :

$$T_k(u) \in W_0^1 L_\varphi(\Omega) \quad \text{for all} \quad k > 0, \quad \text{and} \quad H(x, u, \nabla u) \in L^1(\Omega), \tag{5.1}$$

and

$$\lim_{h\to\infty} \frac{1}{h} \int_{\{|u|\leq h\}} a(x, u, \nabla u) \cdot \nabla u \, dx = 0, \tag{5.2}$$

such that u verifying

$$\int_\Omega a(x,u,\nabla u)\cdot(S'(u)v\nabla u+\nabla vS(u))\,dx+\int_\Omega H(x,u,\nabla u)S(u)v\,dx$$
$$=\int_\Omega f(x)S(u)v\ dx+\int_\Omega F\cdot(S'(u)v\nabla u+\nabla vS(u))\,dx$$
$$+\int_\Omega \phi(x,u)\cdot(S'(u)v\nabla u+\nabla vS(u))\,dx+\int_\Omega \Phi(u)\cdot(S'(u)v\nabla u+\nabla vS(u))\,dx, \tag{5.3}$$

for any $v\in C_0^\infty(\Omega)$ and for all functions $S(\cdot)\in C_c^1(\mathbb{R})$ with a compact support.

Theorem 5.1. Assuming that (4.2) − (4.8) hold true, then there exists at least one renormalized solution for the nonlinear elliptic problem (4.1).

Proof of the Theorem 5.1.
Approximate problem :
Let $f_n(x)$ be a sequence in $L^1(\Omega)\cap L^\infty(\Omega)$ of smooth function such that $f_n(x)\to f(x)$ strongly in $L^1(\Omega)$ and $|f_n(x)|\le|f(x)|$ (for example $f_n=T_n(f)$). Then, the approximate problem can be formulated by

$$\begin{cases} -\operatorname{div}(a(x,u_n,\nabla u_n))+H_n(x,u_n,\nabla u_n)=f_n-\operatorname{div}(F)-\operatorname{div}(\phi_n(x,u_n))-\operatorname{div}(\Phi_n(u_n)) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{5.4}$$

where :

$$H_n(x,s,\xi)=\frac{H(x,s,\xi)}{\frac{1}{n}+|H(x,s,\xi)|},$$

and

$$\phi_n(x,s)=T_n(\phi(x,s))\quad\text{and}\quad\Phi_n(s)=T_n(\Phi(s)).$$

Defining the operator $L_n: D(L_n)\subset W_0^1L_\varphi(\Omega)\mapsto W^{-1}L_{\bar\varphi}(\Omega)$, by

$$\begin{aligned}\langle L_nu,v\rangle &=\int_\Omega a(x,u,\nabla u)\cdot\nabla v\,dx+\int_\Omega H_n(x,u,\nabla u)v\,dx\\ &\quad-\int_\Omega \phi_n(x,u)\cdot\nabla v\,dx-\int_\Omega \Phi_n(u)\cdot\nabla v\,dx,\end{aligned}$$

for any $u,v\in W_0^1L_\varphi(\Omega)$.
Thanks to (Theorem 1, [14]), the approximate problem (5.4) admits at least one weak solution denoted u_n belonging to $W_0^1L_\varphi(\Omega)$ i.e.

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla v\ dx+\int_\Omega H_n(x,u_n,\nabla u_n)v\ dx=\int_\Omega f_n(x)v\ dx\\ +\int_\Omega F\cdot\nabla v\ dx+\int_\Omega \phi_n(x,u_n)\cdot\nabla v\ dx+\int_\Omega \Phi_n(u_n)\cdot\nabla v\ dx.\end{aligned} \tag{5.5}$$

.
A priori estimates :

By taking $T_k(u_n)$ as a test function in (5.5), we get

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\,dx+\int_\Omega H_n(x,u_n,\nabla u_n)T_k(u_n)\,dx=\int_\Omega f_n(x)T_k(u_n)\,dx\\+\int_\Omega F\cdot\nabla T_k(u_n)\,dx+\int_\Omega \phi_n(x,u_n)\cdot\nabla T_k(u_n)\,dx+\int_\Omega \Phi_n(u_n)\cdot\nabla T_k(u_n)\,dx.\end{aligned}\tag{5.6}$$

For the first term on the left-hand side of (5.6), according to (4.4), we get

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\,dx\geq\alpha\int_\Omega\varphi(x,|\nabla T_k(u_n)|)\,dx+\int_\Omega\varphi(x,|T_k(u_n)|)\,dx.\tag{5.7}$$

Taking into account the condition (4.5) we have

$$\int_\Omega H_n(x,u_n,\nabla u_n)T_k(u_n)\,dx\geq 0.\tag{5.8}$$

For the first term on the right-hand side of (5.6), we find

$$\Big|\int_\Omega f_n(x)T_k(u_n)\,dx\Big|\leq k\|f(x)\|_{L^1(\Omega)}.\tag{5.9}$$

Moreover, Thanks to Young's inequality we get

$$\begin{aligned}\Big|\int_\Omega F(x)\cdot\nabla T_k(u_n)\,dx\Big|&=\frac{\alpha^2}{2(\alpha+2)}\int_\Omega\frac{2(\alpha+2)}{\alpha^2}|F(x)||\nabla T_k(u_n)|\,dx\\&\leq\frac{\alpha^2}{2(\alpha+2)}\int_\Omega\bar\varphi\big(x,\frac{2(\alpha+2)}{\alpha^2}|F(x)|\big)\,dx+\frac{\alpha^2}{2(\alpha+2)}\int_\Omega\varphi\big(x,|\nabla T_k(u_n)|\big)\,dx.\end{aligned}\tag{5.10}$$

Having in mind (4.8), it yields

$$\Big|\int_\Omega\phi_n(x,u_n)\cdot\nabla T_k(u_n)\,dx\Big|\leq\int_\Omega|\gamma(x)||\nabla T_k(u_n)|\,dx+\int_\Omega\bar{\varphi_x}^{-1}\big(Q(x,|T_k(u_n)|)\big)|\nabla T_k(u_n)|\,dx.\tag{5.11}$$

For the first term on the right-hand side of (5.11), similarly as in (5.10) we get

$$\int_\Omega|\gamma(x)||\nabla T_k(u_n)|\,dx\leq\frac{\alpha^2}{2(\alpha+2)}\int_\Omega\bar\varphi\big(x,\frac{2(\alpha+2)}{\alpha^2}|\gamma(x)|\big)\,dx+\frac{\alpha^2}{2(\alpha+2)}\int_\Omega\varphi\big(x,|\nabla T_k(u_n)|\big)\,dx.\tag{5.12}$$

Concerning the last term on the right-hand side of (5.11). Thanks to $\dfrac{\alpha}{\alpha+1}<1$ and the convexity of the function $\varphi(x,\cdot)$ we find

$$\begin{aligned}&\int_\Omega\bar{\varphi_x}^{-1}\big(Q(x,|T_k(u_n)|)\big)|\nabla T_k(u_n)|\,dx\\&=\int_\Omega\frac{\alpha+1}{\alpha}\bar{\varphi_x}^{-1}\big(Q(x,|T_k(u_n)|)\big)\frac{\alpha}{\alpha+1}|\nabla T_k(u_n)|\,dx\\&\leq\int_\Omega\bar\varphi\big(x,\frac{\alpha+1}{\alpha}\bar{\varphi_x}^{-1}\big(Q(x,|T_k(u_n)|)\big)\big)\,dx+\frac{\alpha}{\alpha+1}\int_\Omega\varphi\big(x,|\nabla T_k(u_n)|\big)\,dx.\end{aligned}\tag{5.13}$$

In the light of $\bar{\varphi}$ satisfies Δ_2-condition then for $q > 0$ such that $\frac{\alpha+1}{\alpha} \leq 2^q$ we obtain that

$$\begin{aligned}
&\int_\Omega \bar{\varphi}\big(x, \frac{\alpha+1}{\alpha}\bar{\varphi}_x{}^{-1}\big(Q(x,|T_k(u_n)|)\big)\big)\,dx \\
&\leq \frac{\alpha+1}{2^q\alpha}\int_\Omega \bar{\varphi}\big(x, 2^q\bar{\varphi}_x{}^{-1}\big(Q(x,|T_k(u_n)|)\big)\big)\,dx \\
&\leq \frac{\alpha+1}{2^q\alpha}\Big[l^q\int_\Omega Q(x,|T_k(u_n)|)\,dx + (l^{q-1}+\cdots+l+1)\int_\Omega M(x)\,dx\Big] \\
&\leq C_1\int_\Omega Q(x,|T_k(u_n)|)\,dx + C_2.
\end{aligned} \tag{5.14}$$

Moreover, due to $\delta \prec\prec \varphi$ we have for any $\epsilon > 0$, we have

$$Q(x,|T_k(u_n)|) \leq \varphi(x,\epsilon|T_k(u_n)|) + h(x)$$

Now, select ϵ such that $0 < \epsilon \leq \frac{1}{2(C_1+1)} < 1$, then it follows that

$$\begin{aligned}
C_1\int_\Omega Q(x,|T_k(u_n)|)\,dx &\leq (C_1+1)\int_\Omega \varphi(x,\epsilon|T_k(u_n)|)\,dx + C_1\int_\Omega h(x)\,dx \\
&\leq \frac{1}{2}\int_\Omega \varphi(x,|T_k(u_n)|)\,dx + C_3.
\end{aligned} \tag{5.15}$$

Since $u_n = 0$ on $\partial\Omega$. We set

$$\Psi_n(t) = \int_0^t \Phi_n(\tau)\,d\tau,$$

it follows that $\Psi_n(0) = 0$ and $\Psi_n(\cdot)$ is $C^1(\mathbb{R}^N)$. In view of the Green formula we obtain

$$\begin{aligned}
\int_\Omega \Phi_n(u_n)\cdot\nabla T_k(u_n)\,dx &= \int_\Omega \operatorname{div}\big(\Psi_n(T_k(u_n))\big)\,dx \\
&= \int_\Omega \Psi_n(T_k(u_n))\cdot\vec{\eta}\,d\sigma = \sum_{i=1}^N\int_{\partial\Omega}\Psi_i, n(T_k(u_n))\cdot\eta_i\,d\sigma = 0.
\end{aligned} \tag{5.16}$$

From the estimations (5.6) and (5.7) − (5.16), we obtain

$$\begin{aligned}
&\alpha\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx + \frac{1}{2}\int_\Omega \varphi(x,|T_k(u_n)|)\,dx \\
&\leq C_0k + C_2 + \frac{\alpha^2}{2(\alpha+2)}\int_\Omega \bar{\varphi}\big(x,\frac{2(\alpha+2)}{\alpha^2}|F|\big)\,dx + \frac{\alpha^2}{2(\alpha+2)}\int_\Omega \bar{\varphi}\big(x,\frac{2(\alpha+2)}{\alpha^2}|\gamma(x)|\big)\,dx \\
&\quad + \big(\frac{\alpha^2}{\alpha+2} + \frac{\alpha}{\alpha+1}\big)\int_\Omega \varphi\big(x,|\nabla T_k(u_n)|\big)\,dx + C_3.
\end{aligned} \tag{5.17}$$

Since $F \in (E_{\bar{\varphi}}(\Omega))^N$ and $\gamma \in E_{\bar{\varphi}}(\Omega)$ it can be deduced that

$$(\alpha - \frac{\alpha^2}{(\alpha+2)} - \frac{\alpha}{\alpha+1}) \int_\Omega \varphi(x, |\nabla T_k(u_n)|)\, dx + \int_\Omega \varphi(x, |T_k(u_n)|)\, dx \leq C_0 k + C_4. \tag{5.18}$$

with $\alpha - \dfrac{\alpha^2}{(\alpha+2)} - \dfrac{\alpha}{\alpha+1} > 0$ and the constants C_0, C_4 are positives and independent on k and n. Therefore, $(T_k(u_n))_n$ is a uniformly bounded sequence in $W_0^1 L_\varphi(\Omega)$, then there exists a subsequence still denoted $(T_k(u_n))_{n\in\mathbb{N}}$ and a measurable function $\rho_k \in W_0^1 L_\varphi(\Omega)$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup \rho_k \text{ weakly in } W_0^1 L_\varphi(\Omega), \\ T_k(u_n) \to \rho_k \text{ strongly in } E_\varphi(\Omega) \quad \text{and a.e in} \quad \Omega. \end{cases} \tag{5.19}$$

Now, we shall show that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω. Indeed, in the light of (5.18), we have

$$\begin{aligned} \inf_{x\in\Omega} \varphi(x,k)\ \operatorname{meas}(\{|u_n| > k\}) &\leq \int_{\{|u_n|>k\}} \varphi(x, |T_k(u_n)|)\, dx \\ &\leq C_5 k \quad \text{for any} \quad k \geq 1, \end{aligned} \tag{5.20}$$

it follows that

$$\operatorname{meas}(\{|u_n| > k\}) \leq \frac{C_5 k}{\inf_{x\in\Omega} \varphi(x,k)} \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty. \tag{5.21}$$

For any $\delta > 0$, it's clear that

$$\operatorname{meas}\{|u_n - u_m| > \delta\} \leq \operatorname{meas}\{|u_n| > k\} + \operatorname{meas}\{|u_m| > k\} + \operatorname{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}. \tag{5.22}$$

Using the same argument as in [6], we conclude that $(u_n)_n$ is a Cauchy sequence in measure, then $u_n \longrightarrow u$ a.e in Ω, for a subsequence. Consequently, we have

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^1 L_\varphi(\Omega), \\ T_k(u_n) \to T_k(u) \text{ strongly in } E_\varphi(\Omega) \quad \text{and a.e in} \quad \Omega. \end{cases} \tag{5.23}$$

Some regularity results :
This section aims to establish the following convergence

$$\limsup_{n\to\infty} \lim_{h\to\infty} \frac{1}{h} \int_{\{|u_n|\leq h\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n\, dx = 0. \tag{5.24}$$

By choosing $\dfrac{T_h(u_n)}{h}$ as a test function for the approximate problem (5.5), we get

$$\begin{aligned} &\frac{1}{h} \int_{\{|u_n|\leq h\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n\, dx + \frac{1}{h} \int_\Omega H_n(x, u_n, \nabla u_n) T_h(u_n)\, dx \\ &= \frac{1}{h} \int_\Omega f_n(x) T_h(u_n)\, dx + \frac{1}{h} \int_{\{|u_n|\leq h\}} F \cdot \nabla T_h(u_n)\, dx \\ &\quad + \frac{1}{h} \int_{\{|u_n|\leq h\}} \phi_n(x, u_n) \cdot \nabla T_h(u_n)\, dx + \frac{1}{h} \int_{\{|u_n|\leq h\}} \Phi_n(u_n) \cdot \nabla T_h(u_n)\, dx. \end{aligned}$$

Having in mind of (4.5), it yields

$$\frac{1}{h}\int_\Omega H_n(x,u_n,\nabla u_n)T_h(u_n)\,dx \geq \int_{\{h<|u_n|\}} |H_n(x,u_n,\nabla u_n)|\,dx \geq 0. \tag{5.25}$$

Thanks to the fact that $f_n \to f$ strongly in $L^1(\Omega)$, and since $\dfrac{T_h(u_n)}{h} \rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$ as n,h tend to infinity, then

$$\frac{1}{h}\int_\Omega f_n(x)T_h(u_n)\,dx \longrightarrow 0 \quad \text{as} \quad n,h\to\infty. \tag{5.26}$$

On the other hand, in view of Young's inequality, we have

$$\begin{aligned}\Big|\frac{1}{h}\int_{\{|u_n|\leq h\}} F\cdot\nabla T_h(u_n)\,dx\Big| \leq{}& \frac{\alpha^2}{2h(\alpha+2)}\int_\Omega \bar{\varphi}\big(x,\frac{2(\alpha+2)}{\alpha^2}|F|\big)\,dx\\ &+\frac{\alpha^2}{2h(\alpha+2)}\int_{\{|u_n|\leq h\}}\varphi(x,|\nabla T_h(u_n)|)\,dx,\end{aligned} \tag{5.27}$$

Moreover, by utilizing (4.8) we can derive

$$\begin{aligned}&\Big|\frac{1}{h}\int_{\{|u_n|\leq h\}}\phi_n(x,u_n)\cdot\nabla T_h(u_n)\,dx\Big|\\ &\leq \frac{1}{h}\int_{\{|u_n|\leq h\}}|\gamma(x)||\nabla T_h(u_n)|\,dx+\frac{1}{h}\int_{\{|u_n|\leq h\}}\bar{\varphi_x}^{-1}\big(Q(x,|u_n|)\big)|\nabla T_h(u_n)|\,dx,\end{aligned} \tag{5.28}$$

then, it follows that

$$\begin{aligned}\frac{1}{h}\int_{\{|u_n|\leq h\}}|\gamma(x)||\nabla T_h(u_n)|\,dx \leq{}& \frac{\alpha^2}{2h(\alpha+2)}\int_\Omega\bar{\varphi}\big(x,\frac{2(\alpha+2)}{\alpha^2}|\gamma(x)|\big)\,dx\\ &+\frac{\alpha^2}{2h(\alpha+2)}\int_{\{|u_n|\leq h\}}\varphi(x,|\nabla T_h(u_n)|)\,dx.\end{aligned} \tag{5.29}$$

Recall that $\gamma(\cdot)\in E_{\bar{\varphi}}(\Omega)$ and $F\in(E_{\bar{\varphi}}(\Omega))^N$, we obtain

$$\frac{\alpha^2}{2h(\alpha+2)}\int_\Omega\bar{\varphi}\big(x,\frac{2(\alpha+2)}{\alpha^2}|F|\big)\,dx\longrightarrow 0 \quad\text{as}\quad h\to\infty, \tag{5.30}$$

and

$$\frac{\alpha^2}{2h(\alpha+2)}\int_\Omega\bar{\varphi}\big(x,\frac{2(\alpha+2)}{\alpha^2}|\gamma(x)|\big)\,dx\longrightarrow 0 \quad\text{as}\quad h\to\infty. \tag{5.31}$$

Similarly as in (5.13) and (5.15) we can show that

$$\begin{aligned}
&\frac{1}{h}\int_{\{|u_n|\le h\}} \bar{\varphi}_x^{-1}\big(Q(x,|u_n|)\big)|\nabla T_h(u_n)|\,dx\\
&\le \frac{1}{h}\int_{\{|u_n|\le h\}} \bar{\varphi}\Big(x,\frac{\alpha+1}{\alpha}\bar{\varphi}_x^{-1}\big(Q(x,|u_n|)\big)\Big)\,dx + \frac{\alpha}{h(\alpha+1)}\int_{\{|u_n|\le h\}} \varphi\big(x,|\nabla T_h(u_n)|\big)\,dx\\
&\le \frac{C_4}{h}\int_{\{|u_n|\le h\}} Q(x,|T_h(u_n)|)\,dx + \frac{C_5}{h}\int_{\{|u_n|\le h\}} M(x)\,dx\\
&\quad + \frac{\alpha}{h(\alpha+1)}\int_{\{|u_n|\le h\}} \varphi\big(x,|\nabla T_h(u_n)|\big)\,dx\\
&\le \frac{\alpha}{2h}\int_{\{|u_n|\le h\}} \varphi(x,|T_h(u_n)|)\,dx + \frac{C_4}{h}\int_\Omega h(x)\,dx + \frac{C_5}{h}\int_\Omega M(x)\,dx\\
&\quad + \frac{\alpha}{h(\alpha+1)}\int_{\{|u_n|\le h\}} \varphi\big(x,|\nabla T_h(u_n)|\big)\,dx.
\end{aligned} \tag{5.32}$$

Thanks to the fact that $M(\cdot), h(\cdot) \in L^1(\Omega)$, by applying Lebesgue's dominated convergence theorem we obtain

$$\frac{C_4}{h}\int_\Omega h(x)\,dx + \frac{C_5}{h}\int_\Omega M(x)\,dx \longrightarrow 0 \quad \text{as} \quad h\to\infty. \tag{5.33}$$

Using the same argument as in (5.16), we obtain

$$\frac{1}{h}\int_{\{|u_n|\le h\}} \Phi_n(u_n)\cdot\nabla T_h(u_n)\,dx = 0. \tag{5.34}$$

By combining (5.28) and (5.26) – (5.34), we establish that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\{|u_n|\le h\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx = 0. \tag{5.35}$$

and in view of (4.4) one has

$$\frac{1}{h}\int_{\{|u_n|\le h\}} \varphi\big(x,|\nabla T_{2h}(u_n)|\big)\,dx + \frac{1}{h}\int_{\{|u_n|\le h\}} \varphi\big(x,|T_h(u_n)|\big)\,dx \longrightarrow 0 \quad \text{as} \quad n,h\to\infty. \tag{5.36}$$

Furthermore, in view of (5.25) one has

$$\int_{\{h<|u_n|\}} |H_n(x,u_n,\nabla u_n)|\,dx \longrightarrow 0 \quad \text{as} \quad n,h\to\infty. \tag{5.37}$$

On the other hand, in view of the growth condition (4.8) and Young's inequality we have

$$\begin{aligned}
&\Big|\frac{1}{h}\int_{\{|u_n|\le h\}}\phi_n(x,u_n)\cdot\nabla T_h(u_n)\,dx\Big|\\
&\le \frac{1}{h}\int_{\{|u_n|\le h\}}|\gamma(x)||\nabla T_h(u_n)|\,dx+\frac{1}{h}\int_{\{|u_n|\le h\}}\bar{\varphi_x}^{-1}\big(\varphi(x,|u_n|)\big)|\nabla T_h(u_n)|\,dx\\
&\le \frac{\alpha^2}{2h(\alpha+2)}\int_\Omega\bar{\varphi}\big(x,\frac{2(\alpha+2)}{\alpha^2}|\gamma(x)|\big)\,dx+\frac{1}{h}C(\alpha)\int_{\{|u_n|\le h\}}\varphi(x,|\nabla T_h(u_n)|)\,dx\\
&\quad+\frac{1}{h}\int_{\{|u_n|\le h\}}\bar{\varphi}\big(x,\frac{\alpha+1}{\alpha}\bar{\varphi_x}^{-1}\big(\varphi(x,|u_n|)\big)\big)\,dx.
\end{aligned}\tag{5.38}$$

Combining (5.31) and (5.32) − (5.36), we concluded that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\{|u_n|\le h\}}\phi_n(x,u_n)\cdot\nabla T_h(u_n)\,dx=0. \tag{5.39}$$

Almost everywhere convergence of gradients :

Lemma 5.2. *Let $(u_n)_n$ be a sequence of weak solutions for the approximate problems (5.5), under the hypothesis (4.2)−(4.4), the sequence $\big(a(x,T_k(u_n),\nabla T_k(u_n))\big)_n$ is uniformly bounded in $(L_{\bar{\varphi}}(\Omega))^N$.*

The proof of Lemma 5.2 follows the same argument used in [12].

Now, let $h\ge\tau\ge k>0$, we define the following functions

$$R_h(r)=1-\frac{|T_{2h}(r)-T_h(r)|}{h}\quad\text{and}\quad\Theta(r)=re^{a_0^2r^2}\quad\text{for any}\quad r\in\mathbb{R}.$$

with $a_0=\dfrac{b(k)}{\alpha}$, we point out that $\Theta'(r)-a_0|\Theta(r)|\ge\dfrac{1}{2}$ for all $r\in\mathbb{R}$.
In view of Lemma 3.1, there exists a sequence ν_m in $\mathbb{D}(\Omega)$ such that

$$T_k(\nu_m)\longrightarrow T_k(u)\quad\text{modularly in}\quad W_0^1L_\varphi(\Omega)\quad\text{as } m\to\infty. \tag{5.40}$$

we set $z_{n,m}=(T_k(r)-T_k(\nu_m))$ for any $r\in\mathbb{R}$.
For $\tau>0$, we denote $\varepsilon_i(n)$ $\big($respectively $\varepsilon_i(n,h)$, $\varepsilon_i(n,m,h)$ and $\varepsilon_i(n,m,h,\tau)\big)$ some real functions go to zero as $n\to\infty$.

By selecting $\Theta(z_{n,m})R_h(u_n)$ as a test function in (5.5), we obtain

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\cdot(\nabla T_k(u_n)-\nabla T_k(\nu_m))\Theta^{'}(z_{n,m})R_h(u_n)\,dx\\
&\quad-\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n sign(u_n)R_h^{'}(u_n)\Theta(z_{n,m})\,dx\\
&\quad+\int_\Omega H_n(x,u_n,\nabla u_n)\Theta(z_{n,m})R_h(u_n)\,dx\\
&=\int_\Omega f_n(x)\Theta(z_{n,m})R_h(u_n)\;dx+\int_\Omega F\cdot\nabla\big(\Theta(z_{n,m})R_h(u_n)\big)\;dx\\
&\quad+\int_\Omega \phi_n(x,u_n)\cdot\nabla\big(\Theta(z_{n,m})R_h(u_n)\big)\,dx+\int_\Omega \Phi_n(u_n)\cdot\nabla\big(\Theta(z_{n,m})R_h(u_n)\big)\,dx.
\end{aligned}\tag{5.41}$$

Concerning the first term on the right-hand side of (5.41), we have $f_n(x)\to f(x)$ strongly in $L^1(\Omega)$ and since $\Theta(z_{n,m})R_h(u_n)\rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$ as $n,m\to\infty$, then

$$\int_\Omega f_n(x)\Theta(z_{n,m})R_h(u_n)\;dx\to 0\quad\text{as}\quad n,m\to\infty.\tag{5.42}$$

For the second term on the right-hand side of (5.41), we have

$$\begin{aligned}
\int_\Omega F\cdot\nabla\big(\Theta(z_{n,m})R_h(u_n)\big)\;dx&=\int_\Omega F\cdot(\nabla T_k(u_n)-\nabla T_k(\nu_m))\Theta^{'}(z_{n,m})R_h(u_n)\;dx\\
&\quad-\int_\Omega F\cdot\nabla u_n sign(u_n)R_h^{'}(u_n)\Theta(z_{n,m})\;dx.
\end{aligned}\tag{5.43}$$

Thanks to (5.23) we have $(\nabla T_k(u_n)-\nabla T_k(\nu_m))\rightharpoonup 0$ weakly in $(L_\varphi(\Omega))^N$ when $n,m\to\infty$. On the one hand, one has $F\Theta^{'}(z_{n,m})R_h(u_n)\to F\Theta^{'}(z)R_h(u)$ strongly in $(E_{\bar\varphi}(\Omega))^N$, then it follows that

$$\int_\Omega F\cdot(\nabla T_k(u_n)-\nabla T_k(\nu_m))\Theta^{'}(z_{n,m})R_h(u_n)\;dx\longrightarrow 0\text{ as }n,m\longrightarrow\infty.\tag{5.44}$$

In view of $|F|\in E_{\bar\varphi}(\Omega)$ and (5.36), then by applying Young's inequality we can show that

$$\begin{aligned}
&\left|\int_\Omega F\cdot\nabla u_n sign(u_n)R_h^{'}(u_n)\Theta(z_{n,m})\;dx\right|\\
&\le\frac{\Theta(2k)}{h}\int_{\{h<|u_n|\le 2h\}}|F||\nabla T_{2h}(u_n)|\;dx\\
&\le\frac{\Theta(2k)}{h}\int_{(h<|u_n|\le 2h)}\bar\varphi\big(x,|F|\big)\;dx+\frac{\Theta(2k)}{h}\int_{\{h<|u_n|\le 2h\}}\varphi\big(x,|\nabla T_{2h}(u_n)|\big)\;dx\\
&\longrightarrow 0\quad\text{as}\quad n,h\to\infty.
\end{aligned}\tag{5.45}$$

Regarding the third term on the right-hand side of (5.41), one has

$$\begin{aligned}
&\int_{\Omega} \phi_n(x,u_n)\cdot\nabla\big(\Theta(z_{n,m})R_h(u_n)\big)\,dx \\
&= \int_{\Omega} \phi_n(x,u_n)\cdot\left(\nabla T_k(u_n)-\nabla T_k(\nu_m)\right)\Theta^{'}(z_{n,m})R_h(u_n)\,dx \\
&\quad - \int_{\Omega} \phi_n(x,u_n)\cdot\nabla u_n sign(u_n)R_h^{'}(u_n)\Theta(z_{n,m})\,dx.
\end{aligned} \tag{5.46}$$

For the first term on the right-hand side of (5.46), we have $\phi_n(x,u_n)R_h(u_n)\Theta^{'}(z_{n,m}) = \phi_n(x,T_{2h}(u_n))R_h(u_n)\Theta^{'}(z_{n,m})$ is bounded in $(E_{\bar{\varphi}}(\Omega))^N$, thus by using Lebesgue's convergence theorem it yields that

$$\phi_n(x,T_{2h}(u_n))R_h(u_n)\Theta^{'}(z_{n,m}) \to \phi(x,T_{2h}(u))R_h(u)\Theta^{'}(z) \text{ strongly in } (E_{\bar{\varphi}}(\Omega))^N,$$

having in mind that $\nabla\left(T_k(u_n)-T_k(\nu_m)\right) \rightharpoonup 0$ weakly in $(L_{\varphi}(\Omega))^N$, therefore

$$\int_{\Omega} \phi_n(x,u_n)\cdot\left(\nabla T_k(u_n)-\nabla T_k(\nu_m)\right)\Theta^{'}(z_{n,m})R_h(u_n)\,dx \longrightarrow 0 \text{ as } n,m\to\infty. \tag{5.47}$$

Moreover, by using the result (5.39) we get

$$\begin{aligned}
&\left|\int_{\Omega} \phi_n(x,u_n)\cdot\nabla u_n sign(u_n)R_h^{'}(u_n)\Theta(z_{n,m})\,dx\right| \\
&\leq \frac{\Theta(2k)}{h}\int_{\{h<|u_n|\leq 2h\}} |\phi_n(x,T_{2h}(u_n))||\nabla u_n|\,dx \longrightarrow 0 \text{ as } n,h\to\infty.
\end{aligned} \tag{5.48}$$

Concerning the fourth term on the right-hand side of (5.41), one has

$$\begin{aligned}
&\int_{\Omega} \Phi_n(u_n)\cdot\nabla\big(\Theta(z_{n,m})R_h(u_n)\big)\,dx \\
&= \int_{\{|u_n|\leq 2h\}} \Phi_n(u_n)\cdot(\nabla T_k(u_n)-\nabla T_k(\nu_m))\Theta^{'}(z_{n,m})R_h(u_n)\,dx \\
&\quad - \frac{1}{h}\int_{\{h<|u_n|\leq 2h\}} \Phi_n(u_n)\cdot\nabla u_n sign(u_n)\Theta(z_{n,m})\,dx.
\end{aligned} \tag{5.49}$$

In view of $(\nabla T_k(u_n)-\nabla T_k(\nu_m)) \rightharpoonup 0$ weakly in $(L_{\varphi}(\Omega))^N$ and due to $\Phi_n(T_{2h}(u_n))$ $R_h(u_n)\Theta^{'}(z_{n,m})$ converges to $\Phi(T_{2h}(u))R_h(u)\Theta^{'}(z)$ strongly in $(E_{\bar{\varphi}}(\Omega))^N$, we obtain

$$\int_{\{|u_n|\leq 2h\}} \Phi_n(u_n)\cdot(\nabla T_k(u_n)-\nabla T_k(\nu_m))\Theta^{'}(z_{n,m})R_h(u_n)\,dx \longrightarrow 0 \text{ as } n,m\to\infty. \tag{5.50}$$

Similarly, we have $\Phi_n(u_n)\cdot\nabla u_n\chi_{\{2<|u_n|\leq 2h\}} \in L^1(\Omega)$ and using the fact that $\Theta(z_{n,m}) \rightharpoonup 0$ weak$-*$ in $L^{\infty}(\Omega)$, it can be deduced that

$$\frac{1}{h}\int_{\{h<|u_n|\leq 2h\}} \Phi_n(u_n)\cdot\nabla u_n sign(u_n)\Theta(z_{n,m})\,dx \longrightarrow 0 \text{ as } n,m,h\to\infty. \tag{5.51}$$

Now, we turn to the first term on the left-hand side of (5.41), we have $R_h(u_n)=1$ on the subset $\{|u_n|\leq k\}$ then

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\cdot(\nabla T_k(u_n)-\nabla T_k(\nu_m))\Theta^{'}(z_{n,m})R_h(u_n)\,dx\\
&=\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(\nu_m))\Theta^{'}(z_{n,m})\,dx\\
&\qquad-\int_{\{k<|u_n|\leq 2h\}}a(x,T_{2h}(u_n),\nabla T_{2h}(u_n))\cdot\nabla T_k(\nu_m)\Theta^{'}(z_{n,m})R_h(u_n)\,dx.
\end{aligned}\tag{5.52}$$

For the second term on the right-hand side of (5.52, thanks to Lemma (5.2) there exists a measurable function $\psi_{2h}\in(L_{\bar\varphi}(\Omega))^N$ such that $a(x,T_{2h}(u_n),\nabla T_{2h}(u_n))$ converges to ψ_{2h} for $\sigma\Big((L_{\bar\varphi}(\Omega))^N,(E_\varphi(\Omega))^N\Big)$. Moreover, by using Lebesgue's convergence theorem we have

$$\nabla T_k(\nu_m)\Theta^{'}(z_{n,m})R_h(u_n)\chi_{\{k<|u_n|\leq 2h\}}\to\nabla T_k(\nu_m)\Theta^{'}(z_m)R_h(u)\chi_{\{k<|u|\leq 2h\}}$$

strongly in $(E_\varphi(\Omega))^N$, it yields that

$$\begin{aligned}
&\int_{\{k<|u_n|\leq 2h\}}a(x,T_{2h}(u_n),\nabla T_{2h}(u_n))\cdot\nabla T_k(\nu_m)\Theta^{'}(z_{n,m})R_h(u_n)\,dx\\
&\longrightarrow\int_{\{k<|u|\leq 2h\}}\psi_{2h}\cdot\nabla T_k(\nu_m)\Theta^{'}(z_m)R_h(u)\,dx,
\end{aligned}\tag{5.53}$$

and since $T_k(\nu_m)\Theta^{'}(z_m)R_h(u)\rightharpoonup T_k(u)\Theta^{'}(z)R_h(u)$ weakly in $W_0^1L_\varphi(\Omega)$ we get

$$\begin{aligned}
&\int_{\{k<|u|\leq 2h\}}\psi_{2h}\cdot\nabla T_k(\nu_m)\Theta^{'}(z_m)R_h(u)\,dx\\
&\longrightarrow\int_{\{k<|u|\leq 2h\}}\psi_{2h}\cdot\nabla T_k(u)\Theta^{'}(z)R_h(u)\,dx=0\quad\text{as}\quad m\to\infty.
\end{aligned}\tag{5.54}$$

Regarding the second term on the left-hand side of (5.41), we have

$$\begin{aligned}
&\left|\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_nR_h^{'}(u_n)\Theta(z_{n,m})\,dx\right|\\
&\leq\frac{\Theta(2k)}{h}\int_{\{h<|u_n|\leq 2h\}}a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx\longrightarrow 0\text{ as }n,h\to\infty.
\end{aligned}\tag{5.55}$$

For the third integral on the left-hand side of (5.41), since u_n and $\Theta(z_{n,m})$ have the same sign on $\{|u_n|\geq k\}$, thus we use (4.5) one has

$$\int_\Omega H_n(x,u_n,\nabla u_n)\Theta(z_{n,m})R_h(u_n)\,dx\geq\int_{\{|u_n|\leq k\}}H_n(x,u_n,\nabla u_n)\Theta(z_{n,m})R_h(u_n)\,dx.\tag{5.56}$$

In view of (4.4) and (4.6) it follows that

$$\begin{aligned}
&\left| \int_{\{|u_n|\le k\}} H_n(x, T_k(u_n), \nabla T_k(u_n))\Theta(z_{n,m}) R_h(u_n)\, dx \right| \\
&\le \int_{\{|u_n|\le k\}} \big(h_0(x) + b(T_k(u_n))\varphi(x, |\nabla T_k(u_n)|)\big)|\Theta(z_{n,m})|\, dx \\
&\le \varepsilon_1(n,m) + \frac{b(k)}{\alpha}\int_{\{|u_n|\le k\}} a(x, T_k(u_n), \nabla T_k(u_n))\cdot\nabla T_k(u_n)|\Theta(z_{n,m})|\, dx.
\end{aligned} \tag{5.57}$$

By combining (5.41) and (5.42) – (5.57), we conclude that

$$\begin{aligned}
&\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\cdot(\nabla T_k(u_n) - \nabla T_k(\nu_m))\Theta'(z_{n,m})\, dx \\
&\qquad - \frac{b(k)}{\alpha}\int_{\{|u_n|\le k\}} a(x, T_k(u_n), \nabla T_k(u_n))\cdot\nabla T_k(u_n)|\Theta(z_{n,m})|\, dx \le \varepsilon_1(n,m,h).
\end{aligned} \tag{5.58}$$

Putting $H_\tau = \{x\in\Omega \quad |\nabla T_k(u)| \le \tau\}$, then for the first term on the left-hand side of (5.58) can be written as

$$\begin{aligned}
&\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\cdot(\nabla T_k(u_n) - \nabla T_k(\nu_m))\Theta'(z_{n,m})\, dx \\
&= \int_\Omega \big(a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u), \nabla T_k(u)\chi_{H_\tau})\big)\cdot\nabla\left(T_k(u_n) - T_k(u)\chi_{H_\tau}\right)\Theta'(z_{n,m})\, dx \\
&\quad + \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\cdot(\nabla T_k(u)\chi_{H_\tau} - \nabla T_k(\nu_m))\,\Theta'(z_{n,m})\, dx \\
&\quad + \int_\Omega a(x, T_k(u_n), \nabla T_k(u)\chi_{H_\tau})\cdot(\nabla T_k(u_n) - \nabla T_k(u)\chi_{H_\tau})\,\Theta'(z_{n,m})\, dx.
\end{aligned} \tag{5.59}$$

Concerning the second term on the right-hand side of (5.59), one has $(T_k(u)\chi_{H_\tau} - T_k(\nu_m))$ belongs to $(E_\varphi(\Omega))^N$, and in view of Lemma 5.2, there exists a measurable function $\rho_k \in (L_{\bar\varphi}(\Omega))^N$ such that $a(x, T_k(u_n), \nabla T_k(u_n))$ converges to ρ_k weakly in $(L_{\bar\varphi}(\Omega))^N$, that gives

$$\begin{aligned}
&\left|\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\cdot\nabla\left(T_k(u)\chi_{H_\tau} - T_k(\nu_l)\right)\Theta'(z_{n,m})\, dx\right| \\
&= \Theta'(2k)\int_\Omega |a(x, T_k(u_n), \nabla T_k(u_n))||\nabla\left(T_k(u)\chi_{H_\tau} - T_k(\nu_l)\right)|\, dx \\
&\longrightarrow \Theta'(2k)\int_\Omega |\rho_k||\nabla\left(T_k(u)\chi_{H_\tau} - T_k(\nu_l)\right)|\, dx.
\end{aligned} \tag{5.60}$$

By using convergence (5.40), we have

$$\int_\Omega |\rho_k||\nabla\left(T_k(u)\chi_{H_\tau} - T_k(\nu_l)\right)|\, dx \longrightarrow \int_\Omega |\rho_k||\nabla\left(T_k(u)\chi_{H_\tau} - T_k(u)\right)|\, dx \quad \text{as} \quad m\to\infty. \tag{5.61}$$

Taking into account to Lebesgue's dominated convergence theorem, we deduce that

$$\int_\Omega |\rho_k||\nabla\left(T_k(u)\chi_{H_\tau} - T_k(u)\right)|\,dx \longrightarrow 0 \quad \text{as} \quad \tau\to\infty. \tag{5.62}$$

Regarding the last term on the right-hand side of (5.59), one has $a(x, T_k(u_n), \nabla T_k(u)\chi_{H_\tau})$ is uniformly bounded in $(E_{\bar\varphi}(\Omega))^N$. Moreover, we use (4.2) and Lebesgue's convergence dominated theorem we conclude that

$$a(x, T_k(u_n), \nabla T_k(u)\chi_{H_\tau}) \to a(x, T_k(u), \nabla T_k(u)\chi_{H_\tau}) \quad \text{strongly in } (E_{\bar\varphi}(\Omega))^N$$

On the other hand, thanks to (5.23) we obtain

$$\begin{aligned}
&\left|\int_\Omega a(x, T_k(u_n), \nabla T_k(u)\chi_{H_\tau})\cdot(\nabla T_k(u_n) - \nabla T_k(u)\chi_{H_\tau})\,\Theta'(z_{n,m})\,dx\right| \\
&\longrightarrow \Theta'(2k)\int_\Omega |a(x, T_k(u), \nabla T_k(u)\chi_{H_\tau})||\nabla\left(T_k(u) - T_k(u)\chi_{H_\tau}\right)|\,dx \quad \text{as} \quad n\to\infty \\
&= \Theta'(z_{n,m})\int_\Omega |a(x, T_k(u), 0)||\nabla T_k(u)\chi_{\Omega\setminus H_\tau}|\,dx = 0.
\end{aligned} \tag{5.63}$$

We can set the second integral on the left-hand side of (5.58) as follows

$$\begin{aligned}
&\frac{b(k)}{\alpha}\int_{\{|u_n|\le k\}} a(x, T_k(u_n), \nabla T_k(u_n))\cdot\nabla T_k(u_n)|\Theta(z_{n,m})|\,dx \\
&= \frac{b(k)}{\alpha}\int_\Omega \big(a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u), \nabla T_k(u)\chi_{H_\tau})\big)\cdot\nabla\left(T_k(u_n) - T_k(u)\chi_{H_\tau}\right)|\Theta(z_{n,m})|\,dx \\
&\quad + \frac{b(k)}{\alpha}\int_{\{|u_n|\le k\}} a(x, T_k(u_n), \nabla T_k(u_n))\cdot\nabla T_k(u)\chi_{H_\tau}|\Theta(z_{n,m})|\,dx \\
&\quad + \frac{b(k)}{\alpha}\int_{\{|u_n|\le k\}} a(x, T_k(u_n), \nabla T_k(u)\chi_{H_\tau})\cdot(\nabla T_k(u_n) - \nabla T_k(u)\chi_{H_\tau})|\Theta(z_{n,m})|\,dx.
\end{aligned} \tag{5.64}$$

Concerning the first integral on the right-hand side of (5.64), we have $a(x, T_k(u_n), \nabla T_k(u_n))\cdot \nabla T_k(u)\chi_{H_\tau}$ converges strongly to $T_k(u)\chi_{H_\tau}\rho_k$ in $L^1(\Omega)$ and since $|\Theta(z_{n,m})| \rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$ as n, m tend to infinity, therefore we establish that

$$\frac{b(k)}{\alpha}\int_{\{|u_n|\le k\}} a(x, T_k(u_n), \nabla T_k(u_n))\cdot\nabla T_k(u)\chi_{H_\tau}|\Theta(z_{n,m})|\,dx = \varepsilon_3(n,m). \tag{5.65}$$

Similarly as in (5.63), we show that

$$\frac{b(k)}{\alpha}\int_{\{|u_n|\le k\}} a(x, T_k(u_n), \nabla T_k(u)\chi_{H_\tau})\cdot(\nabla T_k(u_n) - \nabla T_k(u)\chi_{H_\tau})|\Theta(z_{n,m})|\,dx = \varepsilon_4(n,m,\tau). \tag{5.66}$$

We combine (5.59) and (5.60)-(5.66), we establish

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega}\left(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u),\nabla T_k(u)\chi_{H_\tau})\right)\cdot\nabla\left(T_k(u_n)-T_k(u)\chi_{H_\tau}\right)\,dx\\
&\leq\int_{\Omega}\left(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u),\nabla T_k(u)\chi_{H_\tau})\right)\cdot\nabla\left(T_k(u_n)-T_k(u)\chi_{H_\tau}\right)\\
&\qquad\qquad\times\left(\Theta'(z_{n,m})-\frac{b(k)}{\alpha}|\Theta(z_{n,m})|\right)\,dx\\
&\longrightarrow 0 \quad \text{as} \quad n,\tau\to\infty.
\end{aligned} \tag{5.67}$$

In view of the Lemma 3.8, we conclude that

$$\begin{cases} T_k\left(u_n\right)\to T_k(u) \text{ modularly in } W_0^1L_\varphi(\Omega)\\ \nabla u_n\to\nabla u \quad \text{a.e in} \quad \Omega.\end{cases} \tag{5.68}$$

and

$$a(x,T_k(u_n),\nabla T_k(u_n))\rightharpoonup a(x,T_k(u),\nabla T_k(u)) \quad \text{weakly in} \quad (L_{\bar\varphi}(\Omega))^N. \tag{5.69}$$

Furthermore, we have

$$a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)\longrightarrow a(x,T_k(u),\nabla T_k(u))\cdot\nabla T_k(u) \quad \text{strongly in} \quad L^1(\Omega). \tag{5.70}$$

The equi-integrability of $(H_n(x,u_n,\nabla u_n))_n$ and passage to the limit : Firstly, we focus on proving the following convergence

$$H_n(x,u_n,\nabla u_n)\longrightarrow H(x,u,\nabla u) \quad \text{strongly in} \quad L^1(\Omega). \tag{5.71}$$

In the light of (5.68), we have

$$H_n(x,u_n,\nabla u_n)\longrightarrow H(x,u,\nabla u) \quad \text{a.e in} \quad \Omega. \tag{5.72}$$

Thus, according to Vitali's theorem it remains to establish that $(H_n(x,u_n,\nabla u_n))_n$ is a uniformly equi-integrability sequence. Which has been proved in [5]. Then, we deduce the convergence (5.71).

Now, let $v\in C_0^\infty(\Omega)$ and $S(\cdot)$ be a function in $C_c^1(\mathbb{R})$ with a compact support, such that $\operatorname{supp}(S)\subset[-M,M]$ for some $M>0$. By taking $S(u_n)v$ as a test function for the approximate problem (5.5), we obtain

$$\begin{aligned}
&\int_{\Omega}a(x,u_n,\nabla u_n)\cdot(\nabla u_nS'(u_n)v+\nabla vS(u_n))\,dx+\int_{\Omega}H_n(x,u_n,\nabla u_n)S(u_n)v\,dx\\
&=\int_{\Omega}f_n(x)S(u_n)v\,dx+\int_{\Omega}F\cdot(\nabla u_nS'(u_n)v+\nabla vS(u_n))\,dx\\
&\quad+\int_{\Omega}\phi_n(x,u_n)\cdot(\nabla u_nS'(u_n)v+\nabla vS(u_n))\,dx+\int_{\Omega}\Phi_n(u_n)\cdot(\nabla u_nS'(u_n)v+\nabla vS(u_n))\,dx.
\end{aligned} \tag{5.73}$$

Thanks to the fact that $\mathrm{supp}(S) \subset [-M, M]$, we have

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n) \cdot (\nabla u_n S'(u_n)v + \nabla v S(u_n))\, dx \\ &= \int_\Omega a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla T_M(u_n) S'(T_M(u_n))v\, dx \\ &\quad + \int_\Omega a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla v S(T_M(u_n))\, dx. \end{aligned} \tag{5.74}$$

and

$$\begin{aligned} &\int_\Omega \phi_n(x, u_n) \cdot (\nabla u_n S'(u_n)v + \nabla v S(u_n))\, dx \\ &= \int_\Omega \phi_n(x, T_M(u_n)) \cdot \nabla T_M(u_n) S'(T_M(u_n))v\, dx \\ &\quad + \int_\Omega \phi_n(x, T_M(u_n)) \cdot \nabla v S(T_M(u_n)\, dx. \end{aligned} \tag{5.75}$$

and

$$\begin{aligned} &\int_\Omega \Phi_n(u_n) \cdot (\nabla u_n S'(u_n)v + \nabla v S(u_n))\, dx \\ &= \int_\Omega \Phi_n(T_M(u_n)) \cdot \nabla T_M(u_n) S'(T_M(u_n))v\, dx + \int_\Omega \Phi_n(T_M(u_n)) \cdot \nabla v S(T_M(u_n)\, dx. \end{aligned} \tag{5.76}$$

Having in mind (5.70), and since $S'(u_n)v \longrightarrow S'(u)v$ weak$-*$ in $(L^\infty(\Omega))^N$. Moreover, in view of (5.69) and since $\nabla v S(T_M(u_n)) \longrightarrow \nabla v S(T_M(u))$ strongly in $(E_\varphi(\Omega))^N$, then we derive that

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega a(x, u_n, \nabla u_n) \cdot (\nabla u_n S'(u_n)v + \nabla v S(u_n))\, dx \\ &= \int_\Omega a(x, T_M(u), \nabla T_M(u)) \cdot \nabla T_M(u) S'(T_M(u))v\, dx \\ &\quad + \int_\Omega a(x, T_M(u), \nabla T_M(u)) \cdot \nabla v S(T_M(u))\, dx \\ &= \int_\Omega a(x, u, \nabla u) \cdot (\nabla u S'(u)v + \nabla v S(u))\, dx. \end{aligned} \tag{5.77}$$

In view of $(\nabla(S(u_n)v)$ is bounded in $(L_\varphi(\Omega))^N$, then by (5.23) $(\nabla T_M(u_n) S'(T_M(u_n))v + \nabla v S(T_M(u_n))) \rightharpoonup (\nabla T_M(u) S'(T_M(u))v + \nabla v S(T_M(u)))$ weakly in $(L_\varphi(\Omega))^N$. On the other hand, by using Lebesgue's dominated theorem $\phi_n(x, T_M(u_n)) \to \phi(x, T_M(u))$ strongly in $(E_{\bar\varphi}(\Omega))^N$, we obtain

$$\begin{aligned} \lim_{n\to\infty} &= \int_\Omega \phi(x, T_M(u_n)) \cdot \big(\nabla T_M(u_n) S'(T_M(u_n))v + \nabla v S(T_M(u_n)\big)\, dx \\ &= \int_\Omega \phi(x, u) \cdot (\nabla u S'(u)v + \nabla v S(u))\, dx. \end{aligned} \tag{5.78}$$

Using the same argument as in (5.78), it yields that

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega \Phi(T_M(u_n))\cdot\big(\nabla T_M(u_n)S'(T_M(u_n))v+\nabla vS(T_M(u_n)\big)\,dx\\ =\int_\Omega \Phi(u)\cdot(\nabla uS'(u)v+\nabla vS(u))\,dx.\end{aligned}\tag{5.79}$$

Moreover, one has F belongs to $(E_{\bar\varphi}(\Omega))^N$, and since $(\nabla T_M(u_n)S'(T_M(u_n))v+\nabla vS(T_M(u_n)))\rightharpoonup(\nabla T_M(u)S'(T_M(u))v+\nabla vS(T_M(u)))$ weakly in $(L_\varphi(\Omega))^N$ it follows that

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega F\cdot(\nabla T_M(u_n)S'(T_M(u_n))v+\nabla vS(T_M(u_n)))\,dx\\ =\int_\Omega F\cdot(\nabla uS'(u)v+\nabla vS(u))\,dx.\end{aligned}\tag{5.80}$$

Thanks to (5.71), we have $H_n(x,u_n,\nabla u_n)$ converges to $H(x,u,\nabla u)$ strongly in $L^1(\Omega)$ and due to $S(u_n)v\rightharpoonup S(u)v$ weak-$*$ in $L^\infty(\Omega)$, it follows that

$$\lim_{n\to\infty}\int_\Omega H_n(x,u_n,\nabla u_n)S(u_n)v\,dx=\int_\Omega H(x,u,\nabla u)S(u)v\,dx.\tag{5.81}$$

Similarly, one has $f_n\to f$ strongly in $L^1(\Omega)$ and since $S(u_n)v\rightharpoonup S(u)v$ weak-$*$ in $L^\infty(\Omega)$ thus it yields

$$\lim_{n\to\infty}\int_\Omega f_n(x)S(u_n)v\,dx=\int_\Omega f(x)S(u)v\,dx.\tag{5.82}$$

Finally, by relation (5.73) and (5.74)-(5.82) we conclude that

$$\begin{aligned}&\int_\Omega a(x,u,\nabla u)\cdot(\nabla uS'(u)v+\nabla vS(u))\,dx+\int_\Omega H(x,u,\nabla u)S(u)v\,dx\\ &=\int_\Omega f(x)S(u)v\ dx+\int_\Omega F\cdot(\nabla uS'(u_n)v+\nabla vS(u))\ dx\\ &\quad+\int_\Omega \phi(x,u)\cdot(\nabla uS'(u)v+\nabla vS(u))\,dx+\int_\Omega \Phi(u)\cdot(\nabla uS'(u)v+\nabla vS(u))\,dx,\end{aligned}\tag{5.83}$$

which conclude the proof of the Theorem 5.1. □

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