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DYNAMICS AND OSCILLATIONS OF A CLASS OF CUSHING EQUATIONS INCLUDING GAMMA-DISTRIBUTED DELAYS WITH A GAP

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Abstract. The stability of dynamical systems in presence of time-delay is a problem of big interest since the presence of a time-delay may induce instabilities, and complex behaviors for the corresponding schemes. In particular, the problem becomes even more difficult in the case when the delays are distributed. Besides, as we all know, many phenomena in nature have oscillatory character and their mathematical models have led to the introduction of certain classes of functions to describe them. Such a class form pseudo almost automorphic functions which a natural generalization of the concept of almost periodicity and almost automorphy. This paper studies the dynamics and oscillations of a class of the Cushing equations including gamma-distributed delays with a gap

$$x'(t) = -\alpha x(t) + \beta(t)\, x(t-\tau) + \lambda(t) \int_0^t g(\theta)\, x(t-\theta)\, d\theta.$$

In particular, sufficient conditions are obtained to ensure the existence and stability of the unique pseudo almost automorphic solution for the Cushing equation with gamma-distributed delays .

1. Introduction

Integrodifferential equations appear quite early in the mathematical development of theoretical population dynamics in the pioneering work of such mathematicians as V. Volterra [23] and V. A. Kostitzin [19]. In their attempts to model the growth of populations by means of differential equations these early investigators were quick to point out that the current growth rate of a population is unlikely to depend only on the current population size or, put another way, that growth rates are unlikely to respond instantaneously (or even "quickly") to changes in population sizes or densities. This led Volterra in particular to include functionals of (Volterra) integral type in what have become the classical differential models of population dynamics and mathematical ecology (equations such as the logistic equation, the famous predator-prey system of Volterra and the well-known Volterra-Lotka competition model). Much of this early work involving integrodifferential equations in population dynamics can be found in the

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recent collection of papers edited by Scudo and Ziegler (1978). Despite this early interest in and recognition of the importance of temporal response delays, the formulation, analysis and use of differential models which attempt to incorporate delays in them has lagged considerably behind that of nondelay models which ignore response delays. Besides, the stability of dynamical systems in presence of time-delay is a problem of recurring interest (see for instance [15], [16], [21]), and the references therein), since the presence of a time-delay may induce instabilities, and complex behaviors for the corresponding schemes. The problem becomes even more difficult in the case when the delays are distributed.

To the best of the authors' knowledge, the first population dynamics model including gamma-distributed delays is due to Cushing [9], and it received a lot of attention starting with the 80s ([4],[5], [8]). The linearized model [8] simply writes as:

$$x'(t) = -\alpha x(t) + \lambda \int_0^t g(\theta)\, x(t-\theta)\, d\theta$$

under suitable initial conditions. Notice that α is the death rate of the population x and λ is the rate of egg-laying (maternity rate). The function g represents the proportion of egg laid at any specific time, that survive and hatch out after time s and where the delay kernel is given by the gamma-distribution law:

$$g(\theta) = \frac{a^{n+1}}{n!}\theta^n e^{-a\theta}.$$

Besides, it is well known, that A narrow distribution will lead to some simple discrete delay system of the form

$$x'(t) = -\alpha x(t) + \beta x(t-\tau),$$

whose dynamics and stability are completely understood (see, for instance, [16] and the references therein). In this paper, we will consider

$$x'(t) = -\alpha x(t) + \beta(t)\, x(t-\tau) + \lambda(t) \int_0^t g(\theta)\, x(t-\theta)\, d\theta \tag{1.1}$$

In 1955, Bochner ([18], [10]) suggested another generalization of the concept of almost periodicity that to say, almost automorphy, which is in relation to some aspects of differential geometry. This concept became a generalization of almost periodicity which is one of the most attractive topics in the qualitative theory of differential equations because of their significance and applications in physics, mathematical biology, control theory, and other related fields. Hence, the main purpose of this paper is to study the existence and attractivity of the pseudo almost automorphic solution for the model (1.1) . Clearly, the research for the solutions pseudo almost automorphic for differential equations are more complicated since the fundamental property of uniform continuity is not satisfied by such functions. The remainder of this paper is organized as follows: In Section 2, we will introduce some necessary notations, definitions and fundamental properties of the space $PAA(\mathbb{R}, \mathbb{R}^n)$ which will be used in the paper. In Section 3, based on different methods and analysis techniques and provides several sufficient conditions ensuring the existence and uniqueness of the pseudo almost automorphic solution for the considered system. Section 4 is devoted to the stability of the

pseudo almost automorphic solution of (1.1). It should be mentionned that The main results include Theorems 3.4, 3.5 and 4.1.

2. Preliminaries

Definition 2.1. [18]A continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$.

Remark 2.2. Note that the function g in definition above is measurable but not necessarily continuous. Moreover, if g is continuous, then f is uniformly continuous. Besides, if the convergence above is uniform in $t \in \mathbb{R}$, then f is almost periodic. Denote by $AA(\mathbb{R}, \mathbb{R}^n)$ the collection of all almost automorphic functions

$$AP(\mathbb{R}, \mathbb{R}^n) \subset AA(\mathbb{R}, \mathbb{R}^n) \subset BC(\mathbb{R}, \mathbb{R}^n),$$

where $AP(\mathbb{R}, \mathbb{R}^n)$ and $BC(\mathbb{R}, \mathbb{R}^n)$ are respectively the collection of all almost periodic functions and the set of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$.

Among others things, almost automorphic functions satisfy the following properties:

Theorem 2.3. [14] *For all $f, f_1, f_2 \in AA(\mathbb{R}, \mathbb{R}^n)$, one has*

(1) $f_1 + f_2 \in AA(\mathbb{R}, \mathbb{R}^n)$.
(2) $\lambda f \in AA(\mathbb{R}, \mathbb{R}^n)$ for any scalar $\lambda \in \mathbb{R}$
(3) $f_\alpha \in AA(\mathbb{R}, \mathbb{R}^n)$ where $f_\alpha : \mathbb{R} \longrightarrow X$ is defined by $f_\alpha(\cdot) = f(\cdot + \alpha)$.
(4) Let's $f \in AA(\mathbb{R}, \mathbb{R}^n)$,the range $R_f := \{f(t), t \in \mathbb{R}\}$ is relatively compact in X, thus f is bounded in norm.
(5) If $f_n \longrightarrow f$ uniformly on $\mathbb{R}$ where $f_n \in AA(\mathbb{R}, \mathbb{R}^n)$, then $f \in AA(\mathbb{R}, \mathbb{R}^n)$.
(6) $(AA(\mathbb{R}, \mathbb{R}^n), \|\cdot\|_\infty)$ is a Banach space.

Example 2.4. A classical example of an almost automorphic function which is not almost periodic, as it is not uniformly continuous, is the function defined by

$$f(t) = \cos\left(\frac{1}{2 + \sin t + \sin \pi t}\right), \quad t \in \mathbb{R}\ .$$

The new concept of pseudo almost automorphy generalizes the one of pseudo almost periodicity, in fact, a pseudo almost automorphic function is the sum of an almost automorphic function and of an ergodic perturbation. These functions were introduced recently by Liang, Xiao and Zhang in [24] and [17]. In the literature, many works are devoted to the existence of almost periodic and almost automorphic solutions for differential equations, but results about pseudo almost

automorphic solutions are rare. The concept of pseudo-almost-automorphy generalizes the one of the pseudo-almost-periodicity and it has been recently introduced in the literature [12],[24].
The notation $AA_0(\mathbb{R},\mathbb{R}^n)$ stands for the spaces of functions

$$PAP_0(\mathbb{R},\mathbb{R}^n)=\left\{f\in BC(\mathbb{R},\mathbb{R}^n)/\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\|f(t)\|\,dt=0\right\}.$$

Definition 2.5. A function $f:\mathbb{R}\to\mathbb{R}^n$ is called pseudo-almost automorphic if it can be decomposed as $f=g+\varphi$, where $g\in AA(\mathbb{R}^n)$ and $\varphi\in PAP_0(\mathbb{R}^n)$. The class of all such functions will be denote by $PAA(\mathbb{R},\mathbb{R}^n)$.

Example 2.6. The function

$$f(t)=\sin\left(\frac{1}{2-\sin t-\sin \pi t}\right)+\frac{1}{\sqrt{1+t^2}}$$

belongs to $PAA(\mathbb{R},\mathbb{R})$.

Among others things, pseudo-almost automorphy functions satisfy the following properties:

Theorem 2.7. [14] *One has*

(1) $(PAA(X),\|\cdot\|_\infty)$ is a Banach space.
(2) $PAA(\mathbb{R},\mathbb{R}^n)$ is a translation invariant closed subspace of $BC(\mathbb{R},\mathbb{R}^n)$ containing the constant functions. Furthermore,

$$PAA(\mathbb{R},\mathbb{R}^n)=AA(\mathbb{R},\mathbb{R}^n)\oplus PAP_0(\mathbb{R},\mathbb{R}^n).$$

In order to generalize, extend, and enrich the above models, let us consider the following class of Cushing equations including gamma-distributed delays with a gap

$$x'(t)=-\alpha(t)\,x(t)+\beta(t)\,x(t-\tau)+\lambda(t)\int_0^t g(\theta)\,x(t-\theta)\,d\theta \tag{1.2}$$

where

$$g(\theta)=\begin{cases}0 & \theta<\sigma\\ \frac{a^{n+1}}{n!}(\theta-\sigma)^n e^{-a(\theta-\sigma)} & \theta\geq\sigma\end{cases}$$

Notice that the choice of delay kernel with some gap is more realistic for the study of population dynamics models based on differential equations [4], [20] and [22]
(H_1) $\alpha(t)>0$.
(H_2) The functions $\beta(\cdot),\lambda(\cdot)$ are pseudo almost periodic.

(H_3) Denote $\sup_{t\in\mathbb{R}}(\lambda(t)+\beta(t))<\widetilde{\alpha}$.

3. Existence and uniqueness of pseudo almost automorphic solution

In this section, we establish some results for the existence, uniqueness of pseudo almost automorphic solution of (1.1).

Lemma 3.1. *If $\varphi \in PAA(\mathbb{R},\mathbb{R})$, then for all $h \in \mathbb{R}$, $\varphi(\cdot - h) \in PAA(\mathbb{R},\mathbb{R})$.*

Proof. The proof is immediate, we omitted.

□

Lemma 3.2. *If $\varphi, \psi \in PAA(\mathbb{R},\mathbb{R})$, then $\varphi \times \psi \in PAA(\mathbb{R},\mathbb{R})$.*

Proof. The proof is immediate, we omitted.

□

Lemma 3.3. *If $x(\cdot) \in PAA(\mathbb{R},\mathbb{R})$, then*

$$t \longmapsto \int_0^t g(\theta)\, x(t-\theta)\, d\theta$$

is also $PAA(\mathbb{R},\mathbb{R})$.

Proof. The proof is immediate, we omitted.

□

Theorem 3.4. *Suppose that assumptions $(H_1),(H_2)$ and (H_3) hold. Define the nonlinear operator Γ by: for each $\varphi \in PAA(\mathbb{R},\mathbb{R})$*

$$(\Gamma x)(t) = \left\{ \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi} \left[\beta(s)\, x(s-\tau) + \lambda(s) \int_0^s g(\theta)\, x(s-\theta)\, d\theta \right] ds \right\}$$

Then Γ maps $PAA(\mathbb{R},\mathbb{R})$ into itself.

Proof. First of all, let us check that Γ is well defined. Indeed, by lemma 1, for all if $x(\cdot) \in PAA(\mathbb{R},\mathbb{R})$ the function $T_h(x) = x(\cdot - h) \in PAA(\mathbb{R},\mathbb{R})$ since $PAA(\mathbb{R},\mathbb{R})$ is a translation invariant closed subspace of $BC(\mathbb{R},\mathbb{R})$. So, the function

$$\chi : s \longmapsto \beta(s)\, x(s-\tau) + \lambda(s) \int_0^s g(\theta)\, x(s-\theta)\, d\theta$$

belongs to $PAA(\mathbb{R},\mathbb{R})$. Consequently we can write

$$\chi = \psi + \varphi$$

where $\psi \in AA(\mathbb{R},\mathbb{R})$ and $\varphi \in PAP_0(\mathbb{R},\mathbb{R})$. So, one can write

$$\begin{aligned}
(\Gamma\chi)(t) &:= \int_{-\infty}^{t} \exp\left(-\int_s^t \alpha(\xi)\, d\xi \right) \chi(s)\, ds \\
&= \int_{-\infty}^{t} \exp\left(-\int_s^t \alpha(\xi)\, d\xi \right) \psi(s)\, ds + \int_{-\infty}^{t} \exp\left(-\int_s^t \alpha(\xi)\, d\xi \right) \varphi(s)\, ds \\
&= (\Gamma\psi)(t) + (\Gamma\varphi)(t).
\end{aligned}$$

Let (s'_n) be a sequence of real numbers. By (H_3) we can extract a subsequence (s_n) of (s'_n) such that

$\lim_{n\to+\infty} \alpha(t+s_n) = \alpha^1(t)$ for all $t \in \mathbb{R}$.

$\lim_{n\to+\infty} \alpha^1(t-s_n) = \alpha(t)$ for all $t, s \in \mathbb{R}$.

$\lim_{n\to+\infty} \psi(t+s_n) = \psi^1(t)$ for all $t \in \mathbb{R}$.

$\lim_{n\to+\infty} \psi^1(t-s_n) = \psi(t)$ for all $t \in \mathbb{R}$.

Pose

$$(\Gamma^1\psi)(t) := \int_{-\infty}^{t} \exp\left(-\int_{s}^{t} \alpha^1(\xi)\, d\xi\right) \chi^1(s)\, ds.$$

It follows

$$\begin{aligned}
(\Gamma\psi)(t+s_n) - (\Gamma^1\psi)(t) &= \int_{-\infty}^{t+s_n} e^{-\int_{s}^{t+s_n} \alpha(\xi)d\xi} \psi(s)\, ds - \int_{-\infty}^{t} \exp^{-\int_{s}^{t} \alpha(\xi)d\xi} \psi^1(s)\, ds \\
&= \int_{-\infty}^{t+s_n} e^{-\int_{s-s_n}^{t} \alpha(\sigma+s_n)d\sigma} \psi(s)\, ds - \int_{-\infty}^{t} e^{-\int_{s}^{t} \alpha(\xi)d\xi} \psi^1(s)\, ds \\
&= \int_{-\infty}^{t} e^{-\int_{u}^{t} \alpha(\sigma+s_n)d\sigma} \psi(s+s_n)\, ds - \int_{-\infty}^{t} e^{-\int_{s}^{t} \alpha(\xi)d\xi} \psi^1(s)\, ds \\
&= \int_{-\infty}^{t} e^{-\int_{u}^{t} d_i(\sigma+s_n)d\sigma} \left(\psi(s+s_n) - \psi^1(s)\right) ds \\
&\quad + \int_{-\infty}^{t} \left(e^{-\int_{u}^{t} \alpha(\sigma+s_n)d\sigma} - e^{-\int_{s}^{t} \alpha(\xi)d\xi}\right) \psi^1(s)\, ds \\
&= \int_{-\infty}^{t} e^{-\int_{u}^{t} \alpha(\sigma+s_n)d\sigma} \psi(s+s_n)\, ds - \int_{-\infty}^{t} e^{-\int_{u}^{t} \alpha(\sigma+s_n)d\sigma} \psi(u)\, du.
\end{aligned}$$

By the Lebesgue Dominated Convergence Theorem we obtain

$$\lim_{n\to+\infty} (\Gamma\psi)(t+s_n) = (\Gamma^1\psi)(t) \text{ for all } t \in \mathbb{R}.$$

The same approach proves that

$$\lim_{n\to+\infty} (\Gamma^1\psi)(t-s_n) = (\Gamma\psi)(t) \text{ for all } t \in \mathbb{R}.$$

Consequently, the function $(\Gamma\psi)$ belongs to $AA(\mathbb{R},\mathbb{R})$. Now, let us show that $(\Gamma\varphi)$ belongs to $PAP_0(\mathbb{R},\mathbb{R})$.

$$\begin{aligned}\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left|\int_{-\infty}^{t}\exp\left(-\int_{s}^{t}\alpha(\xi)\,d\xi\right)\varphi(s)\,ds\right|dt &\leq \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_{-\infty}^{t}e^{\widetilde{\alpha}(t-s)}\left|\varphi(s)\right|ds\right)dt\\ &\leq J+K,\end{aligned}$$

where

$$\begin{aligned}J&=\lim_{T\to+\infty}\tfrac{1}{2T}\int_{-T}^{T}\left(\int_{-T}^{t}e^{\widetilde{\alpha}(t-s)}\left|\varphi(s)\right|ds\right)dt \text{ and}\\ K&=\lim_{T\to+\infty}\tfrac{1}{2T}\int_{-T}^{T}dt\left(\int_{-\infty}^{-T}e^{\widetilde{\alpha}(t-s)}\left|\varphi(s)\right|ds\right).\end{aligned}$$

Now, we shall prove that $J=K=0$

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}\left(\int_{-T}^{t}\left|e^{\widetilde{\alpha}(t-s)}\varphi(s)\right|ds\right)dt &= \frac{1}{2T}\int_{-T}^{T}\left(\int_{-T}^{t}e^{\widetilde{\alpha}(t-s)}\left|\varphi(s)\right|ds\right)dt\\ &\leq \int_{0}^{+\infty}e^{--\widetilde{\alpha}\xi}\left(\frac{1}{2T}\int_{-T-\xi}^{T-\xi}\left|\varphi(u)\right|du\right)d\xi\\ &\leq \int_{0}^{+\infty}e^{--\widetilde{\alpha}\xi}\left(\frac{1}{2T}\int_{-T-\xi}^{T+\xi}\left|\varphi(u)\right|du\right)d\xi.\end{aligned}$$

Since the function $\varphi(\cdot)\in PAP_0(\mathbb{R},\mathbb{R})$ then the function φ_T defined by

$$\varphi_T(\xi)=\frac{T+\xi}{T}\frac{1}{2(T+\xi)}\int_{-T-\xi}^{T+\xi}\left|\varphi(u)\right|du$$

is bounded and satisfy $\lim_{T\to+\infty}\varphi_T(\xi)=0$. Consequently, by the Lebesgue dominated convergence theorem, we obtain

$$J=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_{-T}^{t}e^{\widetilde{\alpha}(t-s)}\left|\varphi(s)\right|ds\right)dt=0.$$

On the other hand, notice that $|\varphi|_\infty=\sup_{t\in\mathbb{R}}|\varphi(t)|<\infty$ then

$$K\leq\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left(\int_{-\infty}^{-T}e^{\widetilde{\alpha}(t-s)}\left|\varphi(s)\right|ds\right)dt\leq\lim_{T\to+\infty}\frac{\sup_{t\in\mathbb{R}}|\varphi(t)|}{\widetilde{\alpha}}e^{--2\widetilde{\alpha}T}=0$$

Consequently, (Γx) belongs to $PAA(\mathbb{R},\mathbb{R}^n)$.

□

Theorem 3.5. *Suppose that assumptions* $(H_1)-(H_3)$ *hold. Then the equation* (1.2) *has a unique pseudo almost automorphic solution.*

Proof. First, we prove that the mapping Γ is a contraction mapping of $PAA(\mathbb{R},\mathbb{R})$. In view of (H_2), for any $x, y \in PAA(\mathbb{R},\mathbb{R})$, we have

$$\begin{aligned}
\|(\Gamma x)(t)-(\Gamma y)(t)\| &= \sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}\left\{\beta(s)x(s-\tau)+\lambda(s)\int_0^s g(\theta)x(s-\theta)\,d\theta\right.\right.\\
&\quad \left.\left.-\beta(s)y(s-\tau)+\lambda(s)\int_0^s g(\theta)y(s-\theta)\,d\theta\right\}ds\right|\\
&\leq \sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}\left\{|\beta(s)|\,|(x(s-\tau))-y(s-\tau))|\right.\\
&\quad \left.+|\lambda(s)|\int_\sigma^s \frac{a^{n+1}}{n!}(\theta-\sigma)^n e^{-a(\theta-\sigma)}|x(s-\theta)-y(s-\theta)|\,d\theta\right\}ds\\
&\leq \sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}\left\{|\beta(s)|\,|\varphi_j(s)-\psi_j(s)|\right.\\
&\quad \left.+\sum_{j=1}^{n}|b_{ij}(s)|\,|\varphi_j(s-\tau_j(s))-\psi_j(s-\tau_j(s))|\right\}ds\\
&\leq \sup_{t\in\mathbb{R}}\left[\frac{|\beta(t)|+|\lambda(t)|}{\widetilde{\alpha}}\right]|x-y|.
\end{aligned}$$

By (H_3) Γ is a contraction mapping. Consequently, Γ possess a unique fixed point $x^*\in PAA(\mathbb{R},\mathbb{R}^n)$ that is $\Gamma(x^*)=x^*$. □

4. The global attractivity of the paa solution

Let $x^*(\cdot)$ the pseudo almost automorphic solution of theorem 1 and $x(\cdot)$ be an arbitrary solution of (1.2) . So, one has

$$\dot{x^*}(t) = -\alpha(t)x^*(t)+\beta(t)x^*(t-\tau)+\lambda(t)\int_0^t g(\theta)x^*(t-\theta)\,d\theta$$

and

$$\dot{x}(t) = -\alpha(t)x(t)+\beta(t)x(t-\tau)+\lambda(t)\int_0^t g(\theta)x(t-\theta)\,d\theta$$

Let us pose for all $1\leq i\leq n$, $z(\cdot)=x(\cdot)-x^*(\cdot)$. Consequently, we obtain

$$\left\{\begin{array}{l}\dot{z_i}(t) = -\alpha(t)z(t)+\beta(t)z(t-\tau)+\lambda(t)\int_0^t g(\theta)z(t-\theta)\,d\theta\end{array}\right. \tag{1.3}$$

Clearly, the pseudo almost automorphic solution $x^*(\cdot)$ of system (1.1) is global attractivity if and only if the equilibrium point O of system (1.3) is global attractivity. So let us study the global attractivity of the equilibrium point O for

system (1.3). In this section, we study locally exponential stability of the unique pseudo almost automorphic solution.

Theorem 4.1. *Suppose that assumptions* $(H_1)-(H_3)$ *hold. Then the equilibrium point* O *of the nonlinear system* (1.3) *is global attractive.*

Proof. First, let us prove that the solution of system (1.3) are uniformly bounded. In other words, there exists $M>0$ such that for all $t\geq 0$ one has

$$|z(t)|\leq M.$$

By the assumption (H_3), $1-r>0$. So for any given continuous function $\theta(\cdot)$, there exists a large number $M>0$, such that

$$|\theta(\cdot)|<M \text{ and } (1-r)M>0.$$

Let κ a real number, $\kappa<1$. We shall prove that for all $t\geq 0$, $||\leq \kappa M$. Suppose the contrary, then there must be some $t'>0$, such that

$$\begin{cases} |z(t')| = \kappa M \\ |z(t)| < \kappa M, \quad 0\leq t\leq t' \end{cases}$$

In view of (H_3) , (H_4) and the equation (2), we have

$$\begin{aligned}
|z(t')| &\leq \left\{ |\theta_i(0)|\, e^{-\int_0^{t'}\alpha(u)du} + \int_0^{t'} e^{-\int_s^{t'}\alpha(u)du}\left(|\beta(t)|\,|z(t-\tau)| + |\lambda(t)|\int_0^t |g(\theta)\,z(t-\theta)|\,d\theta\right) ds \right\} \\
&\leq \left\{ |\theta_i(0)|\, e^{-\underline{\alpha}t} + \int_0^{t'} e^{-\underline{\alpha}(t'-s)}\,|z(s)|\left(\sum_{j=1}^n a_{ij}^+ + L_j^f\,|z_j(s)| + b_{ij}^+ L_j^g\,|z_j(s-\tau_j(s))|\right) ds \right\} \\
&\leq \left\{ hMe^{-\underline{d_i}t'} + \int_0^{t'} e^{-\underline{d_i}(t-s)} hM\left(\sum_{j=1}^n a_{ij}^+ + L_j^f + b_{ij}^+ L_j^g\right) ds \right\} \\
&\leq \left\{ e^{-\underline{d_i}t'} + \int_0^{t'} e^{-\underline{d_i}(t'-s)}\left(\sum_{j=1}^n a_{ij}^+ + L_j^f + b_{ij}^+ L_j^g\right) ds \right\} hM \\
&\leq \left\{ e^{-\underline{d_i}t'} + \left(\frac{\beta_\infty+\lambda_\infty}{\underline{\alpha}}\right)\left(1-e^{-\underline{\alpha}t'}\right)\right\} hM \\
&< hM
\end{aligned}$$

which gives a contradiction. Consequently, for all $t\geq 0$, $||\leq \kappa M$. Let us take $\kappa\longrightarrow 1$, then for all $t\geq 0$, $||\leq M$. Thus, there is a constant $\sigma\geq 0$, such that

$$\limsup_{t\longrightarrow+\infty}|z(t)|=\beta.$$

It follows that

$$\forall\varepsilon>0,\exists t_2<0,\forall t,(t\geq t_2\Longrightarrow\|z(t)\|\leq(1+\varepsilon)\,\beta).$$

$$\begin{aligned}\dot{z}(t)+\alpha(t)z(t) &= \beta(t)z(t-\tau)+\lambda(t)\int_0^t g(\theta)z(t-\theta)\,d\theta\\ &\leq (\beta_\infty+\lambda_\infty)z(t)\\ &\leq \max_{1\leq i\leq n}\left(\sum_{j=1}^n a_{ij}^+L_j^f+b_{ij}^+L_j^g\right)(1+\varepsilon)\beta\end{aligned}$$

So, throug the integration, we obtain the inequality

$$\begin{aligned}|z_i(t)| &\leq |\theta_i(0)|\,e^{-\int_0^t d_i(u)du}+\left\{\max_{1\leq i\leq n}\left(\sum_{j=1}^n a_{ij}^+L_j^f+b_{ij}^+L_j^g\right)(1+\varepsilon)\beta\right\}\int_0^t e^{-\int_s^t d_i(u)du}ds\\ &\leq \|\theta\|\,e^{-\underline{d_i}t}+\left\{\max_{1\leq i\leq n}\left(\sum_{j=1}^n a_{ij}^+L_j^f+b_{ij}^+L_j^g\right)(1+\varepsilon)\beta\right\}\int_0^t e^{-\underline{d_i}(t-s)}ds\\ &\leq \|\theta\|\,e^{-\underline{d_i}t}+\left\{\frac{\max_{1\leq i\leq n}\left(\sum_{j=1}^n a_{ij}^+L_j^f+b_{ij}^+L_j^g\right)}{\underline{d_i}}\right\}(1+\varepsilon)\sigma\left(1-e^{-\underline{d_i}t}\right)\\ &\leq \|\theta\|\,e^{-\underline{d_i}t}+\left\{\max_{1\leq i\leq n}\left(\frac{\sum_{j=1}^n a_{ij}^+L_j^f+b_{ij}^+L_j^g}{\underline{d_i}}\right)\right\}(1+\varepsilon)\sigma\left(1-e^{-\underline{d_i}t}\right)\end{aligned}$$

Hence,

$$\|z(t)\|\leq\max_{1\leq i\leq n}\left[\|\theta\|\,e^{-\underline{d_i}t}+\left\{\max_{1\leq i\leq n}\left(\frac{\sum_{j=1}^n a_{ij}^+L_j^f+b_{ij}^+L_j^g}{\underline{d_i}}\right)\right\}(1+\varepsilon)\beta\left(1-e^{-\underline{d_i}t}\right)\right]$$

In particular,

$$\begin{aligned}\limsup_{t\longrightarrow+\infty}\|z(t)\| &\leq \limsup_{t\longrightarrow+\infty}\max_{1\leq i\leq n}\left[\|\theta\|\,e^{-\underline{d_i}t}+\left\{\max_{1\leq i\leq n}\left(\frac{\sum_{j=1}^n a_{ij}^+L_j^f+b_{ij}^+L_j^g}{\underline{d_i}}\right)\right\}(1+\varepsilon)\beta\left(1-e^{-\underline{d_i}t}\right)\right]\\ &= [r(1+\varepsilon)\sigma]\end{aligned}$$

In other words,

$$\beta\leq r(1+\varepsilon)\beta$$

Passing to limit when $\varepsilon\longrightarrow 0$, we obtain

$$\beta(1-r)\leq 0$$

By condition (H_4), we obtain $\sigma=0$ which imply that

$$\lim_{t\longrightarrow+\infty}\|z(t)\|=\lim_{t\longrightarrow+\infty}\|x_i(t)-x_i^*(t)\|=0,$$

and consequently the proof of this theorem is completed. □

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