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INVERSE SOURCE RECONSTRUCTION WITH L1 FRACTIONAL METHOD

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ABSTRACT. This paper studies an inverse source identification problem for a diffusion model with a time-fractional derivative under Dirichlet boundary conditions. We establish the differentiability of the unknown source and prove the injectivity of the associated input–output mapping, thereby guaranteeing the uniqueness of the reconstruction. A numerical procedure based on an L1 finite-difference approximation is developed to stably compute the source term. Several numerical experiments demonstrate the accuracy and robustness of the proposed approach. computational efficiency are also discussed.

Keywords. Time-fractional diffusion equation, Caputo fractional derivative, Inverse source problem, Finite difference method, L1 scheme.
*2020 Mathematics Subject Classification.*Primary 35R11, Secondary 35R30, 65M06

1. INTRODUCTION

Fractional partial differential equations have increasingly gained prominence as robust frameworks for modelling complex dynamics across diverse scientific and engineering disciplines. Given their theoretical and practical utility, FPDEs now represent a thriving research domain, with comprehensive foundational theories detailed in seminal works such as [16]. The study of fractional-order systems—encompassing ordinary differential equations, partial differential equations, and integral formulations—has thus sparked widespread academic interest. Since many linear and non-linear fractional models lack closed-form analytical solutions, the development of reliable approximation techniques remains a primary research priority.

Various strategies have been established to address these equations, including differential transformation methods [17], iterative schemes [12, 8], and spectral methods [5, 6]. Recent literature also offers extensive surveys documenting numerical methodologies and their convergence characteristics in the context of FPDEs [21].

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Parallel to these advances, inverse source problems (ISPs) have become critical for applications ranging from geophysics and medical imaging to heat conduction and electromagnetic field reconstruction [7, 13]. By facilitating the identification of internal source terms from indirect, often noisy observations, ISPs offer an essential toolkit for data assimilation, system monitoring, and diagnostic control, particularly when direct measurement is infeasible. However, such problems are frequently ill-posed in the Hadamard sense, where slight data perturbations can lead to significant reconstruction errors, thereby necessitating rigorous regularization and refined numerical strategies [15].

The application of fractional models to inverse problems is particularly compelling due to their capacity to capture memory effects and anomalous diffusion—phenomena that standard integer-order models often fail to characterize adequately [24]. While the non-locality of fractional derivatives introduces significant analytical and numerical complexities, several studies have successfully addressed uniqueness, stability, and source reconstruction for time-fractional diffusion equations [25, 20, 11]. In particular, Li and Yamamoto [18] provided a rigorous identifiability analysis under limited measurement data.

Significant progress has been made in the numerical treatment of these problems. The L1 method, for instance, is highly regarded for its simplicity and first-order accuracy in approximating Caputo derivatives on uniform grids [19]. Furthermore, various researchers have proposed enhanced mesh-based schemes for improved accuracy and stability [1, 10, 2], while others have investigated inverse problems for parabolic equations under diverse conditions [3, 22]. Recent contributions [9, 4] have also addressed inverse and fractional differential problems, confirming the growing interest in this research area .

This paper builds upon the framework established in [26] to investigate a time-fractional parabolic PDE with an inverse source term subject to non-homogeneous boundary conditions. We employ an eigenfunction expansion of the associated Sturm–Liouville problem to derive an analytical representation, using Dirichlet-type output data to analyze the time-dependent inverse coefficient problem. Specifically, we define an input–output mapping $\Phi[\cdot] : \mathcal{K} \to C^1[0,T]$ and demonstrate that the distinguishability of this mapping implies the injectivity of the source function $c(t)$. By adopting a Fourier-based approach, we provide a novel construction for this mapping and develop a finite-difference scheme, following the methodology proposed by [24], to obtain stable numerical approximations.

The remainder of the paper is organized as follows. In the next section 2, we introduce several useful definitions that will be needed in the subsequent analysis. Section 3 presents the formulation of the inverse problem and the uniqueness analysis. Section 4 describes the L1 discretization and reconstruction algorithm, provides numerical experiments, and Section 5 concludes the paper.

2. Basic concepts of fractional calculus

This section provides essential concepts and notations that are crucial for understanding the paper's main ideas. For additional information, see [23, 16].

Proposition 2.1. *The Euler gamma function $\Gamma(z)$ satisfies the following properties:*

$$\Gamma(z+1) = z\,\Gamma(z), \tag{2.1}$$

$$\int_0^1 s^{z-1}(1-s)^{w-1}\,ds = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}, \quad z, w > 0. \tag{2.2}$$

Definition 2.2. The Caputo fractional derivative of order α of the function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is defined as

$$^{C}D_0^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \tag{2.3}$$

where $n-1 < \alpha \le n$ and $n = [\alpha]+1$ with $[\alpha]$ the integer part of α.

Definition 2.3. The two-parameter Mittag–Leffler function is defined by the series

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \quad \alpha, \beta, z \in \mathbb{C},\ \Re(\alpha) > 0. \tag{2.4}$$

In particular, if $\beta = 1$, then $E_{\alpha,1}(z) = E_\alpha(z)$; and if $\alpha = \beta = 1$, then $E_1(z) = e^z$.

3. Formulation and analysis of the inverse source problem

We consider the following inverse source problem for a time-fractional partial differential equation with boundary and initial conditions:

$$^{C}D_{0+,t}^{\alpha} u(x,t) - u_{xx}(x,t) + q(x)\,u(x,t) = c(t)\,f(x), \quad 0 < \alpha \le 1,\ (x,t) \in \Omega_T, \tag{3.1}$$

$$u(x,0) = \varphi(x), \qquad 0 < x < 1, \tag{3.2}$$

$$u(0,t) = \psi_1(t), \qquad 0 < t < T, \tag{3.3}$$

$$u(1,t) = \psi_2(t), \qquad 0 < t < T. \tag{3.4}$$

where $\Omega_T = \{(x,t) \in \mathbb{R}^2 : 0 < x < 1, 0 < t \le T\}$. The boundary functions $\psi_1(t)$ and $\psi_2(t)$ belong to $C[0,T]$, and the functions $q(x)$ and $\varphi(x)$ satisfy the following conditions:

(A_1) $q(x) \in C^1[0,1]$, $0 < c_0 \le q(x) < c_1$,
(A_2) $\varphi(x) \in C^2[0,1]$, $\varphi(0) = \psi_1(0)$, $\varphi(1) = \psi_2(0)$.

Under these assumptions, the problem (3.1) has a unique solution $u(x,t) \in C(\Omega_T) \cap W_t^1(0,T) \cap C_x^2(0,1)$ that satisfies the equation, initial, and boundary conditions. Here $W_t^1(0,T)$ denotes the space of functions $\chi \in C^1(0,T)$ with $\chi' \in L(0,T)$.

We seek to reconstruct the temporal source profile $c(t)$ from overspecified Dirichlet measurements $u(x_0,t)$ at fixed location x_0,

$$E(t) = u(x_0, t; c). \tag{3.5}$$

We introduce the forward operator $\Phi : \mathcal{K} \to C[0, T]$ mapping source functions to boundary observations via

$$\Phi[c] = u(x_0, t; c), \quad c \in \mathcal{K} \subset L^1(0, T), \tag{3.6}$$

so that the inverse problem reduces to determining $c(t)$ from $\Phi[c]$. The mapping Φ is said to have the *injectivity property* if

$$\Phi[c_1] \neq \Phi[c_2] \implies c_1(t) \neq c_2(t), \quad \forall c_1, c_2 \in \mathcal{K}. \tag{3.7}$$

With this strategy, we transform the inverse problem of determining the an unknown $c(t)$ into the problem of establishing the invertibility of the input-output mapping $\Phi[\cdot]$. According, we employ the previously defined input-output mapping to rigorously establish the uniqueness of the source function. We state that the mapping $\Phi[.]$ possesses the injectivity property if $\Phi[c_1] \neq \Phi[c_2]$, Then $c_1(t) \neq c_2(t)$ is implied. This property means that the injectivity of Φ guarantees the existence of the inverse mapping on its range. Our analysis uses overspecified measurements $E(t) = u(x_0, t)$ to reconstruct $c(t)$. Therefore, using the input-output mapping mentioned above, we can verify that the source function is unique. We provide an auxiliary function $\Theta(x, t)$ to be able to develop the solution for the parabolic problem (3.1)by the Fourier technique of separation of variables as follows

$$\Theta(x, t) = u(x, t) - (1 - x)\psi_1(t) - x\psi_2(t), \tag{3.8}$$

by which the problem (3.1) is changed to one with homogeneous boundary conditions. As a result, $\Theta(x, t)$ can be used to describe the initial value problem as follows

$$\begin{aligned} {}^C D_t^\alpha \Theta(x, t) - \Theta_{xx}(x, t) &= -q(x)\Theta(x, t) + xq(x)\psi_2(t) + (1 - x)q(x)\psi_1(t) \\ &\quad - x\,{}^C D_t^\alpha \psi_2(t) - (1 - x)\,{}^C D_t^\alpha \psi_1(t) + c(t)f(x), \end{aligned} \tag{3.9}$$

$$\Theta(x, 0) = \varphi(x) - (1 - x)\psi_1(0) - x\psi_2(0), \tag{3.10}$$

$$\Theta(0, t) = \Theta(1, t) = 0. \tag{3.11}$$

For the initial boundary value problem, the specific solution is as follows

$$\begin{aligned} \Theta(x, t) = &\sum_{n=1}^{+\infty} \langle H(\theta), X_n(\theta)\rangle E_{\alpha,1}(-\lambda_n t^\alpha) X_n(x) \\ &+ \sum_{n=1}^{+\infty} \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n) s^\alpha \langle h(\theta, t - s), X_n(\theta)\rangle X_n(x) ds \\ &+ \sum_{n=1}^{+\infty} \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n) s^\alpha c(t - s) \langle f(\theta), X_n(\theta)\rangle ds X_n(x), \end{aligned} \tag{3.12}$$

where

$$H(x) = \varphi(x) - (1 - x)\psi_1(0) - x\psi_1(0), \tag{3.13}$$

and

$$h(x, t) = -q(x)v(x, t) + xq(x)\psi_2(t) + \psi_1(t) - x\psi_1(t) - xD_t^\alpha \psi_2(t) - (1 - x)D_t^\alpha \psi_1(t). \tag{3.14}$$

Moreover

$$< H(\theta), X_n(\theta) >= \int_0^1 X_n(\theta)H(\theta)d\theta, \tag{3.15}$$

$E_{\alpha,\beta}$ being the generalized Mittag-Leffler function defined by (2.4).
Assume that the following Sturm-Liouville [18] has a solution represented by $X_n(x)$

$$\begin{cases} -X_{xx}(x) = \lambda X(x) + q(x)X(x), & 0 < x < 1, \\ X(0) = X(1) = 0, & 0 < t < T. \end{cases} \tag{3.16}$$

The measured output data's Dirichlet type can be expressed in the manner shown below

$$E(t) = \Theta(x_0, t) + (1 - x_0)\psi_1(t) + x_0\psi_2(t). \tag{3.17}$$

In order to arrange (3.12), let us define

$$z_n(t) =< H(\theta), X_n(\theta) > E_{\alpha,1}(-\lambda_n t^\alpha). \tag{3.18}$$

$$w_n(t) = \int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n)s^\alpha < H(\theta, t-s)q_j, X_n(\theta) > ds. \tag{3.19}$$

$$\gamma_n(t) = \int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n)s^\alpha c(t-s) < f(\theta), X_n(\theta) > ds. \tag{3.20}$$

The following form can be used to express the resolution in terms of $z_n(t), w_n(t)$ and $\gamma_n(t)$, see [14] for further information.

The analytical solution of the problem (3.1)-(3.16) in series form is given by

$$\Theta(x, t) = \sum_{n=1}^{\infty} z_n(t)X_n(x) + \sum_{n=1}^{\infty} w_n(t)X_n(x) + \sum_{n=1}^{\infty} \gamma_n(t)X_n(x). \tag{3.21}$$

Therefore,

$$E(t) = \sum_{n=1}^{\infty} z_n(t)X_n(x_0)+\sum_{n=1}^{\infty} w_n(t)X_n(x_0)+\sum_{n=1}^{\infty} \gamma_n(t)X_n(x_0)dx+(1-x_0)\psi_1(t)+x_0\psi_2(t), \tag{3.22}$$

is acquired. Consequently, $E(t)$ can be found analytically as a series representation (3.22) on the right-hand side describes the input-output mapping $\Psi[c]$.

$$\Psi[c](t) : \sum_{n=1}^{\infty} z_n(t)X_n(x_0)+\sum_{n=1}^{\infty} w_n(t)X_n(x_0)+\sum_{n=1}^{\infty} \gamma_n(t)X_n(x_0)dx+(1-x_0)\psi_1(t)+x_0\psi_2(t). \tag{3.23}$$

The relation between the corresponding outputs $E_j(t) = u(x_0, t, c_j), j = 1, 2$ and the source functions $c_1(t), c_2(t) \in \mathcal{K}$ at x_0 is implied by the following lemma.

Lemma 3.1. [22] *Let $c_1(t), c_2(t) \in \mathcal{K}$ be two admissible source terms and let $\upsilon_j(x,t) = \upsilon(x,t;c_j)$, $j = 1,2$, be the corresponding solutions of the direct problem (3.1). Assume that the eigenfunctions $\{X_n\}_{n\geq 1}$ of the associated Sturm–Liouville problem form a complete orthogonal basis in $L^2(0,1)$ and that the series representations of the solutions converge uniformly for $t \in (0,T]$.*

Let the outputs be defined by

$$E_j(t) = u(x_0,t;c_j), \qquad j = 1,2.$$

Then the difference $\Delta E(t) = E_1(t) - E_2(t)$ satisfies the series identity

$$\Delta E(t) = \sum_{n=1}^{\infty} \Delta w_n(t) X_n(x_0) + \sum_{n=1}^{\infty} \Delta\gamma_n(t) X_n(x_0), \tag{3.24}$$

for all $t \in (0,T]$, where

$$\Delta w_n(t) = w_n^1(t) - w_n^2(t), \qquad \Delta\gamma_n(t) = \gamma_n^1(t) - \gamma_n^2(t).$$

Proof. By using the identity (3.22), the measured output data $E_j(t) = \Theta(x_0,t); j = 1,2$ can be expressed as

$$E_1(t) = \sum_{n=1}^{+\infty} z_n^1(t) X_n(x_0) + \sum_{n=1}^{+\infty} w_n^1(t) X_n(x_0) + \sum_{n=1}^{+\infty} \gamma_n^1(t) X_n(x_0) + (1-x_0)\psi_1(t) + x_0\psi_2(t), \tag{3.25}$$

$$E_2(t) = \sum_{n=1}^{+\infty} z_n^2(t) X_n(x_0) + \sum_{n=1}^{+\infty} w_n^2(t) X_n(x_0) + \sum_{n=1}^{+\infty} \gamma_n^2(t) X_n(x_0) + (1-x_0)\psi_1(t) + x_0\psi_2(t), \tag{3.26}$$

respectively, since $z_n^1 = z_n^2$ from the definition of $z_n(t)$. The difference between (3.25)-(3.26) implies the intended outcome. $\square$

We arrive at the following conclusion as a result of the definitions and the lemma.

Corollary 3.2. *Let $c_1, c_2 \in \mathcal{K}$ satisfy $\langle f(x)(c_1(t) - c_2(t)), X_n(x)\rangle = 0$ for all $t \in [0,T]$ and some n. Then $E_1(t) = E_2(t)$ for all $t \in [0,T]$.*

Theorem 3.3. *Assume that hypotheses (A_1)–(A_2) hold. Assume moreover that there exists at least one eigenfunction X_n such that*

$$\langle f, X_n\rangle \neq 0 \quad \text{and} \quad X_n(x_0) \neq 0.$$

Let $\Phi : \mathcal{K} \to C^1[0,T]$ be the input–output mapping defined by (3.23), where the output is given by

$$E(t) = u(x_0,t).$$

Then the mapping Φ is injective on the admissible set $\mathcal{K}$, i.e.

$$\Phi[c_1] = \Phi[c_2] \;\Rightarrow\; c_1(t) = c_2(t), \qquad \forall c_1, c_2 \in \mathcal{K}. \tag{3.27}$$

Proof. Suppose $\Phi[c_1] = \Phi[c_2]$ for $c_1, c_2 \in \mathcal{K}$. Then $E_1(t) = E_2(t)$, i.e., $u_1(x_0,t) = u_2(x_0,t)$ for all $t \in [0,T]$.

Let $w = u_1 - u_2$. Then $w(x_0,t) = 0$ and w satisfies:

$${}^{C}D_t^{\alpha} w - w_{xx} + w = f(x)(c_1(t) - c_2(t)).$$

Expanding $w(x,t) = \sum_{n=1}^{\infty} w_n(t) X_n(x)$, we have for some n:

$$(c_1(t) - c_2(t))\langle f, X_n\rangle = 0.$$

By assumption, there exists n such that $\langle f, X_n\rangle \neq 0$ and $X_n(x_0) \neq 0$. For this n, we must have $c_1(t) - c_2(t) = 0$ for all t, hence $c_1(t) = c_2(t)$. $\square$

4. Numerical procedure

In this section, we discretize the problem (3.1) using the finite difference method. Let the spatial step $h = \dfrac{1}{M}$ and the time step size $\Delta t = \dfrac{T}{N}$, where M, N are positive integers. The domain $[0,1] \times [0,T]$ is divided into a mesh $M \times N$, with grid points

$$x_i = ih,\ t_j = j\Delta t,\ i = 0, ..., M,\ \ j = 0, ..., N.$$

For the Caputo derivative ${}^C D_t^\alpha u(x_i, t_j)$ with $0 < \alpha < 1$ [?], we use the L1 approximation

$$\begin{aligned}
{}^C D_t^\alpha u(x_i, t_j) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_j} (t_j - s)^{-\alpha} \frac{\partial u}{\partial t}(x_i, s) ds && (4.1)\\
&\simeq \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{j-1} \left[(j-k)^{1-\alpha} - (j-\alpha-1)^{1-\alpha}\right] \left[\frac{u_j^{k+1} - u_j^k}{(\Delta t)^\alpha}\right] && (4.2)\\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{j-1} B_k \left[\frac{u_j^{k+1} - u_j^k}{(\Delta t)^\alpha}\right]; j = 1, ..., N. && (4.3)\\
B_k &= (j-k)^{1-\alpha} - (j-k-1)^{1-\alpha}. && (4.4)
\end{aligned}$$

On the other hand, we have

$$u_{xx}(x_i, t_j) = \frac{1}{h^2}\left[u_{i+1}^j - 2u_i^j + u_{i-1}^j\right]; i = 1, ..., M-1. \tag{4.5}$$

Hence, the finite difference approximation for discretizing problem (3.1) is

$$\begin{aligned}
\frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{j-1} B_k \frac{u_i^{k+1} - u_i^k}{(\Delta t)^\alpha} &= \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} - q_i u_i^j + c^j f_i, && (4.6)\\
u_i^0 &= \varphi_i, \quad i = 0, \ldots, M, && (4.7)\\
u_0^j &= u_M^j = 0, \quad j = 0, \ldots, N. && (4.8)
\end{aligned}$$

Expanding the Caputo sum gives from the system (4.6) we get the vector forms

$$\begin{aligned}\sum_{k=0}^{j-1}\left(u_i^{k+1}-u_i^k\right)B_k &= \left(u_i^1-u_i^0\right)B_0+(u_i^2-u_i^1)B_1+(u_i^3-u_i^2)B_2+...\\ &+\left(u_i^j-u_i^{j-1}\right)B_{j-1}\\ &=-B_0u_i^0+\left(B_0-B_1\right)u_i^1+\left(B_1-B_2\right)u_i^2+...\\ &+\left(B_{j-2}-B_{j-1}\right)u_i^{j-1}+B_{j-1}u_i^j\\ &=-B_0u_i^0-\sum_{k=1}^{j-1}\left(B_k-B_{k-1}\right)u_i^k+B_{j-1}u_i^j.\end{aligned}\tag{4.9}$$

Substituting in(4.6) and taking into consideration $B_{j-1}=1$, let $r=(\Delta t)^{\alpha}\Gamma(2-\alpha)$, we get

$$-B_0u_i^0-\sum_{k=1}^{j-1}\left(B_k-B_{k-1}\right)u_i^k+u_i^j=\frac{r}{h^2}\left[u_{i+1}^j-2u_i^j+u_{i-1}^j\right]-rq_iu_i^j+rc^jf_i.\tag{4.10}$$

Introduce vectors

$$u^j=(u_1^j,\ldots,u_{M-1}^j)^T,\quad q=\operatorname{diag}(q_1,\ldots,q_{M-1}),\quad F=(f_1,\ldots,f_{M-1})^T,$$

and the tridiagonal matrix

$$A=\begin{bmatrix}-2 & 1 & 0 & \ldots & 0\\ 1 & -2 & 1 & \ldots & 0\\ 0 & 1 & -2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 1\\ 0 & \ldots & 0 & 1 & -2\end{bmatrix}_{(M-1)\times(M-1)}$$

a square matrix $(M-1)$, u^j is a vector solution of the order $(M-1)$, we get vector form

$$\frac{r}{h^2}Au^j-u^j-rEu^j=-\sum_{k=1}^{j-2}\left(B_k-B_{k-1}\right)u^k-B_0u^0-rc^jF;j=1,...,N.\tag{4.11}$$

Let $D=\dfrac{r}{h^2}A-I_{M-1}-rE$, where D is a matrix of the order $(M-1)$, and

$$E=\begin{bmatrix}q(x_1) & 0 & \cdots & 0\\ 0 & q(x_2) & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \cdots & \cdots & q(x_{M-1})\end{bmatrix},\tag{4.12}$$

the diagonal matrix of the order $(M-1)$. Hence we get the linear system

$$Du^j=-\sum_{k=1}^{j-2}\left(B_k-B_{k-1}\right)u^k-B_0u^0-rc^j;\;j=1,...,N,\tag{4.13}$$

with $u^0 = (u_1, u_2, ..., u_{M-1})^T$. We develop a reconstruction algorithm for predicting and correcting errors, beginning with the use of output measurements $u(x_0, t) = E(t)$, we obtain

$$c(t) = \frac{D^\alpha E(t) - u_{xx}(x_0, t) + q(x_0)E(t)}{f(x_0)}. \tag{4.14}$$

The finite difference approximation of $c(t)$ is

$$c^j = \frac{E^j - \frac{1}{h^2}\left(u^j_{i+1} - 2u^j_i + u^j_{i-1}\right) + q_i E^j}{f_i}, \tag{4.15}$$

where $E^j = D^\alpha u\left(\frac{1}{2}, t\right)$.

The system of equations (4.9) can be solved by the inversion matrix method.

4.1. **Numerical examples.** To demonstrate the practical implementation of the method, we consider the following examples.

Example 4.1. Consider the following test problem for $\alpha = 0.5,$ and $\alpha = 0.75$

$$^C D^\alpha_t u(x, t) - u_{xx}(x, t) + u(x, t) = c(t)\sin(\pi x); \quad 0 < x < 1, 0 < t < 1. \tag{4.16}$$

$$u(x, 0) = 2\sin(\pi x); \quad 0 < x < 1, \tag{4.17}$$

$$u(0, t) = u(1, t) = 0; \quad 0 < t < 1. \tag{4.18}$$

The exact solutions of this problem for $\alpha = 0.5$ and $\alpha = 0.75$ are respectively

$$\{c(t), u(x, t)\} = \left\{\frac{1}{16\sqrt{\pi}}(15t^2\pi + 32\sqrt{t}) + (\pi^2 + 1)(t^{\frac{5}{2}} + t + 2), (t^{\frac{5}{2}} + t + 2)\sin(\pi x)\right\}. \tag{4.19}$$

$$\{c(t), u(x, t)\} = \left\{2.066t^{1.75} + 1.1033t^{0.25} + \left(\pi^2 + 1\right)\left(t^{\frac{11}{4}} + t + 2\right), (t^{\frac{11}{4}} + t + 2)\sin(\pi x)\right\}. \tag{4.20}$$

The following tables illustrates the differences between the numerical and exact solutions, demonstrating the accuracy of the L1 method, for the step sizes $h = 0.025, \Delta t = 0.025, x = 0.5$.

TABLE 1. Numerical results for $\alpha = 0.5$.

t	Numerical solution	Exact solution	Error
0	2.0000	2.0000	00000
0.0250	2.0257	2.0251	0.0006
0.0500	2.0513	2.0506	0.0007
0.0750	2.0771	2.0765	0.0006
0.1000	2.1040	2.1032	0.0008

Table 2. Numerical results for $\alpha = 0.75$.

t	Numerical solution	Exact solution	Error
0	2.0000	2.0000	00000
0.0250	2.0250	2.0251	0.0001
0.0500	2.0503	2.0505	0.0002
0.0750	2.0758	2.0762	0.0004
0.1000	2.1018	2.1025	0.0007

Additionally, the L_∞ and L_2 norms of the error are given in the following table

Table 3. L_∞ and L_2 norms of error for Example 4.1

α	L_∞error	L_2error
0.5	0.0020	0.0088
0.75	0.0123	0.0279

These figures clearly demonstrate excellent agreement between the numerical and exact solutions for $c\,(t)$ and $u(x, T)$.

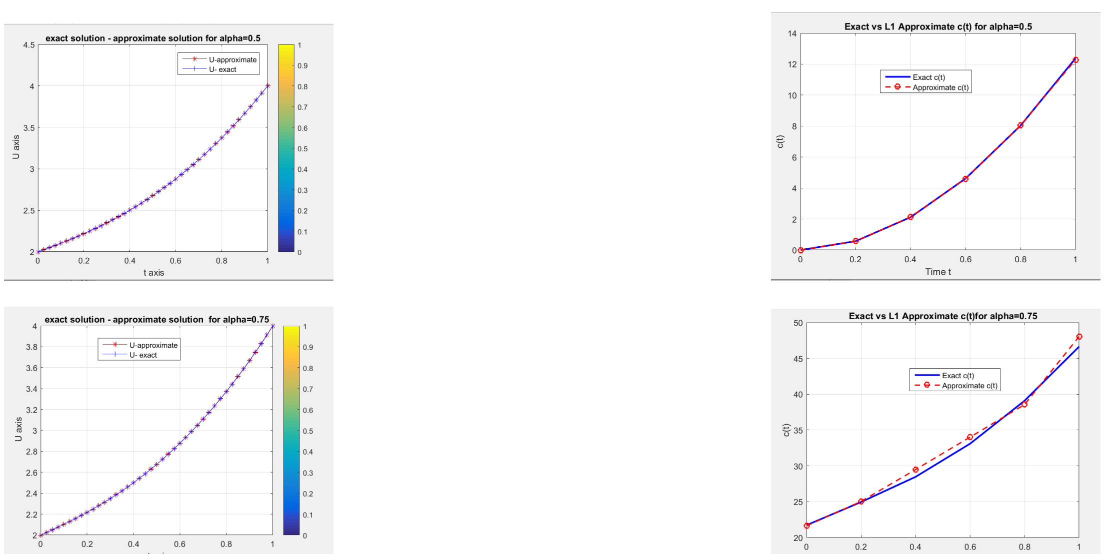

Figure 1. Exact and numerical reconstructions for Example 4.1. Top row: comparison of $u(0.5, t)$ and $c(t)$ for $\alpha = 0.5$. Bottom row: comparison of $u(0.5, t)$ and $c(t)$ for $\alpha = 0.75$.

To further demonstrate the effectiveness of the proposed method, we present the following example.

Example 4.2. Consider the following test problem for $\alpha = 0.5$ and $\alpha = 0.75$

$$\begin{aligned} {}^{C}D_t^{\alpha}u(x,t) - u_{xx}(x,t) + u(x,t) = c(t)\sin(\pi x), &\qquad 0 < x < 1, < t < 1. &(4.21)\\ u(x,0) = 0; &\qquad 0 < x < 1, &(4.22)\\ u(0,t) = u(1,t) = 0; &\qquad 0 < t < 1. &(4.23)\end{aligned}$$

The exact solutions of this problem for $\alpha = 0.5$ and $\alpha = 0.75$ are respectively

$$\{c(t), u(x,t)\} = \left\{\frac{8}{3\sqrt{\pi}}t^{1.5} + (\pi^2+1)t^2, t^2\sin(\pi x)\right\}. \tag{4.24}$$

$$\{c(t), u(x,t)\} = \left\{\frac{2}{\Gamma(2.25)}t^{1.25} + (\pi^2+1)t^2, t^2\sin(\pi x)\right\}. \tag{4.25}$$

The following tables illustrates the differences between the numerical and exact solutions, demonstrating the accuracy of the L1 method, for the step sizes $h = 0.02, \Delta t = 0.01, x = 0.5$.

TABLE 4. Numerical results for $\alpha = 0.5$.

t	Numerical solution	Exact solution	Error
0	2.0000	2.0000	00000
0.0250	2.0257	2.0251	0.0006
0.0500	2.0513	2.0506	0.0007
0.0750	2.0771	2.0765	0.0006
0.1000	2.1040	2.1032	0.0008

TABLE 5. Numerical results for $\alpha = 0.75$

t	Numerical solution	Exact solution	Error
0	2.0000	2.0000	00000
0.0250	2.0250	2.0251	0.0001
0.0500	2.0503	2.0505	0.0002
0.0750	2.0758	2.0762	0.0004
0.1000	2.1018	2.1025	0.0007

Additionally, the L_∞ and L_2 norms of the error are given in the following table

TABLE 6. L_∞ and L_2 norms of error for Example 4.2

α	L_∞error	L_2error
0.5	0.0020	0.0088
0.75	0.0123	0.0279

The excellent match observed in these figures between the numerical and exact solutions for $c(t)$ and $u(x,T)$ validates the effectiveness of the proposed approach.

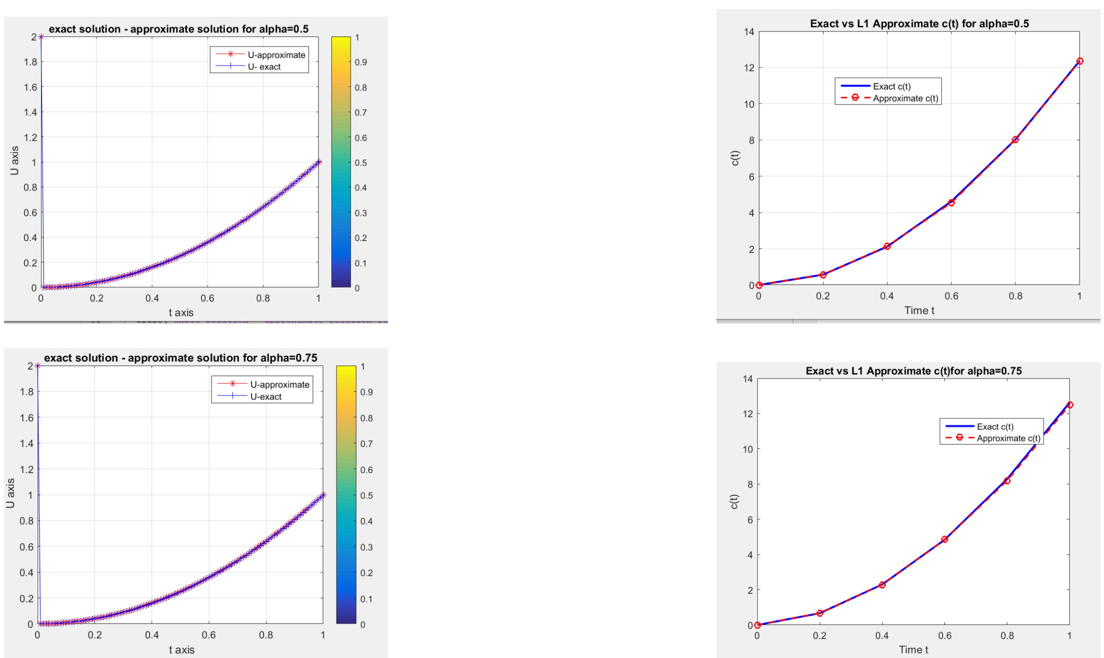

FIGURE 2. Exact and numerical reconstructions for Example 4.2. Top row: comparison of $u(0.5,t)$ and $c(t)$ for $\alpha = 0.5$. Bottom row: comparison of $u(0.5,t)$ and $c(t)$ for $\alpha = 0.75$.

5. CONCLUSION

This paper studied an inverse source problem for a time-fractional diffusion equation from interior measurements. The injectivity of the associated input–output mapping was established, ensuring the uniqueness of the reconstructed source term. A numerical reconstruction based on the L1 finite difference approximation was proposed and validated through test problems with known exact solutions, showing very good agreement between numerical and analytical results. Future work will focus on improving the computational efficiency and extending the method to multidimensional settings.

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