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DETERMINISTIC AND STOCHASTIC BEHAVIORAL COMPARTMENTAL DYNAMICS: STABILITY AND SENSITIVITY ANALYSIS

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Abstract. In this paper, we develop deterministic and stochastic compartmental models to investigate the dynamics of alcohol consumption by incorporating treatment, relapse, and temporary abstinence mechanisms. The population is structured into four interacting compartments: at-risk individuals, active alcohol consumers, individuals undergoing treatment, and abstinent individuals. In the deterministic case, an appropriate Lyapunov function is constructed to derive sufficient conditions for the global asymptotic stability of the equilibria. To capture random environmental and behavioral influences, the model is extended to a stochastic formulation through multiplicative perturbations. Numerical simulations are presented to illustrate the theoretical results.

1. Introduction

Alcohol consumption represents a complex behavioral system influenced by social interactions, individual decision-making, and intervention mechanisms. Understanding its long-term dynamics requires modeling approaches capable of capturing both deterministic structures and stochastic variability [2, 20]. In this context, mathematical modeling and numerical simulation have become powerful tools for understanding the mechanisms underlying alcohol consumption dynamics and for evaluating prevention and treatment strategies. Similar mathematical approaches have been successfully employed in the study of engineering, biological, and epidemiological systems, as well as in the numerical analysis of thermal storage models and nonlinear dynamical systems [3, 17]. For more details, we refer the reader to [21].

Compartmental models, inspired by population dynamics [18, 19], are widely used to describe alcohol consumption behavior. Individuals are divided into categories such as initiating consumption, active consumption, treatment, and abstinence. These models are typically formulated as systems of differential equations, allowing the quantitative analysis of transitions between compartments and the

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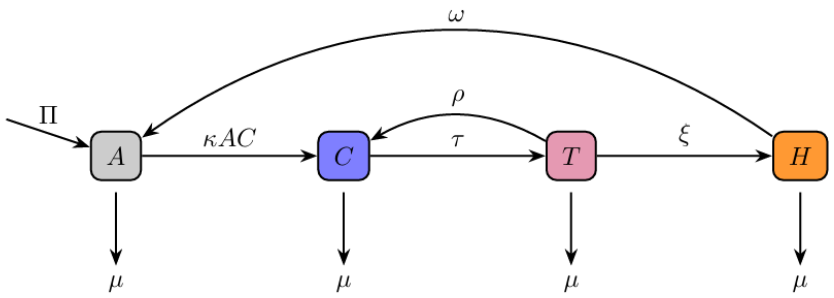


FIGURE 1. Structure of the alcohol consumption model.

assessment of intervention strategies. Moreover, delayed differential equations, reaction–diffusion systems, and semilinear evolution equations have proved effective in describing memory effects and complex dynamical behaviors arising in biological applications [3, 4, 5, 7, 8, 14, 23]. However, these models often assume homogeneous populations and purely deterministic dynamics, which may not accurately reflect the intrinsic randomness observed in real-world alcohol consumption processes [2, 13, 15, 22].

In reality, alcohol consumption is influenced by several random factors, such as societal influence, individual behavior, adherence to treatment, and environmental factors. These factors can significantly impact outcomes, such as continued abstinence or relapse. Therefore, stochastic models based on stochastic differential equations have been developed to better represent these fluctuations and provide a more realistic picture [9, 16].

In addition, it is important to examine how different parameters within the model influence the outcomes. Sensitivity analysis methods, such as partial correlation coefficient (PRCC), are used to identify the most influential factors, such as the prevalence of alcohol consumption or the effectiveness of treatment programs. These studies help refine prevention and intervention strategies. [20].

From a structural perspective, the present work extends classical behavioral compartmental models of alcohol consumption by incorporating the following key features:

(i) A treatment compartment T, representing individuals undergoing rehabilitation or clinical intervention, from which individuals may either achieve sustained abstinence or relapse into active alcohol consumption;
(ii) Temporary abstinence, allowing individuals in the abstinent class H to return to the at-risk class A due to social or behavioral relapse mechanisms;
(iii) Natural demographic processes, including recruitment into the at-risk population and a uniform natural exit rate across all compartments.

In this study, we adopt a three-fold methodological approach:

- A deterministic analysis of the alcohol consumption model, including the stability analysis of the alcohol-free and alcohol-present equilibria using Lyapunov techniques;

- A stochastic formulation of the model with multiplicative noise to investigate the impact of random behavioral and environmental fluctuations on the long-term dynamics;
- A PRCC-based sensitivity analysis to quantify the influence of model parameters on key behavioral outcomes obtained from stochastic simulations.

The alcohol consumption dynamics are described by the following system of differential equations, illustrated schematically in Fig. 1:

$$\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}t} &= \Pi - \mu A - \kappa AC + \omega H, \\
\frac{\mathrm{d}C}{\mathrm{d}t} &= \kappa AC - (\mu + \tau)C + \rho T, \\
\frac{\mathrm{d}T}{\mathrm{d}t} &= \tau C - (\mu + \rho + \xi)T, \\
\frac{\mathrm{d}H}{\mathrm{d}t} &= \xi T - (\mu + \omega)H,
\end{aligned} \tag{1.1}$$

with initial conditions

$$A(0) = a_0 > 0, \quad C(0) = c_0 > 0, \quad T(0) = t_0 > 0, \quad H(0) = h_0 > 0.$$

All model variables and parameters are summarized in Table 1.

Variable/Parameter	Description
$A(t)$	Individuals at risk of alcohol consumption.
$C(t)$	Active alcohol consumers.
$T(t)$	Individuals undergoing treatment or rehabilitation.
$H(t)$	Abstinent individuals (may relapse over time).
Π	Recruitment rate into the at-risk population.
μ	Natural exit rate (applies uniformly to all compartments).
κ	Initiation or social influence rate of alcohol consumption.
τ	Treatment enrollment rate for active consumers.
ρ	Relapse or treatment failure rate.
ξ	Successful treatment (recovery) rate.
ω	Loss of abstinence rate (return to at-risk class).

TABLE 1. Description of parameters and variables of the alcohol consumption model (1.1).

The total population is assumed to be initially positive, and the model ensures biological and behavioral feasibility. Through both analytical and numerical investigations, we analyze the combined effects of treatment, relapse, temporary abstinence, and stochastic perturbations on the persistence or reduction of alcohol consumption, while identifying the most influential parameters using PRCC sensitivity analysis.

The deterministic system (1.1) admits two equilibrium points: the alcohol-free equilibrium

$$E_0 = \left(\frac{\Pi}{\mu}, 0, 0, 0\right),$$

and the alcohol-present equilibrium

$$E^* = (A^*, C^*, T^*, H^*),$$

where all components are strictly positive.
Setting the right-hand sides of system 1.1 equal to zero yields

$$\begin{cases} \Pi - \mu A^* - \kappa A^* C^* + \omega H^* = 0, \\ \kappa A^* C^* - (\mu + \tau) C^* + \rho T^* = 0, \\ \tau C^* - (\mu + \rho + \xi) T^* = 0, \\ \xi T^* - (\mu + \omega) H^* = 0. \end{cases} \tag{1.2}$$

From the last two equations of (1.2), we obtain

$$T^* = \frac{\tau}{\mu + \rho + \xi} C^*, \qquad H^* = \frac{\xi \tau}{(\mu + \omega)(\mu + \rho + \xi)} C^*.$$

Substituting the expression of T^* into the second equation of (1.2) and dividing by $C^* > 0$, we deduce a critical value of the at-risk population,

$$A^* = \frac{(\mu + \tau)(\mu + \rho + \xi) - \rho\tau}{\kappa(\mu + \rho + \xi)}. \tag{1.3}$$

At this stage, the alcohol-present equilibrium E^* is completely determined. We now use the critical quantity A^* to define the threshold parameter.
At the alcohol-free equilibrium E_0, the size of the at-risk population is

$$A_0 = \frac{\Pi}{\mu}.$$

The basic reproduction number is then naturally defined as

$$\mathcal{R}_c = \frac{A_0}{A^*}. \tag{1.4}$$

Using (1.3), this yields

$$\mathcal{R}_c = \frac{\kappa \Pi (\mu + \rho + \xi)}{\mu\big[\mu(\mu + \rho + \xi) + \tau(\mu + \xi)\big]}. \tag{1.5}$$

Then

$$C_* = K\, A_0 \left(1 - \frac{1}{\mathcal{R}_c}\right) = K \frac{\Pi}{\mu} \left(1 - \frac{1}{\mathcal{R}_c}\right),$$

With

$$K = \left(\kappa A_* - \frac{\omega\, \xi\, \tau}{(\mu + \omega)(\mu + \rho + \xi)}\right)^{-1}.$$

Therefore, the alcohol-present equilibrium exists if and only if $\mathcal{R}_c > 1$.

To facilitate the stability analysis around the alcohol-present equilibrium, we introduce the change of variables

$$x_1 = A - A_*, \quad x_2 = C - C_*, \quad x_3 = T - T_*, \quad x_4 = H - H_*,$$

which centers the system at E^* and simplifies the analytical derivations presented in Section 3.

The remainder of this paper is organized as follows. Section 2 is devoted to the deterministic analysis of the alcohol consumption model, including global stability results in the absence of random perturbations. Section 3 introduces the stochastic version of the system and examines the impact of randomness on alcohol consumption dynamics. Section 4 presents numerical simulations that illustrate and validate the theoretical findings. Finally, concluding remarks are provided in the last section.

2. Asymptotic stability analysis without perturbations

In this section, we investigate the long-term behavior of the alcohol consumption model in the absence of random perturbations.

Theorem 2.1. *If* $\mathcal{R}_c > 1$ *and*

$$4C_*^2\kappa^2(\mu+\rho) > \rho^2\tau,$$

then the alcohol-present equilibrium E^* *of system* (1.1) *is globally asymptotically stable in* $\mathbb{R}^4_+$.

Proof. Let us introduce the Lyapunov function $\mathcal{V}$, defined as:

$$\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2, \tag{2.1}$$

where

$$\mathcal{V}_1 = \left(x_2 - C_* \log\left(\frac{C}{C_*}\right)\right) + \frac{\kappa}{2\tau}(x_3+x_4)^2,$$

and

$$\mathcal{V}_2 = \frac{C_*^3\kappa^3(\mu+\rho)}{\mu\left(4C_*^2\kappa^2(\mu+\rho)-\rho^2\tau\right)}(x_1+x_2+x_3+x_4)^2 + \frac{(2\mu+\rho+\omega)\kappa}{2\xi\tau}x_4^2.$$

The time derivative of $\mathcal{V}_1$ along the trajectories of system (1.1) is given by

$$\dot{\mathcal{V}}_1 = \left(1-\frac{C_*}{C}\right)\frac{\mathrm{d}C}{\mathrm{d}t} + \frac{\kappa}{\tau}(x_3+x_4)\left(\frac{\mathrm{d}T}{\mathrm{d}t}+\frac{\mathrm{d}H}{\mathrm{d}t}\right).$$

Using the system equations and the equilibrium relation

$$\kappa A_* C_* - (\mu+\tau)C_* + \rho T_* = 0,$$

we obtain, after straightforward algebraic manipulations,

$$\begin{aligned}\dot{\mathcal{V}}_1 =& \kappa x_2(x_1+x_2+x_3+x_4) - \kappa x_2^2 + \rho x_3\left(\frac{x_2}{C_*}\right) - \rho T\frac{x_2^2}{CC_*} \\ &- \frac{(\mu+\rho)\kappa}{\tau}x_3^2 - \frac{(2\mu+\rho+\omega)\kappa}{\tau}x_3x_4 - \frac{(\mu+\omega)\kappa}{\tau}x_4^2.\end{aligned} \tag{2.2}$$

Similarly, the derivative of $\mathcal{V}_2$ along the trajectories of system (1.1) satisfies

$$\begin{aligned}\dot{\mathcal{V}}_2 = &- \frac{2C_*^3\kappa^3(\mu+\rho)}{C_*\left(4C_*^2\kappa^2(\mu+\rho)-\rho^2\tau\right)}(x_1+x_2+x_3+x_4)^2 \\ &+ \frac{(2\mu+\rho+\omega)\kappa}{\tau}\,x_4x_3 - (\mu+\omega)\frac{(2\mu+\rho+\omega)\kappa}{\xi\tau}\,x_4^2.\end{aligned} \tag{2.3}$$

Combining inequalities (2.2) and (2.3), the time derivative of the Lyapunov function $\mathcal{V}$ can be written as a sum of negative definite quadratic forms, yielding

$$\dot{\mathcal{V}} \leq 0.$$

Moreover,

$$\dot{\mathcal{V}} = 0 \quad \Longleftrightarrow \quad x_1 = x_2 = x_3 = x_4 = 0.$$

Hence, the largest invariant set contained in $\{\dot{\mathcal{V}} = 0\}$ reduces to the singleton $\{E^*\}$. By LaSalle's invariance principle [11], the alcohol-present equilibrium E^* is globally asymptotically stable. □

3. Stochastic extension: Integration of random perturbations

To account for uncertainties inherent in alcohol consumption dynamics, such as variability in social influence and environmental factors, the deterministic model is extended to a stochastic model by incorporating multiplicative random perturbations. These stochastic effects are modeled using stochastic differential equations driven by independent Brownian motions, leading to the following system:

$$\begin{aligned}\mathrm{d}A &= [\Pi - \mu A - \kappa AC + \omega H]\,\mathrm{d}t + \theta_A\,A\,x_1\,\mathrm{d}B_1,\\ \mathrm{d}C &= [\kappa AC - (\mu+\tau)C + \rho T]\,\mathrm{d}t + \theta_C\,C\,x_2\,\mathrm{d}B_2,\\ \mathrm{d}T &= [\tau C - (\mu+\rho+\xi)T]\,\mathrm{d}t + \theta_T\,T\,x_3\,\mathrm{d}B_3,\\ \mathrm{d}H &= [\xi T - (\mu+\omega)H]\,\mathrm{d}t + \theta_H\,H\,x_4\,\mathrm{d}B_4,\end{aligned} \tag{3.1}$$

where

- $\theta_A, \theta_C, \theta_T, \theta_H \geq 0$ denote the noise intensities associated with random behavioral and environmental fluctuations affecting each compartment;
- B_1, B_2, B_3, B_4 are mutually independent one-dimensional standard Brownian motions defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, satisfying the usual conditions.

3.1. Definitions and theorems.

Consider the following n-dimensional stochastic differential system:

$$\mathrm{d}Y(t) = g(t, Y(t))\,\mathrm{d}t + f(t, Y(t))\,\mathrm{d}B(t), \tag{3.2}$$

where $g(t,y) : [t_0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ and $f(t,y) : [t_0,\infty)\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally Lipschitz continuous with respect to y, and $B(t) = (B_1(t),\ldots,B_4(t))^T$ is a 4-dimensional standard Brownian motion.

Let $\mathcal{L}$ denote the infinitesimal generator associated with system (3.2), defined for any function $V(t,y) \in C^{1,2}(\mathbb{R}\times\mathbb{R}^n)$ by

$$\mathcal{L}V = \frac{\partial V}{\partial t} + g^T\frac{\partial V}{\partial y} + \frac{1}{2}\mathrm{tr}\left(f^T\frac{\partial^2 V}{\partial y^2}f\right).$$

In the sequel, we make use of the classical Lyapunov stability theorem and the linearization theorem for stochastic differential equations due to Hasminskii [6]. For additional definitions and results on stochastic stability, we refer the reader to [1].

3.2. **Results.**

Theorem 3.1. *Let* $(A(0), C(0), T(0), H(0)) \in \mathbb{R}^4_+$. *Then system* (3.1) *admits a unique global solution* $(A(t), C(t), T(t), H(t))$ *for* $t \geq 0$, *and this solution remains in* $\mathbb{R}^4_+$ *almost surely.*

Proof. The coefficients of system (3.1) are locally Lipschitz continuous and satisfy the linear growth condition. Hence, for any initial value $(A(0), C(0), T(0), H(0)) \in \mathbb{R}^4_+$, there exists a unique local solution on $[0, \tau_e)$, where τ_e is the explosion time.

To prove that the solution is global, define for each integer $k \geq 1$ the stopping time

$$\tau_k = \inf\left\{t \geq 0 : A(t) \notin \left[\frac{1}{k}, k\right] \text{ or } C(t) \notin \left[\frac{1}{k}, k\right] \text{ or } T(t) \notin \left[\frac{1}{k}, k\right] \text{ or } H(t) \notin \left[\frac{1}{k}, k\right]\right\}.$$

Let $\tau_\infty = \lim_{k\to\infty} \tau_k$. Clearly, $\tau_\infty \leq \tau_e$. We will show that $\tau_\infty = +\infty$ almost surely.

Let us consider the Lyapunov function

$$\begin{aligned}\mathcal{V} = A + A^* - a - a\log\left(\frac{A + A^*}{a}\right) + C + C^* - 1 - \log\left(C + C^*\right)\\ + T + T^* - 1 - \log\left(T + T^*\right) + H + H^* - 1 - \log\left(H + H^*\right),\end{aligned}$$

where

$$a = \frac{\mu}{\kappa}.$$

Applying Itô's formula to $\mathcal{V}$ yields

$$\begin{aligned}\mathcal{L}\mathcal{V}(t) = &\left(1 - \frac{a}{A + A^*}\right)(\Pi - \mu A - \kappa AC + \omega H)\\ &+ \left(1 - \frac{1}{C + C^*}\right)(\kappa AC - (\mu + \tau)C + \rho T)\\ &+ \left(1 - \frac{1}{T + T^*}\right)(\tau C - (\mu + \rho + \xi)T)\\ &+ \left(1 - \frac{1}{H + H^*}\right)(\xi T - (\mu + \omega)H)\\ &+ \frac{1}{2}\frac{a}{(A + A^*)^2}\theta_A^2 (A - A^*)^2 + \frac{1}{2}\frac{1}{(C + C^*)^2}\theta_C^2 (C - C^*)^2\\ &+ \frac{1}{2}\frac{1}{(T + T^*)^2}\theta_T^2 (T - T^*)^2 + \frac{1}{2}\frac{1}{(H + H^*)^2}\theta_H^2 (H - H^*)^2.\end{aligned}$$

Using standard estimates, we obtain

$$\mathcal{L}\mathcal{V}(t) \leq K,$$

where

$$K = \Pi + \frac{\mu^2}{\kappa} + 3\mu + \tau + \rho + \xi + \omega + \frac{1}{2}a\theta_A^2 + \frac{1}{2}\theta_C^2 + \frac{1}{2}\theta_T^2 + \frac{1}{2}\theta_H^2.$$

Integrating from 0 to $\tau_k \wedge T$ and taking expectations yield

$$\mathbb{E}\left[\mathcal{V}(A(\tau_k \wedge T), C(\tau_k \wedge T), T(\tau_k \wedge T), H(\tau_k \wedge T))\right] \leq \mathcal{V}(A(0), C(0), T(0), H(0)) + KT.$$

On the other hand, the function $\mathcal{V}$ tends to infinity whenever any component approaches 0 or $+\infty$. Hence, for any $T > 0$,

$$\lim_{k\to\infty} \mathbb{P}(\tau_k \leq T) = 0.$$

Thus, $\tau_\infty = +\infty$ almost surely, which implies that $\tau_e = +\infty$. Therefore, the solution is global and remains in $\mathbb{R}^4_+$ almost surely.

We now study the stochastic stability of the alcohol-present equilibrium E^*. To do this, we start by analyzing the linearized system of (3.1) at E^*, defined by:

$$\begin{aligned}
\mathrm{d}x_1 &= \left[-\left(\mu + \kappa C_*\right)x_1 - \kappa A_* x_2 + \omega x_4\right]\mathrm{d}t + \theta_A A_* x_1\,\mathrm{d}B_1,\\
\mathrm{d}x_2 &= \left[-\left(\mu + \tau - \kappa A_*\right)x_2 + \kappa C_* x_1 + \rho x_3\right]\mathrm{d}t + \theta_C C_* x_2\,\mathrm{d}B_2,\\
\mathrm{d}x_3 &= \left[-(\mu + \rho + \xi)x_3 + \tau x_2\right]\mathrm{d}t + \theta_T T_* x_3\,\mathrm{d}B_3,\\
\mathrm{d}x_4 &= \left[\xi x_3 - (\mu + \omega)x_4\right]\mathrm{d}t + \theta_H H_* x_4\,\mathrm{d}B_4.
\end{aligned} \tag{3.3}$$

Theorem 3.2. *Suppose that $\mathcal{R}_c > 1$ and that the noise intensities $\theta_A, \theta_C, \theta_T, \theta_H$ are sufficiently small, such that the following conditions are satisfied:*

$$\frac{1}{2}\theta_A^2 < \mu, \tag{3.4}$$

$$\frac{1}{2}\theta_C^2 < \frac{\mu + \left(\frac{2\mu+\xi}{\rho}\right)(\mu + \tau)}{1 + \frac{2\mu+\tau+\xi}{\rho} + \frac{2\mu + \frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa C_*}}, \tag{3.5}$$

$$\frac{1}{2}\theta_T^2 < \frac{\mu + \xi}{1 + \frac{\rho\left(2\mu + \frac{2\mu+\xi}{\rho}(2\mu+\tau)\right)}{\tau\kappa C_*}}, \tag{3.6}$$

$$\frac{1}{2}\theta_H^2 < \mu + \omega, \tag{3.7}$$

then the equilibrium E^* is asymptotically stable in probability.

Proof. We begin by defining the Lyapunov function $\widetilde{V}$ used throughout this proof:

$$\widetilde{V} = m_1\left[\widetilde{V}_1 + \frac{(2\mu+\xi)}{\rho}\widetilde{V}_2 + m_2\widetilde{V}_3 + \frac{2\mu + \frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa C_*}\widetilde{V}_4 + \frac{\rho\left(2\mu + \frac{2\mu+\xi}{\rho}(2\mu+\tau)\right)}{\tau\kappa C_*}\widetilde{V}_5\right] + \widetilde{V}_6,$$

where

$$\widetilde{V}_1 = \frac{1}{2}(x_1 + x_2 + x_3)^2, \quad \widetilde{V}_2 = \frac{1}{2}(x_1 + x_2)^2, \quad \widetilde{V}_3 = \frac{1}{2}x_1^2,$$

$$\widetilde{V}_4 = \frac{1}{2}x_2^2, \quad \widetilde{V}_5 = \frac{1}{2}x_3^2, \quad \widetilde{V}_6 = \frac{1}{2}x_4^2.$$

Moreover, using conditions (3.4)–(3.7) of Theorem 3.2, we have

$$m_1 = 2\max\left\{\frac{3}{4m_3M_1}\left(1+m_2+\frac{2\mu+\xi}{\rho}\right)^2, \frac{3}{4m_3M_2}\left(1+\frac{2\mu+\xi}{\rho}\right)^2, \frac{1}{4m_3M_3}(\xi^2+3)\right\} > 0,$$

$$m_2 = 2m_4\frac{\mu+\frac{2\mu+\xi}{\rho}(\mu+\tau+\xi)-\left(1+\frac{2\mu+\xi}{\rho}+\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa C_*}\right)\frac{1}{2}\theta_C^2}{\kappa A_*} > 0,$$

where

$$m_3 = \frac{\mu+\omega-\frac{1}{2}\theta_H^2}{2(1+\omega^2)} > 0,$$

$$m_4 = \frac{\mu+\kappa C_*-\frac{1}{2}\theta_A^2}{2\kappa A_*} > 0,$$

$$M_1 = \left(1+\frac{2\mu+\xi}{\rho}\right)\mu-\left(1+\frac{2\mu+\xi}{\rho}\right)\frac{1}{2}\theta_A^2+\left(\mu+\kappa C_*-\frac{1}{2}\theta_A^2-\kappa A_* m_4\right)m_2 > 0,$$

$$M_2 = \mu+\frac{2\mu+\xi}{\rho}(\mu+\tau+\xi)-\left(1+\frac{2\mu+\xi}{\rho}+\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa C_*}\right)\frac{1}{2}\theta_C^2-\frac{\kappa A_* m_2}{4m_4} > 0,$$

$$M_3 = \mu+\xi-\left(1+\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa C_*}\frac{\rho}{\tau}\right)\frac{1}{2}\theta_T^2 > 0.$$

Applying Itô's formula to each component of the Lyapunov function and combining the resulting expressions, we obtain

$$\begin{aligned}\mathcal{L}\left(\widetilde{V}_1+\frac{2\mu+\xi}{\rho}\widetilde{V}_2\right) = &-\left(1+\frac{2\mu+\xi}{\rho}\right)\left(\mu-\frac{1}{2}\theta_A^2\right)x_1^2-\left(\mu-\frac{1}{2}\theta_C^2\right)x_2^2\\ &-\frac{2\mu+\xi}{\rho}\left(\mu+\tau+\xi-\frac{1}{2}\theta_C^2\right)x_2^2-\left(\mu+\xi-\frac{1}{2}\theta_T^2\right)x_3^2\\ &+\omega\left[\left(1+\frac{2\mu+\xi}{\rho}\right)x_1+\left(1+\frac{2\mu+\xi}{\rho}\right)x_2+x_3\right]x_4\\ &-\left(2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)\right)x_1x_2.\end{aligned} \tag{3.8}$$

We have

$$\begin{aligned}
&\mathcal{L}\left(\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa C_*}\widetilde{V}_4+\frac{\rho\left(2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)\right)}{\tau\kappa C_*}\widetilde{V}_5\right)\\
=&\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{C_*}A_*-\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa}\rho T_*\left(\frac{x_2}{C_*}\right)^2\\
&+2\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa}\rho T_*\frac{x_2}{C_*}\frac{x_3}{T_*}-\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa}\rho T_*\left(\frac{x_3}{T_*}\right)^2\\
&+\left(2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)\right)x_1x_2+\frac{1}{2}\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa C_*}\theta_C^2x_2^2\\
&+\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa C_*}\frac{\rho}{2\tau}\theta_T^2x_3^2
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathcal{L}\left(\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa C_*}\widetilde{V}_4+\frac{\rho\left(2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)\right)}{\tau\kappa C_*}\widetilde{V}_5\right)\\
=&\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{C_*}A_*-\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa}\rho T_*\left(\frac{x_2}{C_*}-\frac{x_3}{T_*}\right)^2\\
&+\left(2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)\right)x_1x_2+\frac{1}{2}\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa E^*}\theta_C^2x_2^2\\
&+\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa C_*}\frac{\rho}{2\tau}\theta_T^2x_3^2.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{L}\left(\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa C_*}\widetilde{V}_4+\frac{\rho\left(2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)\right)}{\tau\kappa C_*}\widetilde{V}_5\right)\leq&\left(2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)\right)x_1x_2\\
&+\frac{1}{2}\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa E^*}\theta_C^2x_2^2\\
&+\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa C_*}\frac{\rho}{2\tau}\theta_T^2x_3^2.
\end{aligned}\tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\begin{aligned}
&\mathcal{L}\left(\widetilde{V}_1+\frac{2\mu+\xi}{\rho}\widetilde{V}_2+\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa C_*}\widetilde{V}_4+\frac{\rho\left(2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)\right)}{\tau\kappa C_*}\widetilde{V}_5+m_2\widetilde{V}_3\right)\\
&=-\left[\left(1+\frac{2\mu+\xi}{\rho}\right)\mu-\left(1+\frac{2\mu+\xi}{\rho}\right)\frac{1}{2}\theta_A^2\right]x_1^2\\
&\quad-\left[\mu+\frac{2\mu+\xi}{\rho}(\mu+\tau+\xi)-\left(1+\frac{2\mu+\xi}{\rho}+\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa C_*}\right)\frac{1}{2}\theta_C^2\right]x_2^2\\
&\quad-\left[\mu+\xi-\left(1+\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa C_*}\frac{\rho}{\tau}\right)\frac{1}{2}\theta_T^2\right]x_3^2\\
&\quad-m_2\left(\mu+\kappa C_*-\frac{1}{2}\theta_A^2\right)x_1^2-\kappa m_2A_*x_1x_2\\
&\quad+\omega\left[\left(1+m_2+\frac{2\mu+\xi}{\rho}\right)x_1+\left(1+\frac{2\mu+\xi}{\rho}\right)x_2+x_3\right]x_4.
\end{aligned}$$

Using Young's inequality, we have

$$|x_1x_2|\le m_4x_1^2+\frac{1}{4m_4}x_2^2.$$

We obtain

$$\begin{aligned}
&\mathcal{L}\left(\widetilde{V}_1+\frac{2\mu+\xi}{\rho}\widetilde{V}_2+\frac{2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)}{\kappa C_*}\widetilde{V}_4-\frac{\rho\left(2\mu+\frac{2\mu+\xi}{\rho}(2\mu+\tau)\right)}{\tau\kappa C_*}\widetilde{V}_5+m_2\widetilde{V}_3\right)\\
&\le -M_1x_1^2-M_2x_2^2-M_3x_3^2+\omega\left[\left(1+m_2+\frac{2\mu+\xi}{\rho}\right)x_1+\left(1+\frac{2\mu+\xi}{\rho}\right)x_2+x_3\right]x_4.
\end{aligned}\tag{3.10}$$

By the classical Lyapunov stability theorem and the linearization theorem for stochastic differential equations due to Hasminskii [6], the trivial solution of the linearized system (3.3) is asymptotically stable in probability. Consequently, the equilibrium E^* of system (3.1) is asymptotically stable in probability. □

4. Numerical results

In this section, we present numerical simulations carried out using MATLAB to illustrate and compare the dynamical behavior of the deterministic and stochastic versions of the alcohol consumption model. The simulations are used to validate the theoretical stability results established in the previous sections and to highlight the qualitative differences between deterministic trajectories and those influenced by random behavioral and environmental fluctuations.

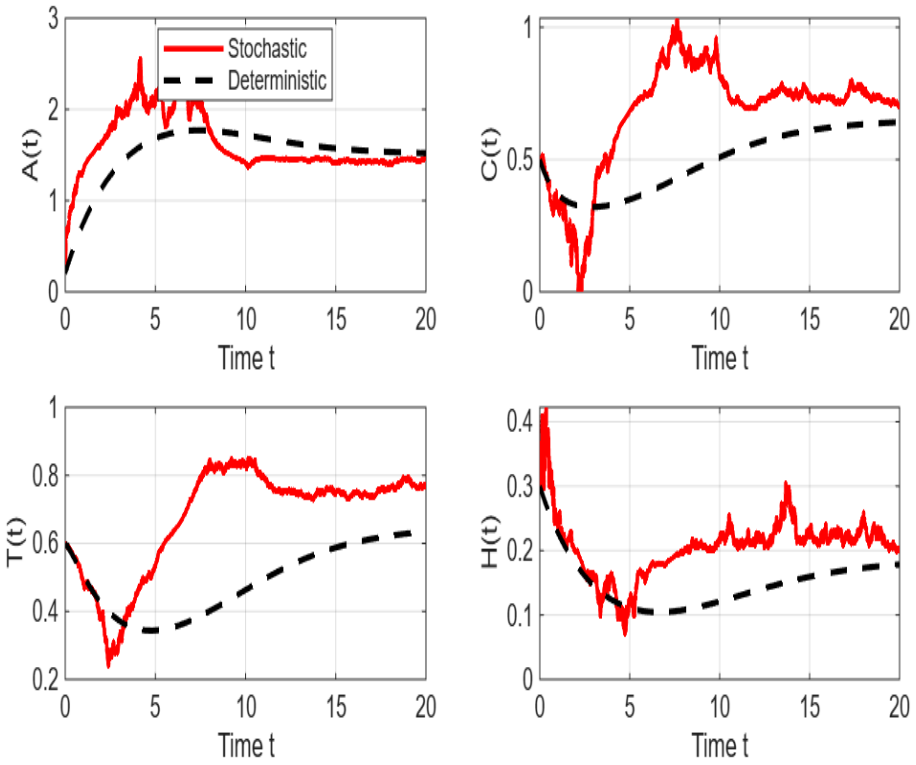


FIGURE 2. Comparison between deterministic and stochastic alcohol consumption dynamics.

We consider the initial conditions $(a_0, c_0, t_0, h_0) = (0.2, 0.5, 0.6, 0.3)$. The parameter values are chosen as

$$\Pi = 0.6,\ \mu = 0.2,\ \kappa = 0.4,\ \omega = 0.5,\ \tau = 0.7,\ \rho = 0.3,\ \xi = 0.2,$$

$$\theta_A = 0.5,\ \theta_C = 0.4,\ \theta_T = 0.2,\ \theta_H = 0.9.$$

For this set of parameters, the system converges to the alcohol-present equilibrium

$$E^* = (A_*, C_*, T_*, H_*) = (1.520,\ 0.641,\ 0.634,\ 0.178).$$

The associated threshold parameter is $\mathcal{R}_c = 2$, indicating the persistence of alcohol consumption within the population. Moreover, the condition $4C_*^2\kappa^2(\mu+\rho) > \rho^2\tau$ is satisfied. According to Theorems 2.1 and 3.2, these results confirm that the alcohol-present equilibrium is globally asymptotically stable in the deterministic framework and asymptotically stable in probability in the stochastic setting. Fig. 2 illustrates the temporal evolution of the model compartments, confirming the theoretical stability results.

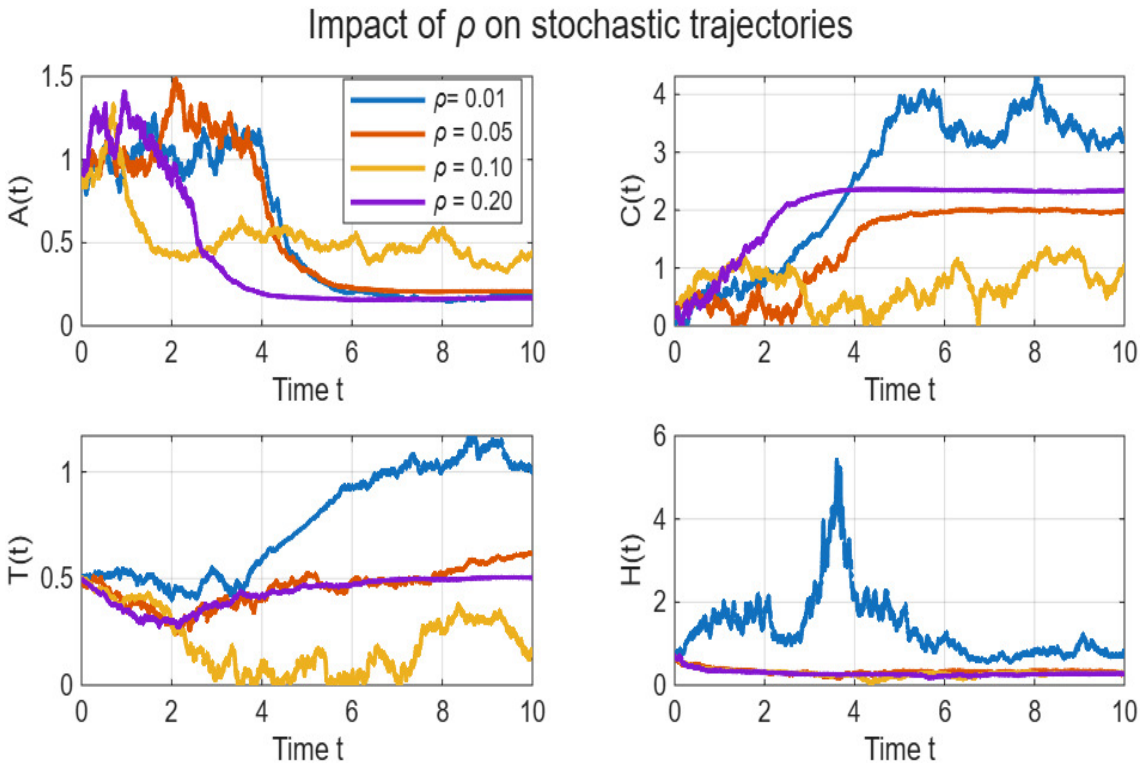


FIGURE 3. Impact of treatment relapse rate ρ on the stochastic dynamics of system (3.1).

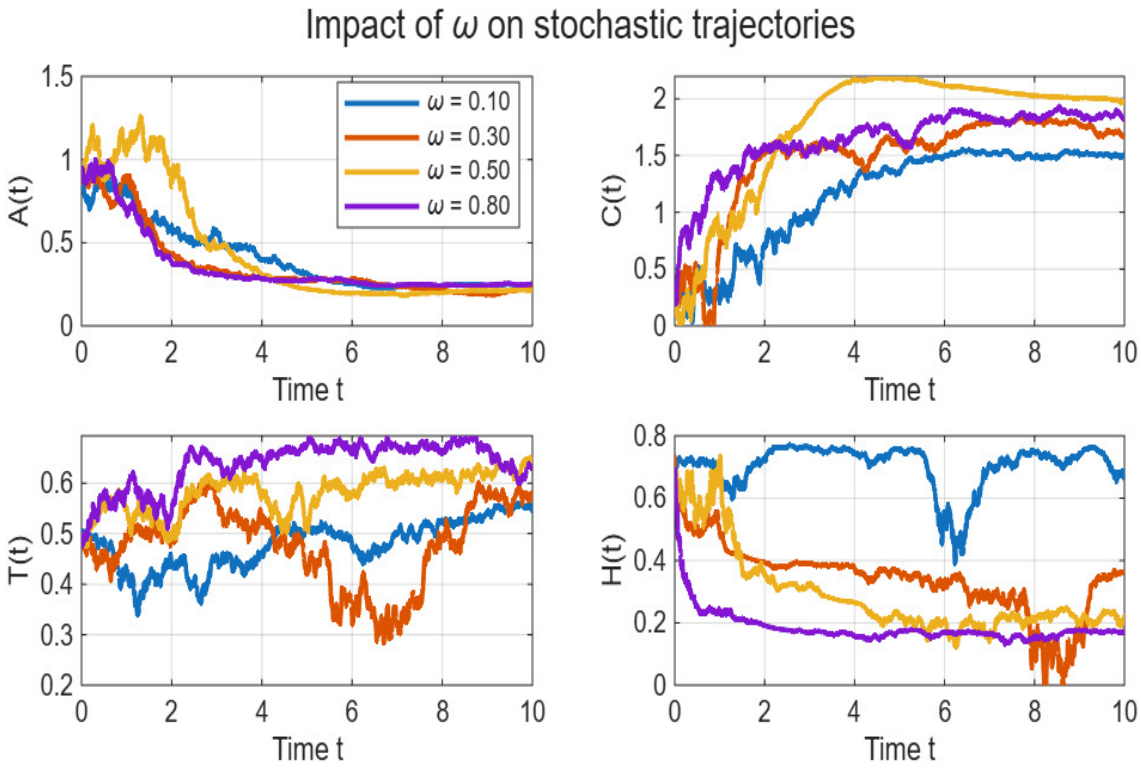


FIGURE 4. Stochastic dynamics of system (3.1).

For Fig. 3 and Fig. 4, the initial conditions are $(a_0, c_0, t_0, h_0) = (0.9, 0.2, 0.5, 0.7)$ with $\Pi = 0.2$, $\mu = 0.06$, $\kappa = 0.7$, $\omega = 0.3$, $\tau = 0.1$, $\xi = 0.2$, $\theta_A = 0.3$, $\theta_C = 0.3$, $\theta_T = 0.3$, $\theta_H = 0.8$, while $\omega = 0.05$ in Fig. 3 and $\rho = 0.01$ in Fig. 4.

The temporal evolution of the compartments in the stochastic alcohol consumption model is depicted in Fig. 3. The relapse parameter ρ represents the rate at which individuals undergoing treatment return to active alcohol consumption due to treatment failure or behavioral relapse. When ρ is small, treatment programs are more effective: the population of active consumers decreases, while the treatment and recovery classes increase, and the at-risk population declines more rapidly. In contrast, larger values of ρ lead to persistent alcohol consumption dynamics, characterized by a higher proportion of active consumers and fewer successful recoveries, as individuals leave treatment prematurely.

Fig. 4 illustrates the influence of the abstinence-loss rate ω, which governs the transition of abstinent individuals back to the at-risk class due to social or behavioral relapse. As ω increases, abstinence is lost more rapidly, leading to a higher long-term level of susceptibility. Consequently, the populations of active alcohol consumers $C(t)$ and individuals undergoing treatment $T(t)$ increase, while the abstinent population $H(t)$ decreases. These results highlight the importance of sustaining recovery and reducing relapse mechanisms in order to mitigate the long-term persistence of alcohol consumption.

Sensitivity Analysis of $\mathcal{R}_c$ using PRCC. To assess the influence of model parameters on the threshold quantity $\mathcal{R}_c$, a sensitivity analysis based on the Partial Rank Correlation Coefficient (PRCC) is performed. As illustrated in Fig. 5, this analysis quantifies the monotonic relationship between each parameter and $\mathcal{R}_c$, while accounting for the simultaneous effects of the remaining parameters.

The results indicate that the social transmission rate κ and the recruitment rate Π are positively correlated with $\mathcal{R}_c$, implying that stronger social influence mechanisms and higher population inflow enhance the persistence of alcohol consumption within the population. In contrast, the natural removal rate μ exhibits the strongest negative impact on $\mathcal{R}_c$, which is expected since higher exit rates reduce the overall prevalence of alcohol consumption.

The treatment relapse rate ρ, which governs the return of individuals from treatment to active alcohol consumption, also shows a significant influence on $\mathcal{R}_c$. An increase in ρ reduces the average duration of treatment effectiveness, thereby affecting the long-term dynamics of alcohol consumption. Moreover, the abstinence-loss rate ω, which governs the transition of abstinent individuals back to the at-risk class, is positively correlated with $\mathcal{R}_c$, as it increases the pool of individuals at risk of resuming alcohol consumption.

Finally, the treatment success rate ξ exhibits a weak negative correlation with $\mathcal{R}_c$, indicating a stabilizing but limited effect on the overall dynamics. These results highlight the relative importance of social transmission and relapse-related mechanisms in driving alcohol consumption persistence.

Overall, the PRCC analysis provides valuable insights for identifying key parameters that can be targeted by prevention and treatment strategies. In particular, reducing the social transmission rate κ and the abstinence-loss rate ω, while enhancing mechanisms that increase the removal rate μ, may significantly lower the value of $\mathcal{R}_c$ and mitigate the persistence of alcohol consumption.

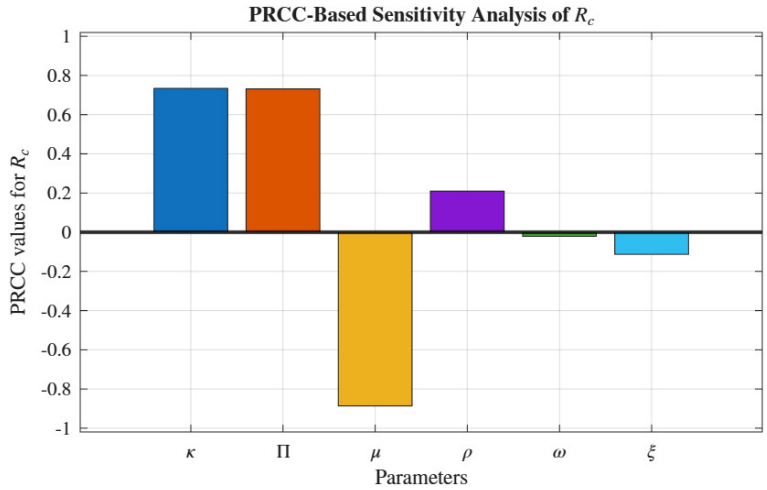


FIGURE 5. PRCC-based sensitivity analysis of the basic reproduction number $\mathcal{R}_c$.

5. Conclusion

In this paper, we formulated and analyzed a deterministic and stochastic model describing the dynamics of alcohol consumption, explicitly incorporating treatment, relapse, and temporary abstinence mechanisms. Lyapunov functions were constructed to establish global asymptotic stability of the equilibria in the deterministic model. In addition, a stochastic extension of the model was formulated by introducing multiplicative noise terms to represent random behavioral and environmental fluctuations. Stochastic Lyapunov techniques were then employed to derive sufficient conditions for stability in probability.

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