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ON SOME PROPERTIES OF DIRICHLET'S ETA FUNCTION

KWARA NANTOMAH

Abstract. In this paper, we establish some inequalities and monotonicity properties involving the Dirichlet's eta function. Thus, we improve and complement some previous results of Cerone and Dragomir and that of Alzer and Kwong. We also provide some new results. The tools employed in proving our results include some inequalities for the Nielsen's beta function, the logarithmic concavity property of the Dirichlet's eta function, the Cauchy-Schwarz integral inequality, and the classical mean value theorem.

Keywords. Dirichlet's eta function; Riemann zeta function; Nielsen's beta function; polygamma function; inequality
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1. Introduction

The Dirichlet's eta function (which is also referred to as the alternating zeta function) is defined as

$$\eta(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^z}, \quad z > 0, \tag{1.1}$$

$$= \frac{1}{\Gamma(z)} \int_0^\infty \frac{u^{z-1}}{e^u + 1} du, \quad z > 0, \tag{1.2}$$

$$= (1 - 2^{1-z})\zeta(z), \quad z > 0,\, z \neq 1 \tag{1.3}$$

where $\Gamma(z)$ is the Euler gamma function and $\zeta(z)$ is the Riemann zeta function which are defined as

$$\Gamma(z) = \int_0^\infty u^{z-1} e^{-u} du, \quad z > 0, \tag{1.4}$$

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}, \quad z > 1. \tag{1.5}$$

In effect, the formula (1.3) extends the definition of $\zeta(z)$ to a larger domain. The Dirichlet eta function plays a remarkable role in areas such as analytic number

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* Corresponding author

theory, probability theory, mathematical physics and mathematical analysis. For this reason, it has been studied by many researchers along different paths.

In 1975, van de Lune [14, Theorem 3] showed that the function $\eta(z)$ is strictly increasing, by using some lengthy procedure. In 2019, Qi [12] deduced the monotonicity of $\eta(z)$ by using a straightforward argument. In 1998, Wang [15] established that the function $\eta(z)$ is logarithmically concave on $(0,\infty)$.

In 2009, Cerone and Dragomir [5] deduced that

$$\frac{(1-2^{1-z})(1-2^{-1-z})}{(1-2^{-z})} \leq \frac{\eta(z)\eta(z+2)}{\eta^2(z+1)}, \quad z>1, \tag{1.6}$$

and conjectured that

$$\frac{\eta(z)\eta(z+2)}{\eta^2(z+1)} \leq 1, \quad z>1. \tag{1.7}$$

In 2015, Alzer and Kwong [2] improved the logarithmic concavity property of $\eta(z)$ by proving that $\eta(z)$ is actually strictly concave on $(0,\infty)$. They further established several interesting inequalities involving $\eta(z)$. Later in the same year, Adell and Lekuona [1] also proved that the generalized function $\eta_a(z)$ is strictly concave on $(0,\infty)$, where $\eta_1(z)=\eta(z)$. Stojiljković [13] also provided an integral representation for a remainder sum of the Dirichlet's eta function and further studied some of its properties.

Motivated by the outstanding works [5] and [2], the goal of this paper is to improve and complement the results (1.6) and (1.7) in addition to some other new results.

In the mean time, we recall the following definitions which shall be required to establish some of our results.

The classical Nielsen's beta function is denoted by any of the following equivalent forms. See [4], [6], [11, p.16].

$$\beta(z) = \int_0^\infty \frac{e^{-zt}}{1+e^{-t}}dt, \tag{1.8}$$

$$= \sum_{k=0}^\infty \frac{(-1)^k}{k+z}, \tag{1.9}$$

$$= \frac{1}{2}\left\{\psi\left(\frac{z+1}{2}\right) - \psi\left(\frac{z}{2}\right)\right\}, \tag{1.10}$$

for $z>0$, where $\psi(z)=\frac{d}{dz}\ln\Gamma(z)$ is the digamma function. Its derivatives are respectively given as (see [8, 9, 10])

$$\beta^{(s)}(z) = (-1)^s \int_0^\infty \frac{t^s e^{-zt}}{1+e^{-t}}dt, \quad s\in\mathbb{N}_0, \tag{1.11}$$

$$= (-1)^s s! \sum_{k=0}^\infty \frac{(-1)^k}{(k+z)^{s+1}}, \quad s\in\mathbb{N}_0, \tag{1.12}$$

$$= \frac{1}{2^{s+1}}\left\{\psi^{(s)}\left(\frac{z+1}{2}\right) - \psi^{(s)}\left(\frac{z}{2}\right)\right\}, \quad s\in\mathbb{N}_0, \tag{1.13}$$

where $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$, $\psi^{(0)}(z) \equiv \psi(z)$ and $\beta^{(0)}(z) = \beta(z)$. In specific,

$$\begin{aligned}\beta^{(s)}(1) &= (-1)^s s! \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^{s+1}} = (-1)^s s! \eta(s+1), \quad s \in \mathbb{N}_0, \\ &= (-1)^s s! \left(1 - \frac{1}{2^s}\right) \zeta(s+1), \quad s \in \mathbb{N}.\end{aligned} \tag{1.14}$$

We are now well positioned to present our results. We shall do that in the next section.

2. Results and Discussion

Theorem 2.1. *Given that $r \in \mathbb{N}$, then the inequalities*

$$\frac{\eta(r+1)}{\eta(r)} \leq \frac{r+1}{r} \frac{\eta(r+2)}{\eta(r+1)}, \tag{2.1}$$

$$\frac{r+1}{2r} \frac{\eta(r+2)}{\eta(r+1)} \leq \frac{\eta(r+1)}{\eta(r)}, \tag{2.2}$$

are satisfied.

Proof. In [7], [10] and [16], it has been established that for $z > 0$ and $s \in \mathbb{N}_0$, the inequalities

$$\beta^{(s)}(z)\beta^{(s+2)}(z) \geq (\beta^{(s+1)}(z))^2 \tag{2.3}$$

and

$$\beta^{(s)}(z)\beta^{(s+2)}(z) \leq 2(\beta^{(s+1)}(z))^2 \tag{2.4}$$

hold. Considering the particular case where $z = 1$ in (2.3) and (2.4), and using (1.14), we respectively obtain

$$\frac{s+2}{s+1} \eta(s+1)\eta(s+3) \geq \eta^2(s+2) \tag{2.5}$$

and

$$\frac{s+2}{s+1} \eta(s+1)\eta(s+3) \leq 2\eta^2(s+2). \tag{2.6}$$

Now, by replacing s with $r - 1$ in (2.5) and (2.6), we respectively obtain the results (2.1) and (2.2). □

In the following theorem, we extend (2.1) to all positive real numbers.

Theorem 2.2. *The inequality*

$$\frac{\eta(z+1)}{\eta(z)} \leq \frac{z+1}{z} \frac{\eta(z+2)}{\eta(z+1)}, \tag{2.7}$$

holds for $z > 0$.

Proof. The Cauchy-Schwarz inequality for integrals is given as

$$\left(\int_a^b K_1(t)K_2(t)dt\right)^2 \leq \int_a^b [K_1(t)]^2\,dt \int_a^b [K_2(t)]^2\,dt \tag{2.8}$$

where $K_1(t)$ and $K_2(t)$ are continuous functions on $[a,b]$. Let $K_1(t) = ((h(t)[p(t)]^u)^{\frac{1}{2}}$ and $K_2(t) = ((h(t)[p(t)]^v)^{\frac{1}{2}}$, where $h(t)$ and $p(t)$ are non-negative and u and v are real numbers. Then (2.8) becomes

$$\left(\int_a^b h(t)\,[p(t)]^{\frac{u+v}{2}}\,dt\right)^2 \leq \int_a^b h(t)\,[p(t)]^u\,dt \int_a^b h(t)\,[p(t)]^v\,dt. \tag{2.9}$$

By letting $h(t) = 1/(e^t+1)$, $p(t) = t$, $a = 0$, $b = \infty$, $u = z+1$ and $v = z-1$ in (2.9), we obtain

$$\left(\int_0^\infty \frac{t^z}{e^t+1}dt\right)^2 \leq \int_0^\infty \frac{t^{z+1}}{e^t+1}dt \int_0^\infty \frac{t^{z-1}}{e^t+1}dt$$

and as a result of (1.2), we obtain

$$[\eta(z+1)\Gamma(z+1)]^2 \leq \eta(z+2)\Gamma(z+2)\eta(z)\Gamma(z). \tag{2.10}$$

Since $\Gamma(z+1) = z\Gamma(z)$, inequality (2.10) then reduces to (2.7). □

In the following theorem, we provide an improvement of (2.7) by utilizing the logarithmic concavity of $\eta(z)$.

Theorem 2.3. *The inequality*

$$\frac{\eta(z+2)}{\eta(z+1)} < \frac{\eta(z+1)}{\eta(z)}, \tag{2.11}$$

holds for $z > 0$.

Proof. Since $\eta(z)$ is logarithmically concave on $(0,\infty)$, then

$$(\ln \eta(z))'' = \left(\frac{\eta'(z)}{\eta(z)}\right)' < 0$$

which implies that the function $A(z) = \eta'(z)/\eta(z)$ is strictly decreasing. Let

$$T(z) = \frac{\eta(z+1)}{\eta(z)},$$

for all $z > 0$. Then logarithmic differentiation gives

$$\frac{T'(z)}{T(z)} = \frac{\eta'(z+1)}{\eta(z+1)} - \frac{\eta'(z)}{\eta(z)} < 0$$

which implies that $T'(z) < 0$. Thus $T(z)$ is strictly decreasing. Hence we have $T(z+1) < T(z)$ which gives (2.11). □

Remark 2.4. Inequality (2.7) can be rearranged as

$$\frac{z}{z+1} \leq \frac{\eta(z)\eta(z+2)}{\eta^2(z+1)} \tag{2.12}$$

for $z > 0$. Comparing the bound in (2.12) with the bound in (1.6), we observe that

$$h(z) = \frac{(1-2^{1-z})(1-2^{-1-z})}{(1-2^{-z})} - \frac{z}{z+1}$$

is negative if $0 < z < a$ and positive if $0 < z < a$, where $a \approx 2.48811974...$ is the root of $h(z)$. Thus, the bound in (2.12) is better than the bound in (1.6) if $0 < z < a$. Also, the domain $z > 1$ in (1.6) has been extended to $z > 0$ in (2.12).

Remark 2.5. Inequality (2.11) can be rearranged as

$$\frac{\eta(z)\eta(z+2)}{\eta^2(z+1)} < 1 \tag{2.13}$$

for $z > 0$. This proves the conjecture (1.7). Additionally, the domain $z > 1$ in (1.7) has been extended to $z > 0$ in the present results.

Corollary 2.6. *The inequality*

$$1 < \frac{\eta(z+1)}{\eta(z)} < 2\ln 2, \tag{2.14}$$

holds for $z > 0$.

Proof. Since $T(z) = \eta(z+1)/\eta(z)$ is decreasing, then for $0 < z < \infty$, we have

$$1 = \lim_{z\to\infty} T(z) < T(z) < \lim_{z\to 0} T(z) = 2\ln 2$$

which gives (2.14). □

Theorem 2.7. *The inequality*

$$\eta'(t) < \frac{\eta(t)-\eta(s)}{t-s} < \eta'(s) \tag{2.15}$$

holds for $0 < s < t$.

Proof. Since $\eta(z)$ is continuous on $[s,t]$ and differentiable on (s,t), then by the mean value theorem, there exist at least $\lambda \in (s,t)$ such that

$$\frac{\eta(t)-\eta(s)}{t-s} = \eta'(\lambda).$$

Since $\eta'(z)$ is strictly decreasing, then for $\lambda \in (s,t)$, we have $\eta'(t) < \eta'(\lambda) < \eta'(s)$ which gives (2.15). □

Remark 2.8. If $s = z$ and $t = z+1$, then (2.15) reduces to

$$\eta'(z+1) < \eta(z+1) - \eta(z) < \eta'(z) \tag{2.16}$$

for $z > 0$.

Theorem 2.9. *For $s > 0$, $t > 0$ such that $s \neq t$, the inequality*

$$0 < \frac{\eta(t)-\eta(s)}{t-s} < \frac{1}{2}\ln\left(\frac{\pi}{2}\right) \tag{2.17}$$

holds.

Proof. Since $\eta'(z)$ is strictly decreasing on $(0,\infty)$, then we have

$$0 = \lim_{z\to\infty} \eta'(z) < \eta'(z) < \lim_{z\to 0} \eta'(z) = \frac{1}{2}\ln\left(\frac{\pi}{2}\right), \tag{2.18}$$

for all $z > 0$. With no loss of generality, let $s < t$ ($t < s$ will have the same effect). Then by the mean value theorem, there exist at least $k \in (s,t)$ such that

$$\frac{\eta(t)-\eta(s)}{t-s} = \eta'(k). \tag{2.19}$$

Now, combining (2.18) and (2.19) gives the desired result (2.17). □

Remark 2.10. If $s = z$ and $t = z+1$, then (2.17) reduces to

$$0 < \eta(z+1) - \eta(z) < \frac{1}{2}\ln\left(\frac{\pi}{2}\right) \tag{2.20}$$

for $z > 0$.

Remark 2.11. We note that inequality (2.17) was earlier obtained in [2] by a different procedure. Since $\eta'(t) > 0$ for all $t > 0$ and $\eta'(s) < \frac{1}{2}\ln(\pi/2)$ for all $s > 0$, then the bounds in (2.15) are better than the bounds in (2.17). Consequently, the bounds in (2.16) are also better than those in (2.20).

Theorem 2.12. *The inequality*

$$\frac{\eta'(t)}{\eta(t)} < \frac{\ln\eta(t) - \ln\eta(s)}{t-s} < \frac{\eta'(s)}{\eta(s)} \tag{2.21}$$

holds for $0 < s < t$.

Proof. Since $\ln\eta(z)$ is continuous on $[s,t]$ and differentiable on (s,t), then by the mean value theorem, there exist at least $\delta \in (s,t)$ such that

$$\frac{\ln\eta(t) - \ln\eta(s)}{t-s} = \frac{\eta'(\delta)}{\eta(\delta)}.$$

Since $\eta'(z)/\eta(z)$ is strictly decreasing, then for $\delta \in (s,t)$, we have

$$\frac{\eta'(t)}{\eta(t)} < \frac{\eta'(\delta)}{\eta(\delta)} < \frac{\eta'(s)}{\eta(s)}$$

which gives (2.21). □

Remark 2.13. If $s = z$ and $t = z+1$, then (2.21) reduces to the elegant inequality

$$\exp\left(\frac{\eta'(z+1)}{\eta(z+1)}\right) < \frac{\eta(z+1)}{\eta(z)} < \exp\left(\frac{\eta'(z)}{\eta(z)}\right) \tag{2.22}$$

for $z > 0$.

Theorem 2.14. *For* $s > 0$, $t > 0$ *such that* $s \neq t$, *the inequality*

$$0 < \frac{\ln\eta(t) - \ln\eta(s)}{t-s} < \ln\left(\frac{\pi}{2}\right) \tag{2.23}$$

holds.

Proof. Since $\eta'(z)/\eta(z)$ is strictly decreasing on $(0,\infty)$, then we have

$$0=\lim_{z\to\infty}\frac{\eta'(z)}{\eta(z)}<\frac{\eta'(z)}{\eta(z)}<\lim_{z\to 0}\frac{\eta'(z)}{\eta(z)}=\ln\left(\frac{\pi}{2}\right), \tag{2.24}$$

for all $z>0$. With no loss of generality, let $s<t$. Then by the mean value theorem, there exist at least $c\in(s,t)$ such that

$$\frac{\ln\eta(t)-\ln\eta(s)}{t-s}=\frac{\eta'(c)}{\eta(c)} \tag{2.25}$$

Now, combining (2.24) and (2.25) gives the desired result (2.23). □

Remark 2.15. If $s=z$ and $t=z+1$, then (2.23) reduces to

$$1<\frac{\eta(z+1)}{\eta(z)}<\frac{\pi}{2} \tag{2.26}$$

for $z>0$.

Remark 2.16. Since $\eta'(t)/\eta(t)>0$ for all $t>0$ and $\eta'(s)/\eta(s)-\ln(\pi/2)<0$ for all $s>0$, then the bounds in (2.21) are better than those in (2.23). Also the upper bound of (2.14) is better than the upper bound of (2.26).

Theorem 2.17. *For $z>0$, the function*

$$\phi(z)=\frac{\eta(z)}{z} \tag{2.27}$$

is strictly decreasing and consequently, the inequality

$$\frac{\eta'(z)}{\eta(z)}<\frac{1}{z} \tag{2.28}$$

holds.

Proof. It has been established in [2, Theorem 4.9] that for all $s>0$, $t>0$ and $s\neq t$, the inequality

$$\frac{1}{2}<\frac{t\eta(s)-s\eta(t)}{t-s}<1 \tag{2.29}$$

holds. By letting $t=z$, $s=z+c$ where $c>0$ in (2.29), we obtain

$$-c<z\eta(z+c)-(z+c)\eta(z)<-\frac{c}{2}.$$

This implies that

$$\phi(z+c)-\phi(z)=\frac{\eta(z+c)}{z+c}-\frac{\eta(z)}{z}=\frac{z\eta(z+c)-(z+c)\eta(z)}{z(z+c)}<0.$$

Hence $\phi(z)$ is strictly decreasing. By differentiating $\phi(z)$, inequality (2.28) is easily obtained. □

Remark 2.18. The decreasing property of $\phi(z)$ implies the subadditivity property of $\eta(z)$ which has been established in [2].

Remark 2.19. Inequality (2.28) implies that

$$\frac{\eta'(x)\eta(y)-\eta(x)\eta'(y)}{\eta(x)\eta(y)} < \frac{y-x}{xy} \tag{2.30}$$

for any $x>0$, $y>0$, $x \neq y$.

The following results are motivated by the paper [3], where the authors studied the monotonicity properties of the function $f(y)=\psi(1+by)^a/\psi(1+ay)^b$ and its associated inequalities.

Theorem 2.20. *For $z>0$, $r\geq 0$, $a>0$, $b>0$ the function*

$$P(z)=\frac{\eta(r+bz)^a}{\eta(r+az)^b} \tag{2.31}$$

is strictly increasing if $a>b$ and strictly decreasing if $a<b$. Consequently,

$$\eta(r)^{a-b} < \frac{\eta(r+bz)^a}{\eta(r+az)^b} < 1, \tag{2.32}$$

if $a>b$. If $a<b$ then the inequality (2.32) is reversed.

Proof. Let $a>b$. Then logarithmic differentiation yields

$$\frac{P'(z)}{P(z)} = ab\left[\frac{\eta'(r+bz)}{\eta(r+bz)} - \frac{\eta'(r+az)}{\eta(r+az)}\right] > 0,$$

since the function $\eta'(s)/\eta(s)$ is decreasing. Hence $P'(z)>0$ which implies that $P(z)$ is strictly increasing. The monotonicity of $P(z)$ then implies that

$$\eta(r)^{a-b} = \lim_{z\to 0} P(z) < P(z) < \lim_{z\to\infty} P(z) = 1$$

which gives (2.32). The proof for the case where $a<b$ follows the same procedure. □

Remark 2.21. When $r=1$ in (2.32), we obtain

$$(\ln 2)^{a-b} < \frac{\eta(1+bz)^a}{\eta(1+az)^b} < 1. \tag{2.33}$$

Remark 2.22. The monotonicity of $P(z)$ further implies that

$$\frac{\eta(r+by)^a}{\eta(r+ay)^b} \leq \frac{\eta(r+bz)^a}{\eta(r+az)^b} \tag{2.34}$$

for $a>b$ and $0<y\leq z$. If $r=0$ and $b=1$ in (2.34), then we obtain the elegant inequality

$$\left(\frac{\eta(y)}{\eta(z)}\right)^a \leq \frac{\eta(ay)}{\eta(az)} \tag{2.35}$$

for $a>1$ and $0<y\leq z$. If $a<b$, then inequality (2.34) is reversed. Also, if $0<a<1$, then inequality (2.35) is reversed.

Theorem 2.23. *For $z>0$, the inequality*

$$1-p < \eta(pz) - p\eta(z) < \frac{1-p}{2} \tag{2.36}$$

holds if $p>1$. If $0<p<1$, then the inequality is reversed.

Proof. Let $p > 1$ and $\theta(z)$ be defined as $\theta(z) = \eta(pz) - p\eta(z)$. Then

$$\theta'(z) = p\left[\frac{\eta'(pz)}{\eta(pz)} - \frac{\eta'(z)}{\eta(z)}\right] < 0,$$

which means that $\theta(z)$ is decreasing. Therefore,

$$1 - p < \lim_{z\to\infty}\theta(z) < \theta(z) < \lim_{z\to 0}\theta(z) = \frac{1-p}{2} < 0$$

which gives (2.36). The proof for the case $0 < p < 1$ follows the same procedure. □

Conjecture 2.24. We conjecture that, for $z > 0$, the function

$$F(z) = \frac{\eta(z)\eta(z+2)}{\eta^2(z+1)} \tag{2.37}$$

is increasing. This will lead to the following double inequality with constant bounds.

$$\frac{1}{24}\left(\frac{\pi}{\ln 2}\right)^2 < \frac{\eta(z)\eta(z+2)}{\eta^2(z+1)} < 1. \tag{2.38}$$

3. Conclusion

In the present work, we have established some inequalities and monotonicity properties involving the Dirichlet's eta function. In doing that, we have provided improvements and complements of some previous results, We have also established some new results which could pave the way for further research on the function. The advantage of the present work is that, the methods of proof are relatively simple and easy to follow. We also give a conjecture which is expected to be of utmost interest to researchers working on analytical properties of the Dirichlet's eta function.

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Department of Mathematics, School of Mathematical Sciences, University of Technology and Applied Sciences, P. O. Box 24, Navrongo, Upper-East Region, Ghana.

Email address: knantomah@utas.edu.gh