Abstract. In this study, we will give the definition of statistical convergence of a sequence of subspaces of a k-dimensional inner product space and examine some inclusion relations.

1. Introduction and preliminaries

Let $F$ be the field reals or complex numbers and $X$ be a vector space over $F$. An inner product on $X$ is a function $\langle,\rangle: X \times X \to F$ such that

1. $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$, for $a, b \in F$ and $x, y, z \in X$
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for $x, y \in X$
3. $\langle x, x \rangle \geq 0$ for $x \in X$
4. $\langle x, x \rangle = 0$ if and only if $x = 0$.

Recall that every real number $x \in \mathbb{R}$ equals its complex conjugate. Hence for real vector spaces the condition (2) becomes $\langle x, y \rangle = \langle y, x \rangle$ for $x, y \in X$.

A norm can be defined from an inner product via $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in X$.

Let $X$ be a inner product space over $F$. Let $V$ be a subspace of $X$ and $x \in X$. An element $v_0 \in V$ is said to be a best approximation or nearest to $x$ in $X$, if $\|x - v_0\| \leq \|x - v\|$ for all $v \in V$. Suppose $V$ is finite dimensional and $\{v_1, v_2, ..., v_k\}$ is an orthonormal basis of $V$. Then

$$v_0 = \sum_{i=1}^{k} \langle v, v_i \rangle v_i$$

is the best approximation to $v$ in $V$. Let $(X, \langle, \rangle)$ be an inner product space. Let $V$ be a subspace of inner product space $X$ and $\{v_1, v_2, ..., v_k\}$ be an orthonormal basis for $V$. The orthogonal projection $P_V$ from $X$ onto the subspace $V$ is defined by

$$P_V u = \sum_{i=1}^{k} \langle u, v_i \rangle v_i$$

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* Corresponding author.

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For every \( u \in X \).

For example, let \( X = \mathbb{R}^3 \) and \( V = \{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\} \) be the \( x_1x_2 \)-plane in \( \mathbb{R}^3 \). Choose an arbitrary point \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \). Orthogonal projection of \( x \) onto \( V \) is \( P_V x = (x_1, x_2, 0) \). In order to show this, choose an arbitrary point \( v = (v_1, v_2, v_3) \in V \). Then \( x - v = (x_1 - v_1, x_2 - v_2, x_3) \) and \( x - P_V x = (0, 0, x_3) \). So

\[
\|x - v\|^2 = |x_1 - v_1|^2 + |x_2 - v_2|^2 + |x_3|^2 \geq |x_3|^2 = \|x - P_V x\|^2.
\]

Thus \( P_V x \) is closer \( x \) than \( v \), so \( P_V x \) is the orthogonal projection. If we take \( \{e_1, e_2, e_3\} \) as the standard basis of \( \mathbb{R}^3 \), then above example tells us the orthogonal projection of \( x = (x_1, x_2, x_3) = x_1e_1 + x_2e_2 + x_3e_3 \) onto \( V = \text{span}\{e_1, e_2\} \) is \( P_V x = x_1e_1 + x_2e_2 = (x_1, x_2, 0) \). As a second example, let \( X = \mathbb{R}^3 \) and \( V \) be subspace of \( \mathbb{R}^3 \) spanned by the vectors \( (1, 1, 1) \) and \( (1, -2, 1) \). Then

\[
P_V x = \frac{x_1 + x_2 + x_3}{3} (1, 1, 1) + \frac{x_1 - 2x_2 + x_3}{6} (1, -2, 1) = \left( \frac{x_1 + x_3}{2}, x_2, \frac{x_1 + x_3}{2} \right).
\]

(For more information on this topic, see [4],[9]).

Distance or gap between two subspaces \( U \) and \( V \) of \( X \) is defined by

\[
d(U, V) = \sup \{ \inf \{ \|u - v\| : v \in V \} : u \in U, \|u\| = 1 \} = \sup \{ \|u - P_V u\| : u \in U, \|u\| = 1 \} = \sup_{u \in U, \|u\| = 1} \|u - P_V u\|.
\]

Given an inner product space \( (X, <, >) \) of dimension \( k \) or higher (may be infinite). Let \( U_n \) and \( V \) be \( k \)-dimensional subspaces of \( X \) for every \( n \in \mathbb{N} \). Throughout the paper, it is supposed that \( U_n = \text{span}\{u_{1n}, u_{2n}, ..., u_{kn}\} \) and \( V = \text{span}\{v_1, v_2, ..., v_k\} \) be \( k \)-dimensional subspaces of \( X \), where \( \{u_{1n}, u_{2n}, ..., u_{kn}\} \) and \( \{v_1, v_2, ..., v_k\} \) are orthonormal for every \( n \in \mathbb{N} \).

Convergence definition for sequence of subspaces was given in [10] as follows:

Let \( U_n \) and \( V \) be \( k \)-dimensional subspaces of \( X \) for \( n = 1, 2, 3, ... \). A sequence \( (U_n) \) of subspaces of \( X \) is said to be convergent to the subspace \( V \) if

\[
\lim_{n \to \infty} \sup_{u_n \in U_n, \|u_n\| = 1} \|u_n - P_V u_n\| = 0,
\]

Also the authors of [10] proved the following theorem that gives a condition equivalent to that in the definition of convergence.

**Theorem 1.1.** A sequence \( (U_n) \) of subspaces is convergent to the subspace \( V \) if and only if

\[
\lim_{n \to \infty} \|u_{in} - P_V u_{in}\| = 0
\]

for every \( i = 1, 2, ..., k \) where

\[
P_V u_{in} = \sum_{j=1}^{k} < u_{jn}, v_j > v_j.
\]
For example, consider a three-dimensional space $X$ with orthonormal basis vectors $e_1, e_2, e_3$. $U_n$ be one-dimensional subspace spanned by the vector 

$$e_1 + \frac{1}{n} \sin ne_2 + \frac{1}{n} \cos ne_3.$$ 

Then the sequence $(U_n)$ converges to the subspace $V$, spanned by the vector $e_1$. If $W_n$ is an one-dimensional subspace spanned by the vector 

$$\sin ne_2 + \cos ne_3$$

then the sequence $(W_n)$ is not convergent.

In most convergence theories it is advantages to have a criterion that verify convergence without using the value of the limit. For this goal we will introduce a Cauchy convergence criterion.

**Definition 1.2.** Let $U_n$ be $k$-dimensional subspaces of $X$ for $n = 1, 2, 3, ...$. A sequence $(U_n)$ of subspaces of $X$ is said to be Cauchy sequence if

$$\sup_{u_n \in U_n, \|u_n\|=1} \|u_n - P_{U_m} u_n\| < \epsilon$$

for every $m, n > n_0 \in \mathbb{N}$.

We can give another definition for Cauchy sequence equivalent to the Definition 1.1.

**Definition 1.3.** A sequence $(U_n)$ of subspaces is convergent to the subspace $V$ if and only if

$$\|u_{i_n} - P_{U_m} u_{i_n}\| < \epsilon$$

for every $i = 1, 2, ..., k$ and $m, n > n_0 \in \mathbb{N}$.

2. Statistical Convergence

In this section, we will give the definition of statistical convergence of sequences of subspaces of a $k$-dimensional inner product space and examine some inclusion relations.

Let $A \subseteq \mathbb{N}$. Define

$$\delta(A) = \liminf_{m \to \infty} \frac{|A \cap \{1, 2, 3, ..., m\}|}{m} \quad \text{and} \quad \bar{\delta}(A) = \limsup_{m \to \infty} \frac{|A \cap \{1, 2, 3, ..., m\}|}{m}.$$

We call $\delta$ and $\bar{\delta}$ the lower and upper asymptotic densities, respectively. If $\delta(A) = \bar{\delta}(A)$, the common value is referred as asymptotic density and denoted by $\delta(A)$.

If $(x_n)$ is a sequence such that $x_n$ satisfies property P for all $n$ except a set of asymptotic density zero, then we say that $x_n$ satisfies P for ”almost all n”, and we abbreviate this by ”a.a. n.” A sequence $(x_n)$ of real or complex numbers is said to be statistically convergent to the number $a$ if for every $\epsilon > 0$,

$$\delta(\{k : |x_k - a| \geq \epsilon\}) = 0.$$ 

A statistically convergent sequence may be bounded or unbounded. For example, define $x_k = k$ if $k$ is a square and $x_k = 0$ otherwise. Then

$$|\{k \leq n : |x_k - 0| > \epsilon\}| \leq \sqrt{n},$$
so \( st - \lim x_k = 0 \).

The idea of statistical convergence was introduced by Steinhaus in [13] and Fast in [6] and since then has been studied by other authors including [2], [3], [7], [8], [11] and [13]. Over the years and under different names, statistical convergence has been discussed in the mathematical analysis.

**Definition 2.1.** A sequence \((U_n)\) of subspaces is said to be statistically convergent to the subspace \(V\) if for every \(\epsilon > 0\),

\[
\lim_{m \to \infty} \frac{1}{m} \left| \left\{ n \leq m : \sup_{u_n \in U_n, \|u_n\| = 1} \|u_n - P_V u_n\| > \epsilon \right\} \right| = 0
\]

i.e.,

\[
\sup_{u_n \in U_n, \|u_n\| = 1} \|u_n - P_V u_n\| < \epsilon
\]
a.a. \(n\).

In this case we write

\( st - \lim U_n = V \).

Another definition equivalent to this definition is given below:

**Definition 2.2.** A sequence \((U_n)\) of subspaces is said to be statistically convergent to the subspace \(V\) if for every \(\epsilon > 0\),

\[
\lim_{m \to \infty} \frac{1}{m} \left| \left\{ n \leq m : \|u_{i_n} - P_V u_{i_n}\| > \epsilon \right\} \right| = 0
\]

for every \(i = 1, 2, 3, \ldots\).

**Theorem 2.3.** Definition 2.1 and Definition 2.2 are equivalent.

**Proof.** Let

\[
\sup_{u_n \in U_n, \|u_n\| = 1} \|u_n - P_V u_n\| < \epsilon
\]
a.a. \(n\). We can write

\[
\|u_{i_n} - P_V u_{i_n}\| \leq \sup_{u_n \in U_n, \|u_n\| = 1} \|u_n - P_V u_n\| < \epsilon
\]
a.a. \(n\) for every \(i = 1, 2, \ldots, k\), \(u_{i_n} \in U_n\) with \(\|u_{i_n}\| = 1\).

Now, say \(\sum_{i=1}^{k} |\gamma_i| = K\) and suppose that

\[
\|u_{i_n} - P_V u_{i_n}\| < \frac{\epsilon}{K}
\]
a.a. \( n \) for every \( i = 1, 2, \ldots, k \).

\[
\sup_{u_n \in U_n, \|u_n\|=1} \|u_n - PVu_n\| = \sup_{u_n \in U_n, \|u_n\|=1} \|(\gamma_1 u_{1n} + \gamma_2 u_{2n} + \ldots + \gamma_k u_{kn}) - PV(\gamma_1 u_{1n} + \gamma_2 u_{2n} + \ldots + \gamma_k u_{kn})\| \\
\leq \sup_{u_n \in U_n, \|u_n\|=1} \sum_{i=1}^{k} \|\gamma_i u_{in} - PV(\gamma_i u_{in})\| \\
= \sum_{i=1}^{k} |\gamma_i| \|u_{in} - PVu_{in}\| < \epsilon
\]

a.a. \( n \) for every \( i = 1, 2, \ldots, k \). \( \square \)

If \( (U_n) \) is convergent to \( V \) then \( (U_n) \) is statistically convergent to \( V \), but the opposite may not be true.

**Example 2.4.** Let \( X \) be three-dimensional space with orthonormal basis vectors \( e_1, e_2, e_3 \) and

\[
U_n = \begin{cases} 
\text{span}\{\sin ne_2 + \cos ne_3\}, & \text{if } n \text{ is square} \\
\text{span}\{e_1 + \frac{1}{n} \sin ne_2 + \frac{1}{n} \cos ne_3\}, & \text{otherwise.}
\end{cases}
\]

Then the sequence \( (U_n) \) is not convergent but statistically convergent to the subspace \( V \), spanned by the vector \( e_1 \).

**Theorem 2.5.** If \( st - \lim U_n = V \) and \( st - \lim U_n = S \) then \( V = S \).

**Proof.** If possible let \( U \neq S \). Choose \( \epsilon = \frac{\|PVu_n - PSu_n\|}{2} > 0 \). Then

\[
\frac{1}{m} \left\{ n \leq m : \sup_{u_n \in U_n, \|u_n\|=1} \|PVu_n - PSu_n\| > \epsilon \right\} \leq \frac{1}{m} \left\{ n \leq m : \sup_{u_n \in U_n, \|u_n\|=1} \|u_n - PVu_n\| > \frac{\epsilon}{2} \right\} + \frac{1}{m} \left\{ n \leq m : \sup_{u_n \in U_n, \|u_n\|=1} \|u_n - PSu_n\| > \frac{\epsilon}{2} \right\}
\]

which is impossible because the right side limit is equal to zero but not left side limit as \( m \to \infty \). \( \square \)

**Definition 2.6.** A sequence \( (U_n) \) of subspaces is said to be statistically Cauchy if for every \( \epsilon > 0 \) there exists a number \( N(= N(\epsilon)) \) such that

\[
\lim_{m \to \infty} \frac{1}{m} \left\{ n \leq m : \sup_{u_n \in U_n, \|u_n\|=1} \|u_n - PU_Nu_n\| > \epsilon \right\} = 0
\]

i.e.,

\[
\sup_{u_n \in U_n, \|u_n\|=1} \|u_n - PU_Nu_n\| < \epsilon
\]

a.a. \( n \).
Another definition equivalent to this definition can be given as:

**Definition 2.7.** A sequence \((U_n)\) of subspaces is said to be statistically Cauchy sequence if for every \(\epsilon > 0\) there exists a number \(N(\epsilon)\) such that

\[
\lim_{m \to \infty} \frac{1}{m} |\{n \leq m : \|u_n - P_{U_n} u_n\| > \epsilon\}| = 0
\]

for every \(i = 1, 2, 3, \ldots\).

**Theorem 2.8.** If a sequence \((U_n)\) of subspaces is statistically convergent, then \((U_n)\) is a statistically Cauchy sequence.

**Proof.** Suppose that \(st - \lim(U_n) = V\) and \(\epsilon > 0\). Then

\[
\sup_{u_n \in U_n, \|u_n\|=1} \|u_n - P_V u_n\| < \frac{\epsilon}{2}
\]
a.a. \(n\), and if \(N\) is chosen such that

\[
\sup_{u_n \in U_n, \|u_n\|=1} \|P_{U_n} u_n - P_V u_n\| < \frac{\epsilon}{2}
\]
a.a. \(n\). Then

\[
\sup_{u_n \in U_n, \|u_n\|=1} \|u_n - P_{U_n} u_n\| < \frac{\epsilon}{2} + \sup_{u_n \in U_n, \|u_n\|=1} \|P_{U_n} u_n - P_V u_n\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]
a.a. \(n\). Hence \((U_n)\) is a statistically Cauchy sequence. \(\square\)

**Theorem 2.9.** Let \((U_n)\) and \((L_n)\) be two sequences of subspaces. If \((L_n)\) is a convergent sequence such that \(U_n = L_n\) a.a. \(n\), then \((U_n)\) is statistically convergent.

**Proof.** Suppose that \(U_n = L_n\) a.a. \(n\) and \(L_n \to V\). From the inclusion

\[
\{n \leq m : \sup_{u_n \in U_n, \|u_n\|=1} \|u_n - P_V u_n\| \geq \epsilon\} \\
\subseteq \{n \leq m : \sup_{u_n \in U_n, \|u_n\|=1} \|u_n - P_V u_n\| \geq \epsilon\} \\
\cup \{n \leq m : \sup_{\ell_n \in L_n, \|\ell_n\|=1} \|\ell_n - P_V \ell_n\| \geq \epsilon\},
\]

for \(\epsilon > 0\), we get

\[
\lim_{m \to \infty} |\{n \leq m : \sup_{u_n \in U_n, \|u_n\|=1} \|u_n - P_V u_n\| \geq \epsilon\}| \\
\leq \lim_{m \to \infty} |\{n \leq m : \sup_{u_n \in U_n, \|u_n\|=1} \|u_n - P_V u_n\| \geq \epsilon\}| \\
+ \lim_{m \to \infty} |\{n \leq m : \sup_{\ell_n \in L_n, \|\ell_n\|=1} \|\ell_n - P_V \ell_n\| \geq \epsilon\}| = 0.
\]

Consequently we have \(st - \lim U_n = V\). \(\square\)

**Theorem 2.10.** \(st - \lim U_n = V\) if and only if \(st - \lim \sum_{j=1}^{k} u_i, v_j > 2 = 1\) for every \(i = 1, 2, 3, \ldots, k\).
Proof. Since \( \{v_1, v_2, ..., v_k\} \) is orthonormal basis, with an easy calculation we get
\[
\|u_{i_n} - P_V u_{i_n}\|^2 = 1 - \sum_{j=1}^{k} <u_{i_n}, v_j>^2
\]
for every \( i = 1, 2, 3, ..., k \) and hence
\[
\frac{1}{m} |\{n \leq m : \|u_{i_n} - P_V u_{i_n}\| > \epsilon\}| = \frac{1}{m} \left\{ n \leq m : \left|1 - \sum_{j=1}^{k} <u_{i_n}, v_j>^2 \right| > \epsilon^2 \right\}.
\]
So, we have if \( st - \lim U_n = V \) if and only if \( st - \lim \sum_{j=1}^{k} <u_{i_n}, v_j>^2 = 1 \) for every \( i = 1, 2, 3, ..., k, \). \( \square \)

References

Department of Mathematics, AFYON KOCATEPE UNIVERSITY, 03200, AFYONKARAHISAR, TURKEY
Email address: fnuray@aku.edu.tr