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ROTATION-COORDINATE DENSITY ESTIMATION ON SPHERES

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Abstract. We study a coordinate-based kernel density estimator for spherical directional data. The construction represents each point of the sphere by a selected rotation and performs smoothing with a distance inherited from the rotation group. The method is formulated as a section-based procedure, not as a fully intrinsic rotation-invariant estimator. We derive local bias, variance, mean squared error and consistency properties away from the polar coordinate singularities. To address numerical instability near the poles, we add a locally normalized pole-corrected version inspired by boundary correction techniques. Simulations show that this correction substantially reduces polar errors.

Keywords. Directional statistics, spherical data, kernel density estimation, rotation coordinates.
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1. Introduction

Statistical analysis on manifolds is an important part of modern data analysis, since many observations naturally take values on nonlinear spaces rather than on Euclidean domains. Directional data, axial observations and orientation measurements arise in geology, meteorology, astronomy, structural biology, computer vision and shape analysis. Among the corresponding sample spaces, the two-dimensional sphere $\mathbb{S}^2$ is one of the most important examples; see, for instance, [13, 12, 1, 17].

Nonparametric density estimation on manifolds has been studied from several viewpoints. Spectral approaches based on the Laplace–Beltrami operator go back to [7]. Kernel methods on Riemannian manifolds were developed, among others, by [18], while tangent-space and exponential-map based approaches were further investigated in [10]. These intrinsic constructions are geometrically natural, but they may involve geometric quantities such as normal coordinates, exponential maps, geodesic distances and volume-density corrections. Such quantities are explicit on standard spaces such as spheres and tori, but may be difficult to compute

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in more involved geometric settings. Recent contributions in kernel-based nonparametric estimation also include adaptive recursive density estimation under right censoring [15] and local linear estimation of conditional density and mode under censoring and truncation [5]. Although these works address different statistical settings, they reflect the continuing role of kernel smoothing and asymptotic analysis in modern nonparametric estimation.

The present paper studies a different, more specific construction on $\mathbb{S}^2$. We use a rotation-induced coordinate section of the quotient representation

$$\mathbb{S}^2 \simeq SO(3)/SO(2).$$

The idea is to associate each point of the sphere with a selected rotation in $SO(3)$, and then to smooth the data through the ambient distance inherited from the rotation group. In the explicit calculations we use the section

$$s(\theta, \phi) = R_z(\phi)R_x(\theta), \qquad (\theta, \phi) \in [0, \pi] \times [0, 2\pi),$$

which sends the north pole to the point $F(\theta, \phi) = s(\theta, \phi)e_3$. Its image is a two-dimensional surface $\Sigma \subset SO(3)$.

It is important to state the scope of the method precisely. The estimator studied here is not a fully intrinsic or rotation-invariant spherical kernel estimator. It depends on the chosen section, or equivalently on the chosen system of rotation coordinates. The north pole is used only as a convenient reference point for the formulas; a different reference point would lead to a different section. Thus the estimator should be understood as a rotation-coordinate estimator rather than as a substitute for intrinsic rotation-invariant methods in directional statistics.

This section-based viewpoint also explains the main numerical difficulty. Away from the poles, the spherical density can be lifted to a density on Σ, estimated there, and then brought back to the sphere by a Jacobian correction. This asymptotic Jacobian correction contains a factor involving $1/\sin\theta$, which becomes unstable near the two polar coordinate singularities. To address this issue, we introduce a pole-corrected normalization inspired by boundary correction and endpoint correction methods for kernel density estimators [16, 19, 8, 14, 4, 3, 20, 21]. The correction replaces the asymptotic Jacobian factor by the exact local kernel mass computed with respect to the spherical surface measure.

The contributions of the paper are as follows:

- we formulate a rotation-coordinate estimator on $\mathbb{S}^2$ using a section of the quotient map $SO(3) \to SO(3)/SO(2) \simeq \mathbb{S}^2$;
- we derive the induced metric on the surface of selected rotations and show explicitly that, in the chosen coordinates, it is flat;
- we establish local asymptotic expansions for the bias, variance and mean squared error away from the polar singularities;
- we introduce a pole-corrected normalization which avoids the explicit singular factor near the poles and is asymptotically equivalent to the original rotation-coordinate estimator away from them;
- we provide simulations showing the practical effect of this correction in global and polar-cap errors.

The remainder of the paper is organized as follows. Section 2 presents the geometric framework and the surface Σ. Section 3 introduces the statistical model, the original rotation-coordinate estimator and its pole-corrected version. Section 4 gives the local asymptotic analysis. Section 5 reports the simulation study. Section 6 concludes the paper.

2. Geometric framework

Let the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ be defined by

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

The rotation group is

$$SO(3) = \{A \in GL(3, \mathbb{R}) \mid A^\top A = I,\ \det A = 1\}.$$

The group $SO(3)$ acts transitively on $\mathbb{S}^2$. This action is not free: the stabilizer of the north pole $e_3 = (0, 0, 1)$ is the subgroup of rotations around the z-axis, which is isomorphic to $SO(2)$. Hence

$$\mathbb{S}^2 \simeq SO(3)/SO(2).$$

In what follows, we use an explicit section of this quotient map in order to obtain concrete rotation coordinates. For $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$, consider the rotation matrices

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \qquad R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and define the two-dimensional subset

$$\Sigma = \{R_z(\phi)R_x(\theta) : \theta \in [0, \pi],\ \phi \in [0, 2\pi)\} \subset SO(3).$$

This surface will serve as the geometric parameter space throughout the paper. Its link with the sphere is described by the mapping

$$F : [0, \pi] \times [0, 2\pi) \to \mathbb{S}^2, \qquad F(\theta, \phi) = R_z(\phi)R_x(\theta)e_3,$$

where $e_3 = (0, 0, 1)$ is the north pole of the sphere. A direct computation yields

$$R_x(\theta)e_3 = \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix},$$

and therefore

$$F(\theta, \phi) = \begin{pmatrix} \sin\theta\,\sin\phi \\ -\sin\theta\,\cos\phi \\ \cos\theta \end{pmatrix}.$$

Hence F is the usual spherical parametrization, up to a harmless sign convention in the azimuthal variable, and its image is the whole sphere $\mathbb{S}^2$. As usual in spherical coordinates, the parameter ϕ is not identifiable at the poles $\theta = 0$ and $\theta = \pi$. For this reason, the pointwise asymptotic considerations developed later are understood away from these singular points.

We now equip Σ with the geometry inherited from $SO(3)$. Let $\mathfrak{so}(3)$ denote the Lie algebra of $SO(3)$, and consider the standard basis

$$A_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad A_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

On $\mathfrak{so}(3)$, we consider the positive definite bilinear form

$$\langle X, Y \rangle = -\operatorname{tr}(XY), \qquad X, Y \in \mathfrak{so}(3),$$

which is a positive multiple of the negative Killing form and therefore induces a bi-invariant Riemannian metric on $SO(3)$. Through the parametrization

$$\Psi(\theta, \phi) = R_z(\phi) R_x(\theta),$$

this metric induces a Riemannian structure on Σ.

Using left trivialization, we first observe that

$$\Psi^{-1}\partial_\theta \Psi = R_x(-\theta)R_z(-\phi)\, R_z(\phi)\, \partial_\theta R_x(\theta) = R_x(-\theta)\partial_\theta R_x(\theta) = A_x.$$

Similarly,

$$\Psi^{-1}\partial_\phi \Psi = R_x(-\theta)R_z(-\phi)\, \partial_\phi R_z(\phi) R_x(\theta) = R_x(-\theta) A_z R_x(\theta).$$

Since the metric is Ad-invariant, it follows that

$$\big\langle R_x(-\theta)A_z R_x(\theta), R_x(-\theta)A_z R_x(\theta)\big\rangle = \langle A_z, A_z \rangle.$$

Now,

$$A_x^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad A_z^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that

$$\langle A_x, A_x \rangle = -\operatorname{tr}(A_x^2) = 2, \qquad \langle A_z, A_z \rangle = -\operatorname{tr}(A_z^2) = 2.$$

Moreover,

$$\langle A_x, R_x(-\theta)A_z R_x(\theta)\rangle = \langle R_x(\theta)A_x R_x(-\theta), A_z\rangle = \langle A_x, A_z\rangle = -\operatorname{tr}(A_x A_z) = 0.$$

Therefore, in the coordinates (θ, ϕ), the induced metric tensor on Σ is diagonal and takes the form

$$g_\Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \text{that is,} \qquad g_\Sigma = 2(d\theta^2 + d\phi^2).$$

Its determinant is equal to 4, and the associated area element is thus

$$dA_\Sigma = \sqrt{\det(g_\Sigma)}\, d\theta\, d\phi = 2\, d\theta\, d\phi.$$

Consequently,

$$\operatorname{Area}(\Sigma) = \int_0^\pi \int_0^{2\pi} 2\, d\phi\, d\theta = 4\pi^2.$$

For the statistical construction developed below, we also need a distance on Σ. Let $d_{SO(3)}$ be the geodesic distance on $SO(3)$ associated with the above bi-invariant metric. Since the norm of the infinitesimal generator of a rotation with angular speed 1 is $\sqrt{2}$, one has, for any $R_1, R_2 \in SO(3)$,

$$d_{SO(3)}(R_1, R_2) = \sqrt{2}\,\alpha(R_1^\top R_2),$$

where $\alpha(Q) \in [0, \pi]$ denotes the rotation angle of Q. Equivalently,

$$d_{SO(3)}(R_1, R_2) = \sqrt{2}\ \arccos\left(\frac{\operatorname{tr}(R_1^\top R_2) - 1}{2}\right).$$

We then define a distance on Σ by restriction,

$$d_\Sigma(R_1, R_2) := d_{SO(3)}(R_1, R_2), \qquad R_1, R_2 \in \Sigma.$$

Remark 2.1. Since $d_{SO(3)}$ is a metric on $SO(3)$, its restriction immediately defines a metric on Σ. It is worth stressing, however, that this distance is not the intrinsic geodesic distance on the sphere $\mathbb{S}^2$. Rather, it is the ambient geodesic distance inherited from $SO(3)$ and restricted to the parameter surface Σ. This extrinsic viewpoint is the coordinate choice on which the kernel construction below is based; it should not be confused with an intrinsic rotation-invariant estimator on the sphere.

For materials on Riemannian geometry, Lie groups and Lie algebras, we refer to [2, 11, 9, 6].

3. Statistical model and estimator

Let $X_1, \ldots, X_n$ be independent and identically distributed random variables taking values on $\mathbb{S}^2$, with density f with respect to the spherical surface measure

$$d\sigma(\theta, \phi) = \sin\theta\, d\theta\, d\phi.$$

Using the section introduced in the previous section, we write

$$X_i = F(\theta_i, \phi_i), \qquad R_i := R_z(\phi_i) R_x(\theta_i) \in \Sigma, \qquad i = 1, \ldots, n.$$

The distribution of R_i is absolutely continuous with respect to the induced area element $dA_\Sigma = 2\, d\theta\, d\phi$. Its density g is given by

$$g(\theta, \phi) = \frac{f(\theta, \phi)\sin\theta}{2}.$$

Indeed,

$$g(\theta, \phi) dA_\Sigma = \frac{f(\theta, \phi)\sin\theta}{2}\, 2 d\theta d\phi = f(\theta, \phi) d\sigma(\theta, \phi).$$

Thus, away from the polar coordinate singularities,

$$f(\theta, \phi) = \frac{2g(\theta, \phi)}{\sin\theta}, \qquad \theta \in (0, \pi).$$

Let $K : [0, \infty) \to \mathbb{R}$ be a bounded radial kernel and define

$$m_0(K) := \int_{\mathbb{R}^2} K(\|u\|) du, \qquad m_2(K) := \int_{\mathbb{R}^2} u_1^2 K(\|u\|) du,$$

and

$$R(K) := \int_{\mathbb{R}^2} K(\|u\|)^2 du.$$

We assume

$$0 < m_0(K) < \infty, \qquad m_2(K) < \infty, \qquad R(K) < \infty.$$

For $R \in \Sigma$, the kernel estimator of the lifted density g is

$$\widehat{g}_h(R) = \frac{1}{nh^2 m_0(K)} \sum_{i=1}^{n} K\left(\frac{d_\Sigma(R_i, R)}{h}\right). \tag{3.1}$$

The corresponding original rotation-coordinate estimator of the spherical density is

$$\widehat{f}_h^O(\theta, \phi) = \frac{2}{\sin\theta}\, \widehat{g}_h\big(R_z(\phi)R_x(\theta)\big), \qquad \theta \in (0, \pi). \tag{3.2}$$

This estimator is coordinate-based. It depends on the chosen section $(\theta, \phi) \mapsto R_z(\phi)R_x(\theta)$ and is therefore not invariant under arbitrary rotations of the sphere. This dependence is a structural limitation of the present construction.

Pole-corrected normalization. The factor $2/\sin\theta$ in (3.2) is the asymptotic Jacobian correction associated with the chosen rotation coordinates. It is appropriate away from the poles, but becomes unstable near $\theta = 0$ and $\theta = \pi$. These points are not boundaries of $\mathbb{S}^2$, but they play the role of boundary points for the chosen coordinate section. Following the spirit of boundary corrections for kernel density estimation [8, 3, 4, 20, 21], we replace the asymptotic normalization by the exact local kernel mass on the sphere.

For $(\theta, \phi) \in [0, \pi] \times [0, 2\pi)$, define

$$B_h(\theta, \phi) = \int_0^\pi \int_0^{2\pi} K\left(\frac{d_\Sigma(R_z(\psi)R_x(t), R_z(\phi)R_x(\theta))}{h}\right) \sin t \, d\psi \, dt. \tag{3.3}$$

The pole-corrected estimator is then

$$\widehat{f}_h^{PC}(\theta, \phi) = \frac{\dfrac{1}{n}\displaystyle\sum_{i=1}^{n} K\left(\frac{d_\Sigma(R_i, R_z(\phi)R_x(\theta))}{h}\right)}{B_h(\theta, \phi)}. \tag{3.4}$$

This definition avoids the explicit singular division by $\sin\theta$. It keeps the same distance d_Σ and the same rotation-coordinate smoothing, but normalizes by the actual kernel mass available on the sphere.

Proposition 3.1. *Fix $\varepsilon > 0$. Uniformly for $\theta \in [\varepsilon, \pi - \varepsilon]$,*

$$B_h(\theta, \phi) = \frac{h^2}{2} m_0(K) \sin\theta + o(h^2), \qquad h \to 0.$$

Consequently, away from the poles, $\widehat{f}_h^{PC}$ is asymptotically equivalent to the original rotation-coordinate estimator $\widehat{f}_h^O$.

Proof. The proof follows from the same local change of variables used in the bias calculation. For $\theta \in [\varepsilon, \pi - \varepsilon]$, the local expansion of d_Σ gives

$$t = \theta + \frac{h}{\sqrt{2}} u_1, \qquad \psi = \phi + \frac{h}{\sqrt{2}} u_2,$$

so that $dt\, d\psi = (h^2/2) du$. Since $\sin t = \sin\theta + o(1)$ uniformly on compact subsets away from the poles, (3.3) yields

$$B_h(\theta, \phi) = \frac{h^2}{2} \sin\theta \int_{\mathbb{R}^2} K(\|u\|) du + o(h^2),$$

which proves the result. □

The asymptotic results in the next section are stated for the original rotation-coordinate estimator $\widehat{f}_h^O$. By Proposition 3.1, they also describe the leading behaviour of the pole-corrected estimator on compact subsets away from the poles. Near the poles, the corrected estimator should be used in numerical work because it avoids the unstable factor $1/\sin\theta$.

4. Asymptotic properties

Throughout this section, we work under the following assumptions:

(A1) The density f is of class C^2 on $\mathbb{S}^2$. Consequently, g is of class C^2 on every compact subset of $(0, \pi) \times [0, 2\pi)$.

(A2) The kernel K is bounded, radial and integrable, with finite second moment.

(A3) The bandwidth satisfies

$$h = h_n \to 0, \qquad nh^2 \to \infty.$$

4.1. **Bias.** The first step consists in describing the local behavior of the distance d_Σ in the coordinates (θ, ϕ). This provides the Euclidean approximation needed for the kernel expansion.

Proposition 4.1. *Fix* $(\theta_0, \phi_0) \in (0, \pi) \times [0, 2\pi)$ *and set*

$$R_0 = R_z(\phi_0) R_x(\theta_0).$$

Then, as $(\theta, \phi) \to (\theta_0, \phi_0)$,

$$d_\Sigma\big(R_z(\phi)R_x(\theta), R_0\big)^2 = 2\Big[(\theta - \theta_0)^2 + (\phi - \phi_0)^2\Big] + O\big(\|(\theta - \theta_0, \phi - \phi_0)\|^4\big).$$

Proof. Put $R(\theta, \phi) = R_z(\phi)R_x(\theta)$, $\Delta\theta = \theta - \theta_0$ and $\Delta\phi = \phi - \phi_0$. A direct computation gives

$$\mathrm{tr}(R_0^\top R(\theta, \phi)) = \cos(\Delta\phi) + \cos(\Delta\theta) + \cos\theta_0 \cos(\theta_0 + \Delta\theta)\big(1 - \cos(\Delta\phi)\big).$$

Expanding the trigonometric terms at the origin yields

$$\mathrm{tr}(R_0^\top R(\theta, \phi)) = 3 - (\Delta\theta^2 + \Delta\phi^2) + O\big(\|(\Delta\theta, \Delta\phi)\|^4\big).$$

Since

$$d_\Sigma(R(\theta, \phi), R_0) = \sqrt{2}\, \arccos\left(\frac{\mathrm{tr}(R_0^\top R(\theta, \phi)) - 1}{2}\right)$$

and $\arccos(1-u) = \sqrt{2u} + O(u^{3/2})$, the asserted expansion follows after squaring. $\square$

Proposition 4.1 shows that, locally, the ambient distance d_Σ is equivalent to the Euclidean norm in the coordinates (θ, ϕ), up to the scaling factor $\sqrt{2}$. This is the key ingredient in the bias expansion.

Theorem 4.2. *Fix* $(\theta_0, \phi_0) \in (\varepsilon, \pi - \varepsilon) \times [0, 2\pi)$ *and let*

$$R_0 = R_z(\phi_0) R_x(\theta_0).$$

Under assumptions (A1)–(A3),

$$\mathbb{E}\big[\widehat{g}_h(R_0)\big] = g(\theta_0, \phi_0) + \frac{h^2 m_2(K)}{4 m_0(K)} \big(\partial_{\theta\theta} g + \partial_{\phi\phi} g\big)(\theta_0, \phi_0) + o(h^2).$$

Proof. Using $dA_\Sigma = 2 d\theta d\phi$, the expectation is

$$\frac{1}{h^2 m_0(K)} \int_0^{2\pi} \int_0^{\pi} K\left(\frac{d_\Sigma(R_z(\phi)R_x(\theta), R_0)}{h}\right) g(\theta, \phi)\, 2 d\theta d\phi.$$

The kernel localizes the integral near (θ_0, ϕ_0). With $\theta = \theta_0 + h u_1/\sqrt{2}$ and $\phi = \phi_0 + h u_2/\sqrt{2}$, Proposition 4.1 gives $d_\Sigma/h = \|u\| + o(1)$ and $2 d\theta d\phi = h^2 du$. A second-order Taylor expansion of g gives

$$g\left(\theta_0 + \frac{h}{\sqrt{2}} u_1, \phi_0 + \frac{h}{\sqrt{2}} u_2\right) = g_0 + \frac{h}{\sqrt{2}} \nabla g_0 \cdot u + \frac{h^2}{4} u^\top H_g u + o(h^2 \|u\|^2).$$

The odd terms vanish by radial symmetry of K, while $\int u_1^2 K(\|u\|) du = \int u_2^2 K(\|u\|) du = m_2(K)$ and $\int K(\|u\|) du = m_0(K)$. This gives the stated expansion. $\square$

The preceding result can now be transferred to the original density f on the sphere.

Theorem 4.3. *Under the assumptions of Theorem 4.2,*

$$\mathbb{E}\big[\widehat{f}_h^O(\theta_0, \phi_0)\big] = f(\theta_0, \phi_0) + \frac{h^2 m_2(K)}{4 m_0(K)} \mathcal{L} f(\theta_0, \phi_0) + o(h^2),$$

where

$$\mathcal{L} f = \partial_{\theta\theta} f + \partial_{\phi\phi} f + 2 \cot\theta\, \partial_\theta f - f.$$

Proof. Recall that

$$g(\theta, \phi) = \frac{1}{2} f(\theta, \phi) \sin\theta, \qquad \widehat{f}_h^O(\theta, \phi) = \frac{2}{\sin\theta} \widehat{g}_h(\theta, \phi).$$

Thus,

$$\mathbb{E}\big[\widehat{f}_h^O(\theta_0, \phi_0)\big] = \frac{2}{\sin\theta_0} \mathbb{E}\big[\widehat{g}_h(\theta_0, \phi_0)\big].$$

Applying Theorem 4.2, we obtain

$$\mathbb{E}\big[\widehat{f}_h^O(\theta_0, \phi_0)\big] = \frac{2}{\sin\theta_0} \left[g_0 + \frac{h^2 m_2(K)}{4 m_0(K)} (g_{\theta\theta} + g_{\phi\phi})_0 + o(h^2) \right].$$

Since $2g_0/\sin\theta_0 = f(\theta_0,\phi_0)$, it remains to compute
$$\frac{2}{\sin\theta}(g_{\theta\theta}+g_{\phi\phi}).$$

From
$$g = \frac{1}{2} f \sin\theta,$$
we derive
$$g_\theta = \frac{1}{2}\big(f_\theta \sin\theta + f\cos\theta\big),$$
hence
$$g_{\theta\theta} = \frac{1}{2}\big(f_{\theta\theta}\sin\theta + 2f_\theta\cos\theta - f\sin\theta\big).$$
Also,
$$g_{\phi\phi} = \frac{1}{2} f_{\phi\phi}\sin\theta.$$
Adding these two expressions,
$$g_{\theta\theta}+g_{\phi\phi} = \frac{1}{2}\sin\theta\,(f_{\theta\theta}+f_{\phi\phi}-f) + f_\theta\cos\theta.$$
Therefore,
$$\frac{2}{\sin\theta}(g_{\theta\theta}+g_{\phi\phi}) = f_{\theta\theta}+f_{\phi\phi}-f+2\cot\theta\, f_\theta.$$
Substituting this identity in the preceding expansion proves the result. □

Remark 4.4. The differential operator
$$\mathcal{L}f = f_{\theta\theta}+f_{\phi\phi}+2\cot\theta\, f_\theta - f$$
is not the Laplace–Beltrami operator on $\mathbb{S}^2$. This again reflects the fact that the estimator is built from the ambient geometry of $SO(3)$ restricted to Σ, rather than from the intrinsic spherical metric.

4.2. **Variance, MSE and related consequences.** We next turn to the stochastic fluctuations of the estimator. The first result concerns the variance of the lifted estimator on Σ.

Theorem 4.5. *Under assumptions* (A1)–(A3),
$$\operatorname{Var}\big(\widehat{g}_h(R_0)\big) = \frac{g(\theta_0,\phi_0)}{nh^2}\frac{R(K)}{m_0(K)^2} + o\left(\frac{1}{nh^2}\right).$$

Proof. Let
$$Y_i = \frac{1}{h^2 m_0(K)} K\left(\frac{d_\Sigma(R_i,R_0)}{h}\right), \qquad \widehat{g}_h(R_0) = n^{-1}\sum_{i=1}^n Y_i.$$
Then $\operatorname{Var}(\widehat{g}_h(R_0)) = n^{-1}\operatorname{Var}(Y_1)$. The term $(\mathbb{E}Y_1)^2$ is of order one and is negligible compared with h^{-2}. Using the same local change of variables as in the bias calculation,
$$\mathbb{E}Y_1^2 = \frac{1}{h^4 m_0(K)^2}\int_\Sigma K^2\left(\frac{d_\Sigma(R,R_0)}{h}\right) g(R)\,dA_\Sigma(R) = \frac{g(\theta_0,\phi_0)}{h^2 m_0(K)^2}R(K) + o(h^{-2}).$$
Multiplication by $1/n$ proves the result. □

Passing back to the sphere immediately yields the corresponding variance expansion for $\widehat{f}_h^O$.

Theorem 4.6. *Under the assumptions of Theorem 4.5,*

$$\operatorname{Var}\big(\widehat{f}_h^O(\theta_0, \phi_0)\big) = \frac{2f(\theta_0, \phi_0)}{nh^2 \sin\theta_0} \frac{R(K)}{m_0(K)^2} + o\left(\frac{1}{nh^2}\right).$$

Proof. Since

$$\widehat{f}_h^O(\theta_0, \phi_0) = \frac{2}{\sin\theta_0} \widehat{g}_h(\theta_0, \phi_0),$$

we have

$$\operatorname{Var}\big(\widehat{f}_h^O(\theta_0, \phi_0)\big) = \frac{4}{\sin^2\theta_0} \operatorname{Var}\big(\widehat{g}_h(\theta_0, \phi_0)\big).$$

Apply Theorem 4.5 and use

$$g(\theta_0, \phi_0) = \frac{1}{2} f(\theta_0, \phi_0) \sin\theta_0.$$

This gives

$$\operatorname{Var}\big(\widehat{f}_h^O(\theta_0, \phi_0)\big) = \frac{4}{\sin^2\theta_0} \left[\frac{1}{nh^2} \frac{1}{2} f(\theta_0, \phi_0) \sin\theta_0 \frac{R(K)}{m_0(K)^2} + o\left(\frac{1}{nh^2}\right)\right],$$

which simplifies to the announced formula. □

Combining the bias and variance expansions yields the usual pointwise mean squared error behavior.

Corollary 4.7. *For each fixed* $(\theta_0, \phi_0) \in (\varepsilon, \pi - \varepsilon) \times [0, 2\pi)$,

$$\operatorname{MSE}\big(\widehat{f}_h^O(\theta_0, \phi_0)\big) = O(h^4) + O\left(\frac{1}{nh^2}\right).$$

Proof. The mean squared error decomposes as

$$\operatorname{MSE}(\widehat{f}_h^O) = \operatorname{Bias}(\widehat{f}_h^O)^2 + \operatorname{Var}(\widehat{f}_h^O).$$

Theorem 4.3 gives a bias of order $O(h^2)$, while Theorem 4.6 gives a variance of order $O((nh^2)^{-1})$. The result follows immediately. □

We also obtain pointwise consistency.

Theorem 4.8. *Let* $(\theta_0, \phi_0) \in (\varepsilon, \pi - \varepsilon) \times [0, 2\pi)$ *be fixed. Under assumptions* (A1)–(A3),

$$\widehat{f}_h^O(\theta_0, \phi_0) \xrightarrow{\mathbb{P}} f(\theta_0, \phi_0).$$

Proof. By Theorem 4.3, the bias tends to zero because $h \to 0$. By Theorem 4.6, the variance tends to zero because $nh^2 \to \infty$. Therefore, for every $\eta > 0$, Chebyshev's inequality yields

$$\mathbb{P}\big(|\widehat{f}_h^O(\theta_0, \phi_0) - f(\theta_0, \phi_0)| > \eta\big) \le \frac{\operatorname{Var}(\widehat{f}_h^O(\theta_0, \phi_0))}{\eta^2} + \frac{\operatorname{Bias}(\widehat{f}_h^O(\theta_0, \phi_0))^2}{\eta^2},$$

and both terms tend to zero. □

Finally, balancing the leading terms h^4 and $(nh^2)^{-1}$ in the mean squared error yields the usual two-dimensional optimal bandwidth order

$$h_{\text{opt}} \asymp n^{-1/6},$$

with corresponding pointwise optimal rate

$$\text{MSE} \asymp n^{-2/3}.$$

5. Simulation study

The simulation study illustrates the behaviour of the rotation-coordinate estimator and the effect of the pole correction introduced in (3.4). We compare three estimators: the original rotation-coordinate estimator $\widehat{f}_h^O$, the pole-corrected estimator $\widehat{f}_h^{PC}$, and a classical spherical kernel estimator $\widehat{f}_h^{cl}$ based on the geodesic angle on the sphere.

All experiments use von Mises–Fisher densities and samples of size $n = 300$. Four models are considered. The first two are unimodal densities centered at the north and south poles, both with concentration parameter 12. These models directly test the behaviour at the two singularities of the chosen coordinate section. The third model is a symmetric bimodal mixture with equal weights and concentration parameter 14, centered at $(0, 0, 1)$ and $(0, 1, 0)$. The fourth model is an asymmetric three-component mixture with weights $0.5, 0.3, 0.2$, concentrations $16, 10, 20$, and centers $(0, 0, 1)$, $(1, 0, 0)$, and $(0, 1, 0)$, respectively.

The same radial Gaussian kernel is used for the three estimators. The distance d_Σ is evaluated through the trace formula

$$d_\Sigma(R_1, R_2) = \sqrt{2}\,\arccos\left(\frac{\operatorname{tr}(R_1^\top R_2) - 1}{2}\right).$$

The denominator $B_h(\theta, \phi)$ in (3.3) is computed by numerical quadrature on the spherical grid. All estimated densities are finally normalized numerically so that their integral over the sphere is equal to one. In addition to the global integrated squared error, we compute local integrated squared errors on polar caps of angular radius 15°. These local errors measure the improvement near the coordinate singularities.

Model	h	Original	Pole-corrected	Classical
North-pole unimodal	0.22	1.114650	0.059753	0.085423
South-pole unimodal	0.22	1.205559	0.042756	0.071408
Bimodal	0.20	1.213900	0.046471	0.057949
Asymmetric mixture	0.22	1.104897	0.033193	0.049858

Table 1. Global integrated squared errors for the original rotation-coordinate, pole-corrected and classical estimators.

The global errors reported in Table 1 show a substantial improvement after the pole correction. The original rotation-coordinate estimator is strongly affected by the factor $1/\sin\theta$, especially in models with mass near one of the poles. Replacing this factor by the exact local kernel mass stabilizes the procedure and yields considerably smaller errors in all four experiments. The classical spherical estimator remains the intrinsic rotation-invariant benchmark, but in these selected simulations the pole-corrected version gives smaller integrated errors.

Model	Cap	Original	Pole-corrected	Classical
North-pole unimodal	North	1.075904	0.034389	0.057110
South-pole unimodal	South	1.168263	0.026029	0.050834
Bimodal	North	1.085834	0.009126	0.015399
Asymmetric mixture	North	1.039015	0.012308	0.022964

TABLE 2. Polar-cap integrated squared errors near the relevant coordinate singularities. The cap radius is $15°$.

Table 2 focuses on the regions where the original asymptotic normalization is most unstable. The correction sharply reduces the polar-cap error. For example, in the north-pole unimodal model, the polar-cap error decreases from 1.075904 to 0.034389. Similarly, in the south-pole model, the error near the south pole decreases from 1.168263 to 0.026029. These results support the role of B_h as a pole-stabilizing normalization.

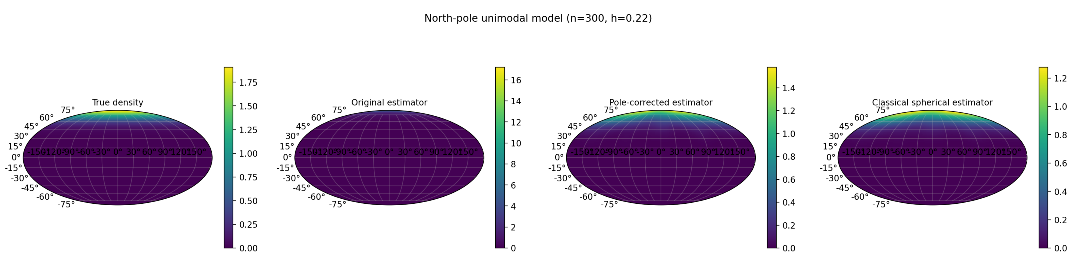

FIGURE 1. North-pole unimodal model: true density, original rotation-coordinate estimator, pole-corrected estimator and classical spherical estimator.

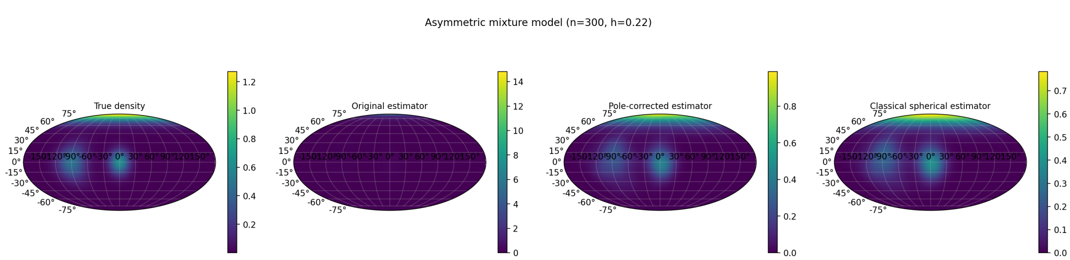

FIGURE 2. Asymmetric mixture model: true density, original rotation-coordinate estimator, pole-corrected estimator and classical spherical estimator.

Figures 1 and 2 show the same phenomenon visually. The original estimator exhibits artificial inflation near the polar coordinate singularities, while the pole-corrected estimator preserves the rotation-coordinate construction and removes the most visible instability. These simulations should not be interpreted as a general dominance result over intrinsic spherical estimators. Rather, they show that the proposed correction directly addresses the numerical weakness of the original section-based estimator.

6. Conclusion

We have studied a coordinate-based kernel density estimator on the sphere obtained from a rotation-induced section of the quotient representation $\mathbb{S}^2 \simeq SO(3)/SO(2)$. The construction lifts the density estimation problem to a two-dimensional surface of selected rotations, performs smoothing with the ambient distance inherited from the rotation group, and then returns to the spherical density.

The revised presentation clarifies the exact nature of the method. The induced metric on the selected surface is flat in the chosen rotation coordinates. Consequently, the original rotation-coordinate estimator is locally equivalent to a two-dimensional Euclidean kernel estimator in these coordinates, followed by a spherical Jacobian correction. The estimator is therefore not an intrinsic rotation-invariant spherical estimator; it is a section-based estimator whose output depends on the chosen coordinate section.

We derived local bias, variance, mean squared error and pointwise consistency away from the polar coordinate singularities. The variance expansion explains why the original asymptotic-Jacobian form becomes unstable near the poles. To address this issue, we introduced a pole-corrected normalization inspired by boundary and endpoint correction methods for kernel density estimation. This correction replaces the singular Jacobian factor by the exact local kernel mass over the sphere. It is asymptotically equivalent to the original rotation-coordinate estimator away from the poles, while the simulations show that it greatly reduces polar-cap errors in finite samples.

Future work may investigate multi-section versions that combine several reference points on the sphere, data-driven bandwidth selection, and extensions to more general homogeneous spaces where section-based constructions may offer an alternative to explicit volume-density corrections.

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