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TIME-FRACTIONAL WAVE-DIFFUSION-WAVE EQUATION: INVERSE PROBLEM IN A CYLINDRICAL DOMAIN

ERKINJON KARIMOV[1*] and MUZAFFAR TOSHPULATOV [2]

Abstract. This paper addresses the inverse source problem for a mixed fractional wave-diffusion-wave equation posed in a cylindrical domain. The governing equation involves a time-dependent variable-order fractional derivative, which enables the model to effectively capture temporal transitions between wave-like and diffusive behaviors. The solution is constructed in the form of a Fourier-Bessel series. By employing the method of separation of variables together with fundamental properties of Bessel functions, we analyze the uniform convergence of the resulting infinite series. This analysis ultimately leads to a rigorous proof of the existence of a solution

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1. Introduction

In this section, we briefly outline the motivation behind our study of the wave-diffusion-wave process. We first explain how gas flow phenomena can be modeled using wave and diffusion equations. Next, we provide a concise justification for adopting a model that combines wave-diffusion-wave equations, arguing that this framework offers a more accurate representation of the underlying physical processes. Finally, we describe our approach to investigating the inverse source problem associated with the resulting mixed-type equation.

1.1. **Gas flow in porous media.**

Classical diffusion models based on Fick's law assume that pressure disturbances propagate at infinite speed, an assumption that is physically unrealistic for gas flow in tight geological formations such as shale reservoirs. Experimental observations instead reveal anomalous transport behavior, commonly referred to as anomalous diffusion. In gas flow through porous media, anomalous diffusion arises from the complex and heterogeneous structure of the medium, causing particle transport to deviate from classical Fickian dynamics and exhibit a nonlinear dependence on time. The occurrence

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* Corresponding author

and specific type of anomalous diffusion depend on various properties of the porous medium [1], including porosity, specific surface area, and fractal dimension parameters. Consequently, the mathematical description of such processes extends beyond classical partial differential equations. Fractional-order diffusion equations, which replace traditional integer-order time and space derivatives with fractional derivatives, provide an effective modeling framework. This approach accurately captures memory effects and complex interactions within the system [2].

1.2. **Why wave-diffusion-wave equations?** When gas is injected into a porous medium or a pressure pulse is generated, the resulting disturbance propagates as a wave at very short time scales, characterized by a finite propagation speed. As time progresses, the disturbance gradually spreads and exhibits diffusive behavior. This transition from wave-like to diffusive dynamics is precisely captured by wave-diffusion-wave equations.

In particular, for a fractional time order $\alpha \in (1, 2)$, the governing equation continuously interpolates between the classical wave equation as $\alpha \to 2^-$ and the diffusion equation as $\alpha \to 1^+$ [3]. Such transitional behavior has been experimentally observed in gas dynamics, for instance, in isothermal gas dissolution processes [4]. Consequently, employing a single fractional-order equation that encompasses both regimes provides a more realistic and unified modeling framework than switching between separate wave and diffusion models.

However, in some physical scenarios, the governing dynamics may temporarily reduce to a purely diffusive process before reverting to wave-like behavior. To account for such situations, we consider a time-switched system of wave-diffusion-wave equations, which allows for successive transitions between wave and diffusion regimes over different time intervals.

1.3. **Inverse source problem and our approach.** We investigate the inverse source problem for a fractional-order wave-diffusion-wave equation, which generalizes classical second-order hyperbolic and parabolic models. Related inverse problems for fractional evolution equations have been studied in, for example, [5, 6]. In the present work, the time-fractional derivative is defined in the sense of the regularized Prabhakar (Prabhakar-Caputo) operator, a nonlocal fractional derivative characterized by a three-parameter Mittag-Leffler kernel [7, 8]. This operator extends the classical Caputo derivative by incorporating additional degrees of freedom, allowing for a more accurate representation of complex memory effects, multi-scale relaxation phenomena, and temporal heterogeneity inherent in anomalous transport processes.

From a physical perspective, inverse source problems of this type arise in applications such as subsurface gas flow, where it is necessary to identify the location, intensity, or temporal profile of a gas leakage or injection source based on indirect boundary or interior measurements.

To solve the problem, we employ the method of separation of variables, which reduces the original inverse problem to a sequence of fractional differential equations with respect to time and corresponding spatial eigenvalue problems. The spatial components are associated with a Sturm-Liouville system, while the temporal components involve fractional operators with Prabhakar-type memory kernels. The resulting solution is expressed as an infinite series expansion in terms of the eigenfunctions of the spatial operator. We derive explicit analytical representations for the unknown source function and establish sufficient conditions for the existence of a solution. Furthermore, by utilizing the asymptotic properties of special functions and appropriate estimates for the Mittag-Leffler kernel, we prove the uniform convergence of the obtained series solution.

2. Formulation of the problem

We consider the following mixed equation

$$f(r)=\begin{cases} {}^{PC}D_{0t}^{\alpha_1,\beta_1,\gamma_1,\delta}u(t,r)-\Delta u(t,r), & (t,r)\in\Omega_1,\\ {}^{PC}D_{T_1t}^{\alpha_2,\beta_2,\gamma_2,\delta}u(t,r)-\Delta u(t,r), & (t,r)\in\Omega_2,\\ {}^{PC}D_{T_2t}^{\alpha_3,\beta_3,\gamma_3,\delta}u(t,r)-\Delta u(t,r), & (t,r)\in\Omega_3\end{cases} \tag{2.1}$$

in a cylindrical domain $\Omega=\Omega_1\cup\Omega_2\cup\Omega_3\cup N_1\cup N_2$.

Here $N_1=\{(t,r),t=T_1,0<r<1\}\,,N_2=\{(t,r),t=T_2,0<r<1\}\,,$ $\Omega_1=\{(t,r):0<r<1,0<t<T_1\}\,,\Omega_2=\{(t,r):0<r<1,T_1<t<T_2\}\,,$ $\Omega_3=\{(t,r):0<r<1,T_2<t<T\}\,,\ T,T_1,T_2,r,\delta,\alpha_i,\gamma_i,\beta_i(i=1,2,3)\in\mathbb{R}$ such that $0<T_1<T_2<T,1<\beta_1\le 2,0<\beta_2\le 1,1<\beta_3\le 2$, $\Delta u(t,r)$ is the Laplaci,an which can be written in polar coordinates as

$$\Delta u(t,r)=u_{rr}+\frac{1}{r}u_r,$$

$${}^{PC}D_{at}^{\alpha,\beta,\gamma,\delta}y(t)={}^{P}I_{at}^{\alpha,m-\beta,-\gamma,\delta}\frac{d^m}{dt^m}y(t)$$

is the regularized Prabhakar fractional derivative [9], $m-1<\beta\le m\ \ (m\in\mathbb{N})$,

$${}^{P}I_{at}^{\alpha,\beta,\gamma,\delta}y(t)=\int_a^t(t-\xi)^{\beta-1}E_{\alpha,\beta}^{\gamma}\left[\delta(t-\xi)^{\alpha}\right]y(\xi)d\xi,t>a \tag{2.2}$$

is the Prabhakar fractional integral and $E_{\alpha,\beta}^{\gamma}(z)=\sum\limits_{k=0}^{\infty}\dfrac{(\gamma)_k z^k}{\Gamma(\alpha k+\beta)k!}$ is the Prabhakar function [7].

Direct problem. Find a *regular solution* of (2.1) in Ω that satisfies the following

- initial condition

$$u(0,r)=\varphi(r),0\le r\le 1, \tag{2.3}$$

- boundary conditions

$$\left[r\frac{\partial u(t,r)}{\partial r}\right]_{r=0}=0,u(t,r)|_{r=1}=0,0\le t\le T, \tag{2.4}$$

- transmitting conditions

$$\lim_{t\to T_1+0}{}^{PC}D_{T_1t}^{\alpha_2,\beta_2,\gamma_2,\delta}u(t,r)=\lim_{t\to T_1-0}u_t(t,r),\quad 0\le r\le 1, \tag{2.5}$$

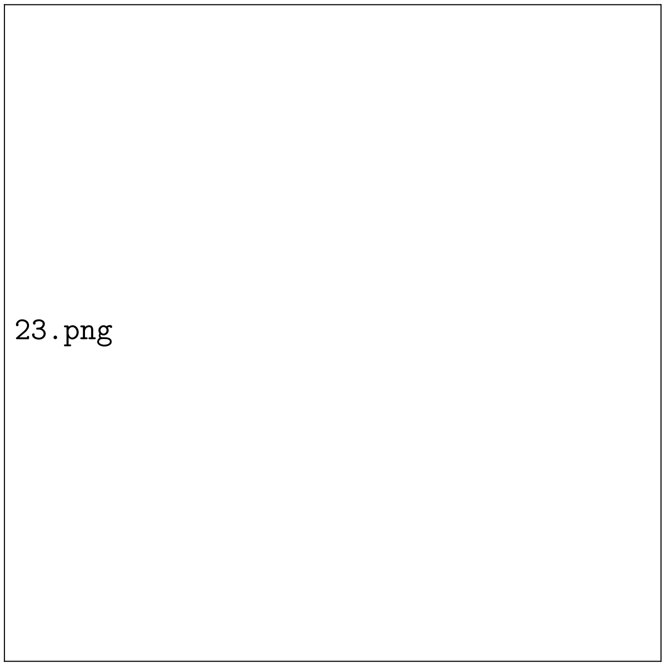

Figure 1: Mixed domain Ω.

$$\lim_{t\to T_2+0} u_t(t,r) = \lim_{t\to T_2-0} {}^{PC}D_{T_1 t}^{\alpha_2,\beta_2,\gamma_2,\delta} u(t,r), \quad 0<r<1. \tag{2.6}$$

Here $f(r), \varphi(r)$ are given functions.

A similar direct problem for an almost similar equation in a rectangular domain was investigated by us in [10].

Definition 2.1. Function $u(t,r)$ is called a *regular solution* of (2.1) in Ω, if

$$u(t,r)\in C(\overline{\Omega}), u_{rr}(t,r)\in C(\Omega), {}^{PC}D_{0t}^{\alpha_1,\beta_1,\gamma_1,\delta}u(t,r)\in C(\Omega_1),$$

$${}^{PC}D_{T_1 t}^{\alpha_2,\beta_2,\gamma_2,\delta}u(t,r)\in C(\Omega_2\cup J_1\cup J_2), {}^{PC}D_{T_2 t}^{\alpha_3,\beta_3,\gamma_3,\delta}u(t,r)\in C(\Omega_3)$$

and satisfies (2.1) in Ω.

If the function $f(r)$ is unknown, we can formulate the following inverse problem by imposing an additional condition on the time variable.

Inverse source problem. To find a pair of functions $\{u(t,r), f(r)\}$ satisfying (2.1)-(2.6) together with the additional time measurement

$$u(\xi,r)=\psi(r), T_1<\xi\le T_2, \tag{2.7}$$

where $\psi(r)$ is a given function.

Using the Fourier method, we seek a solution for the homogeneous case of equation (2.1) in the following form:

$$u(t,r)=R(r)T(t). \tag{2.8}$$

Substituting (2.8) into (2.1) at $f(r)\equiv 0$, we get the following equalities:

$$\frac{{}^{PC}D_{0t}^{\alpha_1,\beta_1,\gamma_1,\delta}T(t)}{T(t)} = \frac{R'(r)}{rR(r)} + \frac{R''(r)}{R(r)} = -\mu^2, \qquad 0 < t < T_1,$$

$$\frac{{}^{PC}D_{T_1t}^{\alpha_2,\beta_2,\gamma_2,\delta}T(t)}{T(t)} = \frac{R'(r)}{rR(r)} + \frac{R''(r)}{R(r)} = -\mu^2, \qquad T_1 < t < T_2,$$

$$\frac{{}^{PC}D_{T_2t}^{\alpha_3,\beta_3,\gamma_3,\delta}T(t)}{T(t)} = \frac{R'(r)}{rR(r)} + \frac{R''(r)}{R(r)} = -\mu^2, \qquad T_2 < t < T.$$

Here μ is a real number. It is easy to see that from these equalities one can get

$$r^2R''(r) + rR'(r) + (\mu^2 r^2)R(r) = 0. \tag{2.9}$$

The resulting equation is the Bessel equation [11]. The solution of (2.9) satisfying the conditions $rR'(r)|_{r=0} = 0, R(1) = 0$ has the form

$$R_k = J_0(\mu_k r), k = 1, 2, 3, ...,$$

where J_0 is zero-order Bessel functions of the first kind [11]. Due to the asymptotic form of the Bessel function for the case $\nu = 0$ [11], the value of μ_k can be written as

$$\mu_k = k\pi - \frac{\pi}{4}.$$

Let us note several inverse source problems for a classic combination of wave and diffusion equations, called parabolic-hyperbolic equations. In [12], the inverse source problem was investigated for a classical parabolic-hyperbolic equation; later on in [13], a similar problem was studied for a degenerate parabolic-hyperbolic equation with nonlocal boundary conditions.

We would also like to note several works in which inverse source problems for a combination of fractional wave and sub-diffusion equations were investigated. In [14], the inverse source problem was studied for a system of sub-diffusion equations and wave equation with a uniformly elliptic operator in space and Caputo fractional-order operator in the time variable. The uniqueness of the solution to the nonlocal inverse source problem for the mixed-type equation was proved in [15], and another inverse source problem with different nonlocal conditions for such an equation was targeted in [16]. In [17], the authors consider a space-dependent inverse source problem for a combination of a time-fractional pseudo-parabolic equation and a hyperbolic equation with mixed derivatives. The time-dependent inverse source problem for a system of time-fractional wave and diffusion equations was considered in [18]. We also note the work [5], in which the inverse source problem for a parabolic-hyperbolic equation with a time-fractional derivative was studied in a cylindrical domain. Similar inverse source problems for parabolic-hyperbolic equations with loaded terms were investigated in [19] (linear) and in [20] (nonlinear).

3. Formal solution.

We search for the solution to the problem in the form of a Fourier-Bessel series as follows:

$$u(t,r)=\sum_{k=1}^{\infty} {}_1U_k(t)J_0(\mu_k r),\quad (t,r)\in\Omega_1, \tag{3.1}$$

$$u(t,r)=\sum_{k=1}^{\infty} {}_2U_k(t)J_0(\mu_k r),\quad (t,r)\in\Omega_2, \tag{3.2}$$

$$u(t,r)=\sum_{k=1}^{\infty} {}_3U_k(t)J_0(\mu_k r),\quad (t,r)\in\Omega_3. \tag{3.3}$$

Here ${}_1U_k(t), {}_2U_k(t), {}_3U_k(t)$ are unknown functions to be found.

Substituting (3.1),(3.2),(3.3) into the equation (2.1), we obtain the following fractional differential equations:

$${}^{PC}D_{0t}^{\alpha_1,\beta_1,\gamma_1,\delta}\,{}_1U_k(t)+\lambda_k\cdot{}_1U_k(t)=f_k,\quad 0<t<T_1, \tag{3.4}$$

$${}^{PC}D_{T_1t}^{\alpha_2,\beta_2,\gamma_2,\delta}\,{}_2U_k(t)+\lambda_k\cdot{}_2U_k(t)=f_k,\quad T_1\le t\le T_2, \tag{3.5}$$

$${}^{PC}D_{T_2t}^{\alpha_3,\beta_3,\gamma_3,\delta}\,{}_3U_k(t)+\lambda_k\cdot{}_3U_k(t)=f_k,\quad T_2<t<T. \tag{3.6}$$

The value of λ_k, which appears in differential equations (10), (11), and (12), is equal to

$$\lambda_k=-\left(4k-1\right)^2\left(\frac{\pi}{4}\right)^2,$$

f_k are the Fourier-Bessel coefficients of the function

$$f(r)=\sum_{k=1}^{\infty} f_k J_0(\mu_k r),\quad (t,r)\in\Omega \tag{3.7}$$

given by

$$f_k=\frac{2}{J_1^2(\mu_k)}\int\limits_0^1 f(r)J_0(\mu_k r)r\,dr.$$

For convenience in our calculations, we will continue to write this value as λ_k instead of substituting it.

General solutions of (3.4), (3.5) and (3.6) can be written as follows [22]:

$$\begin{gathered}{}_1U_k\left(t\right)={}_1A_k+{}_1A_k\cdot\lambda_k\cdot t^{\beta_1}\cdot\Gamma\left(\gamma_1\right)\cdot E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k t^{\beta_1}\\ \delta t^{\alpha_1}\end{array}\right)+\\ +{}_2A_k t+{}_2A_k\cdot\lambda_k\cdot t^{\beta_1+1}\Gamma\left(\gamma_1\right)\cdot E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k t^{\beta_1}\\ \delta t^{\alpha_1}\end{array}\right)+\\ +\Gamma\left(\gamma_1\right)\cdot f_k\cdot t^{\beta_1}\cdot E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k t^{\beta_1}\\ \delta t^{\alpha_1}\end{array}\right),\end{gathered} \tag{3.8}$$

$$\begin{aligned} {}_2U_k(t) &= {}_3A_k + {}_3A_k\lambda_k (t-T_1)^{\beta_2}\,\Gamma(\gamma_2)\times \\ &\times\ E_2\left(\begin{array}{c|c} \gamma_2,\gamma_2,1;1,0 & \lambda_k (t-T_1)^{\beta_2} \\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta (t-T_1)^{\alpha_2} \end{array}\right)+ \\ &+\ \Gamma(\gamma_2)\, f_k (t-T_1)^{\beta_2} E_2\left(\begin{array}{c|c} \gamma_2,\gamma_2,1;1,0 & \lambda_k (t-T_1)^{\beta_2} \\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta (t-T_1)^{\alpha_2} \end{array}\right), \end{aligned} \tag{3.9}$$

$$\begin{aligned} &{}_3U_k(t) = {}_4A_k + {}_5A_k (t-T_2) + {}_4A_k\lambda_k (t-T_2)^{\beta_3}\,\Gamma(\gamma_3)\times \\ &\times E_2\left(\begin{array}{c|c} \gamma_3,\gamma_3,1;1,0 & \lambda_k (t-T_2)^{\beta_3} \\ \beta_3+1,\beta_3,\alpha_3;\gamma_3,\gamma_3;1,1 & \delta (t-T_2)^{\alpha_3} \end{array}\right) + {}_5A_k\lambda_k (t-T_2)^{\beta_3+1}\,\Gamma(\gamma_3)\times \\ &\times E_2\left(\begin{array}{c|c} \gamma_3,\gamma_3,1;1,0 & \lambda_k (t-T_2)^{\beta_3} \\ \beta_3+2,\beta_3,\alpha_3;\gamma_3,\gamma_3;1,1 & \delta (t-T_2)^{\alpha_3} \end{array}\right) + \Gamma(\gamma_3)\,(t-T_2)^{\beta_3} f_k\times \\ &\times E_2\left(\begin{array}{c|c} \gamma_3,\gamma_3,1;1,0 & \lambda_k (t-T_2)^{\beta_3} \\ \beta_3+1,\beta_3,\alpha_3;\gamma_3,\gamma_3;1,1 & \delta (t-T_2)^{\alpha_3} \end{array}\right). \end{aligned} \tag{3.10}$$

Here

$$\begin{aligned} &E_2\left(\begin{array}{c|c} \gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2 & x \\ \delta_1,\alpha_3,\beta_2;\delta_2,\alpha_4;\delta_3,\beta_3 & y \end{array}\right) = \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\Gamma(\gamma_1+\alpha_1 m+\beta_1 n)\,\Gamma(\gamma_2+\alpha_2 m)}{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\delta_1+\alpha_3 m+\beta_2 n)}\cdot\frac{x^m}{\Gamma(\delta_2+\alpha_4 m)}\cdot\frac{y^n}{\Gamma(\delta_3+\beta_3 n)} \end{aligned}$$

is a bivariate Mittag-Leffler function [21], ${}_1A_k, {}_2A_k, {}_3A_k, {}_4A_k, {}_5A_k$ are unknown constants to be found.

Using $\lim\limits_{t\to T_1^-} {}_1U_k(t) = \lim\limits_{t\to T_1^+} {}_2U_k(t)$ (which follows from regularity condition $u(t,r)\in C(\overline{\Omega})$ and (3.1)-(3.2)) and due to (3.8), (3.9) we will obtain

$$\begin{aligned} &{}_1A_k + {}_2A_kT_1 + \Gamma(\gamma_1)\,{}_1A_k\cdot\lambda_k\cdot T_1^{\beta_1} E_2\left(\begin{array}{c|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1} \end{array}\right)+ \\ &+\Gamma(\gamma_1)\,{}_2A_k\cdot\lambda_k\cdot T_1^{\beta_1+1} E_2\left(\begin{array}{c|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1} \end{array}\right)+ \\ &+\Gamma(\gamma_1)\cdot f_k\cdot T_1^{\beta_1} E_2\left(\begin{array}{c|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1} \end{array}\right) = {}_3A_k. \end{aligned} \tag{3.11}$$

Now we use the condition (2.3) which can be written as ${}_1U_k(0)=\varphi_k$, where

$$\varphi_k = \frac{2}{J_1(\mu_k)}\int_0^1 r\varphi(r)J_0(\mu_k r)dr.$$

This will give us ${}_1A_k=\varphi_k$. To use condition (2.5), we need to take the derivative of ${}_1U_k(t)$ with respect to the variable t. After performing the calculations, we obtain the following expression for the derivative of ${}_1U_k(t)$:

$$ {}_1U_k'(t) = {}_2A_k + \Gamma(\gamma_1)\,{}_1A_k \cdot \lambda_k \cdot t^{\beta_1-1} E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array} \middle| \begin{array}{c} \lambda_k t^{\beta_1} \\ \delta t^{\alpha_1} \end{array} \right) + $$
$$ + \Gamma(\gamma_1)\,{}_2A_k \cdot \lambda_k \cdot t^{\beta_1} E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array} \middle| \begin{array}{c} \lambda_k t^{\beta_1} \\ \delta t^{\alpha_1} \end{array} \right) + $$
$$ + \Gamma(\gamma_1) \cdot f_k \cdot t^{\beta_1-1} E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array} \middle| \begin{array}{c} \lambda_k t^{\beta_1} \\ \delta t^{\alpha_1} \end{array} \right). $$

Now we need to find $\lim\limits_{t\to T_1^+} {}^{PC}D_{T_1 t}^{\alpha_2,\beta_2,\gamma_2,\delta}\,{}_2U_k(t)$. For this aim, we let $t \to T_1^+$ and from the equation (3.9) considering ${}_2U_k(T_1) = {}_3A_k$, we will get

$$ \lim_{t\to T_1^+} {}^{PC}D_{T_1 t}^{\alpha_2,\beta_2,\gamma_2,\delta}\,{}_2U_k(t) = \lim_{t\to T_1^+}\left[f_k - \lambda_k \cdot {}_2U_k(t) \right] = f_k - \lambda_k \cdot {}_3A_k. $$

Hence, the transmitting condition (2.5) gives us

$$ f_k - \lambda_k \cdot {}_3A_k = {}_2A_k + {}_1A_k \cdot \lambda_k \cdot E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) \times $$
$$ \times T_1^{\beta_1-1}\Gamma(\gamma_1) + {}_2A_k \lambda_k T_1^{\beta_1} \Gamma(\gamma_1) E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) + $$
$$ + \Gamma(\gamma_1) \cdot f_k \cdot T_1^{\beta_1-1} E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right). \tag{3.12} $$

Substituting (3.11) into the left-hand side of (3.12), then dividing the resulting equality by λ_k, and provided that $\Delta_k \neq 0$ and, we obtain ${}_2A_k$ as follows (see A1 in the Appendix section):

$$ \begin{aligned} {}_2A_k = \frac{1}{\Delta_k}\Bigg[& \frac{f_k}{\lambda_k} - {}_1A_k - {}_1A_k \cdot \lambda_k \cdot E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) \times \\ & \times T_1^{\beta_1} \cdot \Gamma(\gamma_1) - {}_1A_k T_1^{\beta_1-1} \Gamma(\gamma_1) E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) - \\ & - \Gamma(\gamma_1) f_k T_1^{\beta_1} E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) - \\ & - \frac{\Gamma(\gamma_1) f_k T_1^{\beta_1-1}}{\lambda_k} E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) \Bigg]. \end{aligned} \tag{3.13} $$

Here

$$ \begin{aligned} \Delta_k = & \frac{1}{\lambda_k} + T_1 + T_1^{\beta_1} \cdot \Gamma(\gamma_1) E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) + \\ & + T_1^{\beta_1+1} \cdot \lambda_k \cdot \Gamma(\gamma_1) E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right). \end{aligned} \tag{3.14} $$

Lemma 3.1. [23] *If $\beta_1 > \gamma_1, \alpha_1 > 0$, then the equality $\lim\limits_{k\to\infty} \Delta_k = T_1 > 0$ holds.*

This lemma allows us to state that Δ_k can be zero only at a countable number of k.

Now, let us start finding the coefficients ${}_4A_k$ and ${}_5A_k$. For this aim we will use $u(t,r) \in C(\overline{\Omega})$ which gives us:

$$\lim_{t\to T_2^-} {}_2U_k(t) = \lim_{t\to T_2^+} {}_3U_k(t). \tag{3.15}$$

Based on relation (3.15) and equalities (3.9), (3.10), we find the coefficient ${}_4A_k$ as follows:

$$\begin{aligned}
&{}_3A_k + {}_3A_k\lambda_k\left(T_2-T_1\right)^{\beta_2}\Gamma\left(\gamma_2\right)\times\\
&\times E_2\left(\begin{array}{c|c} \gamma_2,\gamma_2,1;1,0 & \lambda_k\left(T_2-T_1\right)^{\beta_2}\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(T_2-T_1\right)^{\alpha_2}\end{array}\right)+\\
&+\Gamma\left(\gamma_2\right)f_k\left(T_2-T_1\right)^{\beta_2}E_2\left(\begin{array}{c|c} \gamma_2,\gamma_2,1;1,0 & \lambda_k\left(T_2-T_1\right)^{\beta_2}\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(T_2-T_1\right)^{\alpha_2}\end{array}\right) = {}_4A_k.
\end{aligned} \tag{3.16}$$

To use the transmitting condition (2.6), we need the limit value of ${}^{PC}D_{T_1t}^{\alpha_2,\beta_2,\gamma_2,\delta}u(t,r)$ as $t\to T_2^-$. For this aim, we pass to the limit as $t\to T_2^-$ and from the equation (3.5) we obtain:

$$\begin{aligned}
\lim_{t\to T_2^-} {}^{PC}D_{T_1t}^{\alpha_2,\beta_2,\gamma_2,\delta}\,{}_2U_k(t) &= \lim_{t\to T_2^-} -\lambda_k\cdot{}_2U_k(t)+f_k = -\lambda_k\lim_{t\to T_2^-}{}_2U_k(t)+f_k =\\
&= -\lambda_k\cdot{}_4A_k+f_k.
\end{aligned} \tag{3.17}$$

To find ${}_3U_k'(t)$,we differentiate (3.10) and obtain

$$\begin{aligned}
&{}_3U_k'\left(t\right) = {}_4A_k\lambda_k\left(t-T_2\right)^{\beta_3-1}\Gamma\left(\gamma_3\right)\times\\
&\times E_2\left(\begin{array}{c|c} \gamma_3,\gamma_3,1;1,0 & \lambda_k\left(t-T_2\right)^{\beta_3}\\ \beta_3,\beta_3,\alpha_3;\gamma_3,\gamma_3;1,1 & \delta\left(t-T_2\right)^{\alpha_3}\end{array}\right)+{}_5A_k+{}_5A_k\lambda_k\left(t-T_2\right)^{\beta_3}\Gamma(\gamma_3)\times\\
&\times E_2\left(\begin{array}{c|c} \gamma_3,\gamma_3,1;1,0 & \lambda_k\left(t-T_2\right)^{\beta_3}\\ \beta_3+1,\beta_3,\alpha_3;\gamma_3,\gamma_3;1,1 & \delta\left(t-T_2\right)^{\alpha_3}\end{array}\right)+\\
&+\Gamma(\gamma_3)\left(t-T_2\right)^{\beta_3-1}\times f_k\cdot E_2\left(\begin{array}{c|c} \gamma_3,\gamma_3,1;1,0 & \lambda_k\left(t-T_2\right)^{\beta_3}\\ \beta_3,\beta_3,\alpha_3;\gamma_3,\gamma_3;1,1 & \delta\left(t-T_2\right)^{\alpha_3}\end{array}\right).
\end{aligned} \tag{3.18}$$

Based on the transmitting condition (2.6) and equations (3.17),(3.18) we deduce that

$$ {}_5A_k = -\lambda_k\cdot{}_4A_k+f_k. \tag{3.19}$$

4. Inverse source problem.

Using condition (2.7) and equality (3.9), we can write down the following:

$$\begin{aligned}
\psi_k = {}&{}_3A_k\lambda_k\left(\xi-T_1\right)^{\beta_2}\Gamma\left(\gamma_2\right)E_2\left(\begin{array}{c|c} \gamma_2,\gamma_2,1;1,0 & \lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)+\\
&+{}_3A_k+\Gamma\left(\gamma_2\right)f_k\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{c|c} \gamma_2,\gamma_2,1;1,0 & \lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right).
\end{aligned}$$

From this equation, we determine f_k:

$$f_k = \frac{\psi_k - {}_3A_k}{\Gamma(\gamma_2)(\xi - T_1)^{\beta_2} E_2 \left(\begin{array}{c|c} \gamma_2, \gamma_2, 1; 1, 0 & \lambda_k (\xi - T_1)^{\beta_2} \\ \beta_2 + 1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 & \delta (\xi - T_1)^{\alpha_2} \end{array} \right)} - \\ - \frac{{}_3A_k \lambda_k (\xi - T_1)^{\beta_2} \Gamma(\gamma_2) E_2 \left(\begin{array}{c|c} \gamma_2, \gamma_2, 1; 1, 0 & \lambda_k (\xi - T_1)^{\beta_2} \\ \beta_2 + 1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 & \delta (\xi - T_1)^{\alpha_2} \end{array} \right)}{\Gamma(\gamma_2)(\xi - T_1)^{\beta_2} E_2 \left(\begin{array}{c|c} \gamma_2, \gamma_2, 1; 1, 0 & \lambda_k (\xi - T_1)^{\beta_2} \\ \beta_2 + 1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 & \delta (\xi - T_1)^{\alpha_2} \end{array} \right)}. \tag{4.1}$$

We will introduce the following notation:

$${}_1M_k = \Gamma(\gamma_2)(\xi - T_1)^{\beta_2} E_2 \left(\begin{array}{c|c} \gamma_2, \gamma_2, 1; 1, 0 & \lambda_k (\xi - T_1)^{\beta_2} \\ \beta_2 + 1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 & \delta (\xi - T_1)^{\alpha_2} \end{array} \right). \tag{4.2}$$

As a result, we can rewrite equation (4.1) as follows:

$$f_k = \frac{\psi_k - {}_3A_k (1 + \lambda_k \cdot {}_1M_k)}{{}_1M_k} \text{ or } {}_3A_k (1 + \lambda_k \cdot {}_1M_k) + f_k \cdot {}_1M_k = \psi_k. \tag{4.3}$$

Note that here ${}_1M_k \neq 0$, if $\alpha_2 = 1$ and $\gamma_2 = \beta_2$ (see for the proof in [24]).

Based on equation (3.11), we can write ${}_3A_k$ in a simpler form as follows:

$${}_3A_k = {}_1A_k \left(1 + \lambda_k T_1^{\beta_1} \Gamma(\gamma_1) E_2 \left(\begin{array}{c|c} \gamma_1, \gamma_1, 1; 1, 0 & \lambda_k T_1^{\beta_1} \\ \beta_1 + 1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 & \delta T_1^{\alpha_1} \end{array} \right) \right) + \\ + {}_2A_k \left(T_1 + \lambda_k T_1^{\beta_1 + 1} \Gamma(\gamma_1) E_2 \left(\begin{array}{c|c} \gamma_1, \gamma_1, 1; 1, 0 & \lambda_k T_1^{\beta_1} \\ \beta_1 + 2, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 & \delta T_1^{\alpha_1} \end{array} \right) \right) + \\ + \Gamma(\gamma_1) f_k T_1^{\beta_1} E_2 \left(\begin{array}{c|c} \gamma_1, \gamma_1, 1; 1, 0 & \lambda_k T_1^{\beta_1} \\ \beta_1 + 1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 & \delta T_1^{\alpha_1} \end{array} \right). \tag{4.4}$$

For convenience, we will introduce the following notations:

$${}_2M_k = T_1^{\beta_1} \cdot \Gamma(\gamma_1) \cdot E_2 \left(\begin{array}{c|c} \gamma_1, \gamma_1, 1; 1, 0 & \lambda_k T_1^{\beta_1} \\ \beta_1 + 1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 & \delta T_1^{\alpha_1} \end{array} \right), \tag{4.5}$$

$${}_3M_k = T_1 + T_1^{\beta_1 + 1} \cdot \lambda_k \cdot \Gamma(\gamma_1) \cdot E_2 \left(\begin{array}{c|c} \gamma_1, \gamma_1, 1; 1, 0 & \lambda_k T_1^{\beta_1} \\ \beta_1 + 2, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 & \delta T_1^{\alpha_1} \end{array} \right). \tag{4.6}$$

Let us rewrite the form of equation (4.4):

$${}_2A_k \cdot {}_3M_k - {}_3A_k + f_k \cdot {}_2M_k = -{}_1A_k (1 + \lambda_k \cdot {}_2M_k) \tag{4.7}$$

We simplify (3.12) as follows and introduce the notation:

$${}_1A_k \cdot \lambda_k \cdot T_1^{\beta_1 - 1} \Gamma(\gamma_1) E_2 \left(\begin{array}{c|c} \gamma_1, \gamma_1, 1; 1, 0 & \lambda_k T_1^{\beta_1} \\ \beta_1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 & \delta T_1^{\alpha_1} \end{array} \right) + {}_2A_k \times \\ \times \left(1 + \lambda_k \cdot T_1^{\beta_1} \Gamma(\gamma_1) E_2 \left(\begin{array}{c|c} \gamma_1, \gamma_1, 1; 1, 0 & \lambda_k T_1^{\beta_1} \\ \beta_1 + 1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 & \delta T_1^{\alpha_1} \end{array} \right) \right) + \lambda_k \cdot {}_3A_k + \\ + f_k \left(\Gamma(\gamma_1) T_1^{\beta_1 - 1} E_2 \left(\begin{array}{c|c} \gamma_1, \gamma_1, 1; 1, 0 & \lambda_k T_1^{\beta_1} \\ \beta_1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 & \delta T_1^{\alpha_1} \end{array} \right) - 1 \right) = 0,$$

$$ {}_4M_k = T_1^{\beta_1 - 1} \cdot \Gamma\left(\gamma_1\right) \cdot E_2 \left(\begin{array}{c} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right). \tag{4.8} $$

As a result, we have the following expression:

$$ {}_2A_k(1 + \lambda_k \cdot {}_2M_k) + {}_3A_k \lambda_k + f_k({}_4M_k - 1) = -\lambda_k \cdot {}_1A_k \cdot {}_4M_k \tag{4.9} $$

Using equations (4.3),(4.7), and (4.9), we form a system of equations with respect to unknown coefficients and then, based on condition (2.3), we substitute φ_k instead of ${}_1A_k$.

$$ \begin{cases} {}_3A_k\left(1 + \lambda_k \cdot {}_1M_k\right) + f_k \cdot {}_1M_k = \psi_k, \\ {}_2A_k \cdot {}_3M_k - {}_3A_k + f_k \cdot {}_2M_k = -\varphi_k\left(1 + \lambda_k \cdot {}_2M_k\right), \\ {}_2A_k(1 + \lambda_k \cdot {}_2M_k) + {}_3A_k \lambda_k + f_k({}_4M_k - 1) = -\lambda_k \cdot \varphi_k \cdot {}_4M_k. \end{cases} \tag{4.10} $$

From the resulting system of equations (4.10), we determine the unknowns ${}_2A_k, {}_3A_k$ and f_k using Cramer's formulas.

In this case, we first identify the main determinant of the system. It is known that showing that this determinant is different from zero means that the system has a unique solution.

$$ \widetilde{\Delta}_k = \begin{vmatrix} 0 & 1 + \lambda_k \cdot {}_1M_k & {}_1M_k \\ {}_3M_k & -1 & {}_2M_k \\ 1 + \lambda_k \cdot {}_2M_k & \lambda_k & {}_4M_k - 1 \end{vmatrix} = {}_1M_k + {}_2M_k + {}_3M_k + 2\lambda_k \cdot {}_1M_k \times $$
$$ \times {}_2M_k + \lambda_k \cdot {}_2M_k^2 + \lambda_k^2 \cdot {}_1M_k \cdot {}_2M_k^2 + 2\lambda_k \cdot {}_1M_k \cdot {}_3M_k - {}_3M_k \cdot {}_4M_k - \lambda_k \cdot {}_1M_k \times $$
$$ \times {}_3M_k \cdot {}_4M_k. \tag{4.11} $$

The expression (4.11) we rewrite as follows (see A2 in the Appendix section):

$$ \widetilde{\Delta}_k = \Gamma\left(\gamma_2\right)\left(\xi - T_1\right)^{\beta_2} E_2 \left(\begin{array}{c} \gamma_2, \gamma_2, 1; 1, 0 \\ \beta_2 + 1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k \left(\xi - T_1\right)^{\beta_2} \\ \delta\left(\xi - T_1\right)^{\alpha_2} \end{array} \right) + T_1 + $$
$$ + T_1^{\beta_1} \Gamma(\gamma_1) E_2 \left(\begin{array}{c} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1 + 1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) + T_1^{\beta_1 + 1} \lambda_k \Gamma(\gamma_1) \times $$
$$ \times E_2 \left(\begin{array}{c} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1 + 2, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) + 2\lambda_k T_1^{\beta_1} \Gamma\left(\gamma_1\right) \Gamma\left(\gamma_2\right) \times $$
$$ \times \left(\xi - T_1\right)^{\beta_2} E_2 \left(\begin{array}{c} \gamma_2, \gamma_2, 1; 1, 0 \\ \beta_2 + 1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k \left(\xi - T_1\right)^{\beta_2} \\ \delta\left(\xi - T_1\right)^{\alpha_2} \end{array} \right) \times $$
$$ \times E_2 \left(\begin{array}{c} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1 + 1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) + \Gamma^2\left(\gamma_1\right) \lambda_k T_1^{2\beta_1} \times $$
$$ \times \left(E_2 \left(\begin{array}{c} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1 + 1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) \right)^2 + \lambda_k^2 \cdot \Gamma\left(\gamma_2\right) \cdot T_1^{2\beta_1} \times $$
$$ \times \Gamma^2\left(\gamma_1\right)\left(\xi - T_1\right)^{\beta_2} E_2 \left(\begin{array}{c} \gamma_2, \gamma_2, 1; 1, 0 \\ \beta_2 + 1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k \left(\xi - T_1\right)^{\beta_2} \\ \delta\left(\xi - T_1\right)^{\alpha_2} \end{array} \right) \times $$
$$ \times \left(E_2 \left(\begin{array}{c} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1 + 1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) \right)^2 + 2\lambda_k \Gamma\left(\gamma_2\right)\left(\xi - T_1\right)^{\beta_2} T_1 \times $$

$$\times E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)+2\lambda_k^2\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\times$$
$$\times\left(\xi-T_1\right)^{\beta_2}T_1^{\beta_1+1}E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)\times$$
$$\times E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)-T_1^{\beta_1}\Gamma(\gamma_1)\times$$
$$\times E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha}\end{array}\right)-\Gamma^2(\gamma_1)\lambda_kT_1^{2\beta_1}\times$$
$$\times E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)-T_1^{\beta_1}\Gamma(\gamma_1)\Gamma(\gamma_2)\lambda_k\times$$
$$\times\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)\times$$
$$\times E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)-\lambda_k^2\Gamma^2(\gamma_1)\Gamma(\gamma_2)T_1^{2\beta_1}\left(\xi-T_1\right)^{\beta_2}\times$$
$$\times E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right). \tag{4.12}$$

Lemma 4.1. *If* $\beta_1>\gamma_1,\beta_2>\gamma_2,\alpha_1>0,\alpha_2>0$, *then the equality* $\lim\limits_{k\to\infty}\widetilde{\Delta}_k=T_1>0$ *holds.*

Proof. Based on the expression of $\widetilde{\Delta}_k$ given above, we can write the following

$$\lim_{k\to\infty}\widetilde{\Delta}_k=\Gamma\left(\gamma_2\right)\lim_{k\to\infty}E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)\times$$
$$\times\left(\xi-T_1\right)^{\beta_2}+T_1^{\beta_1}\Gamma(\gamma_1)\lim_{k\to\infty}E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha}\end{array}\right)+$$
$$+T_1+T_1\Gamma(\gamma_1)\lim_{k\to\infty}\left[\lambda_kT_1^{\beta_1}\right]E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+$$
$$+2T_1^{\beta_1}\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\lim_{k\to\infty}E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times\lim_{k\to\infty}\left[\lambda_k\left(\xi-T_1\right)^{\beta_2}\right]E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)+$$
$$+\Gamma^2\left(\gamma_1\right)T_1^{\beta_1}\lim_{k\to\infty}\left[\lambda_kT_1^{\beta_1}\right]E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$

$$\times \lim_{k\to\infty} E_2 \left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1+1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) + \Gamma^2(\gamma_1)\Gamma(\gamma_2) T_1^{\beta_1} \times$$

$$\times \lim_{k\to\infty} \left[\lambda_k (\xi - T_1)^{\beta_2}\right] E_2 \left(\begin{array}{l} \gamma_2, \gamma_2, 1; 1, 0 \\ \beta_2+1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k (\xi - T_1)^{\beta_2} \\ \delta (\xi - T_1)^{\alpha_2} \end{array} \right) \times$$

$$\times \lim_{k\to\infty} \left[\lambda_k T_1^{\beta_1}\right] E_2 \left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1+1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) \times$$

$$\lim_{k\to\infty} E_2 \left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1+1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) + 2\Gamma(\gamma_2) T_1 \times$$

$$\times \lim_{k\to\infty} \left[\lambda_k (\xi - T_1)^{\beta_2}\right] E_2 \left(\begin{array}{l} \gamma_2, \gamma_2, 1; 1, 0 \\ \beta_2+1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k (\xi - T_1)^{\beta_2} \\ \delta (\xi - T_1)^{\alpha_2} \end{array} \right) +$$

$$+ 2\Gamma(\gamma_1)\Gamma(\gamma_2) T_1 \lim_{k\to\infty} \left[\lambda_k T_1^{\beta_1}\right] E_2 \left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1+2, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) \times$$

$$\times \lim_{k\to\infty} \left[\lambda_k (\xi - T_1)^{\beta_2}\right] E_2 \left(\begin{array}{l} \gamma_2, \gamma_2, 1; 1, 0 \\ \beta_2+1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k (\xi - T_1)^{\beta_2} \\ \delta (\xi - T_1)^{\alpha_2} \end{array} \right) -$$

$$\times \Gamma(\gamma_1) T_1^{\beta_1} \lim_{k\to\infty} E_2 \left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha} \end{array} \right) - \Gamma^2(\gamma_1) T_1^{\beta_1} \times$$

$$\times \lim_{k\to\infty} E_2 \left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) \times$$

$$\times \lim_{k\to\infty} \left[\lambda_k T_1^{\beta_1}\right] E_2 \left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1+2, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) - \Gamma(\gamma_1)\Gamma(\gamma_2) T_1^{\beta_1} \times$$

$$\times \lim_{k\to\infty} \left[\lambda_k (\xi - T_1)^{\beta_2}\right] E_2 \left(\begin{array}{l} \gamma_2, \gamma_2, 1; 1, 0 \\ \beta_2+1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k (\xi - T_1)^{\beta_2} \\ \delta (\xi - T_1)^{\alpha_2} \end{array} \right) \times$$

$$\times \lim_{k\to\infty} E_2 \left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) -$$

$$- \Gamma^2(\gamma_1)\Gamma(\gamma_2) T_1^{\beta_1} \lim_{k\to\infty} E_2 \left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1+2, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) \times$$

$$\times \lim_{k\to\infty} \left[\lambda_k (\xi - T_1)^{\beta_2}\right] E_2 \left(\begin{array}{l} \gamma_2, \gamma_2, 1; 1, 0 \\ \beta_2+1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k (\xi - T_1)^{\beta_2} \\ \delta (\xi - T_1)^{\alpha_2} \end{array} \right) \times$$

$$\times \lim_{k\to\infty} \left[\lambda_k T_1^{\beta_1}\right] E_2 \left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right).$$

Now, we will use the following integral representations for E_2:

$$E_2 \left(\begin{array}{l} \gamma_2, \gamma_2, 1; 1, 0 \\ \beta_2+1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k(\xi - T_1)^{\beta_2} \\ \delta(\xi - T_1)^{\alpha_2} \end{array} \right) = \frac{1}{\Gamma(\gamma_2)} \int_0^1 \int_0^\infty \zeta^{\frac{\beta_2+1}{2}-1} \eta^{\gamma_2-1} e^{-\eta} \times$$

$$\times (1-\zeta)^{\frac{\beta_2+1}{2}-1} e^{\frac{\beta_2+1}{2}, \gamma_2}_{\beta_2, -\gamma_2} \left(\lambda_k(\xi - T_1)^{\beta_2} \eta^{\gamma_2} \xi^{\beta_2}\right) e^{\frac{\beta_2+1}{2}, 1}_{\alpha_2, -1} \left(\delta(\xi - T_1)^{\alpha_2}(1-\zeta)^{\alpha_2}\eta\right) d\zeta d\eta,$$

$$E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array}\middle|\begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array}\right)=\frac{1}{\Gamma(\gamma_1)}\int\limits_0^1\int\limits_0^\infty \zeta^{\frac{\beta_1}{2}-1}\times$$

$$\times\,\eta^{\gamma_1-1}(1-\zeta)^{\frac{\beta_1}{2}-1}e^{-\eta}\,e^{\frac{\beta_1}{2},\gamma_1}_{\beta_1,-\gamma_1}\left(\lambda_k T_1^{\beta_1}\zeta^{\beta_1}\eta^{\gamma_1}\right)\,e^{\frac{\beta_1}{2},1}_{\alpha_1,-1}\left(\delta T_1^{\alpha_2}(1-\zeta)^{\alpha_2}\eta\right)d\zeta d\eta,$$

$$E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array}\middle|\begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array}\right)=\frac{1}{\Gamma(\gamma_1)}\int\limits_0^1\int\limits_0^\infty \zeta^{\frac{\beta_1+1}{2}-1}\times$$

$$\times\,\eta^{\gamma_1-1}(1-\zeta)^{\frac{\beta_1+1}{2}-1}e^{-\eta}\,e^{\frac{\beta_1+1}{2},\gamma_1}_{\beta_1,-\gamma_1}\left(\lambda_k T_1^{\beta_1}\zeta^{\beta_1}\eta^{\gamma_1}\right)\,e^{\frac{\beta_1+1}{2},1}_{\alpha_1,-1}\left(\delta T_1^{\alpha_1}(1-\zeta)^{\alpha_1}\eta\right)d\zeta d\eta,$$

$$E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array}\middle|\begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array}\right)=\frac{1}{\Gamma\left(\gamma_1\right)}\int\limits_0^1\int\limits_0^\infty \zeta^{\frac{\beta_1}{2}}\eta^{\gamma_1-1}(1-\zeta)^{\frac{\beta_1}{2}}\times$$

$$\times\,e^{-\eta}\,e^{\frac{\beta_1}{2}+1,\gamma_1}_{\beta_1,-\gamma_1}\left(\lambda_k T_1^{\beta_1}\eta^{\gamma_1}\zeta^{\beta_1}\right)e^{\frac{\beta_1}{2}+1,1}_{\alpha_1,-1}\left(\delta T_1^{\alpha_1}(1-\zeta)^{\alpha_1}\eta\right)d\zeta d\eta.$$

Then we will use the following asymptotic formulas [25]:

$$\lim_{|z|\to\infty} e^{\mu,\delta}_{\alpha,\beta}(z)=0,\quad \lim_{|z|\to\infty} z e^{\mu,\delta}_{\alpha,\beta}(z)=-\frac{1}{\Gamma\left(\mu-\alpha\right)\Gamma\left(\delta+\beta\right)}$$

to get

$$\lim_{k\to\infty}\widetilde{\Delta}_k=T_1>0.$$

□

We have demonstrated that the limit of $\widetilde{\Delta}_k$ as k approaches infinity is non-zero and positive. This provides that $\widetilde{\Delta}_k$ can be zero only at a countable number of k.

Now we will determine the unknown coefficients ${}_2A_k$, ${}_3A_k$ and the unknown function f_k (see A3 in the Appendix section):

$${}_2A_k=\frac{{}_2\Delta_k}{\widetilde{\Delta}_k}=\frac{1}{\widetilde{\Delta}_k}\Bigg[\psi_k-\psi_k T_1^{\beta_1-1}\Gamma(\gamma_1)E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array}\middle|\begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array}\right)+$$

$$+\,\varphi_k T_1^{\beta_1-1}\Gamma(\gamma_1)E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array}\middle|\begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array}\right)-\varphi_k-2\lambda_k\varphi_k\Gamma\left(\gamma_2\right)\times$$

$$\times\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{c} \gamma_2,\gamma_2,1;1,0 \\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 \end{array}\middle|\begin{array}{c} \lambda_k\left(\xi-T_1\right)^{\beta_2} \\ \delta\left(\xi-T_1\right)^{\alpha_2} \end{array}\right)-\Gamma(\gamma_1)\lambda_k\varphi_k T_1^{\beta_1}\times$$

$$\times\,E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array}\middle|\begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array}\right)-2\lambda_k^2\varphi_k\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}\times$$

$$\times\,T_1^{\beta_1}E_2\left(\begin{array}{c} \gamma_2,\gamma_2,1;1,0 \\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 \end{array}\middle|\begin{array}{c} \lambda_k\left(\xi-T_1\right)^{\beta_2} \\ \delta\left(\xi-T_1\right)^{\alpha_2} \end{array}\right)\times$$

$$\times\,E_2\left(\begin{array}{c} \gamma_1,\gamma_1,1;1,0 \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 \end{array}\middle|\begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array}\right)-$$

$$-\lambda_k\psi_k T_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\Bigg], \tag{4.13}$$

$$\begin{aligned}
{}_3A_k=\frac{{}_3\Delta_k}{\widetilde{\Delta}_k}=\frac{1}{\widetilde{\Delta}_k}\Bigg[&\psi_k T_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+\\
&+\lambda_k\psi_k T_1^{2\beta_1}\Gamma^2(\gamma_1)\times\left(E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\right)^2-\\
&-\lambda_k\varphi_k\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)T_1^{\beta_1}\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times\\
&\times E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)-\lambda_k^2T_1^{2\beta_1}\Gamma\left(\gamma_2\right)\times\\
&\times\varphi_k\Gamma^2\left(\gamma_1\right)\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times\\
&\times E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times\\
&\times E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)+\varphi_k\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}\times\\
&\times E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)+2\varphi_k\lambda_k\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\times\\
&\times T_1^{\beta_1}\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)\times\\
&\times E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+\varphi_k\lambda_k^2\Gamma^2\left(\gamma_1\right)\times\\
&\times\Gamma\left(\gamma_2\right)T_1^{2\beta_1}\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)\times\\
&\times\left(E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\right)^2-\psi_k T_1^{\beta_1}\Gamma(\gamma_1)\times\\
&\times E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)-\psi_k T_1^{2\beta_1}\Gamma^2(\gamma_1)\lambda_k\times\\
&\times E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha}\end{array}\right)\times\\
&\times E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+\psi_k T_1+\\
&+\psi_k\lambda_k T_1^{\beta_1+1}\Gamma(\gamma_1)E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\Bigg],
\end{aligned} \tag{4.14}$$

$$
\begin{aligned}
f_k = \frac{{}_f\Delta_k}{\widetilde{\Delta}_k} = \frac{1}{\widetilde{\Delta}_k}\Bigg[& -2\lambda_k\varphi_k T_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+\\
&+\psi_k-\varphi_k-\lambda_k^2\varphi_k T_1^{2\beta_1}\Gamma^2(\gamma_1)\left(E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\right)^2-\\
&-\lambda_k\varphi_k\Gamma(\gamma_2)(\xi-T_1)^{\beta_2}E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k(\xi-T_1)^{\beta_2}\\ \delta(\xi-T_1)^{\alpha_2}\end{array}\right)-\\
&-2\lambda_k^2\varphi_k T_1^{\beta_1}\Gamma(\gamma_1)\Gamma(\gamma_2)(\xi-T_1)^{\beta_2}E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times\\
&\times E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k(\xi-T_1)^{\beta_2}\\ \delta(\xi-T_1)^{\alpha_2}\end{array}\right)-\lambda_k^3 T_1^{2\beta_1}\varphi_k\Gamma^2(\gamma_1)\times\\
&\times\Gamma(\gamma_2)(\xi-T_1)^{\beta_2}E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k(\xi-T_1)^{\beta_2}\\ \delta(\xi-T_1)^{\alpha_2}\end{array}\right)\times\\
&\times\left(E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha}\end{array}\right)\right)^2+\psi_k\lambda_k T_1^{\beta_1}+\\
&+\psi_k\lambda_k^2 T_1^{\beta_1+1}\Gamma(\gamma_1)E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+\psi_k\lambda_k\times\\
&\times T_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+\lambda_k\varphi_k\Gamma(\gamma_1)T_1^{\beta_1}\times\\
&\times E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+\lambda_k^2\varphi_k\times\\
&\times\Gamma^2(\gamma_1)T_1^{2\beta_1}E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times\\
&\times E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+\lambda_k^2\varphi_k\Gamma(\gamma_1)\Gamma(\gamma_2)\times\\
&\times(\xi-T_1)^{\beta_2}T_1^{\beta_1}E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k(\xi-T_1)^{\beta_2}\\ \delta(\xi-T_1)^{\alpha_2}\end{array}\right)\times\\
&\times E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha}\end{array}\right)+\lambda_k^3\varphi_k\Gamma^2(\gamma_1)\times\\
&\times\Gamma(\gamma_2)T_1^{2\beta_1}E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k(\xi-T_1)^{\beta_2}\\ \delta(\xi-T_1)^{\alpha_2}\end{array}\right)\times\\
&\times E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times\\
&\times E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha}\end{array}\right)\Bigg].
\end{aligned}
\tag{4.15}
$$

Now we substitute (7.3) and (7.4) into equation (3.16), after certain simplifications we determine the coefficient ${}_4A_k$, as follows (see A4 in Appendix section):

$$
\begin{aligned}
{}_4A_k = \frac{1}{\widetilde{\Delta}_k}\Bigg[& \psi_k T_1^{\beta_1}\Gamma(\gamma_1) E_2\left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1+1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) + \\
& + \lambda_k \psi_k T_1^{2\beta_1}\Gamma^2(\gamma_1) \left(E_2\left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1+1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right)\right)^2 - \\
& - \lambda_k \varphi_k \Gamma(\gamma_1)\Gamma(\gamma_2) T_1^{\beta_1} (\xi - T_1)^{\beta_2} E_2\left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) \times \\
& \times E_2\left(\begin{array}{l} \gamma_2, \gamma_2, 1; 1, 0 \\ \beta_2+1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k (\xi - T_1)^{\beta_2} \\ \delta (\xi - T_1)^{\alpha_2} \end{array} \right) - \lambda_k^2 T_1^{2\beta_1}\Gamma(\gamma_2) \times \\
& \times \varphi_k \Gamma^2(\gamma_1) (\xi - T_1)^{\beta_2} E_2\left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) \times \\
& \times E_2\left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1+2, \beta_1, \alpha; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) \times \\
& \times E_2\left(\begin{array}{l} \gamma_2, \gamma_2, 1; 1, 0 \\ \beta_2+1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k (\xi - T_1)^{\beta_2} \\ \delta (\xi - T_1)^{\alpha_2} \end{array} \right) + \varphi_k \Gamma(\gamma_2) (\xi - T_1)^{\beta_2} \times \\
& \times E_2\left(\begin{array}{l} \gamma_2, \gamma_2, 1; 1, 0 \\ \beta_2+1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k (\xi - T_1)^{\beta_2} \\ \delta (\xi - T_1)^{\alpha_2} \end{array} \right) + 2\varphi_k \lambda_k \Gamma(\gamma_1)\Gamma(\gamma_2) \times \\
& \times T_1^{\beta_1} (\xi - T_1)^{\beta_2} E_2\left(\begin{array}{l} \gamma_2, \gamma_2, 1; 1, 0 \\ \beta_2+1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k (\xi - T_1)^{\beta_2} \\ \delta (\xi - T_1)^{\alpha_2} \end{array} \right) \times \\
& \times \left(E_2\left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1+1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right)\right)^2 - \psi_k T_1^{\beta_1}\Gamma(\gamma_1) \\
& \times E_2\left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) + \psi_k \lambda_k T_1^{\beta_1+1}\Gamma(\gamma_1) \times \\
& \times E_2\left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1+2, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) - \lambda_k \psi_k T_1^{2\beta_1}\Gamma^2(\gamma_1) \times \\
& \times E_2\left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) \times \\
& \times E_2\left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1+2, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha} \end{array} \right) + \psi_k T_1 + \\
& + 2\psi_k T_1^{\beta_1} (T_2 - T_1)^{\beta_2} \lambda_k E_2\left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1+1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) \times \\
& \times \Gamma(\gamma_1)\Gamma(\gamma_2) E_2\left(\begin{array}{l} \gamma_2, \gamma_2, 1; 1, 0 \\ \beta_2+1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k (T_2 - T_1)^{\beta_2} \\ \delta (T_2 - T_1)^{\alpha_2} \end{array} \right) \times \\
& + \lambda_k^2 \psi_k T_1^{2\beta_1} (T_2 - T_1)^{\beta_2} \left(E_2\left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1+1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right)\right)^2 \times \\
& \times \Gamma^2(\gamma_1)\Gamma(\gamma_2) E_2\left(\begin{array}{l} \gamma_2, \gamma_2, 1; 1, 0 \\ \beta_2+1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k (T_2 - T_1)^{\beta_2} \\ \delta (T_2 - T_1)^{\alpha_2} \end{array} \right) - \lambda_k \psi_k \times \\
& \times T_1^{\beta_1} (T_2 - T_1)^{\beta_2} \Gamma(\gamma_1)\Gamma(\gamma_2) E_2\left(\begin{array}{l} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{array} \middle| \begin{array}{c} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{array} \right) \times
\end{aligned}
$$

$$\times E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(T_2-T_1\right)^{\beta_2}\\ \delta\left(T_2-T_1\right)^{\alpha_2}\end{array}\right)\times$$
$$-\lambda_k^2T_1^{2\beta_1}\psi_k\Gamma^2(\gamma_1)\Gamma(\gamma_2)E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times\left(T_2-T_1\right)^{\beta_2}E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(T_2-T_1\right)^{\beta_2}\\ \delta\left(T_2-T_1\right)^{\alpha_2}\end{array}\right)+$$
$$+\psi_k\Gamma(\gamma_2)\left(T_2-T_1\right)^{\beta_2}E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(T_2-T_1\right)^{\beta_2}\\ \delta\left(T_2-T_1\right)^{\alpha_2}\end{array}\right)-$$
$$-\varphi_k\Gamma(\gamma_2)\left(T_2-T_1\right)^{\beta_2}E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(T_2-T_1\right)^{\beta_2}\\ \delta\left(T_2-T_1\right)^{\alpha_2}\end{array}\right)-$$
$$-2\lambda_k\varphi_k\left(T_2-T_1\right)^{\beta_2}T_1^{\beta_1}\Gamma(\gamma_1)\Gamma(\gamma_2)E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(T_2-T_1\right)^{\beta_2}\\ \delta\left(T_2-T_1\right)^{\alpha_2}\end{array}\right)-$$
$$-\lambda_k^2\varphi_k\left(T_2-T_1\right)^{\beta_2}T_1^{2\beta_1}E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(T_2-T_1\right)^{\beta_2}\\ \delta\left(T_2-T_1\right)^{\alpha_2}\end{array}\right)\times$$
$$\times\Gamma^2(\gamma_1)\Gamma(\gamma_2)\left(E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\right)^2+$$
$$+\lambda_k\varphi_k\Gamma(\gamma_1)\Gamma(\gamma_2)T_1^{\beta_1}E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times\left(T_2-T_1\right)^{\beta_2}E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(T_2-T_1\right)^{\beta_2}\\ \delta\left(T_2-T_1\right)^{\alpha_2}\end{array}\right)+$$
$$+\lambda_k^2\varphi_kT_1^{2\beta_1}\Gamma^2(\gamma_1)\Gamma(\gamma_2)E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times\left(T_2-T_1\right)^{\beta_2}E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(T_2-T_1\right)^{\beta_2}\\ \delta\left(T_2-T_1\right)^{\alpha_2}\end{array}\right). \tag{4.16}$$

By substituting (7.4) and (4.16) into the equation (3.19), we determine the coefficient ${}_5A_k$ as follows:

$${}_5A_k=\frac{1}{\widetilde{\Delta}_k}\Bigg[-2\lambda_k\varphi_k\Gamma(\gamma_2)E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)\times$$
$$\times\left(\xi-T_1\right)^{\beta_2}-4\lambda_k^2\varphi_kE_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$

$$\times \Gamma(\gamma_1)\Gamma(\gamma_2)E_2\left(\begin{array}{l|c} \gamma_2,\gamma_2,1;1,0 & \lambda_k\left(\xi-T_1\right)^{\beta_2} \\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(\xi-T_1\right)^{\alpha_2} \end{array}\right)\times$$

$$\times T_1^{\beta_1}\left(\xi-T_1\right)^{\beta_2}-2\lambda_k^3\varphi_k E_2\left(\begin{array}{l|c} \gamma_2,\gamma_2,1;1,0 & \lambda_k\left(\xi-T_1\right)^{\beta_2} \\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(\xi-T_1\right)^{\alpha_2} \end{array}\right)\times$$

$$\times T_1^{2\beta_1}\Gamma^2(\gamma_1)\Gamma(\gamma_2)\left(\xi-T_1\right)^{\beta_2}\left(E_2\left(\begin{array}{l|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1} \end{array}\right)\right)^2$$

$$+4\lambda_k^2\varphi_k\Gamma(\gamma_1)\Gamma(\gamma_2)E_2\left(\begin{array}{l|c} \gamma_2,\gamma_2,1;1,0 & \lambda_k\left(\xi-T_1\right)^{\beta_2} \\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(\xi-T_1\right)^{\alpha_2} \end{array}\right)\times$$

$$\times T_1^{\beta_1}\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{l|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1} \end{array}\right)+\lambda_k^3\varphi_k T_1^{\beta_1}\times$$

$$\times \Gamma^2(\gamma_1)\left(\left(\xi-T_1\right)^{\beta_2}+T_1^{\beta_1}\right)E_2\left(\begin{array}{l|c} \gamma_2,\gamma_2,1;1,0 & \lambda_k\left(\xi-T_1\right)^{\beta_2} \\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(\xi-T_1\right)^{\alpha_2} \end{array}\right)\times$$

$$\times \Gamma(\gamma_2)E_2\left(\begin{array}{l|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1} \end{array}\right)\times$$

$$\times E_2\left(\begin{array}{l|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1} \end{array}\right)-\lambda_k^2\varphi_k\Gamma(\gamma_1)\Gamma(\gamma_2)\times$$

$$\times T_1^{\beta_1}\left(T_2-T_1\right)^{\beta_2}E_2\left(\begin{array}{l|c} \gamma_2,\gamma_2,1;1,0 & \lambda_k\left(T_2-T_1\right)^{\beta_2} \\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(T_2-T_1\right)^{\alpha_2} \end{array}\right)\times$$

$$\times E_2\left(\begin{array}{l|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1} \end{array}\right)-\lambda_k^3\varphi_k\Gamma^2(\gamma_1)\Gamma(\gamma_2)T_1^{2\beta_1}\times$$

$$\times\left(T_2-T_1\right)^{\beta_2}E_2\left(\begin{array}{l|c} \gamma_2,\gamma_2,1;1,0 & \lambda_k\left(T_2-T_1\right)^{\beta_2} \\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(T_2-T_1\right)^{\alpha_2} \end{array}\right)\times$$

$$\times E_2\left(\begin{array}{l|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1} \end{array}\right)\times$$

$$\times E_2\left(\begin{array}{l|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1} \end{array}\right)+\psi_k-\varphi_k-$$

$$-2\lambda_k\varphi_k T_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\begin{array}{l|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1} \end{array}\right)-$$

$$-\lambda_k^2\varphi_k T_1^{2\beta_1}\Gamma^2(\gamma_1)\left(E_2\left(\begin{array}{l|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1} \end{array}\right)\right)^2\times$$

$$+\lambda_k\varphi_k T_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\begin{array}{l|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1} \end{array}\right)+$$

$$+\lambda_k^2\varphi_k T_1^{2\beta_1}\Gamma^2(\gamma_1)E_2\left(\begin{array}{l|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1} \end{array}\right)\times$$

$$\times E_2\left(\begin{array}{l|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha} \end{array}\right)-\lambda_k^2\psi_k T_1^{2\beta_1}\Gamma^2(\gamma_1)\times$$

$$\times\left(E_2\left(\begin{array}{l|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1} \end{array}\right)\right)^2+\lambda_k\psi_k\times$$

$$\times E_2\left(\begin{array}{c|c}\gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1}\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)T_1^{\beta_1}\Gamma(\gamma_1)+$$

$$+\lambda_k^2\psi_k T_1^{2\beta_1}\Gamma^2(\gamma_1)E_2\left(\begin{array}{c|c}\gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1}\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)\times$$

$$\times E_2\left(\begin{array}{c|c}\gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1}\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)-2\lambda_k^2\psi_k\Gamma(\gamma_1)\times$$

$$\times\Gamma(\gamma_2)T_1^{\beta_1}\left(T_2-T_1\right)^{\beta_2}E_2\left(\begin{array}{c|c}\gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1}\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)\times$$

$$\times E_2\left(\begin{array}{c|c}\gamma_2,\gamma_2,1;1,0 & \lambda_k\left(T_2-T_1\right)^{\beta_2}\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(T_2-T_1\right)^{\alpha_2}\end{array}\right)-\lambda_k^3\psi_k\Gamma^2(\gamma_1)\Gamma(\gamma_2)\times$$

$$\times T_1^{2\beta_1}\left(T_2-T_1\right)^{\beta_2}\left(E_2\left(\begin{array}{c|c}\gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1}\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)\right)^2\times$$

$$\times E_2\left(\begin{array}{c|c}\gamma_2,\gamma_2,1;1,0 & \lambda_k\left(T_2-T_1\right)^{\beta_2}\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(T_2-T_1\right)^{\alpha_2}\end{array}\right)+$$

$$+\lambda_k^2\psi_k T_1^{\beta_1}E_2\left(\begin{array}{c|c}\gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1}\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)\times$$

$$\times E_2\left(\begin{array}{c|c}\gamma_2,\gamma_2,1;1,0 & \lambda_k\left(T_2-T_1\right)^{\beta_2}\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(T_2-T_1\right)^{\alpha_2}\end{array}\right)+$$

$$\times\Gamma(\gamma_1)\Gamma(\gamma_2)\left(T_2-T_1\right)^{\beta_2}+\lambda_k^3\psi_k E_2\left(\begin{array}{c|c}\gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1}\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)\times$$

$$\times\Gamma(\gamma_2)T_1^{2\beta_1}\left(T_2-T_1\right)^{\beta_2}\Gamma^2(\gamma_1)E_2\left(\begin{array}{c|c}\gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1}\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)\times$$

$$\times E_2\left(\begin{array}{c|c}\gamma_2,\gamma_2,1;1,0 & \lambda_k\left(T_2-T_1\right)^{\beta_2}\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(T_2-T_1\right)^{\alpha_2}\end{array}\right)-$$

$$-\lambda_k\psi_k\Gamma(\gamma_2)\left(T_2-T_1\right)^{\beta_2}E_2\left(\begin{array}{c|c}\gamma_2,\gamma_2,1;1,0 & \lambda_k\left(T_2-T_1\right)^{\beta_2}\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(T_2-T_1\right)^{\alpha_2}\end{array}\right)+$$

$$+\lambda_k\varphi_k\left(T_2-T_1\right)^{\beta_2}\Gamma(\gamma_2)E_2\left(\begin{array}{c|c}\gamma_2,\gamma_2,1;1,0 & \lambda_k\left(T_2-T_1\right)^{\beta_2}\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(T_2-T_1\right)^{\alpha_2}\end{array}\right)-$$

$$-2\lambda_k^2\varphi_k T_1^{\beta_1}\left(T_2-T_1\right)^{\beta_2}E_2\left(\begin{array}{c|c}\gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1}\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)\times$$

$$\times\Gamma(\gamma_1)\Gamma(\gamma_2)E_2\left(\begin{array}{c|c}\gamma_2,\gamma_2,1;1,0 & \lambda_k\left(T_2-T_1\right)^{\beta_2}\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(T_2-T_1\right)^{\alpha_2}\end{array}\right)+\lambda_k^3\varphi_k T_1^{2\beta_1}\times$$

$$\times\left(T_2-T_1\right)^{\beta_2}\Gamma^2(\gamma_1)\Gamma(\gamma_2)\left(E_2\left(\begin{array}{c|c}\gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1}\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)\right)^2\times$$

$$\times E_2\left(\begin{array}{c|c}\gamma_2,\gamma_2,1;1,0 & \lambda_k\left(T_2-T_1\right)^{\beta_2}\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(T_2-T_1\right)^{\alpha_2}\end{array}\right)\Bigg]. \tag{4.17}$$

Next, to demonstrate the uniform convergence of solutions expressed as a multiple series, we will use the following estimates, which are based on the estimate for the bivariate Mittag-Leffler type function $E_2(x, y)$, as presented in [23]:

$$\left| E_2 \left(\begin{matrix} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{matrix} \middle| \begin{matrix} \lambda_k t^{\beta_1} \\ \delta t^{\alpha_1} \end{matrix} \right) \right| \leq C_1, \tag{4.18}$$

$$\left| E_2 \left(\begin{matrix} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1 + 1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{matrix} \middle| \begin{matrix} \lambda_k t^{\beta_1} \\ \delta t^{\alpha_1} \end{matrix} \right) \right| \leq C_2, \tag{4.19}$$

$$\left| E_2 \left(\begin{matrix} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1 + 2, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{matrix} \middle| \begin{matrix} \lambda_k t^{\beta_1} \\ \delta t^{\alpha_1} \end{matrix} \right) \right| \leq C_3, \tag{4.20}$$

$$\left| E_2 \left(\begin{matrix} \gamma_2, \gamma_2, 1; 1, 0 \\ \beta_2 + 1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 \end{matrix} \middle| \begin{matrix} \lambda_k (t - T_1)^{\beta_2} \\ \delta (t - T_1)^{\alpha_2} \end{matrix} \right) \right| \leq C_4, \tag{4.21}$$

$$\left| E_2 \left(\begin{matrix} \gamma_3, \gamma_3, 1; 1, 0 \\ \beta_3 + 1, \beta_3, \alpha_3; \gamma_3, \gamma_3; 1, 1 \end{matrix} \middle| \begin{matrix} \lambda_k (t - T_2)^{\beta_3} \\ \delta (t - T_2)^{\alpha_3} \end{matrix} \right) \right| \leq C_5, \tag{4.22}$$

$$\left| E_2 \left(\begin{matrix} \gamma_3, \gamma_3, 1; 1, 0 \\ \beta_3 + 2, \beta_3, \alpha_3; \gamma_3, \gamma_3; 1, 1 \end{matrix} \middle| \begin{matrix} \lambda_k (t - T_2)^{\beta_3} \\ \delta (t - T_2)^{\alpha_3} \end{matrix} \right) \right| \leq C_6. \tag{4.23}$$

For convenience, we performed the above calculations without substituting the value of λ_k. Now, to show the uniform convergence of the solution, we take λ_k as $-\mu_k^2$. Here, it is known that $\mu_k = k\pi - \frac{\pi}{4}$.

5. Convergence of infinite series.

Theorem 5.1. [11] *Let $f(x)$ be a function defined on the interval [0,1] such that $f(x)$ is differentiable $2s$ times $s > 1$ and such that*

a) $f(0) = f'(0) = ... = f^{(2s-1)}(0) = 0$,
b) $f^{(2s)}(x)$ is bounded (this derivative may not exist at certain points),
c) $f(1) = f'(1) = ... = f^{(2s-2)}(1) = 0$. Then the following inequality is satisfied by the Fourier-Bessel coefficients of $f(x)$:

$$|c_n| \leq \frac{C}{\lambda^{(2s-\frac{1}{2})}}, (C = const)$$

The proof of Theorem 5.1 can be found in reference [11].

Lemma 5.2. *If $\{\varphi(r), \psi(r)\} \in C^{m+2}[0; 1]$ and the conditions of Theorem 5.1 are satisfied, then the inequalities hold for the special cases of the series*

$$\sum\nolimits_{n=1}^{\infty} \mu_k^m |\varphi_k|, \sum\nolimits_{n=1}^{\infty} \mu_k^m |\psi_k|$$

.

$$\sum_{n=1}^{\infty} |\varphi_k| \leq \frac{C}{\mu_k^{2s-0,5}}, \sum_{n=1}^{\infty} |\psi_k| \leq \frac{C}{\mu_k^{2s-0,5}},$$

$$\sum_{n=1}^{\infty} \mu_k^2 |\varphi_k| \leq \frac{C}{\mu_k^{2s-2,5}}, \sum_{n=1}^{\infty} \mu_k^2 |\psi_k| \leq \frac{C}{\mu_k^{2s-2,5}},$$

$$\sum_{n=1}^{\infty}\mu_k^4\left|\varphi_k\right| \leq \frac{C}{\mu_k^{2s-4,5}}, \sum_{n=1}^{\infty}\mu_k^4\left|\psi_k\right| \leq \frac{C}{\mu_k^{2s-4,5}},$$
$$\sum_{n=1}^{\infty}\mu_k^6\left|\varphi_k\right| \leq \frac{C}{\mu_k^{2s-6,5}}, \sum_{n=1}^{\infty}\mu_k^6\left|\psi_k\right| \leq \frac{C}{\mu_k^{2s-6,5}},$$
$$\sum_{n=1}^{\infty}\mu_k^8\left|\varphi_k\right| \leq \frac{C}{\mu_k^{2s-8,5}}, \sum_{n=1}^{\infty}\mu_k^8\left|\psi_k\right| \leq \frac{C}{\mu_k^{2s-8,5}}.$$

Proof. It is known from the asymptotic of $\mu_k = (k-\frac{1}{4})\pi$ $(\mu_k \sim k)$. Based on this, the series $\sum\limits_{k=1}^{\infty}\frac{C}{\mu_k^p}$ converges when $p>1$. In our case, for the convergence of the highest-order $(m=8)$ series:

$$2s-8,5>1 \implies 2s>9,5$$

That is, if the function is differentiable at least 10 times $(2s=10)$, then convergent majorant series exist for all the series listed above:

$$\mu_k^m\left|\varphi_k\right| \leq \frac{C}{\mu_k^{10-m-0,5}}, \mu_k^m\left|\psi_k\right| \leq \frac{C}{\mu_k^{10-m-0,5}}$$

for each $m\in\{0,2,4,6,8\}$, the exponent is greater than 1 $(p\geq 1.5)$. By the Weierstrass M-test, all these series converge absolutely and uniformly on the interval $[0,1]$. □

To prove a uniform convergence of the infinite series (3.1), let us start with the following estimation

$$|u(t,r)| \leq \sum_{k=1}^{\infty}\left|{}_1U_k(t)J_0(\lambda_k r)\right| \leq \sum_{k=1}^{\infty}\left|{}_1U_k(t)\right|, \quad (\left|J_0(x)\leq 1\right|, \forall x\in\mathbb{R}).$$

To complete this estimation, we need to estimate $|{}_1U_k(t)|$. For this aim, we use estimation of coefficients, and from the estimates in Lemma 5.1, we obtain the following estimate for $u(t,r)$ in Ω_1 as (see for details in A5 of the Appendix section)

$$\sum_{k=1}^{\infty}\left|{}_1U_k(t)\right| \leq \sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-0,5}}\Big(1+2tS_1+2\Gamma(\gamma_1)t^{\beta_1}\varepsilon C_2\Big)+$$
$$+\sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-2,5}}\Big(t\left(S_2+2S_4\right)+\Gamma(\gamma_1)t^{\beta_1}C_2+\Gamma(\gamma_1)t^{\beta_1+1}\times$$
$$\times\left(C_1+S_1\right)C_3+\Gamma(\gamma_1)t^{\beta_1}C_2(\varepsilon T_1+S_4+S_5)\Big)+$$
$$+\sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-4,5}}\Big(tS_3+t^{\beta_1+1}\Gamma(\gamma_1)C_3\left(S_2+2S_4\right)+\Gamma(\gamma_1)t^{\beta_1}\times$$
$$\times C_2\left(S_3+S_6\right)\Big)+\sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-6,5}}\Big(\Gamma(\gamma_1)t^{\beta_1+1}S_3C_3+\Gamma(\gamma_1)t^{\beta_1}S_7C_2\Big).$$

In a similar way we estimate $u(t,r)$ in Ω_2 as

$$|u(t,r)| \leq \sum_{k=1}^{\infty} |{}_2U_k(t)J_0(\lambda_k r)| \leq \sum_{k=1}^{\infty} |{}_2U_k(t)|\,.$$

Using estimates in Lemma 5.1, after certain evaluations (see for details A6 of the Appendix section), we obtain the following estimate:

$$\sum_{k=1}^{\infty} |{}_2U_k(t)| \leq \sum_{k=1}^{\infty} \frac{C}{\mu_k^{2s-0,5}} \left(\frac{S_2}{2} + 2\Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2} \varepsilon C_4(S_4+S_1T_1) \right) +$$
$$+ \sum_{k=1}^{\infty} \frac{C}{\mu_k^{2s-2,5}} \left(\frac{S_3(C_1+2C_2)}{2C_2} + S_9 + \Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2} C_4 \times \right.$$
$$\left. \times \left(2S_4 + S_1T_1 + \frac{S_2}{2} + \varepsilon T_1 + S_5\right) \right) + \sum_{k=1}^{\infty} \frac{C}{\mu_k^{2s-4,5}} \left(S_8 + \Gamma\left(\gamma_2\right) \times \right.$$
$$\left. \times \left(t-T_1\right)^{\beta_2} C_4 \left(\frac{S_3(C_1+2C_2)}{2C_2} + S_3 + S_6 + S_9 \right) \right) +$$
$$+ \sum_{k=1}^{\infty} \frac{C}{\mu_k^{2s-6,5}} \left(\Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2} S_8C_4 + \Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2} S_7C_4 \right).$$

Finally, we will estimate $u(t,r)$ in Ω_3 as follows:

$$|u(t,r)| \leq \sum_{k=1}^{\infty} |{}_3U_k(t)J_0(\lambda_k r)| \leq \sum_{k=1}^{\infty} |{}_3U_k(t)|\,.$$

Estimating coefficients, based on Lemma 5.1, we obtain the following estimate for $u(t,r)$ in Ω_3 (see for details A7 of the Appendix section):

$$\sum_{k=1}^{\infty} |{}_3U_k(t)| \leq \sum_{k=1}^{\infty} \frac{C}{\mu_k^{2s-0,5}} \left(\frac{S_2}{2} + 2S_{10} + 2\left(t-T_2\right)\varepsilon + 2\Gamma\left(\gamma_3\right)\left(t-T_2\right)^{\beta_3} \times \right.$$
$$\left. \times\, \varepsilon C_5 + S_4 + T_1S_1 \right) + \sum_{k=1}^{\infty} \frac{C}{\mu_k^{2s-2,5}} \left(S_{11} + \Gamma\left(\gamma_3\right)\left(t-T_2\right)^{\beta_3} \left((\frac{S_2}{2} + S_{10})C_5 + \right.\right.$$
$$\left. + S_5C_5 + (S_4 + S_1T_1 + S_{10})C_5 + C_5(\varepsilon T_1 + S_4) \right) + S_{13} +$$
$$\left. + (t-T_2)(2S_4 + 2\varepsilon C_6 + S_1T_1 + 2S_{10} + \varepsilon T_1 + \frac{S_2}{2} + S_5) \right) + \sum_{k=1}^{\infty} \frac{C}{\mu_k^{2s-4,5}} \times$$
$$\times \left(S_7 + 2S_{12} + (t-T_2)(S_3 + S_6 + S_{11} + S_{13}) + \Gamma\left(\gamma_3\right)\left(t-T_2\right)^{\beta_3} \left(S_{11}C_5 + \right.\right.$$
$$+ (t-T_2)C_6(\frac{S_2}{2} + S_{10} + S_5) + C_5(S_3 + S_6) + S_{13}C_5 +$$

$$+ (t-T_2)\varepsilon C_6(2S_4+S_1T_1+S_{10}+\varepsilon T_1)\Big)\Bigg) + \sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-6,5}}\times$$
$$\times\Bigg(2(t-T_2)(S_7+S_{12}) + \Gamma(\gamma_3)(t-T_2)^{\beta_3}\Big(C_5(S_7+S_{12})+$$
$$+ (t-T_2)C_6(S_3+S_6+S_{11}) + S_7C_5 + C_5S_{12}+$$
$$+ (t-T_2)S_{13}C_6\Big)\Bigg) + \sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-8,5}}2\Gamma(\gamma_3)(t-T_2)^{\beta_3+1}C_6(S_7+S_{12}).$$

Theorem 5.3. *If $\alpha_2 = 1,\ \gamma_2 = \beta_2, \Delta_k \neq 0, \widetilde{\Delta}_k \neq 0,\ \{\varphi(r), \psi(r)\} \in C^{10}[0;1]$ and the equalities*

$$\varphi^{(i)}(0) = 0, \psi^{(i)}(0) = 0\,(i = \overline{1,9}), \quad \varphi^{(j)}(1) = 0, \psi^{(j)}(1) = 0\,(j = \overline{1,8})$$

are satisfied. The unique solution to problem (2.1)-(2.7) exists and represented by (3.1),(3.2),(3.3), (3.7).

6. Conclusion

In this work, we have investigated an inverse source problem for a mixed wave-diffusion-wave equation involving the regularized Prabhakar fractional derivative in a cylindrical domain. The considered model captures temporal transitions between hyperbolic and parabolic dynamics, which naturally arise in various anomalous transport processes, particularly in gas flow through heterogeneous porous media. By allowing successive changes in the governing time-fractional operator, the model provides a flexible and physically meaningful framework for describing wave-diffusion interactions with memory effects.

The problem was studied under suitable initial, boundary, and transmitting conditions, together with an additional time measurement used to reconstruct the unknown spatial source term. By applying the method of separation of variables, the original problem was reduced to a sequence of fractional ordinary differential equations coupled with a Bessel-type Sturm-Liouville eigenvalue problem. This approach led to an explicit representation of the solution in terms of a Fourier-Bessel series, where the temporal components are expressed via bivariate Mittag-Leffler-type functions associated with the Prabhakar kernel.

A key contribution of the paper lies in the rigorous solvability analysis of the inverse source problem. Under appropriate assumptions on the fractional parameters and compatibility conditions on the given data, we derived closed-form expressions for all unknown Fourier coefficients, including the source term. Moreover, by employing asymptotic properties of Bessel functions and sharp estimates for Mittag-Leffler-type functions, we established the absolute and uniform convergence of the obtained series solution. This convergence analysis guarantees the existence and uniqueness of a regular solution and ensures that the reconstructed source function is well-defined and stable within the adopted functional framework.

The results obtained in this study extend several earlier works on inverse problems for mixed-type and fractional partial differential equations. In particular, the present paper generalizes known results for classical parabolic-hyperbolic and fractional wave-diffusion equations to the setting of Prabhakar-type operators and cylindrical geometry, thereby enriching the theoretical foundation of inverse problems for fractional evolution equations with memory.

From an applied perspective, the developed methodology is relevant to inverse identification problems arising in subsurface flow, viscoelastic media, and other systems where anomalous diffusion and finite-speed wave propagation coexist. The analytical formulas derived here may also serve as benchmarks for numerical algorithms and inverse reconstruction schemes.

Possible directions for future research include the analysis of stability estimates for the inverse source problem, the extension to space-fractional or multidimensional geometries, and the development of efficient numerical methods based on the theoretical results presented in this work. Another promising direction is the investigation of inverse problems with noisy or partial data, which is of significant importance for real-world applications.

Overall, this paper demonstrates that Prabhakar's fractional models provide a powerful and mathematically tractable tool for studying inverse problems in mixed wave-diffusion processes, and it opens the door to further theoretical and applied developments in this area.

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7. Appendix

A1: Finding coefficient ${}_2A_k$. By substituting equation (3.11) into left side of equation (3.12), we obtain the following expression:

$$f_k - \lambda_k \Bigg[{}_1A_k \cdot \lambda_k \cdot T_1^{\beta_1} \cdot \Gamma(\gamma_1) \cdot E_2 \left(\begin{matrix} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1+1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{matrix} \;\middle|\; \begin{matrix} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{matrix} \right) +$$

$$+ {}_2A_k T_1 + {}_2A_k \cdot \lambda_k \cdot T_1^{\beta_1+1} \Gamma(\gamma_1) E_2 \left(\begin{matrix} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1+2, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{matrix} \;\middle|\; \begin{matrix} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{matrix} \right) +$$

$$+ {}_1A_k + \Gamma(\gamma_1) \cdot f_k \cdot T_1^{\beta_1} E_2 \left(\begin{matrix} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1+1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{matrix} \;\middle|\; \begin{matrix} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{matrix} \right) \Bigg] =$$

$$= {}_2A_k + {}_1A_k \cdot \lambda_k \cdot T_1^{\beta_1-1} \cdot \Gamma(\gamma_1) \times E_2 \left(\begin{matrix} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{matrix} \;\middle|\; \begin{matrix} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{matrix} \right) +$$

$$+ {}_2A_k \cdot \lambda_k \cdot T_1^{\beta_1} \cdot \Gamma(\gamma_1) \cdot E_2 \left(\begin{matrix} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1+1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{matrix} \;\middle|\; \begin{matrix} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{matrix} \right) +$$

$$+ \Gamma(\gamma_1) \cdot f_k \cdot T_1^{\beta_1-1} \cdot E_2 \left(\begin{matrix} \gamma_1, \gamma_1, 1; 1, 0 \\ \beta_1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 \end{matrix} \;\middle|\; \begin{matrix} \lambda_k T_1^{\beta_1} \\ \delta T_1^{\alpha_1} \end{matrix} \right).$$

We divide the last equation by λ_k :

$$
{}_2A_k\left[\frac{1}{\lambda_k}+T_1+T_1^{\beta_1}\Gamma\left(\gamma_1\right)E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+\right.
$$

$$
\left.+T_1^{\beta_1+1}\cdot\lambda_k\cdot\Gamma\left(\gamma_1\right)E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\right]=\frac{f_k}{\lambda_k}-
$$

$$
-{}_1A_k-{}_1A_k\cdot\lambda_k\cdot T_1^{\beta_1}\cdot\Gamma\left(\gamma_1\right)E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)-
$$

$$
-{}_1A_kT_1^{\beta_1-1}\Gamma\left(\gamma_1\right)E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)-
$$

$$
-\Gamma\left(\gamma_1\right)\cdot f_k\cdot T_1^{\beta_1}E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)-
$$

$$
-\frac{\Gamma\left(\gamma_1\right)f_k}{\lambda_k}T_1^{\beta_1-1}E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right).
$$

A2: Simplification of $\widetilde{\Delta}_k$. By substituting the notations (4.2),(4.5),(4.6) and (4.8) into (4.11), we find the value of $\widetilde{\Delta}_k$ as follows:

$$
\widetilde{\Delta}_k=\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)+
$$

$$
+T_1+T_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+
$$

$$
+T_1^{\beta_1+1}\cdot\lambda_k\cdot\Gamma(\gamma_1)\cdot E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+
$$

$$
+2\lambda_kT_1^{\beta_1}\Gamma\left(\gamma_1\right)E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)\times
$$

$$
\times\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+
$$

$$
+\Gamma^2\left(\gamma_1\right)\left(E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\right)^2\lambda_kT_1^{2\beta_1}+
$$

$$
+\lambda_k^2\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)\times
$$

$$
\times T_1^{2\beta_1}\Gamma^2\left(\gamma_1\right)\left(E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\right)^2+
$$

$$
+2\lambda_k\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)\times
$$

$$
\times\left(T_1+T_1^{\beta_1+1}\lambda_k\Gamma\left(\gamma_1\right)E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\right)-
$$

$$- \left(T_1 + T_1^{\beta_1+1} \lambda_k \Gamma(\gamma_1) E_2 \left(\begin{array}{c|c} \gamma_1, \gamma_1, 1; 1, 0 & \lambda_k T_1^{\beta_1} \\ \beta_1+2, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 & \delta T_1^{\alpha_1} \end{array} \right) \right) \times$$
$$\times T_1^{\beta_1-1} \Gamma(\gamma_1) E_2 \left(\begin{array}{c|c} \gamma_1, \gamma_1, 1; 1, 0 & \lambda_k T_1^{\beta_1} \\ \beta_1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 & \delta T_1^{\alpha_1} \end{array} \right) -$$
$$- \lambda_k \Gamma(\gamma_2) (\xi - T_1)^{\beta_2} E_2 \left(\begin{array}{c|c} \gamma_2, \gamma_2, 1; 1, 0 & \lambda_k (\xi - T_1)^{\beta_2} \\ \beta_2+1, \beta_2, \alpha_2; \gamma_2, \gamma_2; 1, 1 & \delta (\xi - T_1)^{\alpha_2} \end{array} \right) \times$$
$$\times \left(T_1 + T_1^{\beta_1+1} \lambda_k \Gamma(\gamma_1) E_2 \left(\begin{array}{c|c} \gamma_1, \gamma_1, 1; 1, 0 & \lambda_k T_1^{\beta_1} \\ \beta_1+2, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 & \delta T_1^{\alpha_1} \end{array} \right) \right) \times$$
$$\times T_1^{\beta_1-1} \Gamma(\gamma_1) E_2 \left(\begin{array}{c|c} \gamma_1, \gamma_1, 1; 1, 0 & \lambda_k T_1^{\beta_1} \\ \beta_1, \beta_1, \alpha_1; \gamma_1, \gamma_1; 1, 1 & \delta T_1^{\alpha_1} \end{array} \right).$$

As a result of simplifying similar terms, the value of $\widetilde{\Delta}_k$ can be written as (4.12).

A3: Finding the unknown coefficients ${}_2A_k, {}_3A_k$ and the unknown functionf_k. To accomplish this, we first need to find ${}_2\Delta_k, {}_3\Delta_k, {}_f\Delta_k$.

$$ {}_2A_k = \frac{{}_2\Delta_k}{\widetilde{\Delta}_k}, {}_3A_k = \frac{{}_3\Delta_k}{\widetilde{\Delta}_k}, f_k = \frac{{}_f\Delta_k}{\widetilde{\Delta}_k}. \tag{7.1}$$

To find ${}_2\Delta_k$, we construct the following determinant

$$ {}_2\Delta_k = \begin{vmatrix} \psi_k & 1 + \lambda_k \cdot {}_1M_k & {}_1M_k \\ -\varphi_k (1 + \lambda_k \cdot {}_2M_k) & -1 & {}_2M_k \\ -\lambda_k \cdot \varphi_k \cdot {}_4M_k & \lambda_k & {}_4M_k - 1 \end{vmatrix} = \psi_k (1 - {}_4M_k) -$$
$$- \lambda_k \cdot \varphi_k \cdot {}_4M_k \cdot {}_2M_k (1 + \lambda_k \cdot {}_1M_k) - \varphi_k (1 + \lambda_k \cdot {}_2M_k) \lambda_k \cdot {}_1M_k -$$
$$- \Big(\lambda_k \cdot \varphi_k \cdot {}_1M_k \cdot {}_4M_k - \varphi_k (1 + \lambda_k \cdot {}_2M_k)(1 + \lambda_k \cdot {}_1M_k)({}_4M_k - 1) +$$
$$+ \lambda_k \cdot \psi_k \cdot {}_2M_k \Big) = \psi_k - \varphi_k - {}_4M_k (\psi_k - \varphi_k) - \lambda_k \cdot \varphi_k ({}_1M_k + {}_2M_k) -$$
$$- 2\lambda_k^2 \cdot \varphi_k \cdot {}_1M_k \cdot {}_2M_k - \lambda_k \cdot \varphi_k \cdot {}_1M_k - \lambda_k \cdot \psi_k \cdot {}_2M_k.$$

Now, substituting the designations (4.2), (4.5), (4.6),(4.8) into the last equality and simplifying, we find ${}_2\Delta_k$ as follows:

$$
\begin{aligned}
{}_2\Delta_k = \psi_k - \psi_k T_1^{\beta_1-1}\Gamma(\gamma_1)E_2\left(\begin{array}{c|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)+ \\
+\varphi_k T_1^{\beta_1-1}\Gamma(\gamma_1)E_2\left(\begin{array}{c|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1,\beta_1,\alpha;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)-\varphi_k- \\
-2\lambda_k\varphi_k\Gamma(\gamma_2)(\xi-T_1)^{\beta_2}E_2\left(\begin{array}{c|c} \gamma_2,\gamma_2,1;1,0 & \lambda_k(\xi-T_1)^{\beta_2} \\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta(\xi-T_1)^{\alpha_2}\end{array}\right)- \\
-\lambda_k\varphi_k T_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\begin{array}{c|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)- \\
-2\lambda_k^2\varphi_k\Gamma(\gamma_1)\Gamma(\gamma_2)E_2\left(\begin{array}{c|c} \gamma_2,\gamma_2,1;1,0 & \lambda_k(\xi-T_1)^{\beta_2} \\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta(\xi-T_1)^{\alpha_2}\end{array}\right)\times \\
\times(\xi-T_1)^{\beta_2}T_1^{\beta_1}E_2\left(\begin{array}{c|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)- \\
-\lambda_k\psi_k T_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\begin{array}{c|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right).
\end{aligned}
$$

Now using formula (7.1),we find the coefficient ${}_2A_k$

$$
\begin{aligned}
{}_2A_k = \frac{{}_2\Delta_k}{\widetilde{\Delta}_k} = \frac{1}{\widetilde{\Delta}_k}\Bigg[-\psi_k T_1^{\beta_1-1}\Gamma(\gamma_1)E_2\left(\begin{array}{c|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)+ \\
+\psi_k+\varphi_k T_1^{\beta_1-1}\Gamma(\gamma_1)E_2\left(\begin{array}{c|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)-\varphi_k- \\
-2\lambda_k\varphi_k\Gamma(\gamma_2)(\xi-T_1)^{\beta_2}E_2\left(\begin{array}{c|c} \gamma_2,\gamma_2,1;1,0 & \lambda_k(\xi-T_1)^{\beta_2} \\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta(\xi-T_1)^{\alpha_2}\end{array}\right)- \\
-\lambda_k\varphi_k T_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\begin{array}{c|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)\times \\
-2\lambda_k^2\varphi_k\Gamma(\gamma_1)\Gamma(\gamma_2)E_2\left(\begin{array}{c|c} \gamma_2,\gamma_2,1;1,0 & \lambda_k(\xi-T_1)^{\beta_2} \\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta(\xi-T_1)^{\alpha_2}\end{array}\right)\times \\
\times(\xi-T_1)^{\beta_2}T_1^{\beta_1}E_2\left(\begin{array}{c|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)- \\
-\lambda_k\psi_k T_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\begin{array}{c|c} \gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1} \\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)\Bigg].
\end{aligned} \tag{7.2}
$$

In the same manner, we determine the coefficient ${}_3A_k$:

$$
{}_3\Delta_k = \begin{vmatrix} 0 & \psi_k & {}_1M_k \\ {}_3M_k & -\varphi_k\left(1+\lambda_k\cdot{}_2M_k\right) & {}_2M_k \\ 1+\lambda_k\cdot{}_2M_k & -\lambda_k\cdot\varphi_k\cdot{}_4M_k & {}_4M_k-1 \end{vmatrix} = 0+\psi_k\cdot{}_2M_k\left(1+{}_4M_k\right)-
$$

$$
-\lambda_k\cdot\varphi_k\cdot{}_4M_{k1}M_k\cdot{}_3M_k-\Big(-\varphi_k\left(1+\lambda_k\cdot{}_2M_k\right){}_1M_k\left(1+\lambda_k\cdot{}_2M_k\right)+
$$

$$
+\psi_k\cdot{}_3M_k\left({}_4M_k-1\right)+0\Big)=\psi_k\cdot{}_2M_k+\lambda_k\cdot\psi_k\cdot{}_2M_k^2-
$$

$$
-\lambda_k\cdot\varphi_k\cdot{}_1M_k\cdot{}_3M_k\cdot{}_4M_k+\varphi_k\cdot{}_1M_k+2\lambda_k\cdot\varphi_k\cdot{}_1M_k\cdot{}_2M_k+
$$

$$
+\varphi_k\cdot\lambda_k^2\cdot{}_1M_k\cdot{}_2M_k^2-\psi_k\cdot{}_3M_k\cdot{}_4M_k+\psi_k\cdot{}_3M_k.
$$

Substituting the above notations (4.2), (4.5),(4.6) and (4.8) into the expression obtained in ${}_3\Delta_k$ as

$$
\begin{aligned}
&{}_3\Delta_k=\psi_k T_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\begin{array}{c|c}\gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1}\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)+\\
&+\lambda_k\psi_k T_1^{2\beta_1}\Gamma^2(\gamma_1)\left(E_2\left(\begin{array}{c|c}\gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1}\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)\right)^2-\\
&-\lambda_k\varphi_k\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)T_1^{\beta_1}\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{c|c}\gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1}\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)\times\\
&\times E_2\left(\begin{array}{c|c}\gamma_2,\gamma_2,1;1,0 & \lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)-\lambda_k^2T_1^{2\beta_1}\Gamma\left(\gamma_2\right)\times\\
&\times\varphi_k\Gamma^2\left(\gamma_1\right)\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{c|c}\gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1}\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha}\end{array}\right)\times\\
&\times E_2\left(\begin{array}{c|c}\gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1}\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)\times\\
&\times E_2\left(\begin{array}{c|c}\gamma_2,\gamma_2,1;1,0 & \lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)+\varphi_k\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}\times\\
&\times E_2\left(\begin{array}{c|c}\gamma_2,\gamma_2,1;1,0 & \lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)+2\varphi_k\lambda_k\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\times\\
&\times T_1^{\beta_1}\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{c|c}\gamma_2,\gamma_2,1;1,0 & \lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)\times\\
&\times E_2\left(\begin{array}{c|c}\gamma_1,\gamma_1,1;1,0 & \lambda_k T_1^{\beta_1}\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1 & \delta T_1^{\alpha_1}\end{array}\right)+\varphi_k\lambda_k^2\Gamma^2\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\times\\
&\times T_1^{2\beta_1}\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{c|c}\gamma_2,\gamma_2,1;1,0 & \lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1 & \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)\times
\end{aligned}
$$

$$\times\left(E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\right)^2-$$
$$-\psi_k T_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)-$$
$$-\psi_k\Gamma^2(\gamma_1)\lambda_k E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times T_1^{2\beta_1}E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+\psi_k T_1+$$
$$+\psi_k\lambda_k T_1^{\beta_1+1}\Gamma(\gamma_1)E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right),$$

according to formula (7.1), we find ${}_3A_k$ as

$${}_3A_k=\frac{{}_3\Delta_k}{\widetilde{\Delta}_k}=\frac{1}{\widetilde{\Delta}_k}\Bigg[\psi_k T_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+$$
$$+\lambda_k\psi_k T_1^{2\beta_1}\Gamma^2(\gamma_1)\left(E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\right)^2-$$
$$-\lambda_k\varphi_k\Gamma(\gamma_1)\Gamma(\gamma_2)T_1^{\beta_1}(\xi-T_1)^{\beta_2}E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k(\xi-T_1)^{\beta_2}\\ \delta(\xi-T_1)^{\alpha_2}\end{array}\right)-\lambda_k^2 T_1^{2\beta_1}\Gamma(\gamma_2)\times$$
$$\times\varphi_k\Gamma^2(\gamma_1)(\xi-T_1)^{\beta_2}E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k(\xi-T_1)^{\beta_2}\\ \delta(\xi-T_1)^{\alpha_2}\end{array}\right)+\varphi_k\Gamma(\gamma_2)(\xi-T_1)^{\beta_2}\times$$
$$\times E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k(\xi-T_1)^{\beta_2}\\ \delta(\xi-T_1)^{\alpha_2}\end{array}\right)+2\varphi_k\lambda_k\Gamma(\gamma_1)\Gamma(\gamma_2)\times$$
$$\times T_1^{\beta_1}(\xi-T_1)^{\beta_2}E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k(\xi-T_1)^{\beta_2}\\ \delta(\xi-T_1)^{\alpha_2}\end{array}\right)\times$$
$$\times E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+\varphi_k\lambda_k^2\Gamma^2(\gamma_1)\Gamma(\gamma_2)\times$$
$$\times T_1^{2\beta_1}(\xi-T_1)^{\beta_2}E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k(\xi-T_1)^{\beta_2}\\ \delta(\xi-T_1)^{\alpha_2}\end{array}\right)\times$$
$$\times\left(E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\right)^2-$$
$$-\psi_k T_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)-$$

$$
\begin{aligned}
&-\psi_k T_1^{2\beta_1}\Gamma^2(\gamma_1)\lambda_k E_2\left(\left.\begin{matrix}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{matrix}\,\right|\,\begin{matrix}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha}\end{matrix}\right)\times\\
&\times E_2\left(\left.\begin{matrix}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{matrix}\,\right|\,\begin{matrix}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{matrix}\right)+\psi_k T_1+\\
&+\psi_k\lambda_k T_1^{\beta_1+1}\Gamma(\gamma_1)E_2\left(\left.\begin{matrix}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{matrix}\,\right|\,\begin{matrix}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{matrix}\right)\Bigg]. \qquad (7.3)
\end{aligned}
$$

Now, we will find the unknown function f_k in a similar way:

$$
\begin{aligned}
&f_k=\frac{{}_f\Delta_k}{\widetilde{\Delta}_k}=\frac{1}{\widetilde{\Delta}_k}\Bigg[-2\lambda_k\varphi_k T_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\left.\begin{matrix}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{matrix}\,\right|\,\begin{matrix}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{matrix}\right)-\\
&\psi_k-\varphi_k-\lambda_k^2\varphi_k T_1^{2\beta_1}\Gamma^2(\gamma_1)\left(E_2\left(\left.\begin{matrix}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{matrix}\,\right|\,\begin{matrix}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{matrix}\right)\right)^2-\\
&-\lambda_k\varphi_k\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}E_2\left(\left.\begin{matrix}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{matrix}\,\right|\,\begin{matrix}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{matrix}\right)-\\
&-2\lambda_k^2\varphi_k T_1^{\beta_1}\Gamma\left(\gamma_1\right)E_2\left(\left.\begin{matrix}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{matrix}\,\right|\,\begin{matrix}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{matrix}\right)\times\\
&\times\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}E_2\left(\left.\begin{matrix}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{matrix}\,\right|\,\begin{matrix}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{matrix}\right)-\\
&-\lambda_k^3 T_1^{2\beta_1}\varphi_k\Gamma^2\left(\gamma_1\right)E_2\left(\left.\begin{matrix}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{matrix}\,\right|\,\begin{matrix}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{matrix}\right)\times\\
&\times\left(E_2\left(\left.\begin{matrix}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{matrix}\,\right|\,\begin{matrix}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha}\end{matrix}\right)\right)^2+\psi_k\lambda_k T_1^{\beta_1}+\\
&+\psi_k\lambda_k^2 T_1^{\beta_1+1}\Gamma\left(\gamma_1\right)E_2\left(\left.\begin{matrix}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{matrix}\,\right|\,\begin{matrix}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{matrix}\right)\times\\
&\times\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}+\psi_k\lambda_k E_2\left(\left.\begin{matrix}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{matrix}\,\right|\,\begin{matrix}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{matrix}\right)\times\\
&\times T_1^{\beta_1}\Gamma\left(\gamma_1\right)+\lambda_k\varphi_k\Gamma(\gamma_1)T_1^{\beta_1}E_2\left(\left.\begin{matrix}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{matrix}\,\right|\,\begin{matrix}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{matrix}\right)+\\
&+\lambda_k^2\varphi_k\Gamma^2(\gamma_1)T_1^{2\beta_1}E_2\left(\left.\begin{matrix}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{matrix}\,\right|\,\begin{matrix}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{matrix}\right)\times \qquad (7.4)\\
&\times E_2\left(\left.\begin{matrix}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{matrix}\,\right|\,\begin{matrix}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{matrix}\right)+\lambda_k^2\varphi_k\Gamma(\gamma_1)\Gamma\left(\gamma_2\right)\\
&+\left(\xi-T_1\right)^{\beta_2}T_1^{\beta_1}E_2\left(\left.\begin{matrix}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{matrix}\,\right|\,\begin{matrix}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{matrix}\right)\times\\
&\times E_2\left(\left.\begin{matrix}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{matrix}\,\right|\,\begin{matrix}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha}\end{matrix}\right)+T_1^{2\beta_1}\times\\
&\times\lambda_k^3\varphi_k E_2\left(\left.\begin{matrix}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{matrix}\,\right|\,\begin{matrix}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{matrix}\right)\times
\end{aligned}
$$

$$\times\Gamma^2(\gamma_1)\Gamma(\gamma_2)E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha}\end{array}\right)\Bigg].$$

A4: Simplification of ${}_4A_k$. We substitute (7.3) and (7.4) into equation (3.16):

$$_4A_k=\frac{1}{\widetilde{\Delta}_k}\Bigg[\psi_kT_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+$$
$$+\lambda_k\psi_kT_1^{2\beta_1}\Gamma^2(\gamma_1)\left(E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\right)^2-$$
$$-\lambda_k\varphi_k\Gamma(\gamma_1)\Gamma(\gamma_2)T_1^{\beta_1}(\xi-T_1)^{\beta_2}E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k(\xi-T_1)^{\beta_2}\\ \delta(\xi-T_1)^{\alpha_2}\end{array}\right)-\lambda_k^2T_1^{2\beta_1}\Gamma(\gamma_2)\varphi_k\times$$
$$\times\Gamma^2(\gamma_1)(\xi-T_1)^{\beta_2}E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k(\xi-T_1)^{\beta_2}\\ \delta(\xi-T_1)^{\alpha_2}\end{array}\right)+\varphi_k\Gamma(\gamma_2)\times$$
$$\times(\xi-T_1)^{\beta_2}E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k(\xi-T_1)^{\beta_2}\\ \delta(\xi-T_1)^{\alpha_2}\end{array}\right)+$$
$$+2\varphi_k\lambda_k\Gamma(\gamma_1)\Gamma(\gamma_2)E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k(\xi-T_1)^{\beta_2}\\ \delta(\xi-T_1)^{\alpha_2}\end{array}\right)$$
$$\times T_1^{\beta_1}(\xi-T_1)^{\beta_2}E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+$$
$$+\varphi_k\lambda_k^2\Gamma^2(\gamma_1)\Gamma(\gamma_2)E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k(\xi-T_1)^{\beta_2}\\ \delta(\xi-T_1)^{\alpha_2}\end{array}\right)\times$$
$$\times T_1^{2\beta_1}(\xi-T_1)^{\beta_2}\left(E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\right)^2-$$
$$-\psi_kT_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)-$$
$$-\psi_kT_1^{2\beta_1}\Gamma^2(\gamma_1)\lambda_kE_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times$$
$$\times E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_kT_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+\psi_kT_1+$$

$$
\begin{aligned}
&+\psi_k\lambda_k T_1^{\beta_1+1}\Gamma(\gamma_1)E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\Bigg]+\\
&+\frac{1}{\widetilde{\Delta}_k}\Bigg[\psi_k-\varphi_k-2\lambda_k\varphi_k T_1^{\beta_1}\Gamma(\gamma_1)E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)-\\
&-\lambda_k^2\varphi_k T_1^{2\beta_1}\Gamma^2(\gamma_1)\left(E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\right)^2-\\
&-\lambda_k\varphi_k\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)-\\
&-2\lambda_k^2\varphi_k T_1^{\beta_1}\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)\times\\
&\times\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)-\lambda_k^3 T_1^{2\beta_1}\varphi_k\Gamma^2\left(\gamma_1\right)\times\\
&\times\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)\times\\
&\times\left(E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\right)^2+\psi_k\lambda_k T_1^{\beta_1}+\\
&+\psi_k\lambda_k^2 T_1^{\beta_1+1}\Gamma\left(\gamma_1\right)E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+\\
&+\psi_k\lambda_k T_1^{\beta_1}\Gamma\left(\gamma_1\right)E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+\\
&+\lambda_k\varphi_k\Gamma(\gamma_1)T_1^{\beta_1}E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+\\
&+\lambda_k^2\varphi_k\Gamma^2(\gamma_1)T_1^{2\beta_1}E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\\
&\times E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+\lambda_k^2\varphi_k\Gamma(\gamma_1)\Gamma\left(\gamma_2\right)\times\\
&\times\left(\xi-T_1\right)^{\beta_2}T_1^{\beta_1}E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)\times\\
&\times E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)+T_1^{2\beta_1}\lambda_k^3\varphi_k\Gamma^2(\gamma_1)\times\\
&\times\Gamma(\gamma_2)E_2\left(\begin{array}{l}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}\lambda_k\left(\xi-T_1\right)^{\beta_2}\\ \delta\left(\xi-T_1\right)^{\alpha_2}\end{array}\right)\times\\
&\times E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\times\\
&\times E_2\left(\begin{array}{l}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}\lambda_k T_1^{\beta_1}\\ \delta T_1^{\alpha_1}\end{array}\right)\Bigg].
\end{aligned}
$$

and after simplifying similar terms, the coefficient ${}_4A_k$ can be written as (4.16).

A5: Estimate of $u(t,r)$ in Ω_1. For this aim, first using (3.8) we will obtain

$$\left|{}_1U_k(t)\right| \leq \mu_k^2\left|\varphi_k\right|t^{\beta_1}\Gamma\left(\gamma_1\right)\left|E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}-\mu_k^2t^{\beta_1}\\ \delta t^{\alpha_1}\end{array}\right)\right|+$$
$$+\left|\varphi_k\right|+\mu_k^2\left|{}_2A_k\right|t^{\beta_1+1}\Gamma\left(\gamma_1\right)\left|E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+2,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}-\mu_k^2t^{\beta_1}\\ \delta t^{\alpha_1}\end{array}\right)\right|+$$
$$\left|{}_2A_k\right|t+\Gamma\left(\gamma_1\right)\left|f_k\right|t^{\beta_1}\left|E_2\left(\begin{array}{c}\gamma_1,\gamma_1,1;1,0\\ \beta_1+1,\beta_1,\alpha_1;\gamma_1,\gamma_1;1,1\end{array}\middle|\begin{array}{c}-\mu_k^2t^{\beta_1}\\ \delta t^{\alpha_1}\end{array}\right)\right|.$$

Now using (4.19),(4.20) we get

$$\begin{aligned}\left|{}_1U_k(t)\right| \leq{}& \left|{}_2A_k\right|t+\mu_k^2\left|\varphi_k\right|t^{\beta_1}\Gamma\left(\gamma_1\right)C_2+\mu_k^2\left|{}_2A_k\right|t^{\beta_1+1}\Gamma\left(\gamma_1\right)C_3+\\ &+\left|\varphi_k\right|+\Gamma\left(\gamma_1\right)\left|f_k\right|t^{\beta_1}C_2.\end{aligned}\tag{7.5}$$

Based on (7.2) and (4.18)-(4.20), considering $\frac{1}{|\tilde{\Delta}_k|}<\varepsilon$ (ε is a positive number), we deduce

$$\begin{aligned}\left|{}_2A_k\right| \leq{}& \left|\psi_k\right|\varepsilon+\left|\varphi_k\right|\varepsilon+\left|\psi_k\right|\varepsilon T_1^{\beta_1-1}\Gamma(\gamma_1)C_1+\left|\varphi_k\right|\varepsilon T_1^{\beta_1-1}\Gamma(\gamma_1)C_1+\\ &+2\mu_k^2\left|\varphi_k\right|\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}\varepsilon C_4+\mu_k^2\left|\varphi_k\right|\Gamma(\gamma_1)T_1^{\beta_1}\varepsilon C_2+\\ &+2\mu_k^4\left|\varphi_k\right|\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}T_1^{\beta_1}\varepsilon C_2C_4+\mu_k^2\left|\psi_k\right|T_1^{\beta_1}\Gamma(\gamma_1)\varepsilon C_2.\end{aligned}\tag{7.6}$$

Let us introduce the following notations:

$$S_1=\varepsilon+\varepsilon\Gamma(\gamma_1)T_1^{\beta_1-1}C_1,\quad S_2=2\Gamma(\gamma_2)\left(\xi-T_1\right)^{\beta_2}\varepsilon C_4,$$
$$S_3=2\Gamma(\gamma_1)\Gamma(\gamma_2)T_1^{\beta_1}\left(\xi-T_1\right)^{\beta_2}\varepsilon C_2C_4,\quad S_4=T_1^{\beta_1}\varepsilon\Gamma(\gamma_1)C_2.$$

Using the above notations, we rewrite (7.6)

$$\left|{}_2A_k\right| \leq \left|\varphi_k\right|S_1+\mu_k^2\left|\varphi_k\right|\left(S_2+S_4\right)+\left|\psi_k\right|S_1+\mu_k^2\left|\psi_k\right|S_4+\mu_k^4\left|\varphi_k\right|S_3.\tag{7.7}$$

Similarly, according to (7.4) and (4.18)-(4.21), we obtain the estimate of f_k as follows:

$$\begin{aligned}\left|f_k\right| \leq{}& \left|\psi_k\right|\varepsilon+\left|\varphi_k\right|\varepsilon+2\mu_k^2\left|\varphi_k\right|\Gamma(\gamma_1)T_1^{\beta_1}\varepsilon C_2+\mu_k^4\left|\varphi_k\right|\Gamma^2\left(\gamma_1\right)T_1^{2\beta_1}\varepsilon C_2^2+\\ &+\mu_k^2\left|\varphi_k\right|\Gamma(\gamma_2)T_1^{\beta_1}\left(\xi-T_1\right)^{\beta_2}\varepsilon C_4+2\mu_k^4\left|\varphi_k\right|\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}T_1^{\beta_1}\varepsilon C_2C_4+\\ &+\mu_k^6\left|\varphi_k\right|\Gamma^2\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}T_1^{2\beta_1}\varepsilon C_2^2C_4+\mu_k^2\left|\psi_k\right|\varepsilon T_1+\\ &+\mu_k^4\left|\psi_k\right|T_1^{\beta_1+1}\varepsilon C_3+\mu_k^2\left|\psi_k\right|\Gamma(\gamma_1)T_1^{\beta_1}\varepsilon C_2+\mu_k^2\left|\varphi_k\right|\Gamma(\gamma_1)T_1^{\beta_1}\varepsilon C_1+\\ &+\mu_k^4\left|\varphi_k\right|\Gamma^2(\gamma_1)T_1^{2\beta_1}\varepsilon C_1C_3+\mu_k^4\left|\varphi_k\right|\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}T_1^{\beta_1}\varepsilon C_1C_4+\\ &+\mu_k^6\left|\varphi_k\right|\Gamma^2\left(\gamma_1\right)\Gamma\left(\gamma_2\right)T_1^{2\beta_1}\varepsilon C_1C_3C_4.\end{aligned}\tag{7.8}$$

Let us rewrite (7.8) by introducing the following notations:

$$\begin{aligned}\left|f_k\right| \leq{}& \left|\varphi_k\right|\varepsilon+\mu_k^2\left|\varphi_k\right|S_5+\mu_k^4\left|\varphi_k\right|\left(S_3+S_6\right)+\mu_k^6\left|\varphi_k\right|S_7+\left|\psi_k\right|\varepsilon+\\ &+\mu_k^2\left|\psi_k\right|\left(\varepsilon T_1+S_4\right).\end{aligned}\tag{7.9}$$

where,

$$S_5 = 2T_1^{\beta_1}\Gamma(\gamma_1)\varepsilon C_2 + \Gamma(\gamma_2)\left(\xi - T_1\right)^{\beta_2}\varepsilon C_4 + T_1^{\beta_1}\Gamma(\gamma_1)\varepsilon C_1,$$
$$S_6 = T_1^{2\beta_1}\Gamma^2(\gamma_1)\varepsilon C_2^2 + T_1^{2\beta_1}\Gamma^2(\gamma_1)\varepsilon C_1 C_3 + \Gamma(\gamma_1)\Gamma(\gamma_2)\left(\xi - T_1\right)^{\beta_2} T_1^{\beta_1}\varepsilon C_1 C_4,$$
$$S_7 = \Gamma^2(\gamma_1)\Gamma(\gamma_2)\left(\xi - T_1\right)^{\beta_2} T_1^{2\beta_1}\varepsilon C_2^2 C_4 + \Gamma^2(\gamma_1)\Gamma(\gamma_2) T_1^{2\beta_1}\varepsilon C_1 C_3 C_4$$

Substituting (7.7) and (7.9) into (7.5), we obtain the estimate ${}_1U_k(t)$ taking the following form

$$\begin{aligned}
\left|{}_1U_k(t)\right| \leq{}& \left|\varphi_k\right| + \left|\varphi_k\right| tS_1 + \mu_k^2\left|\varphi_k\right| t\left(S_2 + S_4\right) + \mu_k^4\left|\varphi_k\right| tS_3 + \left|\psi_k\right| tS_1 + \\
&+ \mu_k^2\left|\psi_k\right| tS_4 + \mu_k^2\left|\varphi_k\right| t^{\beta_1}\Gamma(\gamma_1)C_2 + \mu_k^2\left|\varphi_k\right| t^{\beta_1+1}\Gamma(\gamma_1)C_3S_1 + \\
&+ \mu_k^4\left|\varphi_k\right| t^{\beta_1+1}\Gamma(\gamma_1)C_3\left(S_2 + S_4\right) + \mu_k^6\left|\varphi_k\right| t^{\beta_1+1}\Gamma(\gamma_1)C_3S_3 + \\
&+ \mu_k^2\left|\psi_k\right| t^{\beta_1+1}\Gamma(\gamma_1)C_3S_1 + \mu_k^4\left|\psi_k\right| t^{\beta_1+1}\Gamma(\gamma_1)C_3S_4 + \left|\varphi_k\right|\Gamma(\gamma_1)\times \\
&\times t^{\beta_1}\varepsilon C_2 + \mu_k^2\left|\varphi_k\right|\Gamma(\gamma_1)t^{\beta_1}S_5C_2 + \mu_k^4\left|\varphi_k\right| t^{\beta_1}\Gamma(\gamma_1)C_2\left(S_3 + S_6\right) + \\
&+ \mu_k^6\left|\varphi_k\right| t^{\beta_1}\Gamma(\gamma_1)C_2S_7 + \left|\psi_k\right|\Gamma(\gamma_1)t^{\beta_1}\varepsilon C_2 + \mu_k^2\left|\psi_k\right| t^{\beta_1}\Gamma(\gamma_1)C_2\times \\
&\times\left(\varepsilon T_1 + S_4\right).
\end{aligned}$$

After simplification, we have the following

$$\begin{aligned}
&\sum_{k=1}^{\infty}\left|{}_1U_k(t)\right| \leq \sum_{k=1}^{\infty}\left|\varphi_k\right|\Big(1 + tS_1 + \Gamma(\gamma_1)t^{\beta_1}\varepsilon C_2\Big) + \sum_{k=1}^{\infty}\mu_k^2\left|\varphi_k\right|\Big(t\left(S_2 + S_4\right) + \\
&+ \Gamma(\gamma_1)t^{\beta_1}C_2 + \Gamma(\gamma_1)t^{\beta_1+1}C_1C_3 + \Gamma(\gamma_1)t^{\beta_1}C_2S_5\Big) + \\
&+ \sum_{k=1}^{\infty}\mu_k^4\left|\varphi_k\right|\Big(tS_3 + t^{\beta_1+1}\Gamma(\gamma_1)C_3\left(S_2 + S_4\right) + \Gamma(\gamma_1)t^{\beta_1}C_2\left(S_3 + S_6\right)\Big) + \\
&+ \sum_{k=1}^{\infty}\mu_k^6\left|\varphi_k\right|\Big(\Gamma(\gamma_1)t^{\beta_1+1}S_3C_3 + \Gamma(\gamma_1)t^{\beta_1}S_7C_2\Big) + \sum_{k=1}^{\infty}\left|\psi_k\right|\Big(tS_1 + \Gamma(\gamma_1)t^{\beta_1}\varepsilon C_2\Big) + \\
&+ \sum_{k=1}^{\infty}\mu_k^2\left|\psi_k\right|\Big(tS_4 + t^{\beta_1+1}\Gamma(\gamma_1)S_1C_3 + \Gamma(\gamma_1)t^{\beta_1}C_2(\varepsilon T_1 + S_4)\Big) + \\
&+ \sum_{k=1}^{\infty}\mu_k^4\left|\psi_k\right|\Gamma(\gamma_1)t^{\beta_1+1}S_4C_3. \qquad (7.10)
\end{aligned}$$

From the estimates in Lemma 5.1 , we obtain the following estimate for ${}_1U_k(t)$

$$\begin{aligned}
\sum_{k=1}^{\infty}\left|{}_1U_k(t)\right| \leq{}& \sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-0,5}}\Big(1 + 2tS_1 + 2\Gamma(\gamma_1)t^{\beta_1}\varepsilon C_2\Big) + \\
&+ \sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-2,5}}\Big(t\left(S_2 + 2S_4\right) + \Gamma(\gamma_1)t^{\beta_1}C_2 + \Gamma(\gamma_1)t^{\beta_1+1}\left(C_1 + S_1\right)C_3 +
\end{aligned}$$

$$+\Gamma(\gamma_1)t^{\beta_1}C_2(\varepsilon T_1+S_4+S_5)\Big)+\sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-4,5}}\Big(tS_3+t^{\beta_1+1}\Gamma(\gamma_1)C_3\times$$

$$\times\left(S_2+2S_4\right)+\Gamma(\gamma_1)t^{\beta_1}C_2\left(S_3+S_6\right)\Big)+\sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-6,5}}\Big(\Gamma(\gamma_1)t^{\beta_1+1}S_3C_3+$$

$$+\Gamma(\gamma_1)t^{\beta_1}S_7C_2\Big).$$

A6: Estimate of $u(t,r)$ in Ω_2. Now we will demonstrate the convergence of the solution ${}_2U_k(t)$:

$$\left|{}_2U_k\left(t\right)\right|\leq\mu_k^2\left|{}_3A_k\right|\Gamma\left(\gamma_2\right)\left|E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}-\mu_k^2\left(t-T_1\right)^{\beta_2}\\ \delta\left(t-T_1\right)^{\alpha_2}\end{array}\right)\right|\times$$
$$\times\left(t-T_1\right)^{\beta_2}+\left|{}_3A_k\right|+\Gamma\left(\gamma_2\right)\left|f_k\right|\left(t-T_1\right)^{\beta_2}\times$$
$$\times\left|E_2\left(\begin{array}{c}\gamma_2,\gamma_2,1;1,0\\ \beta_2+1,\beta_2,\alpha_2;\gamma_2,\gamma_2;1,1\end{array}\middle|\begin{array}{c}-\mu_k^2\left(t-T_1\right)^{\beta_2}\\ \delta\left(t-T_1\right)^{\alpha_2}\end{array}\right)\right|. \tag{7.11}$$

Using the estimate (4.21), we rewrite (7.11) as follows.

$$\left|{}_2U_k\left(t\right)\right|\leq\left|{}_3A_k\right|+\mu_k^2\left|{}_3A_k\right|\Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2}C_4+\Gamma\left(\gamma_2\right)\left|f_k\right|\left(t-T_1\right)^{\beta_2}C_4. \tag{7.12}$$

To demonstrate the uniform convergence of the solution ${}_2U_k\left(t\right)$, we need to find an estimate of ${}_3A_k$, which we determine as follows. For this, we use estimates (4.18)-(4.21) and $\frac{1}{|\tilde{\Delta}_k|}<\varepsilon$:

$$\left|{}_3A_k\right|\leq\left|\psi_k\right|\varepsilon T_1^{\beta_1}\Gamma(\gamma_1)\varepsilon C_2+\mu_k^2\left|\psi_k\right|\varepsilon T_1^{2\beta_1}\Gamma^2(\gamma_1)C_2^2+$$
$$+\mu_k^2\left|\varphi_k\right|\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}T_1^{\beta_1}C_1C_4\varepsilon+\mu_k^4\left|\varphi_k\right|\times$$
$$\times\Gamma^2\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}T_1^{2\beta_1}C_1C_3C_4\varepsilon+\left|\varphi_k\right|\Gamma\left(\gamma_2\right)C_4\varepsilon\left(\xi-T_1\right)^{\beta_2}+$$
$$+2\mu_k^2\left|\varphi_k\right|\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)T_1^{\beta_1}\left(\xi-T_1\right)^{\beta_2}C_2C_4\varepsilon+\mu_k^4\left|\varphi_k\right|\Gamma^2\left(\gamma_1\right)\times$$
$$\times\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}T_1^{2\beta_1}C_2^2C_4\varepsilon+\left|\psi_k\right|\varepsilon T_1^{\beta_1}\Gamma(\gamma_1)\varepsilon C_1+\mu_k^2\left|\psi_k\right|\varepsilon\times$$
$$\times T_1^{2\beta_1}\Gamma^2(\gamma_1)C_1C_3+\left|\psi_k\right|\varepsilon T_1+\mu_k^2\left|\psi_k\right|\varepsilon T_1^{\beta_1+1}\Gamma(\gamma_1)C_3.$$

As a result of the simplifications, we obtain the following:

$$\left|{}_3A_k\right|\leq\left|\varphi_k\right|\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}\varepsilon C_4+\mu_k^2\left|\varphi_k\right|\Bigg(\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\times$$
$$\times\left(\xi-T_1\right)^{\beta_2}T_1^{\beta_1}C_1C_4\varepsilon+2\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}T_1^{\beta_1}C_2C_4\varepsilon\Bigg)+$$
$$+\mu_k^4\left|\varphi_k\right|\Bigg(\Gamma^2\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}T_1^{2\beta_1}C_1C_3C_4\varepsilon+\Gamma^2\left(\gamma_1\right)\left(\xi-T_1\right)^{\beta_2}\times$$
$$\times\Gamma\left(\gamma_2\right)T_1^{2\beta_1}\varepsilon C_2^2C_4\Bigg)+\left|\psi_k\right|\Bigg(T_1^{\beta_1}\Gamma\left(\gamma_1\right)C_2\varepsilon+T_1^{\beta_1}\Gamma\left(\gamma_1\right)C_1\varepsilon+T_1\varepsilon\Bigg)+$$

$$+\mu_k^2\left|\psi_k\right|\left(T_1^{2\beta_1}\Gamma^2\left(\gamma_1\right)C_2^2\varepsilon+T_1^{2\beta_1}\Gamma^2\left(\gamma_1\right)C_1C_3\varepsilon+T_1^{\beta_1+1}\Gamma\left(\gamma_1\right)C_3\varepsilon\right). \tag{7.13}$$

We will introduce the following notations:

$$S_8=\Gamma^2\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}T_1^{2\beta_1}C_4\varepsilon\Big(C_1C_3+C_2^2\Big),$$
$$S_9=T_1^{2\beta_1}\Gamma^2\left(\gamma_1\right)C_2^2\varepsilon+T_1^{2\beta_1}\Gamma^2\left(\gamma_1\right)C_1C_3\varepsilon+T_1^{\beta_1+1}\Gamma\left(\gamma_1\right)C_3\varepsilon.$$

Based on these notations, we will rewrite (7.13) as

$$\left|{}_3A_k\right|\leq\left|\varphi_k\right|\frac{S_2}{2}+\mu_k^2\left|\varphi_k\right|\frac{S_3(C_1+2C_2)}{2C_2}+\mu_k^4\left|\varphi_k\right|S_8+\left|\psi_k\right|\left(S_4+S_1T_1\right)+$$
$$+\mu_k^2\left|\psi_k\right|S_9. \tag{7.14}$$

We substitute (7.14) into (7.12) and obtain the following estimate

$$\sum_{k=1}^{\infty}\left|{}_2U_k(t)\right|\leq\sum_{k=1}^{\infty}\left|\varphi_k\right|\left(\frac{S_2}{2}+\Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2}\varepsilon C_4\right)+\sum_{k=1}^{\infty}\mu_k^2\left|\varphi_k\right|\times$$
$$\left(\frac{S_3(C_1+2C_2)}{2C_2}+\left(t-T_1\right)^{\beta_2}\Gamma\left(\gamma_2\right)\frac{S_2C_4}{2}+\Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2}C_4S_5\right)+$$
$$+\sum_{k=1}^{\infty}\mu_k^4\left|\varphi_k\right|\left(S_8+\Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2}C_4\frac{S_3(C_1+2C_2)}{2C_2}+\right.$$
$$\left.+\Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2}C_4\Big(S_3+S_6\Big)\right)+\sum_{k=1}^{\infty}\mu_k^6\left|\varphi_k\right|\times$$
$$\times\left(\Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2}S_8C_4+\Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2}S_7C_4\right)+$$
$$+\sum_{k=1}^{\infty}\left|\psi_k\right|\left(\left(S_4+S_1T_1\right)+\Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2}\varepsilon C_4\right)+$$
$$+\sum_{k=1}^{\infty}\mu_k^2\left|\psi_k\right|\left(S_9+\Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2}C_4(S_4+S_1T_1)+\right.$$
$$\left.+\Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2}C_4\Big(\varepsilon T_1+S_4\Big)\right)+\sum_{k=1}^{\infty}\mu_k^4\left|\psi_k\right|\Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2}C_4S_9.$$

From the estimates in Lemma 5.1 , we obtain the following estimate for $u(t,r)$ in Ω_2:

$$\sum_{k=1}^{\infty}\left|{}_2U_k(t)\right|\leq\sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-0,5}}\left(\frac{S_2}{2}+2\Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2}\varepsilon C_4(S_4+S_1T_1)\right)+$$

$$+\sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-2,5}}\Bigg(\frac{S_3(C_1+2C_2)}{2C_2}+S_9+\Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2}C_4\times$$
$$(2S_4+S_1T_1+\frac{S_2}{2}+\varepsilon T_1+S_5)\Bigg)+\sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-4,5}}\Bigg(S_8+\Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2}\times$$
$$\times C_4\Bigg(\frac{S_3(C_1+2C_2)}{2C_2}+S_3+S_6+S_9\Bigg)\Bigg)+$$
$$+\sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-6,5}}\Bigg(\Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2}S_8C_4+\Gamma\left(\gamma_2\right)\left(t-T_1\right)^{\beta_2}S_7C_4\Bigg).$$

A7: Estimate of $u(t,r)$ in Ω_3. Let us estimate ${}_3U_k(t)$:

$$\left|{}_3U_k(t)\right|\le\left|{}_4A_k\right|+\left|{}_5A_k\right|\left(t-T_2\right)+\mu_k^2\left|{}_4A_k\right|\left(t-T_2\right)^{\beta_3}\Gamma\left(\gamma_3\right)\times$$
$$\times\left|E_2\left(\begin{array}{c}\gamma_3,\gamma_3,1;1,0\\ \beta_3+1,\beta_3,\alpha_3;\gamma_3,\gamma_3;1,1\end{array}\;\middle|\;\begin{array}{c}-\mu_k^2\left(t-T_2\right)^{\beta_3}\\ \delta\left(t-T_2\right)^{\alpha_3}\end{array}\right)\right|+\mu_k^2\left|{}_5A_k\right|\times$$
$$\times\left(t-T_2\right)^{\beta_3+1}\Gamma(\gamma_3)\left|E_2\left(\begin{array}{c}\gamma_3,\gamma_3,1;1,0\\ \beta_3+2,\beta_3,\alpha_3;\gamma_3,\gamma_3;1,1\end{array}\;\middle|\;\begin{array}{c}-\mu_k^2\left(t-T_2\right)^{\beta_3}\\ \delta\left(t-T_2\right)^{\alpha_3}\end{array}\right)\right|+$$
$$+\Gamma(\gamma_3)\left(t-T_2\right)^{\beta_3}\left|f_k\right|\left|E_2\left(\begin{array}{c}\gamma_3,\gamma_3,1;1,0\\ \beta_3+1,\beta_3,\alpha_3;\gamma_3,\gamma_3;1,1\end{array}\;\middle|\;\begin{array}{c}-\mu_k^2\left(t-T_2\right)^{\beta_3}\\ \delta\left(t-T_2\right)^{\alpha_3}\end{array}\right)\right|.$$

Based on estimates (4.18) through (7.2), we can write the following

$$\left|{}_3U_k(t)\right|\le\left|{}_4A_k\right|+\left|{}_5A_k\right|\left(t-T_2\right)+\mu_k^2\left|{}_4A_k\right|\left(t-T_2\right)^{\beta_3}\Gamma\left(\gamma_3\right)C_5+$$
$$+\mu_k^2\left|{}_5A_k\right|\left(t-T_2\right)^{\beta_3+1}\Gamma(\gamma_3)C_6+\Gamma(\gamma_3)\left(t-T_2\right)^{\beta_3}\left|f_k\right|C_5. \tag{7.15}$$

We also calculate the estimates of the coefficients ${}_4A_k$ and ${}_5A_k$. First, using estimates (4.18)-(4.21), we determine the estimate of coefficient ${}_4A_k$ as follows:

$$\left|{}_4A_k\right|\le\left|\varphi_k\right|\Big(\Gamma\left(\gamma_2\right)\left(\xi-T_1\right)^{\beta_2}\varepsilon C_4+\Gamma\left(\gamma_2\right)\left(T_2-T_1\right)^{\beta_2}\varepsilon C_4\Big)+$$
$$+\mu_k^2\left|\varphi_k\right|\Big(\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)T_1^{\beta_1}\left(\xi-T_1\right)^{\beta_2}\varepsilon C_1C_4++2\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\times$$
$$\times\left(\xi-T_1\right)^{\beta_2}T_1^{\beta_1}\varepsilon C_2C_4+2\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\left(T_2-T_1\right)^{\beta_2}T_1^{\beta_1}\varepsilon C_2C_4+$$
$$+\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)\left(T_2-T_1\right)^{\beta_2}T_1^{\beta_1}\varepsilon C_1C_4\Big)+\mu_k^4\left|\varphi_k\right|\Big(\Gamma^2\left(\gamma_1\right)\times$$
$$\times\Gamma\left(\gamma_2\right)T_1^{2\beta_1}\left(\xi-T_1\right)^{\beta_2}\varepsilon C_2^2C_4+\Gamma^2\left(\gamma_1\right)\Gamma\left(\gamma_2\right)T_1^{\beta_1}\times$$
$$\times\left(\xi-T_1\right)^{\beta_2}\varepsilon C_1C_3C_4+\Gamma^2\left(\gamma_1\right)\Gamma\left(\gamma_2\right)T_1^{2\beta_1}\left(T_2-T_1\right)^{\beta_2}\varepsilon C_2^2C_4+$$
$$\Gamma^2\left(\gamma_1\right)\Gamma\left(\gamma_2\right)T_1^{2\beta_1}\left(T_2-T_1\right)^{\beta_2}\varepsilon C_1C_3C_4\Big)+\left|\psi_k\right|\Big(T_1^{\beta_1}\Gamma\left(\gamma_1\right)\times$$
$$\times\varepsilon C_2+T_1^{\beta_1}\Gamma\left(\gamma_1\right)\varepsilon C_1+T_1\varepsilon+\Gamma\left(\gamma_2\right)\left(T_2-T_1\right)^{\beta_2}\times$$
$$\times\varepsilon C_4\Big)+\mu_k^2\left|\psi_k\right|\Big(T_1^{\beta_1+1}\Gamma\left(\gamma_1\right)\varepsilon C_3+T_1^{2\beta_1}\Gamma^2\left(\gamma_1\right)\varepsilon C_1C_3+$$

$$
\begin{aligned}
&+2\Gamma(\gamma_1)\Gamma(\gamma_2)(T_2-T_1)^{\beta_2}T_1^{\beta_1}\varepsilon C_2C_4+T_1^{2\beta_1}\Gamma^2(\gamma_1)\varepsilon C_2^2+\\
&+\Gamma(\gamma_1)\Gamma(\gamma_2)(T_2-T_1)^{\beta_2}T_1^{\beta_1}\varepsilon C_1C_4\Big)+\mu_k^4|\psi_k|\times\\
&\times\Big(\Gamma^2(\gamma_1)\Gamma(\gamma_2)T_1^{2\beta_1}(T_2-T_1)^{\beta_2}\varepsilon C_2^2C_4+\Gamma^2(\gamma_1)\Gamma(\gamma_2)T_1^{2\beta_1}\times\\
&\times(T_2-T_1)^{\beta_2}\varepsilon C_1C_3C_4\Big).
\end{aligned}
\tag{7.16}
$$

Let us introduce the following notations:

$$
\begin{aligned}
&S_{10}=\Gamma(\gamma_2)(T_2-T_1)^{\beta_2}\varepsilon C_4,\\
&S_{11}=\Gamma(\gamma_1)\Gamma(\gamma_2)T_1^{\beta_1}\varepsilon C_4(2C_2+C_1)\left((\xi-T_1)^{\beta_2}+(T_2-T_1)^{\beta_2}\right),\\
&S_{12}=T_1^{2\beta_1}\Gamma^2(\gamma_1)\Gamma(\gamma_2)(T_2-T_1)^{\beta_2}\varepsilon\left(C_2^2C_4+C_1C_3C_4\right),\\
&S_{13}=T_1^{\beta_1+1}\Gamma(\gamma_1)\varepsilon C_3+T_1^{2\beta_1}\Gamma^2(\gamma_1)\varepsilon C_1C_3+2\Gamma(\gamma_1)\Gamma(\gamma_2)T_1^{\beta_1}(T_2-T_1)^{\beta_2}\times\\
&\times\varepsilon C_2C_4+\Gamma(\gamma_1)\Gamma(\gamma_2)T_1^{\beta_1}(T_2-T_1)^{\beta_2}\varepsilon C_1C_4+T_1^{2\beta_1}\Gamma^2(\gamma_1)\varepsilon C_2^2.
\end{aligned}
$$

We rewrite (7.16) according to the corresponding notations

$$
\begin{aligned}
&|{}_4A_k|\le|\varphi_k|\left(\frac{S_2}{2}+S_{10}\right)+\mu_k^2|\varphi_k|S_{11}+\mu_k^4|\varphi_k|(S_7+S_{12})+\\
&+|\psi_k|(S_4+T_1S_1+S_{10})+\mu_k^2|\psi_k|S_{13}+\mu_k^4|\psi_k|S_{12}.
\end{aligned}
\tag{7.17}
$$

By substituting the estimates (7.9) and (7.17) into (3.19), we determine the estimate of ${}_5A_k$ as follows

$$
\begin{aligned}
&|{}_5A_k|\le|\varphi_k|\varepsilon+\mu_k^2|\varphi_k|\left(\frac{S_2}{2}+S_5+S_{10}\right)+\mu_k^4|\varphi_k|(S_3+S_6+S_{11})+\\
&+\mu_k^6|\varphi_k|(2S_7+S_{12})+|\psi_k|\varepsilon+\mu_k^2|\psi_k|(2S_4+T_1S_1+S_{10}+\varepsilon T_1)+\\
&+\mu_k^4|\psi_k|S_{13}+\mu_k^6|\psi_k|S_{12}.
\end{aligned}
\tag{7.18}
$$

Substituting the estimates (7.9),(7.17) and (7.18) into (7.15), we obtain the following estimate for the solution ${}_3U_k(t)$

$$
\begin{aligned}
&\sum_{k=1}^{\infty}|{}_3U_k(t)|\le\sum_{k=1}^{\infty}|\varphi_k|\left(\frac{S_2}{2}+S_{10}+(t-T_2)\varepsilon+\Gamma(\gamma_3)(t-T_2)^{\beta_3}\varepsilon C_5\right)+\\
&+\sum_{k=1}^{\infty}\mu_k^2|\varphi_k|\Bigg(S_{11}+(t-T_2)(\frac{S_2}{2}+S_5+S_{10})+\\
&+\Gamma(\gamma_3)(t-T_2)^{\beta_3}\left((\frac{S_2}{2}+S_{10})C_5+(t-T_2)\varepsilon C_6+S_5C_5\right)\Bigg)+\\
&+\sum_{k=1}^{\infty}\mu_k^4|\varphi_k|\Bigg(S_7+S_{12}+(t-T_2)(S_3+S_6+S_{11})+\Gamma(\gamma_3)(t-T_2)^{\beta_3}\Big(S_{11}C_5+\\
&+(t-T_2)C_6(\frac{S_2}{2}+S_{10}+S_5)+C_5(S_3+S_6)\Big)\Bigg)+\sum_{k=1}^{\infty}\mu_k^6|\varphi_k|\times
\end{aligned}
$$

$$\left((t-T_2)(2S_7+S_{12})+\Gamma\left(\gamma_3\right)\left(t-T_2\right)^{\beta_3}\left(C_5(S_7+S_{12})+\right.\right.$$

$$\left.\left.+(t-T_2)C_6(S_3+S_6+S_{11})+S_7C_5\right)\right)+\sum_{k=1}^{\infty}\mu_k^8\left|\varphi_k\right|\times$$

$$\times\Gamma\left(\gamma_3\right)\left(t-T_2\right)^{\beta_3+1}C_6(2S_7+S_{12})+\sum_{k=1}^{\infty}\left|\psi_k\right|\left(S_4+T_1S_1+\right.$$

$$\left.+S_{10}+(t-T_2)\varepsilon+\left(t-T_2\right)^{\beta_3}\Gamma\left(\gamma_3\right)C_5\varepsilon\right)+\sum_{k=1}^{\infty}\mu_k^2\left|\psi_k\right|\times$$

$$\times\left(S_{13}+(t-T_2)(2S_4+S_1T_1+S_{10}+\varepsilon T_1)+\Gamma\left(\gamma_3\right)\left(t-T_2\right)^{\beta_3}\times\right.$$

$$\left.\times\left((S_4+S_1T_1+S_{10})C_5+(t-T_2)\varepsilon C_6+C_5(\varepsilon T_1+S_4)\right)\right)+$$

$$+\sum_{k=1}^{\infty}\mu_k^4\left|\psi_k\right|\left(S_{12}+S_{13}(t-T_2)+\Gamma\left(\gamma_3\right)\left(t-T_2\right)^{\beta_3}\times\right.$$

$$\left.\times\left(S_{13}C_5+(t-T_2)\varepsilon C_6(2S_4+S_1T_1+S_{10}+\varepsilon T_1)\right)\right)+$$

$$+\sum_{k=1}^{\infty}\mu_k^6\left|\psi_k\right|\left((t-T_2)S_{12}+\Gamma\left(\gamma_3\right)\left(t-T_2\right)^{\beta_3}\left(C_5S_{12}+\right.\right.$$

$$\left.\left.+(t-T_2)S_{13}C_6\right)\right)+\sum_{k=1}^{\infty}\mu_k^8\left|\psi_k\right|\left(t-T_2\right)^{\beta_3+1}\Gamma\left(\gamma_3\right)S_{12}C_6.$$

From the estimates in Lemma 5.1, we obtain the following estimate for ${}_3U_k(t)$

$$\sum_{k=1}^{\infty}\left|{}_3U_k(t)\right|\leq\sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-0,5}}\left(\frac{S_2}{2}+2S_{10}+2\left(t-T_2\right)\varepsilon+\right.$$

$$\left.+2\Gamma\left(\gamma_3\right)\left(t-T_2\right)^{\beta_3}\varepsilon C_5+S_4+T_1S_1\right)+\sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-2,5}}\times$$

$$\times\left(S_{11}+\Gamma\left(\gamma_3\right)\left(t-T_2\right)^{\beta_3}\left((\frac{S_2}{2}+S_{10})C_5+S_5C_5+\right.\right.$$

$$\left.+(S_4+S_1T_1+S_{10})C_5+C_5(\varepsilon T_1+S_4)\right)+S_{13}+(t-T_2)\times$$

$$\left.\times\left(2S_4+2\varepsilon C_6+S_1T_1+2S_{10}+\varepsilon T_1+\frac{S_2}{2}+S_5\right)\right)+\sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-4,5}}\times$$

$$\times\left(S_7+2S_{12}+(t-T_2)(S_3+S_6+S_{11}+S_{13})+\Gamma\left(\gamma_3\right)\left(t-T_2\right)^{\beta_3}\times\right.$$
$$\times\left(S_{11}C_5+(t-T_2)C_6(\frac{S_2}{2}+S_{10}+S_5)+C_5(S_3+S_6)+\right.$$
$$\left.\left.+S_{13}C_5+(t-T_2)\varepsilon C_6(2S_4+S_1T_1+S_{10}+\varepsilon T_1)\right)\right)+\sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-6,5}}\times$$
$$\times\left(2(t-T_2)(S_7+S_{12})+\Gamma\left(\gamma_3\right)\left(t-T_2\right)^{\beta_3}\left(C_5(S_7+S_{12})+\right.\right.$$
$$+(t-T_2)C_6(S_3+S_6+S_{11})+S_7C_5+C_5S_{12}+$$
$$\left.\left.+(t-T_2)S_{13}C_6\right)\right)+\sum_{k=1}^{\infty}\frac{C}{\mu_k^{2s-8,5}}2\Gamma\left(\gamma_3\right)\left(t-T_2\right)^{\beta_3+1}C_6(S_7+S_{12}).$$

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[1] Ghent University, Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent, Belgium.

Email address: erkinjon.karimov@ugent.be

[2] Andijan State University, Department of Algebra and Analysis, Andijan, Uzbekistan.

Email address: muzaffar.toshpulatov89@gmail.com