Gulf Journal of Mathematics
Vol 23, Issue 2 (2026) 1-10

ISSN: 2309-4966
https://doi.org/10.56947/n8dw2419

A DOUBLY CRITICAL ELLIPTIC PROBLEM WITH SUBMANIFOLD SINGULARITIES

ABDOURAHMANE DIATTA[1] and EL HADJI ABDOULAYE THIAM[2*]

Abstract. Let $N \geq 4$ and Ω be a bounded domain in $\mathbb{R}^N$ containing a smooth closed submanifold Σ of dimension k with $2 \leq k \leq N-2$. We study the existence of positive solutions to a doubly critical Hardy–Sobolev elliptic equation whose singular weights are powers of the distance to Σ, with two distinct critical exponents $2s_1^*$ and $2s_2^*$ corresponding to the parameters s_1 and s_2 with $0 \leq s_2 < s_1 < 2$. Using the mountain pass lemma and asymptotic expansions of the energy functional in Fermi coordinates, we prove that a positive solution exists whenever a geometric quantity involving the mean curvature H, the scalar curvature R_g of Σ, and the potential h satisfies a negativity condition at some point of Σ.

Keywords. Hardy–Sobolev inequality; doubly critical exponents; submanifold singularities; mountain pass lemma; Fermi coordinates; mean curvature.
2020 Mathematics Subject Classification. Primary 35J20; Secondary 35B33, 53C21.

1. Introduction

Elliptic equations involving Hardy–Sobolev critical nonlinearities arise at the intersection of geometric analysis, nonlinear functional analysis, and the study of singular phenomena in partial differential equations. They reveal how geometric singularities—points, curves, or higher-dimensional submanifolds—influence compactness, concentration, and the formation of extremals.

A central theme is the interplay between singular weights (powers of the distance to a set), critical exponents, and the geometry of the singular submanifold. In the unweighted case, the critical equation $-\Delta u = u^{2^*-1}$ is invariant under scaling and exhibits a profound lack of compactness. The seminal work of Brezis and Nirenberg [3] showed that a lower-order perturbation may restore compactness and produce nontrivial solutions. Their approach has inspired an extensive literature on compactness recovery through potentials, boundary geometry, curvature, or weighted nonlinearities.

In the presence of singularities the landscape becomes significantly richer. For point singularities, the works of Ghoussoub and Robert [9, 10] established the

Date: Received: May 6, 2026; Revised: Jun 4, 2026; Accepted: Jun 10, 2026.
* Corresponding author

decisive role of geometric invariants such as the boundary mean curvature and the Hardy singular mass. More recent contributions extended the analysis to singularities on curves or sets of higher codimension, showing that tangential and normal geometric contributions enter the energy expansion in subtle and indispensable ways.

When the singular set is a smooth compact submanifold of intermediate dimension, new geometric phenomena arise and the analysis requires a refined understanding of how curvature, the second fundamental form, and variations in the induced metric influence weighted Sobolev inequalities. The interaction between the geometry of the submanifold and the concentration of solutions is significantly more involved than in the point or curve cases.

In this work we investigate a doubly critical elliptic problem of Hardy–Sobolev type, where the singularity is supported on a smooth compact submanifold

$$\Sigma \subset \Omega \subset \mathbb{R}^N, \qquad N \geq 4,$$

of dimension $2 \leq k \leq N-2$. Let $\rho_\Sigma(x) = \mathrm{dist}(x, \Sigma)$ and $0 \leq s_2 < s_1 < 2$. We study positive solutions to

$$-\Delta u + h(x)\, u = \lambda\, \rho_\Sigma^{-s_1}\, u^{2^*_{s_1}-1} + \rho_\Sigma^{-s_2}\, u^{2^*_{s_2}-1} \quad \text{in } \Omega, \qquad u \in H^1_0(\Omega), \tag{1.1}$$

where $2^*_s = \frac{2(N-s)}{N-2}$ is the Hardy–Sobolev critical exponent, the potential $h \in C(\Omega)$ is such that $-\Delta + h$ is coercive, and $\lambda > 0$.

Our main result is the following.

Theorem 1.1. *Let $N \geq 4$, $0 \leq s_2 < s_1 < 2$, Ω be a bounded domain of $\mathbb{R}^N$, and $\Sigma \subset \Omega$ be a smooth submanifold of dimension $2 \leq k \leq N-2$. Let $h \in C(\Omega)$ be such that $-\Delta + h$ is coercive. Then there exist constants A_{N,λ,s_1,s_2} and B_{N,λ,s_1,s_2}, depending only on N, λ, s_1, s_2, such that if there exists $y_0 \in \Sigma$ with*

$$A_{N,\lambda,s_1,s_2}\, H^2(y_0) + B_{N,\lambda,s_1,s_2}\, R_g(y_0) + h(y_0) < 0,$$

then problem (1.1) admits a nonnegative solution $u \in H^1_0(\Omega) \setminus \{0\}$. Here H and R_g denote, respectively, the norm of the mean curvature and the scalar curvature of Σ.

The distinctive feature of (1.1) is the presence of two distinct Hardy–Sobolev critical nonlinearities. This double criticality creates a delicate competition between two scaling regimes and significantly complicates the variational analysis. In addition, the singularity lies along a submanifold, which introduces expansions of the metric in Fermi-type coordinates, curvature tensors, and other local invariants.

Our study connects to and extends several lines of research. For point singularities, the results of Ghoussoub–Robert and others establish the role of curvature and the Hardy singular mass in determining extremals for Hardy–Sobolev inequalities. For curve singularities ($k = 1$), the works of Fall–Thiam [8], Ijaodoro–Thiam [13], and Ciss–Diatta–Thiam [6] show that curvature of the curve strongly influences concentration behavior. When $k \geq 2$, new anisotropic effects arise due to the geometry of the submanifold, and expansions require the full second fundamental form and scalar curvature contributions. Problems involving two critical nonlinearities are already highly nontrivial even without singularities; when combined with Hardy weights, the difficulty increases dramatically. Related problems and techniques also appear in recent work published in the Gulf Journal of Mathematics [1, 14].

The proof combines variational methods, blow-up and concentration analysis, and precise geometric expansions in tubular neighborhoods of Σ. We study the variational functional associated with (1.1) and compare its critical levels with the best constants of the corresponding limiting problem on $\mathbb{R}^N$. We construct a family of highly concentrated test functions built from the ground state of the limiting problem, rescaled in Fermi coordinates around a point $y_0 \in \Sigma$. We perform detailed asymptotic expansions of the Dirichlet energy and weighted critical integrals, revealing explicit contributions from the mean curvature, second fundamental form, scalar curvature, and the value $h(y_0)$. Finally, we identify a geometric quantity whose sign determines whether the concentration mechanism lowers the variational level below the critical threshold, allowing the Mountain Pass Theorem to yield a nontrivial solution.

The main achievements of this paper are as follows. We provide precise metric expansions in Fermi-type coordinates around Σ, keeping all geometric contributions up to order $O(\varepsilon^2)$. We construct new anisotropic test functions adapted to the geometry of the problem, incorporating both tangential and normal directions. We identify explicit geometric conditions guaranteeing that the variational level lies below the critical threshold, thereby proving existence of a positive solution. We obtain, for the first time, an existence result for a doubly critical Hardy–Sobolev equation with a singularity distributed along a submanifold of dimension $k \geq 2$.

The paper is organised as follows. Section 2 introduces the geometric framework: we set up Fermi-type coordinates around Σ and derive the expansion of the induced metric in tubular neighborhoods of Σ up to order $O(\varepsilon^2)$. Section 3 is devoted to the proof of the main result. In Section 3.1 we establish the variational framework and carry out the Palais–Smale analysis. Section 3.2 is dedicated to the construction of the test functions and the asymptotic expansion of the associated energy. Finally, Section 3.3 completes the proof of Theorem 1.1 by combining the geometric condition with the variational argument.

2. Fermi Coordinates and Local Metric Expansion Near Σ

Let $\Omega \subset \mathbb{R}^N$, $N \geq 4$, be a bounded domain, and let Σ be a smooth closed submanifold of Ω of dimension k with $2 \leq k \leq N-2$. Fix a point $y_0 \in \Sigma$ and choose an orthonormal basis $(E_a)_{1\leq a\leq k}$ of $T_{y_0}\Sigma$. For $r > 0$ sufficiently small, there exists a local parametrisation of Σ near y_0:

$$f : B_{\mathbb{R}^k}(0,r) \longrightarrow \Sigma, \qquad t \longmapsto f(t) = \exp^{\Sigma}_{y_0}\left(\sum_{a=1}^{k} t_a E_a\right),$$

where $\exp^{\Sigma}_{y_0}$ denotes the exponential map of Σ at y_0.

Consider a smooth orthonormal frame field $(E_i(f(t)))_{k+1\leq i\leq N}$ on the normal bundle of Σ, such that $(E_\alpha(f(t)))_{1\leq\alpha\leq N}$ forms an oriented orthonormal basis of

$\mathbb{R}^N$ for every t, with $E_i(f(0)) = E_i$. Set $Q_r := B_{\mathbb{R}^k}(0,r) \times B_{\mathbb{R}^{N-k}}(0,r)$. For $r > 0$ sufficiently small, we parametrise a neighbourhood of $y_0 = F(0,0)$ by

$$F : Q_r \longrightarrow \Omega, \qquad (t,z) \longmapsto F(t,z) = f(t) + \sum_{i=k+1}^{N} z_i\, E_i(f(t)).$$

Let $\rho_\Sigma : \Omega \to \mathbb{R}$ denote the distance to Σ. In these local coordinates,

$$\rho_\Sigma(F(t,z)) = |z|, \qquad \text{for all } (t,z) \in Q_r. \tag{2.1}$$

For $a, b = 1, \dots, k$ and $i, j = k+1, \dots, N$, we introduce smooth functions $\Gamma^i_{ab}(f(t))$ and $\beta^i_{ja}(f(t))$ defined through

$$dE_i\left(\frac{\partial f}{\partial t_a}\right) = -\sum_{b=1}^{k} \Gamma^i_{ab}\, \frac{\partial f}{\partial t_b} + \sum_{\substack{j=k+1 \\ j\neq i}}^{N} \beta^i_{ja}\, E_j, \tag{2.2}$$

where Γ^i_{ab} encodes the components of the second fundamental form of Σ in $\mathbb{R}^N$, and β^i_{ja} represents the torsion coefficients. The antisymmetry relation $\beta^i_{ja}(f(t)) = -\beta^j_{ia}(f(t))$ holds.

The norms of the second fundamental form and of the mean curvature of Σ are defined by

$$\Gamma := \left(\sum_{a,b=1}^{k} \sum_{i=k+1}^{N} (\Gamma^i_{ab})^2\right)^{1/2}, \qquad H := \left(\sum_{i=k+1}^{N} \left(\sum_{a=1}^{k} \Gamma^i_{aa}\right)^2\right)^{1/2}.$$

We now derive the expansion of the metric induced by the parametrisation F_{y_0}. For $(t,z) \in Q_r$, set

$$g_{ab}(x) = \partial_{t_a} F_{y_0}(x) \cdot \partial_{t_b} F_{y_0}(x), \quad g_{ai}(x) = \partial_{t_a} F_{y_0}(x) \cdot \partial_{z_i} F_{y_0}(x), \quad g_{ij}(x) = \partial_{z_i} F_{y_0}(x) \cdot \partial_{z_j} F_{y_0}(x).$$

Lemma 2.1. *For all $x = (t,z) \in Q_r$,*

$$g_{ab}(x) = \delta_{ab} - 2\sum_{i=k+1}^{N} z_i\, \Gamma^i_{ab} + \sum_{i,j=k+1}^{N} \sum_{c=1}^{k} z_i z_j\, \Gamma^i_{ac}\Gamma^j_{bc} + \sum_{i,j=k+1}^{N} \sum_{\substack{l=k+1 \\ l\neq i,\, l\neq j}}^{N} z_i z_j\, \beta^i_{la}\beta^j_{lb}$$
$$- \frac{1}{3}\sum_{c,d=1}^{k} R_{acbd}(x_0)\, t_c t_d + O(|x|^3),$$

$$g_{ia}(x) = \sum_{\substack{j=k+1 \\ j\neq i}}^{N} z_j\, \beta^j_{ia}, \qquad g_{ij}(x) = \delta_{ij}.$$

The functions Γ^i_{ab} and β^j_{ia} are evaluated at $f(t)$.

Lemma 2.2. *For every $x = (t, z) \in Q_r$, the determinant of the metric satisfies*

$$\sqrt{|g|}(x) = 1 - \sum_{i=k+1}^{N} z_i H^i - \frac{1}{2} \sum_{i,j=k+1}^{N} \sum_{c=1}^{k} z_i z_j \, \Gamma^i_{ab} \Gamma^j_{ab} + \sum_{i,j=k+1}^{N} z_i z_j \, H^i H^j$$
$$- \frac{1}{6} \sum_{c,d=1}^{k} \mathrm{Ric}_{cd} \, t_c t_d + O(|x|^3).$$

Moreover, the components of the inverse metric satisfy

$$g^{ab}(x) = \delta_{ab} + 2 \sum_{i=k+1}^{N} z_i \, \Gamma^i_{ab} + 3 \sum_{c=1}^{k} \sum_{i,j=k+1}^{N} z_i z_j \, \Gamma^i_{ac} \Gamma^j_{bc} - \frac{1}{3} \sum_{c,d=1}^{k} R_{acbd}(x_0) \, t_c t_d + O(|x|^3),$$

$$g^{ia}(x) = - \sum_{j=k+1}^{N} z_j \, \beta^j_{ia} - 2 \sum_{c=1}^{k} \sum_{i,j=k+1}^{N} z_i z_j \, \Gamma^i_{ac} \beta^j_{ac} + O(|x|^3),$$

$$g^{ij}(x) = \delta_{ij} + \sum_{c=1}^{k} \sum_{l,m=k+1}^{N} z_l z_m \, \beta^l_{ic} \beta^m_{jc} + O(|x|^3).$$

The functions Γ^i_{ab} and β^j_{ac} are evaluated at $f(t)$.

See Thiam [16] for the proofs of Lemmas 2.1 and 2.2.

3. Proof of the Main Result

3.1. **Variational Framework and Palais–Smale Analysis.** Let $N \geq 4$, $0 \leq s_2 < s_1 < 2$, $\lambda > 0$, Ω be a bounded domain in $\mathbb{R}^N$, and $\Sigma \subset \Omega$ be a smooth closed submanifold. Let $h \in C(\Omega)$ be such that $-\Delta + h$ is coercive. We consider the following problem: find $u \in H^1_0(\Omega)$ positive solution of

$$-\Delta u + hu = \lambda \, \rho_\Sigma^{-s_1} \, u^{2^*_{s_1}-1} + \rho_\Sigma^{-s_2} \, u^{2^*_{s_2}-1} \qquad \text{in } \Omega. \tag{3.1}$$

The energy functional $J : H^1_0(\Omega) \to \mathbb{R}$ associated with (3.1) is

$$J(u) = \frac{1}{2} \int_\Omega \left(|\nabla u|^2 + hu^2 \right) dx - \frac{\lambda}{2^*_{s_1}} \int_\Omega \rho_\Sigma^{-s_1} |u|^{2^*_{s_1}} dx - \frac{1}{2^*_{s_2}} \int_\Omega \rho_\Sigma^{-s_2} |u|^{2^*_{s_2}} dx, \tag{3.2}$$

with variational level

$$c^* := \inf_{u \in H^1_0(\Omega)} \max_{\tau \geq 0} J(\tau u).$$

We say that J satisfies the Palais–Smale condition at level $c \in \mathbb{R}$, denoted $(PS)_c$, if any sequence $(u_n)_n \subset H^1_0(\Omega)$ with $J(u_n) \to c$ and $J'(u_n) \to 0$ in $H^{-1}(\Omega)$ is relatively compact in $H^1(\Omega)$. Let $(u_n)_n \subset H^1_0(\Omega)$ be a $(PS)_c$ sequence. Using the expression of J and its derivative we obtain

$$\langle J'(u_n), u_n \rangle = \int_\Omega (|\nabla u_n|^2 + hu_n^2) \, dx - \lambda \int_\Omega \frac{|u_n|^{2^*_{s_1}}}{\rho_\Sigma^{s_1}} \, dx - \int_\Omega \frac{|u_n|^{2^*_{s_2}}}{\rho_\Sigma^{s_2}} \, dx = o(1).$$

Combining this with the energy relation and using $2^*_{s_2} > 2^*_{s_1}$ (so that $\frac{1}{2^*_{s_1}} - \frac{1}{2^*_{s_2}} > 0$), a straightforward computation gives

$$c \geq \left(\frac{1}{2} - \frac{1}{2^*_{s_1}}\right) \int_\Omega (|\nabla u_n|^2 + h u_n^2)\, dx + o(1),$$

which implies that $(u_n)_n$ is bounded in $H^1_0(\Omega)$. This boundedness allows us to extract weakly convergent subsequences, which will be used to recover compactness under suitable energy constraints.

Due to the lack of compactness of the embedding $H^1_0(\Omega) \hookrightarrow L^{2^*_s}(\Omega, \rho_\Sigma^{-s}\, dx)$, J fails to satisfy the Palais–Smale condition globally. However, J satisfies $(PS)_c$ for any c with $0 < c < \beta^*$, where

$$\beta^* := \max_{\tau \geq 0} \Pi(\tau w) = \Pi(w)$$

is the variational level of the functional

$$\Pi(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2\, dx - \frac{\lambda}{2^*_{s_1}} \int_{\mathbb{R}^N} |z|^{-s_1} |v|^{2^*_{s_1}}\, dx - \frac{1}{2^*_{s_2}} \int_{\mathbb{R}^N} |z|^{-s_2} |v|^{2^*_{s_2}}\, dx.$$

Here Π is the energy functional associated with the limiting Euler–Lagrange equation

$$-\Delta w = \lambda \frac{w^{2^*_{s_1}-1}}{|z|^{s_1}} + \frac{w^{2^*_{s_2}-1}}{|z|^{s_2}} \qquad \text{in } \mathbb{R}^N, \tag{3.3}$$

where $x = (t, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ and $2^*_{s_i} := \frac{2(N-s_i)}{N-2}$, $i = 1, 2$.

Proposition 3.1. *Problem (3.3) admits a positive solution $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. Moreover, w satisfies:*

(i) *$w(x) = \theta(|t|, |z|)$ for some function $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$;*
(ii) *there exist two positive constants c_1 and c_2 such that*

$$\frac{c_1}{1 + |x|^{N-2}} \leq w(x) \leq \frac{c_2}{1 + |x|^{N-2}}, \qquad |\nabla w(x)| \leq \frac{c_2}{1 + |x|^{N-1}}, \qquad \forall\, x \in \mathbb{R}^N.$$

For a complete proof, we refer to [7].

3.2. **Construction of Test Functions and Energy Expansion.** We construct test functions to compare c^* and β^*. Let $\eta \in C^\infty_c(F_{y_0}(Q_{2r}))$ with $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $F_{y_0}(B_r)$. For $\varepsilon > 0$, define $u_\varepsilon : \Omega \to \mathbb{R}$ by

$$u_\varepsilon(y) = \varepsilon^{\frac{2-N}{2}}\, \eta\big(F^{-1}_{y_0}(y)\big)\, w\big(\varepsilon^{-1} F^{-1}_{y_0}(y)\big)\,. \tag{3.4}$$

For $x = (t, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ we have

$$u_\varepsilon(F_{y_0}(x)) = \varepsilon^{\frac{2-N}{2}}\, \eta(x)\, \theta\left(\frac{|t|}{\varepsilon}, \frac{|z|}{\varepsilon}\right),$$

so that $u_\varepsilon \in H^1_0(\Omega)$. Writing $F = F_{y_0}$ for brevity, we rewrite $u_\varepsilon(y) = \varepsilon^{\frac{2-N}{2}} \eta(F^{-1}(y)) W_\varepsilon(y)$ where $W_\varepsilon(y) = w(F^{-1}(y)/\varepsilon)$.

The aim of this section is to expand

$$J(\tau u_\varepsilon) = \frac{\tau^2}{2}\int_\Omega \big(|\nabla u_\varepsilon|^2 + hu^2\big)\,dx - \lambda\frac{\tau^{2^*_{s_1}}}{2^*_{s_1}}\int_\Omega \rho_\Sigma^{-s_1}|u_\varepsilon|^{2^*_{s_1}}\,dx - \frac{\tau^{2^*_{s_2}}}{2^*_{s_2}}\int_\Omega \rho_\Sigma^{-s_2}|u_\varepsilon|^{2^*_{s_2}}\,dx \tag{3.5}$$

as $\varepsilon \to 0$.

Lemma 3.2. *Let $N \geq 4$. Then as $\varepsilon \to 0$,*

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^2\,dy &= \int_{\mathbb{R}^N}|\nabla w|^2\,dx + \varepsilon^2\frac{H^2 - 3R_g(x_0)}{k(N-k)}\int_{Q_{r/\varepsilon}}|z|^2|\nabla_t w|^2\,dx\\ &\quad + \varepsilon^2\frac{R_g(x_0)}{3k^2}\int_{Q_{r/\varepsilon}}|t|^2|\nabla_t w|^2\,dx + \varepsilon^2\frac{R_g(x_0)+H^2}{2(N-k)}\int_{Q_{r/\varepsilon}}|z|^2|\nabla w|^2\,dx\\ &\quad - \varepsilon^2\frac{R_g(x_0)}{6k}\int_{Q_{r/\varepsilon}}|t|^2|\nabla w|^2\,dx + O(\varepsilon^{N-2}).\end{aligned}$$

Proof. By integration by parts, using the fact that $|\nabla\eta|$ and $\Delta\eta$ are supported in $F(Q_{2r})\setminus F(Q_r)$, and the change of variable $y = F(x)/\varepsilon$, we have

$$\int_\Omega |\nabla u_\varepsilon|^2\,dy = \int_{Q_{r/\varepsilon}}|\nabla w|^2_{g_\varepsilon}\sqrt{|g_\varepsilon|}(x)\,dx + O\left(\varepsilon^2\int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}} w^2\,dx + \int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}}|\nabla w|^2\,dx\right).$$

Expanding $|\nabla w|^2_{g_\varepsilon}$ and $\sqrt{|g_\varepsilon|}$ using Lemma 2.2, and using the antisymmetry of the torsion coefficients to cancel the cross-terms (as in (3.6)–(3.9) below), we obtain

$$\sum_{ij=k+1}^{N}\int_{Q_{r/\varepsilon}}[g_\varepsilon^{ij} - \delta_{ij}]\,\partial_{z_i}w\,\partial_{z_j}w\sqrt{|g_\varepsilon|}\,dx = O\left(\int_{Q_{r/\varepsilon}}\varepsilon^3|x|^3|\nabla_z w|^2\,dx\right), \tag{3.6}$$

$$\sum_{i=k+1}^{N}\sum_{a=1}^{k}\int_{Q_{r/\varepsilon}} g_\varepsilon^{ia}\,(\partial_{t_a}w\,\partial_{z_i}w)\sqrt{|g_\varepsilon|}\,dx = O\left(\int_{Q_{r/\varepsilon}}\varepsilon^3|x|^3|\nabla w|^2\,dx\right), \tag{3.7}$$

$$\begin{aligned}\sum_{ab=1}^{k}\int_{Q_{r/\varepsilon}}[g_\varepsilon^{ab} - \delta_{ab}]\,\partial_{t_a}w\,\partial_{t_b}w\sqrt{|g_\varepsilon|}\,dx &= \varepsilon^2\frac{3\Gamma^2 - 2H^2}{k(N-k)}\int_{Q_{r/\varepsilon}}|z|^2|\nabla_t w|^2\,dx\\ &\quad + \varepsilon^2\frac{R_g(x_0)}{3k^2}\int_{Q_{r/\varepsilon}}|t|^2|\nabla_t w|^2\,dx + O\left(\int_{Q_{r/\varepsilon}}\varepsilon^3|x|^3|\nabla_t w|^2\,dx\right),\end{aligned} \tag{3.8}$$

$$\begin{aligned}\int_{Q_{r/\varepsilon}}|\nabla w|^2(\sqrt{|g_\varepsilon|} - 1)\,dx &= \varepsilon^2\frac{H^2 - \frac{1}{2}\Gamma^2}{N-k}\int_{Q_{r/\varepsilon}}|z|^2|\nabla w|^2\,dx\\ &\quad - \varepsilon^2\frac{R_g(x_0)}{6k}\int_{Q_{r/\varepsilon}}|t|^2|\nabla w|^2\,dx + O\left(\int_{Q_{r/\varepsilon}}\varepsilon^3|x|^3|\nabla w|^2\,dx\right).\end{aligned} \tag{3.9}$$

Combining (3.6)–(3.9) and using Proposition 3.1 with the change of variable to estimate the remainder terms gives $\chi(\varepsilon) = O(\varepsilon^{N-2})$. The result follows. $\square$

Lemma 3.3. *For $N \geq 4$, as $\varepsilon \to 0$,*

$$\int_\Omega h\, u_\varepsilon^2\, dy = \varepsilon^2 h(x_0) \int_{Q_{r/\varepsilon}} w^2\, dx + O\left(\varepsilon^2 \delta \int_{Q_{r/\varepsilon}} w^2\, dx\right) + O(\varepsilon^{N-2}).$$

Proof. By the change of variables $y = \varepsilon^{-1} F(x)$ and the continuity of h,

$$\int_\Omega h\, u_\varepsilon^2\, dy = \varepsilon^2 h(y_0) \int_{Q_{r/\varepsilon}} w^2\, dx + O\left(\varepsilon^2 \int_{Q_{r/\varepsilon}} |h(F(\varepsilon x)) - h(y_0)|\, w^2\right) + O\left(\varepsilon^2 \int_{Q_{2r/\varepsilon} \setminus Q_{r/\varepsilon}} w^2\, dx\right).$$

By Proposition 3.1, the last term is $O(\varepsilon^N)$. By continuity of h, for any $\delta > 0$ there exists $r_\delta > 0$ such that $|h(y) - h(y_0)| < \delta$ on $F(Q_{r_\delta})$, which bounds the middle term. The result follows. $\square$

Lemma 3.4. *Let $N \geq 4$ and $s \in [0, 2)$. Then, as $\varepsilon \to 0$,*

$$\int_\Omega \rho_\Sigma^{-s} |u_\varepsilon|^{2_s^*}\, dy = \int_{\mathbb{R}^N} |z|^{-s} |w|^{2_s^*}\, dx + \varepsilon^2 \frac{R_g(x_0) + H^2}{4(N-k)} \int_{Q_{r/\varepsilon}} |z|^{2-s} |w|^{2_s^*}\, dx$$
$$- \varepsilon^2 \frac{R_g(x_0)}{6k} \int_{Q_{r/\varepsilon}} |z|^{-s} |t|^2 |w|^{2_s^*}\, dx + O(\varepsilon^{N-s}).$$

Proof. By the change of variable formula, (2.1) and the definition of u_ε, together with Lemma 2.2 and the Gauss equation $R_g(x_0) = H^2 - \Gamma^2$ (see [12], Chapter 4), the expansion follows by a direct computation. Details are analogous to Lemma 3.2. It easily follows that $\rho(\varepsilon) = O(\varepsilon^{N-s})$ by Proposition 3.1 and polar coordinates. $\square$

3.3. **Proof of Theorem 1.1.** Let $\tau \geq 0$ and $u \in H_0^1(\Omega)$. By Lemmas 3.2, 3.3 and 3.4, for $N \geq 4$,

$$J(\tau u_\varepsilon) = \Pi(\tau w) + \varepsilon^2 \left(H^2(x_0)\, \mathcal{L}_{1,\varepsilon}(\tau, w) + R_g(x_0)\, \mathcal{L}_{2,\varepsilon}(\tau, w) + h(x_0)\, \frac{\tau^2}{2} \int_{Q_{r/\varepsilon}} w^2\, dx \right) + O(\varepsilon^{N-2}).$$

By Proposition 3.1, for $N \geq 5$ all integrals over $Q_{r/\varepsilon}$ in the $\mathcal{L}$ functionals stabilise to finite integrals over $\mathbb{R}^N$, and $O(\varepsilon^{N-2})$ absorbs the tails. Setting

$$A_{N,\lambda,s_1,s_2} := \begin{cases} \dfrac{\mathcal{L}_{1,N}(1, w)}{\displaystyle\int_{\mathbb{R}^N} w^2\, dx} & N \geq 5, \\[2ex] \displaystyle\lim_{\varepsilon \to 0} \frac{\mathcal{L}_{1,4,\varepsilon}(1, w)}{\displaystyle\int_{Q_{r/\varepsilon}} w^2\, dx} & N = 4, \end{cases} \tag{3.10}$$

$$B_{N,\lambda,s_1,s_2} := \begin{cases} \dfrac{\mathcal{L}_{2,N}(1, w)}{\displaystyle\int_{\mathbb{R}^N} w^2\, dx} & N \geq 5, \\[2ex] \displaystyle\lim_{\varepsilon \to 0} \frac{\mathcal{L}_{2,4,\varepsilon}(1, w)}{\displaystyle\int_{Q_{r/\varepsilon}} w^2\, dx} & N = 4. \end{cases} \tag{3.11}$$

Remark 3.5. The limits in (3.10) and (3.11) for $N = 4$ are well-defined and finite. Indeed, by the decay estimates of Proposition 3.1(ii), the function w satisfies $w(x) \le c_2(1 + |x|^{N-2})^{-1}$ and $|\nabla w(x)| \le c_2(1 + |x|^{N-1})^{-1}$. A standard polar coordinate computation then shows that every weighted integral appearing in $\mathcal{L}_{1,4,\varepsilon}(1, w)$ and $\mathcal{L}_{2,4,\varepsilon}(1, w)$ converges absolutely, and that the tails outside $Q_{r/\varepsilon}$ decay at rate $O(\varepsilon^{N-2}) = O(\varepsilon^2)$. Consequently the ratios stabilise as $\varepsilon \to 0$, yielding finite constants A_{N,λ,s_1,s_2} and B_{N,λ,s_1,s_2} that depend only on N, λ, s_1, and s_2.

(where the limits are finite by Remark 3.5), we have since $2^*_{s_2} > 2^*_{s_1}$ that $J(\tau u_\varepsilon)$ has a unique maximum in τ, attained at $\tau_\varepsilon := 1 + o_\varepsilon(1)$, and

$$c^* = \max_{\tau \ge 0} J(\tau u_\varepsilon) := J(\tau_\varepsilon u_\varepsilon) \le \Pi(\tau_\varepsilon w) + \varepsilon^2 \mathcal{G}(\tau_\varepsilon w) < \Pi(\tau_\varepsilon w) \le \Pi(w) = \beta^*,$$

provided

$$A_{N,\lambda,s_1,s_2}\, H^2(y_0) + B_{N,\lambda,s_1,s_2}\, R_g(y_0) + h(y_0) < 0.$$

Since $c^* < \beta^*$, the Mountain Pass Theorem (applied via the $(PS)_{c^*}$ condition) yields a nonnegative $u \in H^1_0(\Omega) \setminus \{0\}$ solving (1.1). This completes the proof of Theorem 1.1. □

Acknowledgement. The authors thank the referees for their careful reading and constructive suggestions, which led to a significant improvement of the manuscript.

References

1. Abassi, T. K., & BRIKI, M.(2026). Critical point theory for the p-Laplacian problems. *Gulf Journal of Mathematics 23* (01), pp. 1-12. https://doi.org/10.56947/j247qj94.
2. Badiale, M., & Serra, E. (2005). Existence and multiplicity results for elliptic problems with critical growth and Hardy potential. *Advances in Differential Equations*, *10*, 753–780.
3. Brezis, H., & Nirenberg, L. (1983). Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Communications on Pure and Applied Mathematics*, *36*, 437–477. https://doi.org/10.1002/cpa.3160360405
4. Caffarelli, L., Kohn, R., & Nirenberg, L. (1984). First order interpolation inequalities with weights. *Compositio Mathematica*, *53*(3), 259–275.
5. Catrina, F., & Wang, Z.-Q. (2001). On the Caffarelli–Kohn–Nirenberg inequalities: sharp constants, existence and nonexistence, and symmetry of extremals. *Communications on Pure and Applied Mathematics*, *54*, 229–258. https://doi.org/10.1002/1097-0312(200102)54:2<229::AID-CPA4>3.0.CO;2-I
6. Ciss, M., Diatta, A., & Thiam, E. H. A. (2025). A nonlinear elliptic PDE with curve singularity on the boundary. *Moroccan Journal of Pure and Applied Analysis*, *11*(2), 181–202. https://doi.org/10.2478/mjpaa-2025-0013
7. Diatta, A., & Thiam, E. H. A. (to appear). A nonlinear PDE with two Hardy–Sobolev critical exponents with one dimensional singularity. *Journal of Mathematical Physics, Analysis, Geometry.*
8. Fall, M. M., & Thiam, E. H. A. (2013). A Hardy–Sobolev inequality with singularity on a curve. *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, *30*(6), 1027–1047. https://doi.org/10.1016/j.anihpc.2012.12.009
9. Ghoussoub, N., & Robert, F. (2006). The effect of curvature on the best constant in the Hardy–Sobolev inequalities. *Geometric and Functional Analysis*, *16*, 1201–1245. https://doi.org/10.1007/s00039-006-0579-2

10. Ghoussoub, N., & Robert, F. (2006). Concentration estimates for Emden–Fowler equations with boundary singularities and critical growth. *Duke Mathematical Journal*, *135*, 1–39. https://doi.org/10.1215/S0012-7094-06-13511-X
11. Ghoussoub, N., & Yuan, L. (2006). Multiple solutions for critical elliptic equations with singularities. *Mathematische Annalen*, *336*, 907–936. https://doi.org/10.1007/s00208-006-0027-6
12. Gray, A. (2004). *Tubes* (2nd ed.). Birkhäuser/Springer Science & Business Media. https://doi.org/10.1007/978-3-0348-7966-8
13. Ijaodoro, I. E., & Thiam, E. H. A. (2021). Influence of an L^p-perturbation on Hardy–Sobolev inequality with singularity a curve. *Opuscula Mathematica*, *41*(2), 187–204. https://doi.org/10.7494/OpMath.2021.41.2.187
14. Ridha, S. , Lalmi, A. L., & Salmi, A. (2026). *Blow-up of solutions to p (x)-Laplacian parabolic equations with general nonlinear source. Gulf Journal of Mathematics 23* (01), pp. 1-16. https://doi.org/10.56947/d4kq7d88.
15. Robert, F., & Vétois, J. (2011). Sign-changing solutions for critical equations with boundary singularities. *Advances in Mathematics*, *227*, 199–234. https://doi.org/10.1016/j.aim.2011.01.013
16. Thiam, E. H. A. (2019). Hardy–Sobolev inequality with higher dimensional singularity. *Analysis*, *39*(3), 79–96. https://doi.org/10.1515/anly-2018-0052
17. Thiam, E. H. A. (2024). Mass effect on an elliptic PDE involving two Hardy–Sobolev critical exponents. *Differential Equations & Applications*, *16*(3), 183–198. https://doi.org/10.7153/dea-2024-16-11
18. Thiam, E. H. A. (to appear). A nonlinear elliptic problem with multiple Hardy–Sobolev critical exponents on manifolds.*Partial Differential Equations and Applications*.
19. Thiam, E. H. A. (to appear). Hardy–Sobolev critical equations with totally geodesic singularities: Existence via the Mountain Pass Theorem. Birkhäuser.

[1] Université Assane Seck de Ziguinchor, UFR des Sciences et Technologies, Département de Mathématiques, Ziguinchor, Sénégal.
Email address: a.diatta20160578@zig.univ.sn

[2] Université Iba Der Thiam de Thiès, UFR des Sciences et Techniques, Département de Mathématiques, Thiès, Sénégal.
Email address: elhadjiabdoulaye.thiam@univ-thies.sn