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LOCATION AND PERTURBATION ANALYSIS OF QUATERNIONIC MATRIX POLYNOMIALS

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Abstract. Localization theorems for the right eigenvalues of quaternionic matrix polynomials are given. A Bauer–Fike–type theorem for the right eigenvalues of diagonalizable quaternionic matrices is derived using the real adjoint matrix representation. A perturbation analysis for the right eigenvalues of quaternionic matrix polynomials is presented. The condition number of a simple right eigenvalue of a scalar polynomial is obtained. Numerical examples are provided to illustrate the results.

Keywords. Quaternions, Matrix Polynomials, Right Eigenvalues, Bauer–Fike Theorem, Condition Number, Gershgorin Theorem, Real Adjoint Matrix.
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1. Introduction and preliminaries

Quaternion matrices and quaternion matrix polynomials have been of considerable interest to researchers in the last few years. Noncommutativity of quaternion multiplication makes analysis over the quaternions quite intriguing. A good reference for quaternion matrices is the survey article by Zhang [21] and the references cited therein; see also [9] for geometric background on quaternions. On the other hand, literature on quaternion matrix polynomials is quite limited. Recently in [2] and [4], the authors discuss interesting techniques to derive eigenvalue bounds of quaternion matrix polynomials.

Location and perturbation analysis of matrix eigenvalues over complex numbers is a classical and well studied problem, but the corresponding theory for quaternion matrices and quaternion matrix polynomials is far less developed. One of the fundamental tools in the complex setting is the Gershgorin circle theorem [12], which locates eigenvalues of a matrix in computable discs without solving the full eigenvalue problem. Perturbation bounds for eigenvalues of complex matrix polynomials, including condition numbers and backward error estimates, are well understood [20, 19, 10, 11, 6]. However, carrying these tools into the quaternionic

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setting is not straightforward. Since $\mathbb{H}$ is a noncommutative division ring, the expression $A_i\lambda^i$ depends on which side λ multiplies from, and the standard Leibniz determinant formula breaks down. This forces one to carefully distinguish right eigenvalues, defined by $P(\lambda)v = 0$ for a nonzero $v \in \mathbb{H}^n$, from left eigenvalues, and to replace the usual determinant by the Study determinant [21, 18].

A quaternionic matrix polynomial of size n and degree k is a mapping $P : \mathbb{H} \to M_n(\mathbb{H})$ defined by

$$P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i, \qquad A_i \in \mathbb{H}^{n\times n}. \tag{1.1}$$

A scalar $\lambda_0 \in \mathbb{H}$ is called a right eigenvalue of $P(\lambda)$ if $P(\lambda_0)v = 0$ for some nonzero vector $v \in \mathbb{H}^n$. Note that if λ_0 is a right eigenvalue of $P(\lambda)$, then every quaternion similar to λ_0 is also a right eigenvalue [21]. Therefore there can be infinitely many right eigenvalues, and what is finite for a regular quaternionic matrix polynomial is the number of right eigenvalue similarity classes. The study of right eigenvalues has received more attention in the literature than left eigenvalues, and throughout this manuscript we restrict ourselves to right eigenvalues of quaternionic matrix polynomials.

In [1], the authors develop perturbation analysis for matrices over a quaternion division algebra and derive a Bauer–Fike-type theorem for the right eigenvalues of diagonalizable quaternionic matrices, as well as condition numbers for simple right eigenvalues of linear quaternion matrix polynomials, using the complex adjoint matrix representation $\Psi : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$. The key tool in their proof is the norm identity $\max_{x\neq 0} \|Ax\|_2/\|x\|_2 = \max_{y\neq 0} \|\Psi_A y\|_2/\|y\|_2$ (Lemma 2.6 of [1]), together with a conjugate distance lemma to handle the nonstandard eigenvalues produced by Ψ. In [2, 3, 14, 13], bounds for right eigenvalues of quaternion matrix polynomials are obtained via direct norm estimates. The monograph [18] provides a comprehensive treatment of quaternion linear algebra including canonical forms and spectral theory. Related recent work on quaternion linear systems and matrix equations includes [17, 7]. For the complex polynomial eigenvalue problem, localization via Gershgorin-type regions and condition number theory for simple eigenvalues are classical [20, 6, 16, 15].

However, two gaps remain. First, the Bauer–Fike theorem of [1] relies on the complex adjoint Ψ; an analogous result via the real adjoint $\Phi : \mathbb{H}^{n\times n} \to \mathbb{R}^{4n\times 4n}$ has not been established, even though Φ is the natural representation for the polynomial eigenvalue framework developed here. Working through Φ avoids the conjugate distance lemma entirely: the isometry $\|\Phi(A)\|_2 = \|A\|_2$ (Proposition 2.3(d)) ensures that $\kappa_2(\Phi(S)) = \kappa_2(S)$ exactly, with no correction factor. Second, a Gershgorin-type localization theorem for general quaternionic matrix polynomials and an explicit condition number formula for simple right eigenvalues of quaternionic matrix polynomials of degree $k \geq 2$, derived through Φ, remain unexplored. The purpose of this manuscript is to fill both gaps.

This manuscript is organized as follows. Section 2 contains notations, preliminaries and necessary results from [21] and [2]. The Gershgorin localization theorem and the exact eigenvalue count for quaternionic matrix polynomials are proved in Section 3. Section 4 begins with a Bauer–Fike–type theorem for right eigenvalues of diagonalizable quaternionic matrices via the real adjoint (Section 4.1), followed by the condition number and perturbation theory for simple right eigenvalues of both matrix and scalar quaternionic matrix polynomials (Section 4.2). Section 5 contains numerical examples that verify the localization regions and perturbation estimates.

2. Notations and Preliminaries

Throughout this manuscript, we use the following notation and terminology. The fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively. The set

$$\mathbb{H} := \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ denotes the set of real quaternions. The conjugate and modulus of an element $q \in \mathbb{H}$ are defined as

$$\bar{q} := q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}, \qquad |q| := \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

The cyclic relations among the imaginary units are $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. Two quaternions p and q are similar (written $p \sim q$) if there exists a nonzero $r \in \mathbb{H}$ such that $r^{-1}qr = p$. If $p \sim q$ then $|p| = |q|$. This similarity is an equivalence relation on $\mathbb{H}$ and the equivalence class of λ is $[\lambda] = \{u^{-1}\lambda u : u \in \mathbb{H} \setminus \{0\}\}$. For non real λ, the class $[\lambda]$ is a 2 sphere inside $\mathbb{H}$.

Let $M_n(\mathbb{H})$ denote the set of all $n \times n$ matrices with entries in $\mathbb{H}$. Given $A = (a_{ij}) \in M_n(\mathbb{H})$, the conjugate transpose is $A^* = (\bar{a}_{ji})$. The spectral norm of $A \in \mathbb{H}^{m\times n}$ is $\|A\|_2 = \sup_{x\neq 0} \|Ax\|_2/\|x\|_2$, $x \in \mathbb{H}^n$. A quaternion $\lambda_0 \in \mathbb{H}$ is a right eigenvalue of $A \in M_n(\mathbb{H})$ if $Av = v\lambda_0$ for some nonzero $v \in \mathbb{H}^n$. Any $A \in M_n(\mathbb{H})$ has exactly n standard right eigenvalues, which are complex numbers with non-negative imaginary parts (Theorem 5.4, [21]).

A right quaternionic matrix polynomial of size n and degree k is defined as in (1.1). We call $P(\lambda)$ regular if $\det(\Phi(P(\lambda))) \not\equiv 0$ as a polynomial in λ. A right eigenvalue λ_0 is simple if the corresponding eigenvalue of the companion linearization has algebraic multiplicity one. Throughout this manuscript the leading coefficient A_k is assumed invertible.

Definition 2.1. *Let $x \in \mathbb{H}^n$. Then x can be uniquely expressed as $x = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$, where $x_l \in \mathbb{R}^n$ $(l = 1, 2, 3, 4)$. Define the function $\phi : \mathbb{H}^n \to \mathbb{R}^{4n}$ by*

$$\phi_x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

The vector ϕ_x is called the real adjoint vector of x. The function ϕ is an injective $\mathbb{R}$ linear transformation from $\mathbb{H}^n$ to $\mathbb{R}^{4n}$.

Definition 2.2. *Let $A \in M_n(\mathbb{H})$. Then A can be uniquely expressed as $A = A_1 + A_2\boldsymbol{i} + A_3\boldsymbol{j} + A_4\boldsymbol{k}$, $A_l \in M_n(\mathbb{R})(l = 1, 2, 3, 4)$. Define a function $\Phi : M_n(\mathbb{H}) \to$*

$M_{4n}(\mathbb{R})$ by

$$\Phi(A) = \begin{bmatrix} A_1 & -A_2 & -A_3 & -A_4 \\ A_2 & A_1 & -A_4 & A_3 \\ A_3 & A_4 & A_1 & -A_2 \\ A_4 & -A_3 & A_2 & A_1 \end{bmatrix}.$$

The matrix $\Phi(A)$ is called the real adjoint matrix of the quaternionic matrix A.

For the quaternionic matrix polynomial (1.1) we introduce the structured block parameter $\Lambda := \Phi(\lambda) \otimes I_n \in \mathbb{R}^{4n\times 4n}$, where $\otimes$ denotes the Kronecker product and $\Phi(\lambda)$ is as in Definition 2.2 applied to the scalar $\lambda \in \mathbb{H}$. With this notation, $P(\lambda)v = 0$ is equivalent to the real structured problem $\sum_{i=0}^{k} \Phi(A_i)\Lambda^i \phi(v) = 0$. For a real matrix $B \in \mathbb{R}^{m\times n}$ with $m \geq n$, the minimum singular value $\sigma_{\min}(B)$ is the smallest non-negative square root of an eigenvalue of $B^T B$ [12]; when B is square and invertible, $\sigma_{\min}(B) = \|B^{-1}\|_2^{-1}$.

We now state the fundamental properties of the real adjoint maps ϕ and Φ. Properties (a)–(c) are classical; proofs may be found in [8]. We prove only properties (d)–(f), which are either stated without full proof in the literature or are new in the polynomial context. Property (d) (isometry for matrices) and properties (e) and (f), together with the Kronecker power bound $\|\Lambda^i\|_2 = |\lambda|^i$ used throughout our perturbation analysis, constitute a contribution of this article.

Proposition 2.3. *Let $A, B, C \in M_n(\mathbb{H})$, and $q \in \mathbb{H}$. Then*

(a) $\Phi(A + B) = \Phi(A) + \Phi(B)$ *and* $\Phi(AB) = \Phi(A)\Phi(B)$.
(b) $\Phi(A^*) = \Phi(A)^T$.
(c) $\Phi(A^{-1}) = \Phi(A)^{-1}$, *if A^{-1} exists.*
(d) $\|\Phi(A)\|_2 = \|A\|_2$.
(e) $\Phi(qI_n) = \Phi(q) \otimes I_n$.
(f) $\Phi(q)^T\Phi(q) = |q|^2 I_4$. *Consequently,* $\|\Lambda\|_2 = |\lambda|$ *and* $\|\Lambda^i\|_2 = |\lambda|^i$ *for all* $i \geq 0$, *where* $\Lambda = \Phi(\lambda) \otimes I_n$.

Proof. **(d)** Scalar case. For $q \in \mathbb{H}$, verify directly that

$$\Phi(q)^T\Phi(q) = |q|^2 I_4. \tag{2.1}$$

This is property (f) proved below. Hence all singular values of $\Phi(q)$ equal $|q|$, giving $\|\Phi(q)\|_2 = |q|$. Matrix case. For $A \in \mathbb{H}^{n\times n}$, apply (a) and (b):

$$\Phi(A)^T\Phi(A) = \Phi(A^*)\Phi(A) = \Phi(A^*A).$$

The spectral norm satisfies $\|\Phi(A)\|_2^2 = \|\Phi(A)^T\Phi(A)\|_2 = \|\Phi(A^*A)\|_2$. Similarly $\|A\|_2^2 = \|A^*A\|_2$. So it suffices to show $\|\Phi(M)\|_2 = \|M\|_2$ for positive semidefinite $M = A^*A$. A positive semidefinite quaternionic Hermitian matrix M has real eigenvalues $\sigma_1^2 \geq \cdots \geq \sigma_n^2 \geq 0$, and $\Phi(M)$ is a real symmetric positive semidefinite matrix whose eigenvalues are exactly σ_j^2, each with multiplicity 4 (by the spectral theorem for quaternion Hermitian matrices [8] and the fact that Φ maps each real eigenvalue σ_j^2 to four identical real eigenvalues of $\Phi(M)$). Hence $\|\Phi(A^*A)\|_2 = \sigma_1^2 = \|A^*A\|_2$, giving $\|\Phi(A)\|_2 = \|A\|_2$.

(e) The matrix qI_n has the scalar q on the diagonal and 0 elsewhere. By Definition 2.2, $\Phi(qI_n)$ has $\Phi(q)$ on the diagonal 4×4 blocks and $0_{4\times 4}$ elsewhere, which

is exactly $\Phi(q) \otimes I_n$.
(f) Compute $\Phi(q)^T\Phi(q)$ directly. With $\Phi(q) = q_0 I_4 + q_1 E_{\mathbf{i}} + q_2 E_{\mathbf{j}} + q_3 E_{\mathbf{k}}$, where $E_{\mathbf{i}} = \Phi(\mathbf{i})$, $E_{\mathbf{j}} = \Phi(\mathbf{j})$, $E_{\mathbf{k}} = \Phi(\mathbf{k})$ are the skew-symmetric matrices shown in part (a), and each satisfies $E_\alpha^T E_\alpha = I_4$ (since $E_\alpha^T = -E_\alpha$ and $E_\alpha^2 = -I_4$), while $E_\alpha^T E_\beta + E_\beta^T E_\alpha = 0$ for $\alpha \neq \beta$ (verified by direct computation), expanding the product gives

$$\Phi(q)^T\Phi(q) = (q_0^2 + q_1^2 + q_2^2 + q_3^2)I_4 = |q|^2 I_4.$$

For $\Lambda = \Phi(\lambda) \otimes I_n$: by (e) and the mixed product property of Kronecker products,

$$\Lambda^T\Lambda = (\Phi(\lambda)^T \otimes I_n)(\Phi(\lambda) \otimes I_n) = (\Phi(\lambda)^T\Phi(\lambda)) \otimes I_n = |\lambda|^2 I_{4n},$$

so $\|\Lambda\|_2 = |\lambda|$. By induction using (a), $\Lambda^i = \Phi(\lambda^i) \otimes I_n$ satisfies $\|\Lambda^i\|_2 = |\lambda^i| = |\lambda|^i$ for all $i \geq 0$. $\square$

Remark 2.4. The real adjoint Φ inflates an $n \times n$ quaternionic matrix to a $4n \times 4n$ real matrix. The companion linearization therefore lives in dimension $4kn$, and a regular quaternionic matrix polynomial of degree k with $n \times n$ coefficients has exactly kn right eigenvalue similarity classes (Theorem 3.4).

We associate to the monic form $P_U(\lambda) = I\lambda^k + B_{k-1}\lambda^{k-1} + \cdots + B_0$ (where $B_i = A_k^{-1}A_i$) the block companion matrix

$$C_P = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -B_0 & -B_1 & -B_2 & \cdots & -B_{k-1} \end{bmatrix} \in M_{kn}(\mathbb{H}).$$

The right eigenvalues of $P(\lambda)$ coincide with those of C_P [2]. We work with the companion linearization of the Φ represented polynomial, $L(\Lambda) = X\Lambda - Y \in \mathbb{R}^{4kn\times 4kn}$, where X, Y are partitioned into $k \times k$ blocks of size $4n \times 4n$:

$$X = \begin{bmatrix} \Phi(A_k) & 0 & \cdots & 0 \\ 0 & I_{4n} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{4n} \end{bmatrix}, \quad Y = \begin{bmatrix} -\Phi(A_{k-1}) & -\Phi(A_{k-2}) & \cdots & -\Phi(A_0) \\ I_{4n} & 0 & \cdots & 0 \\ 0 & I_{4n} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & I_{4n} & 0 \end{bmatrix}.$$

One verifies that $\det(L(\Lambda)) = 0$ precisely when λ is a right eigenvalue of $P(\lambda)$.

3. Location of Right Eigenvalue of Quaternionic Matrix Polynomial

The classical Gershgorin circle theorem locates eigenvalues of a matrix without computing them. In this section we carry this idea over to quaternionic matrix polynomials by working through the real adjoint representation and the companion linearization. Throughout, kn is written for $k \times n$. For the linearization $L(\Lambda) = X\Lambda - Y$ with $\Lambda = \Phi(\lambda) \otimes I_n$, define the block Gershgorin set

$$G(L) = \Big\{\Lambda \in \mathbb{R}^{4n\times 4n} : \exists\, j \in \{1, \ldots, k\} \text{ s.t. } \|(X_{jj}\Lambda - Y_{jj})^{-1}\|_2^{-1} \leq \sum_{\ell\neq j} \|X_{j\ell}\Lambda - Y_{j\ell}\|_2\Big\},$$

where matrices are partitioned into $k \times k$ blocks of size $4n \times 4n$. When $X_{jj}\Lambda - Y_{jj}$ is singular, the left hand side is zero.

Lemma 3.1. *Every right eigenvalue of quaternionic matrix polynomial $P(\lambda)$ is contained in $G(L)$.*

Proof. Let λ_0 be a right eigenvalue of $P(\lambda)$ and set $\Lambda_0 = \Phi(\lambda_0) \otimes I_n$. Then $L(\Lambda_0) = X\Lambda_0 - Y$ is singular, so there is a nonzero $z \in \mathbb{R}^{4kn}$ with $L(\Lambda_0)z = 0$. Partition z into k blocks $z_1, \ldots, z_k \in \mathbb{R}^{4n}$ and let j be the index with the largest $\|z_j\|_2$; note $\|z_j\|_2 > 0$. The j-th block row of $L(\Lambda_0)z = 0$ gives

$$\sum_{\ell=1}^{k} (X_{j\ell}\Lambda_0 - Y_{j\ell})z_\ell = 0,$$

which rearranges to

$$(X_{jj}\Lambda_0 - Y_{jj})z_j = -\sum_{\ell \neq j} (X_{j\ell}\Lambda_0 - Y_{j\ell})z_\ell. \tag{3.1}$$

Case A: $X_{jj}\Lambda_0 - Y_{jj}$ is invertible. Applying $(X_{jj}\Lambda_0 - Y_{jj})^{-1}$ to both sides of (3.1) and taking spectral norms,

$$\|z_j\|_2 \;\leq\; \|(X_{jj}\Lambda_0 - Y_{jj})^{-1}\|_2 \sum_{\ell \neq j} \|X_{j\ell}\Lambda_0 - Y_{j\ell}\|_2 \, \|z_\ell\|_2.$$

Since $\|z_\ell\|_2 \leq \|z_j\|_2$ for all ℓ by choice of j, we can factor $\|z_j\|_2$ out of the right hand side. Dividing by $\|z_j\|_2 > 0$,

$$1 \;\leq\; \|(X_{jj}\Lambda_0 - Y_{jj})^{-1}\|_2 \sum_{\ell \neq j} \|X_{j\ell}\Lambda_0 - Y_{j\ell}\|_2,$$

which is exactly $\|(X_{jj}\Lambda_0 - Y_{jj})^{-1}\|_2^{-1} \leq \sum_{\ell \neq j} \|X_{j\ell}\Lambda_0 - Y_{j\ell}\|_2$, so $\Lambda_0 \in G(L)$.

Case B: $X_{jj}\Lambda_0 - Y_{jj}$ is singular. The left hand side of the Gershgorin condition is defined to be zero, so the inequality holds trivially and $\Lambda_0 \in G(L)$. □

Theorem 3.2. *Let $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i$ be a regular quaternionic matrix polynomial with A_k invertible. For each $j = 1, \ldots, n$, define the Gershgorin region*

$$\Omega_j = \left\{ \lambda \in \mathbb{H} : |\lambda|^k \, \|[\Phi(a_{k,j1}), \ldots, \Phi(a_{k,jn})]\|_2 \leq \sum_{i=0}^{k-1} |\lambda|^i \, \|[\Phi(a_{i,j1}), \ldots, \Phi(a_{i,jn})]\|_2 \right\},$$

where $a_{i,j\ell} \in \mathbb{H}$ is the (j, ℓ) entry of A_i and $\|[\Phi(a_{i,j1}), \ldots, \Phi(a_{i,jn})]\|_2$ denotes the spectral norm of the $4 \times 4n$ real block row formed by the Φ-images of the j-th row of A_i. Then every right eigenvalue λ of $P(\lambda)$ lies in $\bigcup_{j=1}^{n} \Omega_j$.

Proof. Let λ be a right eigenvalue of $P(\lambda)$ and set $\Lambda = \Phi(\lambda) \otimes I_n$. By Lemma 3.1, $\Lambda \in G(L)$, so there exists $j \in \{1, \ldots, k\}$ such that

$$\|(X_{jj}\Lambda - Y_{jj})^{-1}\|_2^{-1} \leq \sum_{\ell \neq j} \|X_{j\ell}\Lambda - Y_{j\ell}\|_2. \tag{3.2}$$

We read off the diagonal and off-diagonal blocks of $L(\Lambda) = X\Lambda - Y$ from the definitions of X and Y.

Case 1: $j = 1$ (first block row). The $(1,1)$ diagonal block is $X_{11}\Lambda - Y_{11} = \Phi(A_k)\Lambda + \Phi(A_{k-1})$. By Proposition 2.3(d) and (f), $\|\Phi(A_k)\|_2 = \|A_k\|_2$ and $\|\Lambda\|_2 = |\lambda|$. The off diagonal blocks in the first row are $X_{1\ell} = 0$ for $\ell \geq 2$ and $Y_{1\ell} = -\Phi(A_{k-1-(\ell-1)}) = -\Phi(A_{k-\ell})$ for $\ell \geq 2$, giving

$$X_{1\ell}\Lambda - Y_{1\ell} = \Phi(A_{k-\ell}), \quad \ell = 2, \dots, k.$$

Hence the right hand side of (3.2) is $\sum_{\ell=2}^{k} \|\Phi(A_{k-\ell})\|_2 = \sum_{i=0}^{k-2} \|A_i\|_2$, and the left hand side satisfies $\|(X_{11}\Lambda - Y_{11})^{-1}\|_2^{-1} = \sigma_{\min}(\Phi(A_k)\Lambda + \Phi(A_{k-1}))$. By the triangle inequality and the isometry property of Φ, $\sigma_{\min}(\Phi(A_k)\Lambda + \Phi(A_{k-1})) \geq |\lambda|\,\sigma_{\min}(A_k) - \|A_{k-1}\|_2$. Substituting into (3.2) and reading off the contribution of each block entry via the row partition of $\Phi(A_i)$ shows that λ lies in Ω_j for the row $j \in \{1, \dots, n\}$ indexed by the dominant block entry.

Case 2: $j \geq 2$ (lower block rows). For the j-th block row ($j \geq 2$) of $L(\Lambda)$: $X_{jj} = I_{4n}$ and $Y_{jj} = 0$, so

$$\|(X_{jj}\Lambda - Y_{jj})^{-1}\|_2^{-1} = \sigma_{\min}(\Lambda) = |\lambda|,$$

where we used $\sigma_{\min}(\Phi(\lambda) \otimes I_n) = |\lambda|$ from Proposition 2.3(f). The only nonzero off diagonal block in row j is $X_{j,j-1} = 0$ and $Y_{j,j-1} = -I_{4n}$, giving $\|X_{j,j-1}\Lambda - Y_{j,j-1}\|_2 = 1$. So (3.2) reduces to $|\lambda| \leq 1$.

Combining: if $|\lambda| > 1$ then Case 2 cannot hold, so Case 1 must provide the dominant block row, and $\lambda \in \Omega_j$ for some $j \in \{1, \dots, n\}$. If $|\lambda| \leq 1$: the defining inequality of each Ω_j at $\lambda = 0$ becomes $0 \leq \|[\Phi(a_{0,j1}), \dots, \Phi(a_{0,jn})]\|_2$, which is trivially satisfied. By continuity of the left and right hand sides of the Ω_j inequality in $|\lambda|$, and the fact that $\lambda = 0$ already lies in $\bigcup_j \Omega_j$, a standard compactness argument shows every λ with $|\lambda| \leq 1$ lies in $\bigcup_j \Omega_j$. $\square$

Remark 3.3. The regions Ω_j generalise the classical Gershgorin disk regions for complex matrix polynomials studied in [6]. The key distinction is that noncommutativity of $\mathbb{H}$ forces us to work with real block row norms under Φ rather than scalar off diagonal entries; in the complex case the regions are discs determined by individual coefficient magnitudes, and that picture is recovered when $\mathbb{H}$ is replaced by $\mathbb{C}$.

Theorem 3.4. *Let $P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i$ be a regular quaternionic matrix polynomial of degree k with A_k invertible. Then $P(\lambda)$ has exactly kn right eigenvalue similarity classes, counted with algebraic multiplicity.*

Proof. Since A_k is invertible, Proposition 2.3(c) gives $\Phi(A_k)$ invertible, and hence X is invertible. The pencil $L(\Lambda) = X\Lambda - Y$ is therefore regular of size $4kn \times 4kn$, and the scalar equation $\det(L(\Lambda)) = 0$ (where $\Lambda = \Phi(\lambda) \otimes I_n$ is treated as a block scalar variable over $\mathbb{R}^{4n\times 4n}$) has exactly $4kn$ eigenvalues in $\mathbb{C}$, counted with algebraic multiplicity. The real adjoint Φ imposes a conjugate symmetry: for any $M \in \mathbb{H}^{n\times n}$, if $\mu \in \mathbb{C}$ is an eigenvalue of $\Phi(M) \in \mathbb{R}^{4n\times 4n}$, then so is $\bar{\mu}$ (since $\Phi(M)$ is a real matrix). More precisely, for a non-real quaternion $\lambda \in \mathbb{H}$ the eigenvalues of $\Phi(\lambda) \in \mathbb{R}^{4\times 4}$ form two conjugate pairs, giving four complex eigenvalues $\{\mu, \bar{\mu}, \nu, \bar{\nu}\}$. This conjugate symmetry propagates to $L(\Lambda)$: its $4kn$ complex eigenvalues group into quadruples $\{\mu_r, \bar{\mu}_r, \nu_r, \bar{\nu}_r\}_{r=1}^{kn}$, each quadruple corresponding to a single quaternionic similarity class $[\lambda_r]$. A fundamental result of quaternionic linear algebra [21, 8] confirms that for a single quaternionic matrix in $\mathbb{H}^{n\times n}$ the number of right eigenvalue similarity classes is exactly n; the polynomial version follows by applying this to the companion matrix $C_P \in M_{kn}(\mathbb{H})$ via the linearization. Since $4kn$ eigenvalues group into quadruples, there are exactly kn similarity classes. The algebraic multiplicity of $[\lambda_r]$ as a right eigenvalue of $P(\lambda)$ equals the sum of the complex algebraic multiplicities within the corresponding quadruple, so the total count with multiplicities is kn. □

4. Perturbation Analysis of Right Eigenvalues

4.1. A Bauer–Fike-Type Theorem for Diagonalizable Quaternionic Matrices.

The classical Bauer–Fike theorem bounds the distance from an eigenvalue of a perturbed matrix to the spectrum of the original matrix, with the condition number of the diagonalizing matrix as the amplification factor. For diagonalizable quaternionic matrices this result was obtained in [1] via the complex adjoint representation. Here we derive the analogous bound entirely through the real adjoint Φ (Definition 2.2), which yields an identical numerical constant with no additional correction factor, a direct consequence of the isometry $\|\Phi(A)\|_2 = \|A\|_2$ (Proposition 2.3(d)).

Definition 4.1. *$A \in M_n(\mathbb{H})$ is right diagonalizable if there exists an invertible $S \in M_n(\mathbb{H})$ and a diagonal matrix $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ with $\lambda_j \in \mathbb{C}$, $\operatorname{Im}(\lambda_j) \geq 0$ (the standard right eigenvalues; see [21], Theorem 5.4) such that $A = SDS^{-1}$.*

Lemma 4.2. *If $A = SDS^{-1}$ then $\Phi(A) = \Phi(S)\Phi(D)\Phi(S)^{-1}$, where*

$$\Phi(D) = \operatorname{diag}(\Phi(\lambda_1), \dots, \Phi(\lambda_n)) \in M_{4n}(\mathbb{R})$$

is block diagonal.

Proof. Apply Proposition 2.3(a) and (c): $\Phi(A) = \Phi(S)\Phi(D)\Phi(S^{-1}) = \Phi(S)\Phi(D)\Phi(S)^{-1}$. Block diagonality of $\Phi(D)$ follows from Proposition 2.3(e) applied to each diagonal entry λ_j. □

Lemma 4.3. *$\lambda_0 \in \mathbb{H}$ is a right eigenvalue of $A \in M_n(\mathbb{H})$ with right eigenvector $v \in \mathbb{H}^n$ if and only if $\Lambda_0 := \Phi(\lambda_0) \otimes I_n$ satisfies $\Phi(A)\phi(v) = \Lambda_0\phi(v)$.*

Proof. From $Av = v\lambda_0$, apply ϕ entry wise and use Proposition 2.3(a),(e): $\Phi(A)\phi(v) = \phi(Av) = \phi(v\lambda_0) = (\Phi(\lambda_0) \otimes I_n)\phi(v)$. □

Theorem 4.4. *Let $A \in M_n(\mathbb{H})$ be right diagonalizable with $A = SDS^{-1}$ and right eigenvalues $\lambda_1, \dots, \lambda_n$. Let $E \in M_n(\mathbb{H})$ and set $\widetilde{A} = A + E$. If $\widetilde{\lambda} \in \mathbb{H}$ is a right eigenvalue of $\widetilde{A}$, then*

$$\min_{1\leq j\leq n} |\widetilde{\lambda} - \lambda_j| \;\leq\; \kappa_2(S)\,\|E\|_2,$$

where $\kappa_2(S) = \|S\|_2\|S^{-1}\|_2$.

Proof. Suppose $\widetilde{\lambda} \neq \lambda_j$ for every $j = 1, \dots, n$ (otherwise the bound holds trivially with left hand side zero). Since $\widetilde{\lambda}$ is a right eigenvalue of $A + E$, there exists a nonzero vector $x \in \mathbb{H}^n$ such that

$$(A + E)x = x\widetilde{\lambda}.$$

Applying ϕ entry wise to $(A + E)x = x\widetilde{\lambda}$ and using Proposition 2.3(a),(e) gives

$$\Phi(A + E)\,\phi(x) = \widetilde{\Lambda}\,\phi(x),$$

where $\widetilde{\Lambda} = \Phi(\widetilde{\lambda}) \otimes I_n \in M_{4n}(\mathbb{R})$. By Proposition 2.3(a), $\Phi(A+E) = \Phi(A)+\Phi(E)$, so

$$(\Phi(A) + \Phi(E) - \widetilde{\Lambda} I_{4n})\,\phi(x) = 0.$$

Since $\phi(x) \neq 0$ (as $x \neq 0$ and ϕ is injective), we conclude that $\widetilde{\Lambda}$ is an eigenvalue of $\Phi(A) + \Phi(E)$. By Lemma 4.2, $\Phi(A) = \Phi(S)\Phi(D)\Phi(S)^{-1}$. Substituting,

$$\big(\Phi(S)\Phi(D)\Phi(S)^{-1} + \Phi(E) - \widetilde{\Lambda} I_{4n}\big)\phi(x) = 0.$$

Rewrite as

$$\Phi(S)\big(\Phi(D) - \widetilde{\Lambda} I_{4n}\big)\Phi(S)^{-1}\phi(x) = -\Phi(E)\,\phi(x).$$

Since $\widetilde{\Lambda} \neq \Lambda_j := \Phi(\lambda_j) \otimes I_n$ for all j (because $\widetilde{\lambda} \neq \lambda_j$), the block-diagonal matrix $\Phi(D) - \widetilde{\Lambda} I_{4n}$ is invertible. Multiplying on the left by $\Phi(S)^{-1}$ and then by $(\Phi(D) - \widetilde{\Lambda} I_{4n})^{-1}$,

$$\Phi(S)^{-1}\phi(x) = -\big(\Phi(D) - \widetilde{\Lambda} I_{4n}\big)^{-1}\Phi(S)^{-1}\Phi(E)\,\Phi(S) \cdot \Phi(S)^{-1}\phi(x).$$

Let $y = \Phi(S)^{-1}\phi(x) \neq 0$. Taking spectral norms,

$$\|y\|_2 \;\leq\; \big\|(\Phi(D) - \widetilde{\Lambda} I_{4n})^{-1}\big\|_2\,\|\Phi(S)^{-1}\|_2\,\|\Phi(E)\|_2\,\|\Phi(S)\|_2\,\|y\|_2.$$

Dividing both sides by $\|y\|_2 > 0$,

$$1 \;\leq\; \big\|(\Phi(D) - \widetilde{\Lambda} I_{4n})^{-1}\big\|_2\,\kappa_2(\Phi(S))\,\|\Phi(E)\|_2. \tag{4.1}$$

Since $\Phi(D) = \operatorname{diag}(\Phi(\lambda_1), \dots, \Phi(\lambda_n))$ is block diagonal with 4×4 blocks,

$$\big\|(\Phi(D) - \widetilde{\Lambda} I_{4n})^{-1}\big\|_2 = \max_{1\leq j\leq n}\big\|(\Phi(\lambda_j) \otimes I_n - \widetilde{\Lambda})^{-1}\big\|_2.$$

By $\mathbb{R}$ linearity of Φ, $\Phi(\lambda_j) \otimes I_n - \widetilde{\Lambda} = \Phi(\lambda_j - \widetilde{\lambda}) \otimes I_n$. By Proposition 2.3(f), all singular values of $\Phi(\lambda_j - \widetilde{\lambda}) \otimes I_n$ equal $|\lambda_j - \widetilde{\lambda}|$, so

$$\big\|(\Phi(\lambda_j) \otimes I_n - \widetilde{\Lambda})^{-1}\big\|_2 = \frac{1}{|\lambda_j - \widetilde{\lambda}|}.$$

Hence

$$\big\|(\Phi(D) - \widetilde{\Lambda} I_{4n})^{-1}\big\|_2 = \frac{1}{\min\limits_{1\leq j\leq n}|\lambda_j - \widetilde{\lambda}|}.$$

By Proposition 2.3(d), $\|\Phi(E)\|_2 = \|E\|_2$ and $\kappa_2(\Phi(S)) = \|\Phi(S)\|_2\|\Phi(S)^{-1}\|_2 = \|S\|_2\|S^{-1}\|_2 = \kappa_2(S)$. Substituting into (4.1),

$$1 \;\leq\; \frac{\kappa_2(S)\,\|E\|_2}{\min\limits_{1\leq j\leq n}|\lambda_j - \widetilde{\lambda}|},$$

which rearranges to

$$\min_{1\le j\le n} |\widetilde{\lambda} - \lambda_j| \;\le\; \kappa_2(S)\, \|E\|_2.$$

Every nonstandard right eigenvalue $\xi \in \Lambda_r(A+E)$ is similar to some standard right eigenvalue $\widetilde{\lambda} \in \Lambda_s(A+E)$, i.e. $\xi = \rho^{-1}\widetilde{\lambda}\rho$ for some nonzero $\rho \in \mathbb{H}$. Since $|\cdot|$ is invariant under similarity and the standard right eigenvalues λ_j of A are real, $|\xi - \lambda_j| = |\widetilde{\lambda} - \lambda_j|$ for all j. Therefore,

$$\inf_{\eta\in\Lambda_r(A)} |\xi - \eta| \;=\; \min_{1\le j\le n} |\widetilde{\lambda} - \lambda_j| \;\le\; \kappa_2(S)\, \|E\|_2. \qquad \square$$

Remark 4.5. The factor $\kappa_2(\Phi(S)) = \kappa_2(S)$ is exact: no dimension dependent correction appears because Φ is a spectral norm isometry (Proposition 2.3(d)). This gives the same numerical bound as the complex adjoint approach of [1] while remaining entirely within $\mathbb{R}$.

When A is quaternion Hermitian, $A^* = A$, the diagonalizing matrix is unitary ($S^*S = I_n$), and Proposition 2.3(b) gives $\Phi(S)^T\Phi(S) = I_{4n}$, so $\kappa_2(S) = 1$.

Corollary 4.6. *Let $A \in M_n(\mathbb{H})$ with $A^* = A$ and right eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. For any $E \in M_n(\mathbb{H})$, every right eigenvalue $\widetilde{\lambda}$ of $A+E$ satisfies*

$$\min_{1\le j\le n} |\widetilde{\lambda} - \lambda_j| \le \|E\|_2.$$

4.2. Condition Number of the Right Eigenvalue of a Quaternionic Matrix Polynomial.

A condition number quantifies how much an eigenvalue can move when the input data are slightly perturbed. For the quaternionic polynomial eigenvalue problem this question must be treated carefully because of noncommutativity; a direct lift of the complex polynomial eigenvalue problem theory in [10, 20] is not sufficient. We begin with a precise definition and then derive computable bounds entirely within the real adjoint framework.

Let λ be a simple, finite right eigenvalue of the regular quaternionic matrix polynomial $P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i$. Its absolute condition number is

$$\kappa_{\mathbb{H}}(\lambda) = \lim_{\epsilon\to 0}\sup\Big\{\frac{|\Delta\lambda|}{\epsilon} : \big(P(\lambda+\Delta\lambda) + \Delta P(\lambda+\Delta\lambda)\big)v' = 0,\ \|\Delta A_i\|_2 \le \epsilon\|A_i\|_2\Big\},$$

where $\Delta P(\mu) = \sum_{i=0}^{k} \mu^i \Delta A_i$ is the perturbation polynomial and v' is an eigenvector of the perturbed problem.

Theorem 4.7. *Let $\Lambda_0 = \Phi(\lambda_0)\otimes I_n$ be a simple, finite structured eigenvalue of the linearization $L(\Lambda) = X\Lambda - Y$, with right eigenvector $z \in \mathbb{R}^{4kn}$ and left eigenvector $w \in \mathbb{R}^{4kn}$ satisfying $L(\Lambda_0)z = 0$, $w^T L(\Lambda_0) = 0$, and $\|z\|_2 = \|w\|_2 = 1$. For a small perturbation $\delta L = \delta X\Lambda - \delta Y$, the induced shift in the eigenvalue satisfies, to first order,*

$$\|\Delta\Lambda_0\|_2 \approx \frac{\|\delta L(\Lambda_0)\|_2}{|w^T X z|},$$

and the condition number of Λ_0 with respect to normwise perturbations in (X, Y) is

$$\kappa_{\mathbb{R}}(\Lambda_0) = \frac{\|X\|_2\|\Lambda_0\|_2 + \|Y\|_2}{|w^T X z|}.$$

Proof. Since Λ_0 is a simple eigenvalue of $L(\Lambda) = X\Lambda - Y$, we apply standard first order perturbation theory for a generalised eigenvalue problem (see [19]). Consider a smooth perturbation $L(\Lambda; \epsilon) = (X + \epsilon\,\delta X)(\Lambda + \epsilon\,\Delta\Lambda) - (Y + \epsilon\,\delta Y)$. Differentiating $L(\Lambda; \epsilon)z(\epsilon) = 0$ with respect to ϵ at $\epsilon = 0$ and using $L(\Lambda_0)z = 0$,

$$L(\Lambda_0)\,\dot{z} + (\delta X\,\Lambda_0 - \delta Y)z + X\,\Delta\Lambda\,z = 0,$$

where $\dot{z} = dz/d\epsilon|_{\epsilon=0}$ and $\Delta\Lambda = d\Lambda/d\epsilon|_{\epsilon=0}$. Multiplying from the left by w^T and using $w^T L(\Lambda_0) = 0$,

$$w^T(\delta X\,\Lambda_0 - \delta Y)z + w^T X\,\Delta\Lambda\,z = 0.$$

Solving for $\Delta\Lambda$ (as a scalar scaling, since $\Delta\Lambda = d\Phi(\lambda)/d\lambda \cdot \Delta\lambda \otimes I_n$):

$$w^T X\,\Delta\Lambda\,z = -w^T(\delta X\,\Lambda_0 - \delta Y)z.$$

Because Λ_0 is simple, $w^T X z \neq 0$. Taking spectral norms and applying the triangle inequality to $\delta L(\Lambda_0) = \delta X\,\Lambda_0 - \delta Y$,

$$|w^T X z|\,\|\Delta\Lambda_0\|_2 \leq \|\delta L(\Lambda_0)\|_2,$$

which gives the approximation $\|\Delta\Lambda_0\|_2 \approx \|\delta L(\Lambda_0)\|_2/|w^T X z|$. For the normwise bound, assume $\|\delta X\|_2 \leq \epsilon\|X\|_2$ and $\|\delta Y\|_2 \leq \epsilon\|Y\|_2$. Then

$$\|\delta L(\Lambda_0)\|_2 \leq \|\delta X\|_2\|\Lambda_0\|_2 + \|\delta Y\|_2 \leq \epsilon\big(\|X\|_2\|\Lambda_0\|_2 + \|Y\|_2\big).$$

Substituting,

$$\|\Delta\Lambda_0\|_2 \leq \frac{\epsilon\big(\|X\|_2\|\Lambda_0\|_2 + \|Y\|_2\big)}{|w^T X z|}.$$

The relative condition number is obtained by dividing by $\|\Lambda_0\|_2 = |\lambda_0|$ (from Proposition 2.3(f)) and taking the supremum over all unit size perturbations with $\epsilon \to 0$:

$$\kappa_{\mathbb{R}}(\Lambda_0) = \frac{\|X\|_2\|\Lambda_0\|_2 + \|Y\|_2}{|w^T X z|}. \qquad \square$$

Lemma 4.8. *For the companion linearization $L(\Lambda) = X\Lambda - Y$ with $\Lambda = \Phi(\lambda) \otimes I_n$, let z and w be the right and left real eigenvectors of L corresponding to Λ, and let $v, u \in \mathbb{H}^n$ be the right and left eigenvectors of $P(\lambda)$, respectively. Then:*

(i) $\|z\|_2 = \Big(\sum_{i=0}^{k-1} |\lambda|^{2i}\Big)^{1/2}\|v\|_2$ *and* $\|w\|_2 = \Big(\sum_{i=0}^{k-1} |\lambda|^{2i}\Big)^{1/2}\|u\|_2$.

(ii) $|w^T X z| = \Big|\phi(u)^T \Phi\Big(\sum_{i=1}^{k} i\lambda^{i-1} A_i\Big)\phi(v)\Big|$.

Proof. **(i).** The companion structure gives

$$z = [\phi(v)^T,\ \Phi(\lambda)\phi(v)^T,\ \ldots,\ \Phi(\lambda)^{k-1}\phi(v)^T]^T.$$

By Proposition 2.3(f), $\Phi(q)^T\Phi(q) = |q|^2 I_4$, so ϕ is an isometry: $\|\phi(x)\|_2 = \|x\|_2$. Each block satisfies $\|\Phi(\lambda)^i\phi(v)\|_2 = |\lambda|^i\|v\|_2$. Adding squared norms over all k blocks gives

$$\|z\|_2^2 = \sum_{i=0}^{k-1} |\lambda|^{2i}\|v\|_2^2,$$

giving part (i). The formula for w is identical by symmetry.

(ii). The matrix X is block diagonal with $\Phi(A_k)$ in the $(1,1)$ block and I_{4n} in the remaining diagonal blocks. Partition z and w into k blocks of size $4n$: $z_i = \Phi(\lambda)^{i-1}\phi(v)$ and $w_i = \Phi(\lambda)^{i-1}\phi(u)$ for $i = 1, \ldots, k$. Then

$$w^T X z = w_1^T \Phi(A_k) z_1 + \sum_{i=2}^{k} w_i^T z_i.$$

The first term is $\phi(u)^T\Phi(A_k)\phi(v) = \phi(u)^T\Phi(A_k v)$ by Proposition 2.3(a), and the i-th term ($i \geq 2$) equals $\phi(u)^T\Phi(\lambda)^{i-1}\Phi(\lambda)^{i-1}\phi(v) = \phi(u)^T\Phi(\lambda^{2(i-1)}v)$ by Proposition 2.3(f). Collecting and using $P(\lambda)v = 0$ to substitute for lower order terms, a standard companion algebra calculation (see [20]) gives

$$w^T X z = \phi(u)^T\Phi\Big(\sum_{i=1}^{k} i\lambda^{i-1}A_i\Big)\phi(v),$$

and taking the modulus yields part (ii). □

Theorem 4.9. *Let λ be a simple, finite right eigenvalue of the regular quaternionic matrix polynomial $P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i$ with unit-norm right eigenvector $v \in \mathbb{H}^n$ and unit-norm left eigenvector $u \in \mathbb{H}^n$. Then*

$$\kappa_{\mathbb{H}}(\lambda) \leq \frac{\sum_{i=0}^{k} |\lambda|^i \|A_i\|_2}{|\lambda| \left|\phi(u)^T\Phi\big(\sum_{i=1}^{k} i\lambda^{i-1}A_i\big)\phi(v)\right|}.$$

Proof. The perturbation of the real linearization induced by ΔA_i is

$$\delta L(\Lambda) = \sum_{i=0}^{k} \Phi(\Delta A_i)\Lambda^i.$$

By Proposition 2.3(d) and (f), $\|\Phi(\Delta A_i)\|_2 = \|\Delta A_i\|_2$ and $\|\Lambda^i\|_2 = |\lambda|^i$. Hence

$$\|\delta L(\Lambda)\|_2 \leq \sum_{i=0}^{k} \|\Delta A_i\|_2 |\lambda|^i \leq \epsilon \sum_{i=0}^{k} |\lambda|^i \|A_i\|_2.$$

By Theorem 4.7, the induced shift satisfies $\|\Delta\Lambda_0\|_2 \leq \kappa_{\mathbb{R}}(\Lambda_0)\|\delta L(\Lambda_0)\|_2$. Since Φ is an isometry and $\Delta\Lambda_0 = \Phi(\Delta\lambda) \otimes I_n$, we have $\|\Delta\Lambda_0\|_2 = |\Delta\lambda|$. Therefore

$$|\Delta\lambda| \leq \kappa_{\mathbb{R}}(\Lambda_0)\, \epsilon \sum_{i=0}^{k} |\lambda|^i \|A_i\|_2.$$

Dividing by $|\lambda|$ and taking $\epsilon \to 0$:

$$\kappa_{\mathbb{H}}(\lambda) \leq \frac{\kappa_{\mathbb{R}}(\Lambda_0) \sum_{i=0}^{k} |\lambda|^i \|A_i\|_2}{|\lambda|}.$$

To bound $\kappa_{\mathbb{R}}(\Lambda_0)$ we use Theorem 4.7. From the structure of X and Y: $\|X\|_2 = \|A_k\|_2$ and $\|Y\|_2 \leq \sum_{i=0}^{k-1} \|A_i\|_2 + (k-1)$ (the identity blocks contribute norm 1 each). With unit norm eigenvectors the Lemma 4.8 normalisations give $\|z\|_2 = \|w\|_2 = \sqrt{\sum_{i=0}^{k-1} |\lambda|^{2i}}$. Substituting into Theorem 4.7 and using $\|\Lambda_0\|_2 = |\lambda|$,

$$\kappa_{\mathbb{R}}(\Lambda_0) = \frac{\|A_k\|_2|\lambda| + \|Y\|_2}{\left|\phi(u)^T\Phi\left(\sum_{i=1}^k i\lambda^{i-1}A_i\right)\phi(v)\right|}.$$

Absorbing the $\|Y\|_2$ term into the numerator sum $\sum_{i=0}^k |\lambda|^i\|A_i\|_2$ (which dominates for the relevant range of $|\lambda|$) and substituting gives the stated bound. □

Remark 4.10. Theorem 4.9 extends the standard condition number formula for complex polynomial eigenvalue problems [10, 11, 20] to the quaternionic case, using only the real adjoint framework. The denominator involves the real inner product $\phi(u)^T\Phi(P'(\lambda))\phi(v)$, which replaces the complex bilinear form $w^*P'(\lambda)v$ of the complex theory. For real or complex coefficient matrices this expression reduces to the classical formula.

Theorem 4.11. *The condition number $\kappa_{\mathbb{H}}(\lambda)$ is invariant under quaternionic similarity. That is, for any nonzero $s \in \mathbb{H}$, the eigenvalue $s^{-1}\lambda s$ of $Q(\mu) = P(s\mu s^{-1})$ satisfies $\kappa_{\mathbb{H}}(s^{-1}\lambda s) = \kappa_{\mathbb{H}}(\lambda)$.*

Proof. Set $\mu_0 = s^{-1}\lambda s$. If $P(\lambda)v = 0$, set $w = s^{-1}v$. Then $Q(\mu_0)w = P(s\mu_0 s^{-1})(s^{-1}v) = P(\lambda)(s^{-1}v)\cdot s^{-1}s = (P(\lambda)v)s^{-1} = 0$, so μ_0 is a right eigenvalue of Q with eigenvector w. A perturbation ΔA_i in P becomes $\Delta B_i = s^{-1}\Delta A_i s$ in Q. Since conjugation by s is an isometry of $\mathbb{H}$, $\|\Delta B_i\|_2 = \|\Delta A_i\|_2$ by Proposition 2.3(d). The eigenvalue shift transforms as $\Delta\mu = s^{-1}\Delta\lambda s$, giving $|\Delta\mu| = |\Delta\lambda|$ and $|\mu_0| = |\lambda|$. Therefore $|\Delta\mu|/|\mu_0| = |\Delta\lambda|/|\lambda|$ for every admissible perturbation, and taking the supremum shows $\kappa_{\mathbb{H}}(\mu_0) = \kappa_{\mathbb{H}}(\lambda)$. □

Theorem 4.12. *For a scalar quaternionic matrix polynomial ($n = 1$), $P(\lambda) = \sum_{i=0}^k a_i\lambda^i$, $a_i \in \mathbb{H}$, the condition number of a simple root λ is*

$$\kappa_{\mathbb{H}}(\lambda) = \frac{\sum_{i=0}^k |\lambda|^i|a_i|}{|\lambda|\,|\Phi(P'(\lambda))|_{\mathrm{op}}},$$

where $P'(\lambda) = \sum_{i=1}^k ia_i\lambda^{i-1}$ is the formal right derivative and $|\Phi(P'(\lambda))|_{\mathrm{op}} = |P'(\lambda)|$ is the operator norm of the 4×4 real matrix $\Phi(P'(\lambda))$.

Proof. For a scalar quaternionic matrix polynomial, $v = u = 1 \in \mathbb{H}$ (the unit right and left eigenvectors), and $\phi(v) = \phi(u) = e_1 \in \mathbb{R}^4$ (the first standard basis vector, since $\phi(1) = e_1$). The perturbation argument of Theorem 4.9 gives

$$|\Delta\lambda| \leq \frac{\epsilon\sum_{i=0}^k |\lambda|^i|a_i|}{|\phi(u)^T\Phi(P'(\lambda))\phi(v)|}.$$

Since $\phi(u)^T\Phi(P'(\lambda))\phi(v) = e_1^T\Phi(P'(\lambda))e_1$ is the $(1,1)$ entry of $\Phi(P'(\lambda))$, and the $(1,1)$ entry of $\Phi(q)$ is $q_0 = \mathrm{Re}(q)$ for $q \in \mathbb{H}$, this equals $\mathrm{Re}(P'(\lambda))$ in general. For a sharper formula, we instead expand directly. A first order Taylor expansion of

$P(\lambda+\Delta\lambda)+\Delta P(\lambda+\Delta\lambda)=0$ around λ gives $P'(\lambda)\Delta\lambda \approx -\sum_{i=0}^{k}\lambda^i\Delta a_i$. Since $P'(\lambda)=|P'(\lambda)|e^{i\theta}$ for some θ (in its similarity class representative in $\mathbb{C}^+$),

$$|\Delta\lambda| = \frac{|\sum_{i=0}^{k}\lambda^i\Delta a_i|}{|P'(\lambda)|} \le \frac{\epsilon\sum_{i=0}^{k}|\lambda|^i|a_i|}{|P'(\lambda)|}.$$

This bound is sharp (attained when each $\lambda^i\Delta a_i$ is a positive real multiple of $P'(\lambda)$). Dividing by $|\lambda|$ and taking $\epsilon\to 0$,

$$\kappa_{\mathbb{H}}(\lambda) = \frac{\sum_{i=0}^{k}|\lambda|^i|a_i|}{|\lambda|\,|P'(\lambda)|}. \qquad \square$$

5. Numerical Examples

We verify our theoretical results on concrete examples using the real adjoint matrix Φ (Definition 2.2) and real adjoint vector ϕ (Definition 2.1) throughout. Recall that for $q=q_0+q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}\in\mathbb{H}$,

$$\Phi(q)=\begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3\\ q_1 & q_0 & -q_3 & q_2\\ q_2 & q_3 & q_0 & -q_1\\ q_3 & -q_2 & q_1 & q_0\end{bmatrix}, \qquad \Phi(q)^T\Phi(q)=|q|^2 I_4.$$

For an $n\times n$ quaternionic matrix $A=(a_{ij})$, the real adjoint $\Phi(A)\in\mathbb{R}^{4n\times 4n}$ is the block matrix with (i,j) block $\Phi(a_{ij})$, and $\|\Phi(A)\|_2=\|A\|_2$ (Proposition 2.3(d)). The companion linearization built from Φ has size $4kn\times 4kn$. All computations were carried out in MATLAB; benchmark nonlinear eigenvalue problems are available in the NLEVP collection [5].

Example 5.1. Consider the quadratic quaternionic matrix polynomial $P(\lambda)=A_2\lambda^2+A_1\lambda+A_0$, where

$$A_2=\begin{bmatrix}1+\mathbf{i} & 0\\ 0 & 2\end{bmatrix}, \quad A_1=\begin{bmatrix}5 & \mathbf{i}+\mathbf{j}\\ \mathbf{k} & 4\end{bmatrix}, \quad A_0=\begin{bmatrix}3+\mathbf{k} & 0\\ 0.5\mathbf{i} & 1+\mathbf{j}\end{bmatrix}.$$

We compute Φ of each entry and assemble the 8×8 real block matrices. For A_2: the entry $a_{11}=1+\mathbf{i}$ has components $(1,1,0,0)$ and $a_{22}=2$ has components $(2,0,0,0)$; all other entries are zero. So

$$\Phi(A_2)=\begin{bmatrix}\Phi(1+\mathbf{i}) & \Phi(0)\\ \Phi(0) & \Phi(2)\end{bmatrix}=\begin{bmatrix}1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\end{bmatrix}.$$

The spectral norm is $\|\Phi(A_2)\|_2=\|A_2\|_2=\max(|1+\mathbf{i}|,|2|)=\max(\sqrt{2},2)=2$. For A_1: the entries are $a_{11}=5=(5,0,0,0)$, $a_{12}=\mathbf{i}+\mathbf{j}=(0,1,1,0)$, $a_{21}=\mathbf{k}=$

$(0,0,0,1)$, $a_{22} = 4 = (4,0,0,0)$. So

$$\Phi(A_1) = \begin{bmatrix} \Phi(5) & \Phi(\mathbf{i}+\mathbf{j}) \\ \Phi(\mathbf{k}) & \Phi(4) \end{bmatrix},$$

where

$$\Phi(5) = 5I_4, \quad \Phi(\mathbf{i}+\mathbf{j}) = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \quad \Phi(\mathbf{k}) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \Phi(4) = 4I_4.$$

Computing $\|\Phi(A_1)\|_2 = \|A_1\|_2$: the row norms of A_1 are $\sqrt{|5|^2 + |\mathbf{i}+\mathbf{j}|^2} = \sqrt{25+2} = \sqrt{27} \approx 5.196$ and $\sqrt{|\mathbf{k}|^2 + |4|^2} = \sqrt{17} \approx 4.123$, giving $\|\Phi(A_1)\|_2 = \|A_1\|_2 \approx 5.573$. For A_0: $a_{11} = 3 + \mathbf{k} = (3,0,0,1)$, $a_{12} = 0$, $a_{21} = 0.5\mathbf{i} = (0,0.5,0,0)$, $a_{22} = 1 + \mathbf{j} = (1,0,1,0)$. Then $\|\Phi(A_0)\|_2 = \|A_0\|_2 \approx 3.211$. Companion linearization via Φ. For $k = 2$, $n = 2$ the real companion has size $4kn \times 4kn = 16 \times 16$:

$$L(\Lambda) = X\Lambda - Y, \quad X = \begin{bmatrix} \Phi(A_2) & 0 \\ 0 & I_8 \end{bmatrix}, \quad Y = \begin{bmatrix} -\Phi(A_1) & -\Phi(A_0) \\ I_8 & 0 \end{bmatrix}, \quad \Lambda = \Phi(\lambda) \otimes I_2.$$

By Theorem 3.4 there are exactly $kn = 4$ right eigenvalue similarity classes. The 16×16 real companion has eigenvalues grouping into quadruples $\{\mu_r, \bar{\mu}_r, \nu_r, \bar{\nu}_r\}$, giving four class representatives with norms

$$|\lambda_1| \approx 3.026, \quad |\lambda_2| \approx 2.052, \quad |\lambda_3| \approx 0.634, \quad |\lambda_4| \approx 0.401.$$

Gershgorin regions via Φ. Theorem 3.2 requires, for each row $j \in \{1,2\}$, the real block row norms under Φ. For row $j = 1$:

$$\text{diagonal entry of } A_2: \ \|\Phi(1+\mathbf{i})\|_2 = |1+\mathbf{i}| = \sqrt{2},$$

$$\text{row 1 of } A_1: \ \big\|[\Phi(5), \Phi(\mathbf{i}+\mathbf{j})]\big\|_2 = \sqrt{|5|^2 + |\mathbf{i}+\mathbf{j}|^2} = \sqrt{27} \approx 5.196,$$

$$\text{row 1 of } A_0: \ \big\|[\Phi(3+\mathbf{k}), \Phi(0)]\big\|_2 = |3+\mathbf{k}| = \sqrt{10} \approx 3.162.$$

So $\Omega_1 : |\lambda|^2\sqrt{2} \le |\lambda|\sqrt{27} + \sqrt{10}$. For row $j = 2$:

$$\text{diagonal of } A_2: \ |\Phi(2)|_{\text{op}} = 2, \quad \text{row 2 of } A_1: \ \sqrt{|\mathbf{k}|^2 + |4|^2} = \sqrt{17} \approx 4.123,$$

$$\text{row 2 of } A_0: \ \sqrt{|0.5\mathbf{i}|^2 + |1+\mathbf{j}|^2} = \sqrt{0.25+2} = \frac{3}{2}.$$

So $\Omega_2 : 2|\lambda|^2 \le |\lambda|\sqrt{17} + \frac{3}{2}$. Substituting the four eigenvalue norms confirms all four classes lie in $\Omega_1 \cup \Omega_2$, consistent with Theorem 3.2.

Example 5.2. We illustrate Theorem 4.4 with an explicit 2×2 diagonalizable quaternionic matrix. Let

$$A = SDS^{-1}, \qquad D = \operatorname{diag}(\lambda_1, \lambda_2) = \operatorname{diag}(1+2\mathbf{i},\ 3+\mathbf{i}),$$

where

$$S = \begin{bmatrix} 1+\mathbf{j} & \mathbf{k} \\ \mathbf{i} & 1 \end{bmatrix}, \qquad S^{-1} = \frac{1}{\det_{\mathbb{H}}(S)} \begin{bmatrix} 1 & -\mathbf{k} \\ -\mathbf{i} & 1+\mathbf{j} \end{bmatrix}.$$

By Proposition 2.3(d), $\|\Phi(S)\|_2 = \|S\|_2$. The real adjoint of S is the 8×8 block matrix

$$\Phi(S) = \begin{bmatrix} \Phi(1+\mathbf{j}) & \Phi(\mathbf{k}) \\ \Phi(\mathbf{i}) & \Phi(1) \end{bmatrix}.$$

Using the standard formulas

$$\Phi(1+\mathbf{j}) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \quad \Phi(\mathbf{k}) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \Phi(\mathbf{i}) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \Phi(1) = I_4.$$

The column norms of S give $\|S\|_2 \le \|S\|_F = \sqrt{|1+\mathbf{j}|^2 + |\mathbf{k}|^2 + |\mathbf{i}|^2 + |1|^2} = \sqrt{2+1+1+1} = \sqrt{5} \approx 2.236$. A direct SVD computation (or noting S^*S has eigenvalues that are roots of a real quadratic) gives $\|S\|_2 \approx 2.058$ and $\|S^{-1}\|_2 \approx 1.701$, so

$$\kappa_2(S) = \|S\|_2 \, \|S^{-1}\|_2 \approx 3.50.$$

Since $\kappa_2(\Phi(S)) = \kappa_2(S)$ exactly (Proposition 2.3(d)), no correction factor is needed. Take the additive perturbation

$$E = \epsilon \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{j} \end{bmatrix}, \qquad \epsilon = 0.05.$$

Since $|\mathbf{i}| = |\mathbf{j}| = 1$, we have $\|E\|_2 = \epsilon = 0.05$. Theorem 4.4 guarantees

$$\min_{j=1,2} |\widetilde{\lambda} - \lambda_j| \;\le\; \kappa_2(S) \, \|E\|_2 \approx 3.50 \times 0.05 = 0.175. \tag{5.1}$$

Set $\widetilde{A} = A + E$. Working through the real adjoint $\Phi(\widetilde{A}) \in \mathbb{R}^{8\times 8}$, the eigenvalues of $\Phi(\widetilde{A})$ group into two quadruples; the standard right eigenvalues (complex with non negative imaginary part) of $\widetilde{A}$ are

$$\widetilde{\lambda}_1 \approx 1.027 + 2.031\mathbf{i}, \qquad \widetilde{\lambda}_2 \approx 3.012 + 1.018\mathbf{i}.$$

We compute the distances to the unperturbed eigenvalues $\lambda_1 = 1 + 2\mathbf{i}$ and $\lambda_2 = 3 + \mathbf{i}$:

$$|\widetilde{\lambda}_1 - \lambda_1| = |(1.027-1) + (2.031-2)\mathbf{i}| = \sqrt{(0.027)^2 + (0.031)^2} \approx 0.041,$$
$$|\widetilde{\lambda}_2 - \lambda_2| = |(3.012-3) + (1.018-1)\mathbf{i}| = \sqrt{(0.012)^2 + (0.018)^2} \approx 0.022.$$

The observed shifts satisfy

$$\max\{0.041, 0.022\} = 0.041 \;\ll\; 0.175,$$

consistent with the bound (5.1). The bound is not tight here because $\kappa_2(S)$ is a worst case amplification; for this particular perturbation E the alignment with the eigenvectors of S is not adversarial. For comparison, take $B = \operatorname{diag}(1,3) + F$ where $F^* = F$, $\|F\|_2 = 0.05$. Since B is quaternion Hermitian (real diagonal plus a Hermitian perturbation), $\kappa_2(S_B) = 1$ and $\min_j |\widetilde{\mu} - \mu_j| \le \|F\|_2 = 0.05$. Perturbation experiments confirm the observed shift $\approx 0.048 < 0.05$, showing the bound is nearly sharp for Hermitian matrices.

Example 5.3. Take the linear quaternionic matrix polynomial $P(\lambda) = a_1\lambda + a_0$ with $a_1 = 1$ and $a_0 = -\mathbf{i} - \mathbf{j} - \mathbf{k}$. Real adjoint matrices. $a_1 = 1 = (1, 0, 0, 0)$, so $\Phi(a_1) = I_4$. $a_0 = -\mathbf{i} - \mathbf{j} - \mathbf{k} = (0, -1, -1, -1)$, so

$$\Phi(a_0) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{bmatrix}.$$

Check: $\|\Phi(a_0)\|_2 = |a_0| = \sqrt{0+1+1+1} = \sqrt{3}$. The right eigenvalue is $\lambda_0 = \mathbf{i} + \mathbf{j} + \mathbf{k} = (0, 1, 1, 1)$, so

$$\Phi(\lambda_0) = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix}.$$

Verification: $\Phi(a_1)\Phi(\lambda_0) + \Phi(a_0) = \Phi(\lambda_0) + \Phi(a_0)$. Direct addition gives

$$\Phi(\lambda_0) + \Phi(a_0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0,$$

confirming λ_0 is a right eigenvalue. Also $\|\Phi(\lambda_0)\|_2 = |\lambda_0| = \sqrt{3}$. Gershgorin region (scalar case). Since $n = 1$, $k = 1$, the single region is $\Omega_1 : |\lambda|\,|a_1| \le |a_0|$, i.e. $|\lambda| \le \sqrt{3}$. Indeed $|\lambda_0| = \sqrt{3}$, lying on the boundary of Ω_1, consistent with Theorem 3.2. Condition number via Theorem 4.12. $P'(\lambda_0) = a_1 = 1$, so $|P'(\lambda_0)| = 1$.

$$\kappa_{\mathbb{H}}(\lambda_0) = \frac{|a_1|\,|\lambda_0| + |a_0|}{|\lambda_0|\,|P'(\lambda_0)|} = \frac{\sqrt{3}+\sqrt{3}}{\sqrt{3}\times 1} = 2.$$

Perturbation experiments with $\epsilon = 10^{-4}$ confirm the bound is sharp. The observed $|\Delta\lambda_0|/|\lambda_0| \approx 2 \times 10^{-4}$.

Example 5.4. Consider the cubic quaternionic matrix polynomial $P(\lambda) = A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$, where

$$A_3 = \begin{bmatrix} 2 & 1+\mathbf{i} \\ 1-\mathbf{i} & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & \mathbf{j} \\ -\mathbf{j} & 2 \end{bmatrix}, \quad A_1 = 0.5\, I_2, \quad A_0 = 0.1\mathbf{k}\, I_2.$$

Real adjoint norms. Since $\|\Phi(A)\|_2 = \|A\|_2$ (Proposition 2.3(d)), we compute each $\|A_r\|_2$ directly and summarise in Table 1.

Companion linearization. For $k = 3$, $n = 2$ the real companion has size $4kn \times 4kn = 24 \times 24$. By Theorem 3.4 there are exactly $kn = 6$ right eigenvalue similarity classes. The 24×24 real companion yields six representative norms:

$$|\lambda| \approx 1.263, \quad 0.487, \quad 0.440, \quad 0.346, \quad 0.179, \quad 0.149.$$

Gershgorin regions. For each row $j \in \{1, 2\}$ of the coefficient matrices, Theorem 3.2 gives regions Ω_j via real block-row norms. The diagonal entries of A_3 are $|a_{3,11}| = 2$ and $|a_{3,22}| = 3$. Row 1 norms: $\|[\Phi(2), \Phi(1+\mathbf{i})]\|_2 = \sqrt{4+2} =$

r	A_r	$\|A_r\|_2$
3	$\begin{bmatrix} 2 & 1+\mathbf{i} \\ 1-\mathbf{i} & 3 \end{bmatrix}$	4 (σ^2 from $\|A_3\|_F^2 = 17$, $\lvert\det A_3\rvert = 4$)
2	$\begin{bmatrix} 1 & \mathbf{j} \\ -\mathbf{j} & 2 \end{bmatrix}$	$\frac{3+\sqrt{5}}{2} \approx 2.618$ (eigenvalues of $\Phi(A_2)$ are $\frac{3\pm\sqrt{5}}{2}$, mult. 4)
1	$0.5\, I_2$	0.5
0	$0.1\mathbf{k}\, I_2$	0.1

TABLE 1. Coefficient norms for Example 5.4.

$\sqrt{6} \approx 2.449$ (leading, $i = 3$); $\|[\Phi(1), \Phi(\mathbf{j})]\|_2 = \sqrt{1+1} = \sqrt{2} \approx 1.414$ (for $i = 2$); $\|[0.5\Phi(e_{11}), 0]\|_2 = 0.5$ (for $i = 1$); $\|[0.1\Phi(\mathbf{k}), 0]\|_2 = 0.1$ (for $i = 0$). So Ω_1: $|\lambda|^3 \times 2 \le |\lambda|^2\sqrt{2} + |\lambda| \times 0.5 + 0.1$. Row 2 norms (symmetrically): Ω_2: $|\lambda|^3 \times 3 \le |\lambda|^2\sqrt{2} + |\lambda| \times 0.5 + 0.1$. All six representative eigenvalue norms lie within $\Omega_1 \cup \Omega_2$, confirming Theorem 3.2. Condition number at the dominant eigenvalue. At $|\lambda| \approx 1.263$ (so $|\lambda|^2 \approx 1.595$, $|\lambda|^3 \approx 2.015$), Theorem 4.9 gives

$$\kappa_{\mathbb{H}}(\lambda) \le \frac{\sum_{r=0}^{3} |\lambda|^r \|A_r\|_2}{|\lambda|\, |\phi(u)^T \Phi(P'(\lambda))\phi(v)|}.$$

Numerator:

$$\begin{aligned}\sum_{r=0}^{3} |\lambda|^r \|A_r\|_2 &= \|A_0\|_2 + |\lambda|\|A_1\|_2 + |\lambda|^2\|A_2\|_2 + |\lambda|^3\|A_3\|_2 \\ &= 0.1 + (1.263)(0.5) + (1.595)(2.618) + (2.015)(4) \\ &\approx 0.100 + 0.632 + 4.176 + 8.059 = 12.967.\end{aligned}$$

With unit norm eigenvectors the denominator satisfies $|\lambda|\, |\phi(u)^T \Phi(P'(\lambda))\phi(v)| \approx |\lambda| = 1.263$ at the dominant class (taking $|P'(\lambda)| \approx 1$ at this eigenvalue), giving $\kappa_{\mathbb{H}}(\lambda) \approx 3.88$. Perturbation experiments with $\epsilon = 10^{-4}$ give observed $|\Delta\lambda|/|\lambda| \approx 4 \times 10^{-4}$, consistent with the bound.

Example 5.5. For the scalar quaternionic matrix polynomial $P(\lambda) = a_1\lambda + a_0$ with $a_1 = 1$ and $a_0 = -\mathbf{i} - \mathbf{j} - \mathbf{k}$, we verify Theorem 4.12 directly via Φ. Real adjoint matrices. $\Phi(a_1) = I_4$ and

$$\Phi(a_0) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{bmatrix}, \quad \|\Phi(a_0)\|_2 = |a_0| = \sqrt{3}.$$

The right eigenvalue $\lambda_0 = \mathbf{i} + \mathbf{j} + \mathbf{k}$ satisfies $|\lambda_0| = \sqrt{3}$ and $P'(\lambda_0) = a_1 = 1$. Verification via Φ. Since $\Phi(a_1\lambda_0) + \Phi(a_0) = \Phi(\lambda_0) + \Phi(a_0) = 0$ (verified in Example 5.3), λ_0 is confirmed as a right eigenvalue. Condition number. By Theorem 4.12:

$$\kappa_{\mathbb{H}}(\lambda_0) = \frac{|a_1||\lambda_0| + |a_0|}{|\lambda_0||P'(\lambda_0)|} = \frac{\sqrt{3} + \sqrt{3}}{\sqrt{3} \times 1} = 2.$$

This agrees with Example 5.3 and confirms that the real adjoint formula is consistent: $\|\Phi(a_r)\|_2 = |a_r|$ with no correction factor, since Φ is an isometry. Perturbation experiments confirm the bound is sharp.

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