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OPIAL INEQUALITIES ON TIMESCALES VIA FRACTIONAL CONFORMABLE CALCULUS

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Abstract. In this work, we prove a new class of weighted inequalities of Opial type via conformable fractional calculus on timescales. The inequalities obtained unify and generalize a number of results previously established in the literature, while also provide significant improvements. Furthermore, we apply our findings to determine the distance between successive zeros of solutions to conformable fractional dynamic equations with damping terms. In addition, we obtain lower bounds for the distance between a zero of a solution and a zero of its derivative in the conformable fractional sense. our results provide suitable conditions for the right and left disfocality, as well as the disconjugacy, of such conformable fractional dynamic equations on compact intervals timescales.

1. Introduction

In 1960, Z.Opial [21] established the following inequality:

$$\int_a^b |f(t)||f'(t)|dt \leq \frac{(b-a)}{4}\int_a^b |f'(t)|^2 dt, \tag{1.1}$$

where f is absolutely continuous function on $[a,b]$ satisfying $f(a) = f(b) = 0$. Since, this classic result has been simplified, generalized, and extended by numerous authors, including Olech [20], Beesack [3], Levinson [17], Mallows [18], Pederson [22]. Notably, if f is absolutely continuous function on $(0,b)$ with $f(0) = 0$, the inequality reduces to:

$$\int_0^b |f(t)||f'(t)|dt \leq \frac{b}{2}\int_0^b |f'(t)|^2 dt. \tag{1.2}$$

Following these foundational works, extensive research has focused on alternative proofs and various generalizations. This includes the development of discrete analogues; for instance, the discrete analogue of equation (1.1) was established in [16], while a discrete version of equation (1.2) was proven by Agarwal and Pang [2]. To naturally unify such continuous and discrete formulations, Hilger [12] introduced the calculus of time scales. These inequalities serve a dual purpose:

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they generalize classical results and provide a powerful analytical tool for investigating the qualitative properties of solutions to dynamic equations. We recall several key results that serve as the primary motivation for the findings presented in this paper. Bohner et al.[7] proved the timescale analogue of equation (1.2) and demonstrated that if $f:[0,b]_{\mathbb{T}}\to\mathbb{R}$ is Δ-differentiable with $f(0)=0$, then

$$\int_0^b |f(t)+f^\sigma(t)||f^\Delta(t)|\Delta t \le b\int_0^b |f^\Delta(t)|^2\Delta t. \tag{1.3}$$

In the same study [7], the investigators demonstrated that if r and q are positive, rd-continuous functions on $[0,b]_{\mathbb{T}}$ such that $\int_0^b \frac{\Delta t}{r(t)}<\infty$, and q is non-increasing, then for any Δ-differentiable function $f:[0,b]_{\mathbb{T}}\to\mathbb{R}$ satisfying the boundary condition $f(0)=0$, we have:

$$\int_0^b q^\sigma(t)|f(t)+f^\sigma(t)||f^\Delta(t)|\Delta t \le \left(\int_0^b \frac{\Delta t}{r(t)}\right)\left(\int_0^b r(t)q(t)|f^\Delta(t)|^2\Delta t\right). \tag{1.4}$$

Following the establishment of inequalities (1.3) and (1.4), a variety of alternative proofs, generalizations, and extensions have appeared in the literature; notable contributions can be found in [14, 27] and the references therein. Specifically, Karpuz et al. [14] established an inequality similar to (1.4) of the form:

$$\int_a^b q(t)|f(t)+f^\sigma(t)||f^\Delta(t)|\Delta t \le K_q(a,b)\int_a^b |f^\Delta(t)|^2\Delta t, \tag{1.5}$$

where q is a positive rd-continuous function on $[a,b]_{\mathbb{T}}$, and $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ is Δ-differentiable with $f(a)=0$, and

$$K_q(a,b)=\left(2\int_a^b q^2(t)(\sigma(t)-a)\Delta t\right)^{\frac{1}{2}}. \tag{1.6}$$

Concurrently, Wong et al. [28] and Srivastava et al. [27] proved that for any positive rd-continuous function r on $[a,b]_{\mathbb{T}}$, and Δ-differentiable function $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ with $f(a)=0$, we have:

$$\int_a^b r(t)|f(t)|^p|f^\Delta(t)|^q\Delta t \le \frac{q}{p+q}(b-a)^p\int_a^b r(t)|f^\Delta(t)|^{p+q}\Delta t, \tag{1.7}$$

where $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ is Δ-differentiable with $f(a)=0$. Additionally, the author in [23] extended this framework to show that if $f:[a,\theta]_{\mathbb{T}}\to\mathbb{R}$ satisfies $f(a)=0$, then

$$\int_a^\theta S(t)|f(t)|^p|f^\Delta(t)|^q\Delta t \le K_1(a,\theta,p,q)\int_a^\theta r(t)|f^\Delta(t)|^{p+q}\Delta t, \tag{1.8}$$

where p,q are positive numbers such that $p+q>1$, the functions r,s are non-negative rd-continuous functions defined on the interval $(a,\theta)_{\mathbb{T}}$, where

$$\begin{aligned} K_1(a,\theta,p,q) = {} & \left(\frac{q}{p+q}\right)^{\frac{q}{p+q}}\left(\int_a^\theta (S(t))^{\frac{q+p}{p}}(r(t))^{-\frac{q}{p}}\right. \\ & \left.\times\left(\int_a^t (r(s))^{-\frac{1}{p+q-1}}\Delta s\right)^{p+q-1}\Delta t\right)^{\frac{p}{p+q}}. \end{aligned} \tag{1.9}$$

If $f(a)$ is replaced by $f(b)$, then

$$\int_{\theta}^{b} S(t)|f(t)|^{p}|f^{\Delta}(t)|^{q}\Delta t \leq K_2(\theta,b,p,q)\int_{\theta}^{b} r(t)|f^{\Delta}(t)|^{p+q}\Delta t, \tag{1.10}$$

where

$$K_2(\theta,b,p,q) = \left(\frac{q}{p+q}\right)^{\frac{q}{p+q}}\left(\int_{\theta}^{b}(S(t))^{\frac{p+q}{p}}(r(t))^{\frac{-q}{p}} \times\left(\int_{t}^{b}(r(s))^{\frac{-1}{p+q-1}}\Delta s\right)^{p+q-1}\Delta t\right)^{\frac{p}{p+q}}. \tag{1.11}$$

More recently, various researchers have investigated fractional integral inequalities using Caputo and Riemann-Liouville approaches; for further details, The reader is referred to [6, 13]. Additionally, in [1, 15] the authors generalized the calculus of fractional order to the conformable calculus. Recently, classical inequalities—including the Opial [25, 26], Hermite-Hadamard [10, 11], and Steffensen inequalities—have been extended using conformable fractional calculus. Motivated by these recent developments, the primary objective of this paper is to establish a series of new weighted conformable fractional inequalities of Opial type on time scales. These results generalize the existing inequalities (1.5)–(1.11). The study of conformable fractional equations on timescales has emerged as a significant area of mathematical research and has recently garnered substantial attention. The fundamental objective is to establish results for dynamic equations where the domain of the unknown function is a timescale $\mathbb{T}$, defined as an arbitrary closed subset of the real numbers $\mathbb{R}$. In this work, we specifically investigate a conformable fractional equation with a damping term

$$T_{\Delta,\alpha}\left(r(t)\left(T_{\Delta,\alpha}x(t)\right)^{\gamma}\right)+p(t)\left(T_{\Delta,\alpha}x(t)\right)^{\gamma}+q(t)\left(x^{\sigma}(t)\right)^{\gamma}=0, t\in[a,b]_{\mathbb{T}}, \tag{1.12}$$

where $\mathbb{T}$ is an arbitrary timescale, $\sigma(t)$ is the forward jump operator on $\mathbb{T}$ which is defined by $\sigma(t) = inf\{s \in \mathbb{T} : s > t\}$, $\gamma \geq 1$ is a quotient of odd positive integers, and r, p, q are rd-continuous functions defined on $\mathbb{T}$ with $r(t) > 0$.

We apply these newly established inequalities to derive several results for the following problems:
(1)- Lower bounds for $b^{\alpha} - a^{\alpha}$: We establish lower bounds for the distance $b^{\alpha} - a^{\alpha}$ for a nontrivial solution x of equation (1.12) that satisfies the conditions $x(a) = T_{\Delta,\alpha}x(b) = 0$ or $T_{\Delta,\alpha}x(a) = x(b) = 0$.
(2-) Estimating the distance between successive zeros: We establish lower bounds for the distance between consecutive generalized zeros of a solution x to equation (1.12).
We assume throughout this work that $\gamma \geq 1$ is a quotient of odd positive integers, the functions r, p, q are rd-continuous defined on the timescale $\mathbb{T}$ satisfying $r(t) > 0$ and the following condition $t^{\alpha-1}\mu(t)|p(t)| \leq \frac{r(t)}{c}$, where $c \geq 1$ is a positive constant. We also impose that $sup\mathbb{T} = \infty$, and denote the timescale interval as $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$.
A solution x of the equation (1.12) is said to have a generalized zero at t if $x(t) = 0$, moreover, x is said to have a generalized zero in the interval $(t, \sigma(t))$

if $x(t)x^{\sigma}(t) < 0$ and $\mu(t) > 0$. The equation (1.12) is said to be dis-conjugate on the timescale interval $[a,b]_{\mathbb{T}}$, if no nontrivial solution of (1.12) possesses more than one generalized zero in $[a,b]_{\mathbb{T}}$. The equation (1.12) is said to be right dis-focal (respectively, left dis-focal) on the timescale $[a,b]_{\mathbb{T}}$ if the solution x of (1.12) satisfying the initial condition $T_{\Delta,\alpha}x(a) = 0$ (respectively, the terminal condition $T_{\Delta,\alpha}x(b) = 0$) has no generalized zeros in $[a,b]_{\mathbb{T}}$.

2. Preliminaries

In this section, we briefly recall the foundational concepts of Δ-conformable fractional calculus on time scales necessary for our subsequent analysis. For an introduction to the Δ-conformable fractional calculus on time scales we refer the reader to [22, 4, 5, 8, 9, 19]. A time scale $\mathbb{T}$ is a non-empty closed subset of the real numbers $\mathbb{R}$, endowed with the topology inherited from the standard topology on $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by:

$$\sigma : \mathbb{T} \to \mathbb{T}, \quad as \quad \sigma(t) = inf\left\{s \in \mathbb{T} : s > t\right\},$$

where $\sigma(t) = t$, if $\mathbb{T}$ has a maximum element t. For any $t \in \mathbb{T}$ the notation $f^{\sigma}(t)$ denotes the composition $f(\sigma(t))$. The graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) := \sigma(t) - t$.

Definition 2.1 (see [5]). Let $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^k$, and $\alpha \in (0,1]$. For $t > 0$, we define the $T_{\Delta,\alpha}(f)(t)$ to be the number (provided it exists) such that, for any $\epsilon > 0$, there exists a neighborhood U of t, such that

$$\left|\left[f^{\sigma}(t) - f(s)\right] t^{1-\alpha} - T_{\Delta,\alpha}(f)(t)\left[\sigma(t) - s\right]\right| \leq \epsilon|\sigma(t) - s|$$

for all $s \in U$.

We call $T_{\Delta,\alpha}(f)(t)$ the Δ-conformable fractional derivative of f of order α at t.

Theorem 2.2 (see [5]). Let $\alpha \in (0,1]$ and $t \in \mathbb{T}^k$. If the function $f : \mathbb{T} \to \mathbb{R}$ is Δ-conformable fractional differentiable of order α at t, then the following relation holds:

$$f^{\sigma}(t) = f(t) + \mu(t)t^{\alpha-1}T_{\Delta,\alpha}f(t). \tag{2.1}$$

The interested reader can find the basic properties of Δ-conformable fractional derivative on timescales in(see [5]).

Theorem 2.3 (see [19]). Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function and $g : \mathbb{T} \to \mathbb{R}$ be Δ-conformable fractional differentiable on $\mathbb{T}$. Then we have

$$\begin{aligned} T_{\Delta,\alpha}(f \circ g)(t) &= \left[\int_0^1 f'\left(g(t) + h\mu(t)t^{\alpha-1}T_{\Delta,\alpha}(g)\right) dh\right] T_{\Delta,\alpha}(g)(t) \\ &= \left[\int_0^1 f'\left(hg^{\sigma}(t) + (1-h)g(t)\right) dh\right] T_{\Delta,\alpha}(g)(t). \end{aligned} \tag{2.2}$$

Definition 2.4 (see [5]). For $0 < \alpha \leq 1$, the Δ-conformable fractional integral of f of order α is defined by

$$\int f(s)\Delta^{\alpha}s = \int f(s)s^{\alpha-1}\Delta s.$$

Definition 2.5 (see [5]). A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is said to be rd-continuous if it is continuous at every right-dense point of $\mathbb{T}$ and its left-sided limits exist at

left-dense points in $\mathbb{T}$.
The interested reader can find the basic properties of Δ-conformable fractional integral on timescales in(see [5]).
The Δ-conformable fractional integration by parts formula is given by:

$$\int_a^b f(t)T_{\Delta,\alpha}g(t)\Delta^\alpha t = [f(t)g(t)]_a^b - \int_a^b T_\alpha(f)(t)g^\sigma(t)\Delta^\alpha t,$$
$$\int_a^b f^\sigma(t)T_{\Delta,\alpha}g(t)\Delta^\alpha t = [f(t)g(t)]_a^b - \int_a^b T_\alpha(f)(t)g(t)\Delta^\alpha t.$$

Theorem 2.6 (see [19]). Let $\alpha \in (0,1]$, and $a, b \in \mathbb{T}$. If $f, g, h : [a,b] \to \mathbb{R}$ are rd-continuous functions, then the Hölder's inequality holds:

$$\int_a^b |f(t)g(t)||h(t)|\Delta^\alpha t \leq \left[\int_a^b |f(t)|^p|h(t)|\Delta^\alpha t\right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(t)|^q|h(t)|\Delta^\alpha t\right]^{\frac{1}{q}}, \quad (2.3)$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

We shall employ inequality (see [2]) to establish the main result of section 3.

$$(a+b)^\lambda \leq 2^{\lambda-1}\left(a^\lambda + b^\lambda\right) \ \textit{where } a, b \geq 0, \lambda \geq 1. \quad (2.4)$$

3. Main results

In this section, we establish our main results. The proofs are established by applying Hölder's inequality and the chain rule within the framework of Δ-conformable fractional calculus on time scales. Throughout this work, we explicitly assume that all involved functions are Δ-conformable fractional differentiable and that all integrals appearing in the subsequent theorems are well-defined and exist.

Theorem 3.1. *Let $\mathbb{T}$ be a timescale with $a, \theta, b \in \mathbb{T}$ such that $a < \theta < b$.Let λ, γ be positive numbers such that $\lambda \geq 1$. Assume that u, v are nonnegative rd-continuous functions on $\mathbb{T}$ such that $\int_a^\theta (u(t))^{\frac{-1}{\lambda+\gamma-1}}\Delta^\alpha t < \infty$ and $\int_\theta^b (u(t))^{\frac{-1}{\lambda+\gamma-1}}\Delta^\alpha t < \infty$. Let $x : [a,b]_\mathbb{T} \to \mathbb{R}$ be a Δ-conformable fractional differentiable function of order $\alpha \in (0,1]$.*
(i-) If $x(a) = 0$, then the following inequality holds on $[a,\theta]_\mathbb{T}$:

$$\int_a^\theta v(t)|x(t) + x^\sigma(t)|^\lambda |T_{\Delta,\alpha}x(t)|^\gamma \Delta^\alpha t \leq C_1(a,\theta,\lambda,\gamma)\int_a^\theta u(t)|T_{\Delta,\alpha}x(t)|^{\lambda+\gamma}\Delta^\alpha t, \quad (3.1)$$

where

$$C_1(a,\theta,\lambda,\gamma) := 2^{\lambda-1} \sup_{t\in[a,\theta]_{\mathbb{T}}} \left\{ \mu^{\lambda}(t) t^{\lambda(\alpha-1)} \frac{v(t)}{u(t)} \right\} + 2^{2\lambda-1} \left(\frac{\gamma}{\lambda+\gamma}\right)^{\frac{\gamma}{\lambda+\gamma}} \left[\left(\int_a^{\theta} v(t)^{\frac{\lambda+\gamma}{\lambda}} u(t)^{-\frac{\gamma}{\lambda}} \right) \times \left(\int_a^t u(s)^{-\frac{1}{\lambda+\gamma-1}} \Delta^{\alpha} s \right)^{\lambda+\gamma-1} \Delta^{\alpha} t \right]^{\frac{\lambda}{\lambda+\gamma}}. \tag{3.2}$$

(ii-) If $x(b)=0$, then the following inequality is satisfied on $[\theta,b]_{\mathbb{T}}$:

$$\int_{\theta}^{b} v(t)|x(t)+x^{\sigma}(t)|^{\lambda} |T_{\Delta,\alpha}x(t)|^{\gamma} \Delta^{\alpha} t \leq C_2(\theta,b,\lambda,\gamma) \int_{\theta}^{b} u(t)|T_{\Delta,\alpha}x(t)|^{\lambda+\gamma} \Delta^{\alpha} t,$$

where

$$C_2(\theta;b;\lambda;\gamma) := 2^{\lambda-1} \sup_{t\in[\theta,b]_{\mathbb{T}}} \left\{ \mu^{\lambda}(t)\cdot t^{\lambda(\alpha-1)} \quad \frac{v(t)}{u(t)} \right\} + 2^{2\lambda-1} \left(\frac{\gamma}{\lambda+\gamma}\right)^{\frac{\gamma}{\lambda+\gamma}} \left[\int_{\theta}^{b} v(t)^{\frac{\lambda+\gamma}{\lambda}} u(t)^{-\frac{\gamma}{\lambda}} \times \quad \left(\int_t^b u(s)^{-\frac{1}{\lambda+\gamma-1}} \Delta^{\alpha} s \right)^{\lambda+\gamma-1} \Delta^{\alpha} t \right]^{\frac{\lambda}{\lambda+\gamma}}.$$

Proof.
(i) Since $x(a)=0$, we acquire

$$|x(t)| = |x(t)-x(a)| = \left| \int_a^t T_{\Delta,\alpha}x(s)\ \Delta^{\alpha} s \right| \leq \int_a^t |T_{\Delta,\alpha}x(s)|\ \Delta^{\alpha} s,$$

Consequently,

$$|x(t)| \leq \int_a^t \frac{1}{(u(s))^{\frac{1}{\lambda+\gamma}}} (u(s))^{\frac{1}{\lambda+\gamma}}\ |T_{\Delta,\alpha}x(s)|\ \Delta^{\alpha} s.$$

By applying (2.3), with indices $\frac{\lambda+\gamma}{\lambda+\gamma-1}$ and $\lambda+\gamma$, we obtain

$$|x(t)| \leq \left(\int_a^t \left(\frac{1}{u(s)^{\frac{1}{\lambda+\gamma}}} \right)^{\frac{\lambda+\gamma}{\lambda+\gamma-1}} \Delta^{\alpha} s \right)^{\frac{\lambda+\gamma-1}{\lambda+\gamma}} \left(\int_a^t \left(u(s)^{\frac{1}{\lambda+\gamma}} |T_{\Delta,\alpha}x(s)| \right)^{\lambda+\gamma} \Delta^{\alpha} s \right)^{\frac{1}{\lambda+\gamma}}$$

$$= \left(\int_a^t \frac{1}{u(s)^{\frac{1}{\lambda+\gamma-1}}} \Delta^{\alpha} s \right)^{\frac{\lambda+\gamma-1}{\lambda+\gamma}} \left(\int_a^t u(s)|T_{\Delta,\alpha}x(s)|^{\lambda+\gamma} \Delta^{\alpha} s \right)^{\frac{1}{\lambda+\gamma}},$$

then for $t\in[a,\theta]_{\mathbb{T}}$, we get

$$|x(t)|^{\lambda} \leq \left(\int_a^t \frac{1}{(u(s))^{\frac{1}{\lambda+\gamma-1}}} \Delta^{\alpha} s \right)^{\frac{\lambda(\lambda+\gamma-1)}{\lambda+\gamma}} \left(\int_a^t u(s)|T_{\Delta,\alpha}x(s)|^{\lambda+\gamma} \Delta^{\alpha} s \right)^{\frac{\lambda}{\lambda+\gamma}}.$$

Using equality (2.1), we obtain

$$x(t)+x^{\sigma}(t)=2x(t)+t^{\alpha-1}\mu(t)T_{\Delta,\alpha}x(t).$$

Applying inequality (2.4), we get

$$|x(t)+x^{\sigma}(t)|^{\lambda}\leq 2^{\lambda-1}\left(2^{\lambda}|x(t)|^{\lambda}+t^{(\alpha-1)\lambda}\mu^{\lambda}(t)|T_{\Delta,\alpha}x(t)|^{\lambda}\right). \tag{3.3}$$

We set

$$y(t):=\int_a^t u(s)|T_{\Delta,\alpha}x(s)|^{\lambda+\gamma}\Delta^{\alpha}s. \tag{3.4}$$

We can see that $y(a)=0$, and

$$T_{\Delta,\alpha}y(t)=u(t)|T_{\Delta,\alpha}x(t)|^{\lambda+\gamma}>0. \tag{3.5}$$

Consequently, we acquire

$$|T_{\Delta,\alpha}x(t)|^{\gamma}=\left(\frac{T_{\Delta,\alpha}y(t)}{u(t)}\right)^{\frac{\gamma}{\lambda+\gamma}}. \tag{3.6}$$

Additionally, since v is nonnegative on $(a,\theta)_{\mathbb{T}}$. It follows from (3.3) and (3.6) that

$$\begin{aligned}
v(t)|x(t)+x^{\sigma}(t)|^{\lambda}|T_{\Delta,\alpha}x(t)|^{\gamma}&\leq 2^{2\lambda-1}\quad v(t)|x(t)|^{\lambda}|T_{\Delta,\alpha}x(t)|^{\gamma}\\
&+2^{\lambda-1}.t^{\lambda(\alpha-1)}\quad\mu^{\lambda}(t)|T_{\Delta,\alpha}x(t)|^{\gamma+\lambda}\quad v(t)\\
\leq 2^{2\lambda-1}\quad v(t)\left(\frac{1}{u(t)}\right)^{\frac{\gamma}{\gamma+\lambda}}&\left(\int_a^t\frac{\Delta^{\alpha}s}{(u(s))^{\frac{1}{\lambda+\gamma-1}}}\right)^{\frac{\lambda(\lambda+\gamma-1)}{\lambda+\gamma}}(y(t))^{\frac{\lambda}{\lambda+\gamma}}\left(T_{\Delta,\alpha}y(t)\right)^{\frac{\gamma}{\gamma+\lambda}}\\
&+2^{\lambda-1}t^{\lambda(\alpha-1)}\mu^{\lambda}(t)\quad v(t)\left(\frac{T_{\Delta,\alpha}y(t)}{u(t)}\right).
\end{aligned} \tag{3.7}$$

Integrating inequality (3.7) from a to θ, we acquire

$$\begin{aligned}
&\int_a^\theta v(t)|x(t)+x^\sigma(t)|^\lambda |T_{\Delta,\alpha}x(t)|^\gamma\,\Delta^\alpha t\\
&\qquad\le 2^{2\lambda-1}\int_a^\theta \frac{v(t)}{u(t)^{\frac{\gamma}{\gamma+\lambda}}}\left(\int_a^t \frac{\Delta^\alpha s}{u(s)^{\frac{1}{\lambda+\gamma-1}}}\right)^{\frac{\lambda(\lambda+\gamma-1)}{\lambda+\gamma}} y(t)^{\frac{\lambda}{\lambda+\gamma}}(T_{\Delta,\alpha}y(t))^{\frac{\gamma}{\gamma+\lambda}}\,\Delta^\alpha t\\
&\qquad\quad + 2^{\lambda-1}\int_a^\theta t^{\lambda(\alpha-1)}\mu^\lambda(t)\frac{v(t)}{u(t)}T_{\Delta,\alpha}y(t)\,\Delta^\alpha t\\
&\qquad\le 2^{2\lambda-1}\int_a^\theta \frac{v(t)}{u(t)^{\frac{\gamma}{\gamma+\lambda}}}\left(\int_a^t \frac{\Delta^\alpha s}{u(s)^{\frac{1}{\lambda+\gamma-1}}}\right)^{\frac{\lambda(\lambda+\gamma-1)}{\lambda+\gamma}} y(t)^{\frac{\lambda}{\lambda+\gamma}}(T_{\Delta,\alpha}y(t))^{\frac{\gamma}{\gamma+\lambda}}\,\Delta^\alpha t\\
&\qquad\quad + 2^{\lambda-1}\max_{t\in[a,\theta]_{\mathbb{T}}}\left(t^{\lambda(\alpha-1)}\mu^\lambda(t)\frac{v(t)}{u(t)}\right)\int_a^\theta T_{\Delta,\alpha}y(t)\,\Delta^\alpha t\\
&\qquad= 2^{2\lambda-1}\int_a^\theta \frac{v(t)}{u(t)^{\frac{\gamma}{\gamma+\lambda}}}\left(\int_a^t \frac{\Delta^\alpha s}{u(s)^{\frac{1}{\lambda+\gamma-1}}}\right)^{\frac{\lambda(\lambda+\gamma-1)}{\lambda+\gamma}} y(t)^{\frac{\lambda}{\lambda+\gamma}}(T_{\Delta,\alpha}y(t))^{\frac{\gamma}{\gamma+\lambda}}\,\Delta^\alpha t\\
&\qquad\quad + 2^{\lambda-1}\max_{t\in[a,\theta]_{\mathbb{T}}}\left(t^{\lambda(\alpha-1)}\mu^\lambda(t)\frac{v(t)}{u(t)}\right)y(\theta).
\end{aligned}$$

By applying (2.3), with conjugate exponents $\frac{\lambda+\gamma}{\lambda}$ and $\frac{\lambda+\gamma}{\gamma}$ to the first integral on the right-hand side, we obtain

$$\begin{aligned}
&\int_a^\theta v(t)|x(t)+x^\sigma(t)|^\lambda |T_{\Delta,\alpha}x(t)|^\gamma\,\Delta^\alpha t\\
&\qquad\le 2^{2\lambda-1}\left(\int_a^\theta \frac{v(t)^{\frac{\lambda+\gamma}{\lambda}}}{u(t)^{\frac{\gamma}{\lambda}}}\left[\int_a^t u(s)^{-\frac{1}{\lambda+\gamma-1}}\Delta^\alpha s\right]^{\lambda+\gamma-1}\Delta^\alpha t\right)^{\frac{\lambda}{\lambda+\gamma}}\\
&\qquad\quad\times\left(\int_a^\theta y(t)^{\frac{\lambda}{\gamma}}T_{\Delta,\alpha}y(t)\,\Delta^\alpha t\right)^{\frac{\gamma}{\gamma+\lambda}}\\
&\qquad\quad + 2^{\lambda-1}\max_{t\in[a,\theta]_{\mathbb{T}}}\left(t^{\lambda(\alpha-1)}\mu^\lambda(t)\frac{v(t)}{u(t)}\right)y(\theta).
\end{aligned}\tag{3.8}$$

From (3.5) and (2.2), it comes that

$$\begin{aligned}
T_{\Delta,\alpha}\left(y(t)^{\frac{\lambda+\gamma}{\gamma}}\right) &= \left(\frac{\lambda+\gamma}{\gamma}\right)T_{\Delta,\alpha}y(t)\int_0^1 [hy^\sigma(t)+(1-h)y(t)]^{\frac{\lambda}{\gamma}}\,dh\\
&\ge \left(\frac{\lambda+\gamma}{\gamma}\right)T_{\Delta,\alpha}y(t)\int_0^1 [hy(t)+(1-h)y(t)]^{\frac{\lambda}{\gamma}}\,dh\\
&= \left(\frac{\lambda+\gamma}{\gamma}\right)T_{\Delta,\alpha}y(t)\,y(t)^{\frac{\lambda}{\gamma}}.
\end{aligned}$$

As a consequence, we acquire

$$(y(t))^{\frac{\lambda}{\gamma}} \quad T_{\Delta,\alpha} y(t) \leq \frac{\gamma}{\lambda+\gamma} \quad T_{\Delta,\alpha}\left((y(t))^{\frac{\lambda+\gamma}{\gamma}}\right). \tag{3.9}$$

Substituting (3.9) into (3.8) and using the fact that $y(a)=0$, we have

$$\begin{aligned}
&\int_a^\theta v(t)|x(t)+x^\sigma(t)|^\lambda |T_{\Delta,\alpha}x(t)|^\gamma \,\Delta^\alpha t \\
&\quad \leq 2^{2\lambda-1}\left(\frac{\gamma}{\lambda+\gamma}\right)^{\frac{\gamma}{\lambda+\gamma}} \left(\int_a^\theta \frac{v(t)^{\frac{\lambda+\gamma}{\lambda}}}{u(t)^{\frac{\gamma}{\lambda}}}\left[\int_a^t \frac{\Delta^\alpha s}{u(s)^{\frac{1}{\lambda+\gamma-1}}}\right]^{\lambda+\gamma-1} \Delta^\alpha t\right)^{\frac{\lambda}{\lambda+\gamma}} \\
&\quad \times \left(\int_a^\theta T_{\Delta,\alpha}\left(y(t)^{\frac{\lambda+\gamma}{\gamma}}\right)\Delta^\alpha t\right)^{\frac{\gamma}{\lambda+\gamma}} + 2^{\lambda-1}\max_{t\in[a,\theta]_{\mathbb{T}}}\left(t^{\lambda(\alpha-1)}\mu^\lambda(t)\frac{v(t)}{v(t)}\right) y(\theta) \\
&\quad = 2^{2\lambda-1}\left(\frac{\gamma}{\lambda+\gamma}\right)^{\frac{\gamma}{\lambda+\gamma}} \left(\int_a^\theta \frac{v(t)^{\frac{\lambda+\gamma}{\lambda}}}{u(t)^{\frac{\gamma}{\lambda}}}\left[\int_a^t \frac{\Delta^\alpha s}{u(s)^{\frac{1}{\lambda+\gamma-1}}}\right]^{\lambda+\gamma-1} \Delta^\alpha t\right)^{\frac{\lambda}{\lambda+\gamma}} y(\theta) \\
&\quad + 2^{\lambda-1}\max_{t\in[a,\theta]_{\mathbb{T}}}\left(t^{\lambda(\alpha-1)}\mu^\lambda(t)\frac{v(t)}{u(t)}\right) y(\theta).
\end{aligned}$$

In view of (3.4), the previous inequality yields

$$\int_a^\theta v(t)|x(t)+x^\sigma(t)|^\lambda |T_{\Delta,\alpha}x(t)|^\gamma \Delta^\alpha t \leq C_1(a,\theta,\lambda,\gamma)\int_a^\theta u(t)|T_{\Delta,\alpha}x(t)|^{\lambda+\gamma}\Delta^\alpha t,$$

which yields the desired inequality (3.1).
(ii) The verification of (ii) follows the same logic as that of (i), substituting the interval $[a,\theta]_{\mathbb{T}}$ with $[\theta,b]_{\mathbb{T}}$.
In the sequel, we suppose there exists a unique solution $\theta \in]a,b[$ to the equation

$$C(\lambda,\gamma) = C_1(a;\theta,\lambda,\gamma) = C_2(\theta,b,\lambda,\gamma),$$

where $C_1(a,\theta,\lambda,\gamma)$ and $C_2(\theta,b,\lambda,\gamma)$ are as defined in Theorem 3.1 and. Since

$$\begin{aligned}
\int_a^b v(t)|x(t)+x^\sigma(t)|^\lambda |T_{\Delta,\alpha}x(t)|^\gamma \Delta^\alpha t &= \int_a^\theta v(t)|x(t)+x^\sigma(t)|^\lambda |T_{\Delta,\alpha}x(t)|^\gamma \Delta^\alpha t \\
&+ \int_\theta^b v(t)|x(t)+x^\sigma(t)|^\lambda |T_{\Delta,\alpha}x(t)|^\gamma \Delta^\alpha t,
\end{aligned}$$

then if we combine (i) and (ii) of Theorem 3.1, we have

Theorem 3.2. *Let $\mathbb{T}$ be a timescale with $a,b\in\mathbb{T}$. Let λ,γ be positive numbers such that $\lambda \geq 1$. Suppose that u,v are nonnegative rd-continuous functions on $(a,b)_{\mathbb{T}}$ such that $\int_a^b (u(t))^{\frac{-1}{\lambda+\gamma-1}}\Delta^\alpha t < \infty$. If $x:[a,b]_{\mathbb{T}}\to\mathbb{R}$ is Δ-conformable fractional differentiable of order $\alpha\in(0,1]$ with $x(a)=x(b)=0$, then we have*

$$\int_a^b v(t)|x(t)+x^\sigma(t)|^\lambda |T_{\Delta,\alpha}x(t)|^\gamma \Delta^\alpha t \leq C(\lambda,\gamma)\int_a^b u(t)|T_{\Delta,\alpha}x(t)|^{\lambda+\gamma}\Delta^\alpha t. \tag{3.10}$$

Theorem 3.3. *Let $\mathbb{T}$ be a timescale with $a, \theta, b \in \mathbb{T}$ such that $a < \theta < b$. Let λ, γ be positive real numbers such that $\lambda + \gamma > 1$. Assume that u, v are non-negative rd-continuous functions on $\mathbb{T}$ such that $\int_a^\theta (u(t))^{\frac{-1}{\lambda+\gamma-1}} \Delta^\alpha t < \infty$ and $\int_\theta^b (u(t))^{\frac{-1}{\lambda+\gamma-1}} \Delta^\alpha t < \infty$.Let $x : [a, b]_{\mathbb{T}} \to \mathbb{R}$ be a Δ-conformable fractional differentiable function of order $\alpha \in (0, 1]$.*

(i-) If $x(a) = 0$, then the following inequality holds on $[a, \theta]_{\mathbb{T}}$

$$\int_a^\theta v(t)|x(t)|^\lambda |T_{\Delta,\alpha} x(t)|^\gamma \Delta^\alpha t \leq K_1(a, \theta, \lambda, \gamma) \int_a^\theta u(t)|T_{\Delta,\alpha} x(t)|^{\lambda+\gamma} \Delta^\alpha t, \quad (3.11)$$

where

$$\begin{aligned} K_1(a, \theta, \lambda, \gamma) = \left(\frac{\gamma}{\lambda+\gamma}\right)^{\frac{\gamma}{\lambda+\gamma}} \Bigg(\int_a^\theta & v(t)^{\frac{\gamma+\lambda}{\lambda}} u(t)^{-\frac{\gamma}{\lambda}} \\ & \times \left[\int_a^t u(s)^{-\frac{1}{\lambda+\gamma-1}} \Delta^\alpha s\right]^{\lambda+\gamma-1} \Delta^\alpha t\Bigg)^{\frac{\lambda}{\lambda+\gamma}}. \end{aligned} \quad (3.12)$$

(ii-) If $x(b) = 0$, then the following inequality holds on $[\theta, b]_{\mathbb{T}}$:

$$\int_\theta^b v(t)|x(t)|^\lambda |T_{\Delta,\alpha} x(t)|^\gamma \Delta^\alpha t \leq K_2(\theta, b, \lambda; \gamma) \int_\theta^b u(t)|T_{\Delta,\alpha} x(t)|^{\lambda+\gamma} \Delta^\alpha t,$$

where

$$K_2(\theta, b, \lambda, \gamma) =$$

$$\left(\frac{\gamma}{\lambda+\gamma}\right)^{\frac{\gamma}{\lambda+\gamma}} \left(\int_\theta^b (v(t))^{\frac{\gamma+\lambda}{\lambda}} (u(t))^{\frac{-\gamma}{\lambda}} \left(\int_t^b (u(s))^{\frac{-1}{\lambda+\gamma-1}} \Delta^\alpha s\right)^{\lambda+\gamma-1} \Delta^\alpha t\right)^{\frac{\lambda}{\lambda+\gamma}}.$$

Proof.
(i) As in Theorem 3.1, we obtain

$$|x(t)|^\lambda \leq \left(\int_a^t (u(s))^{\frac{-1}{\lambda+\gamma-1}} \Delta^\alpha s\right)^{\frac{\lambda(\lambda+\gamma-1}{\lambda+\gamma}} \left(\int_a^t u(s)|T_{\Delta,\alpha} x(s)|^{\lambda+\gamma} \Delta^\alpha s\right)^{\frac{\lambda}{\lambda+\gamma}}. \quad (3.13)$$

We set

$$y(t) := \int_a^t u(s)|T_{\Delta,\alpha} x(s)|^{\lambda+\gamma} \Delta^\alpha s. \quad (3.14)$$

Since $y(a) = 0$, and $T_{\Delta,\alpha} y(t) = u(t)|T_{\Delta,\alpha} x(t)|^{\lambda+\gamma} > 0$, then we obtain

$$|T_{\Delta,\alpha} x(t)|^{\lambda+\gamma} = \frac{T_{\Delta,\alpha} y(t)}{u(t)} \quad \text{and} \quad |T_{\Delta,\alpha} x(t)|^\gamma = \left(\frac{T_{\Delta,\alpha} y(t)}{u(t)}\right)^{\frac{\gamma}{\gamma+\lambda}}. \quad (3.15)$$

Moreover, since v is nonnegative on $(a, \theta)_{\mathbb{T}}$, combining (3.13) and (3.15) yields

$$\begin{aligned} v(t)|x(t)|^\lambda |T_{\Delta,\alpha} x(t)|^\gamma \leq \frac{v(t)}{u(t)^{\frac{\gamma}{\lambda+\gamma}}} & \left(\int_a^t u(s)^{-\frac{1}{\lambda+\gamma-1}} \Delta^\alpha s\right)^{\frac{\lambda(\lambda+\gamma-1)}{\lambda+\gamma}} \\ & \times y(t)^{\frac{\lambda}{\lambda+\gamma}} (T_{\Delta,\alpha} y(t))^{\frac{\gamma}{\lambda+\gamma}}. \end{aligned} \quad (3.16)$$

By integrating the inequality (3.16) from a to θ, we arrive at

$$\int_a^\theta v(t)|x(t)|^\lambda |T_\alpha x(t)|^\gamma \Delta^\alpha t$$
$$\leq \int_a^\theta \frac{v(t)}{(u(t))^{\frac{\gamma}{\gamma+\lambda}}} \left(\int_a^t (u(s))^{\frac{-1}{\gamma+\lambda-1}} \Delta^\alpha s\right)^{\frac{\lambda(\lambda+\gamma-1)}{\lambda+\gamma}} (y(t))^{\frac{\lambda}{\lambda+\gamma}} (T_{\Delta,\alpha} y(t))^{\frac{\gamma}{\lambda+\gamma}} \Delta^\alpha t. \tag{3.17}$$

By applying inequality (2.3) with indices $\frac{\lambda+\gamma}{\lambda}$ and $\frac{\lambda+\gamma}{\gamma}$ to the integral on the right-hand side of (3.17), we acquire

$$\begin{aligned}\int_a^\theta & v(t)|x(t)|^\lambda |T_{\Delta,\alpha} x(t)|^\gamma \Delta^\alpha t \\ &\leq \left(\int_a^\theta \frac{v(t)^{\frac{\lambda+\gamma}{\lambda}}}{u(t)^{\frac{\gamma}{\lambda}}} \left[\int_a^t \frac{\Delta^\alpha s}{u(s)^{\frac{1}{\lambda+\gamma-1}}}\right]^{\lambda+\gamma-1} \Delta^\alpha t\right)^{\frac{\lambda}{\lambda+\gamma}} \\ &\quad \times \left(\int_a^\theta y(t)^{\frac{\lambda}{\gamma}} T_{\Delta,\alpha} y(t)\, \Delta^\alpha t\right)^{\frac{\gamma}{\lambda+\gamma}}.\end{aligned} \tag{3.18}$$

If we substitute (3.13) in (3.18) and using the condition $y(a) = 0$, we acquire

$$\begin{aligned}\int_a^\theta & v(t)|x(t)|^\lambda |T_{\Delta,\alpha} x(t)|^\gamma \Delta^\alpha t \\ &\leq \left(\frac{\gamma}{\gamma+\lambda}\right)^{\frac{\gamma}{\gamma+\lambda}} \left(\int_a^\theta \frac{v(t)^{\frac{\lambda+\gamma}{\lambda}}}{u(t)^{\frac{\gamma}{\lambda}}} \left[\int_a^t u(s)^{-\frac{1}{\lambda+\gamma-1}} \Delta^\alpha s\right]^{\lambda+\gamma-1} \Delta^\alpha t\right)^{\frac{\lambda}{\lambda+\gamma}} \\ &\quad \times \left(\int_a^\theta T_{\Delta,\alpha}\left(y(t)^{\frac{\lambda+\gamma}{\gamma}}\right) \Delta^\alpha t\right)^{\frac{\gamma}{\lambda+\gamma}} \\ &= \left(\frac{\gamma}{\gamma+\lambda}\right)^{\frac{\gamma}{\gamma+\lambda}} \left(\int_a^\theta \frac{v(t)^{\frac{\lambda+\gamma}{\lambda}}}{u(t)^{\frac{\gamma}{\lambda}}} \left[\int_a^t u(s)^{-\frac{1}{\lambda+\gamma-1}} \Delta^\alpha s\right]^{\lambda+\gamma-1} \Delta^\alpha t\right)^{\frac{\lambda}{\lambda+\gamma}} y(\theta).\end{aligned}$$

In view of (3.14), the previous inequality yields

$$\int_a^\theta v(t)|x(t)|^\lambda |T_{\Delta,\alpha} x(t)|^\gamma \Delta^\alpha t \leq K_1(a,\theta,\lambda,\gamma) \int_a^\theta u(t)|T_{\Delta,\alpha} x(t)|^{\lambda+\gamma} \Delta^\alpha t,$$

which establishes the desired inequality (3.11).
(ii) The proof is omitted here as it mirrors the argument for (i), substituting the interval $[a,\theta]_{\mathbb{T}}$ with $[\theta,b]_{\mathbb{T}}$.
In the sequel, we suppose that there exists a unique solution $\theta \in (a,b)_{\mathbb{T}}$ to the equation

$$K(\lambda,\gamma) = K_1(a,\theta,\lambda,\gamma) = K_2(\theta,b,\lambda,\gamma),$$

where $K_1(a,\theta,\lambda,\gamma)$ and $K_2(\theta,b,\lambda,\gamma)$ are defined as in Theorem 3.3 Note that since

$$\int_a^b v(t)|x(t)|^\lambda |T_{\Delta,\alpha}x(t)|^\gamma \,\Delta^\alpha t = \int_a^\theta v(t)|x(t)|^\lambda |T_{\Delta,\alpha}x(t)|^\gamma \,\Delta^\alpha t + \int_\theta^b v(t)|x(t)|^\lambda |T_{\Delta,\alpha}x(t)|^\gamma \,\Delta^\alpha t.$$

If we combine (i) and (ii) of Theorem 3.3, we have the following theorem:

Theorem 3.4. *Let $\mathbb{T}$ be a timescale with $a,b \in \mathbb{T}$. Let λ,γ be positive numbers such that $\lambda+\gamma>1$. Suppose that u,v are non-negative rd-continuous functions on $(a,b)_\mathbb{T}$ such that $\int_a^b (u(t))^{\frac{-1}{\lambda+\gamma-1}}\,\Delta^\alpha t<\infty$. If $x:[a,b]_\mathbb{T}\to\mathbb{R}$ is Δ-conformable fractional differentiable of order $\alpha\in(0,1]$ with $x(a)=x(b)=0$, then we have the following inequality:*

$$\int_a^b v(t)|x(t)|^\lambda |T_{\Delta,\alpha}x(t)|^\gamma \Delta^\alpha t \le K(\lambda,\gamma)\int_a^b u(t)|T_{\Delta,\alpha}x(t)|^{\lambda+\gamma}\Delta^\alpha t. \tag{3.19}$$

In this subsection, we utilize the inequalities established above to derive some results concerning problems (1) and (2).
To simplify the analysis of the results, we define:

$$\Omega(a) = \sup_{t\in[a,b]_\mathbb{T}} \left(t^{\gamma(\alpha-1)}\mu^\gamma(t)\frac{|Q_a(t)|}{r(t)}\right), \textit{with } Q_a(t)=\int_a^t q(s)\Delta^\alpha s.$$

$$\Omega(b) = \sup_{t\in[a,b]_\mathbb{T}} \left(t^{\gamma(\alpha-1)}\mu^\gamma(t)\frac{|Q_b(t)|}{r(t)}\right), \textit{with } Q_b(t)=\int_t^b q(s)\Delta^\alpha s.$$

$$R_a(t)=\int_a^t \frac{\Delta^\alpha s}{(r(s))^{\frac{1}{\gamma}}} \textit{ and } R_b(t)=\int_t^b \frac{\Delta^\alpha s}{(r(s))^{\frac{1}{\gamma}}}.$$

Theorem 3.5. *Suppose that x is a nontrivial solution to (1.12).*

- *If $x(a)=T_{\Delta,\alpha}x(b)=0$, then*

$$2^{2\gamma-2}\Omega(b)+\frac{2^{3\gamma-2}}{(\gamma+1)^{\frac{1}{\gamma+1}}}\left(\int_a^b \frac{|Q_b(t)|^{\frac{\gamma+1}{\gamma}}}{r(t)^{\frac{1}{\gamma}}}(R_a(t))^\gamma\,\Delta^\alpha t\right)^{\frac{\gamma}{\gamma+1}} + \left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}}\left(\int_a^b \frac{|p(t)|^{\gamma+1}}{r(t)^\gamma}(R_a(t))^\gamma\,\Delta^\alpha t\right)^{\frac{1}{\gamma+1}} \ge 1-\frac{1}{c}. \tag{3.20}$$

- *If $T_{\Delta,\alpha}x(a)=x(b)=0$, then*

$$2^{2\gamma-2}\Omega(a)+\frac{2^{3\gamma-2}}{(\gamma+1)^{\frac{1}{\gamma+1}}}\left(\int_a^b \frac{|Q_a(t)|^{\frac{\gamma+1}{\gamma}}}{r(t)^{\frac{1}{\gamma}}}(R_b(t))^\gamma\,\Delta^\alpha t\right)^{\frac{\gamma}{\gamma+1}} + \left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}}\left(\int_a^b \frac{|p(t)|^{\gamma+1}}{r(t)^\gamma}(R_b(t))^\gamma\,\Delta^\alpha t\right)^{\frac{1}{\gamma+1}} \ge 1-\frac{1}{c}. \tag{3.21}$$

Proof. The proof of (3.21) is analogous to the proof of (3.20), so we omit the details, we only prove (3.20).
Without loss of generality, we can assume that $x(t) \geq 0$ in $[a,b]_{\mathbb{T}}$. Multiplying (1.12) by x^σ and integrating by parts yields

$$\begin{aligned}\int_a^b T_{\Delta,\alpha}\left[r(t)(T_{\Delta,\alpha}x(t))^\gamma\right] x^\sigma(t)\,\Delta^\alpha t + \int_a^b p(t)x^\sigma(t)(T_{\Delta,\alpha}x(t))^\gamma\,\Delta^\alpha t \\ = \left[r(t)(T_{\Delta,\alpha}x(t))^\gamma x(t)\right]_a^b - \int_a^b r(t)(T_{\Delta,\alpha}x(t))^{\gamma+1}\,\Delta^\alpha t + \int_a^b p(t)x^\sigma(t)(T_{\Delta,\alpha}x(t))^\gamma\,\Delta^\alpha t \\ = -\int_a^b q(t)(x^\sigma(t))^{\gamma+1}\,\Delta^\alpha t.\end{aligned}$$

Based on the assumption that $x(a) = T_{\Delta,\alpha}x(b) = 0$, we obtain

$$\begin{aligned}-\int_a^b r(t)(T_{\Delta,\alpha}x(t))^{\gamma+1}\,\Delta^\alpha t + \int_a^b p(t)x^\sigma(t)(T_{\Delta,\alpha}x(t))^\gamma\,\Delta^\alpha t \\ = -\int_a^b q(t)(x^\sigma(t))^{\gamma+1}\,\Delta^\alpha t,\end{aligned}$$

It follows that

$$\begin{aligned}\int_a^b r(t)\left(T_{\Delta,\alpha}x(t)\right)^{\gamma+1}\Delta^\alpha t = \int_a^b p(t)x^\sigma(t)\left(T_{\Delta,\alpha}x(t)\right)^\gamma\Delta^\alpha t \\ - \int_a^b T_{\Delta,\alpha}(Q_b(t))\left(x^\sigma(t)\right)^{\gamma+1}\Delta^\alpha t.\end{aligned} \tag{3.22}$$

By integrating the right-hand side of (3.22), we get

$$\begin{aligned}\int_a^b r(t)\left(T_{\Delta,\alpha}x(t)\right)^{\gamma+1}\Delta^\alpha t \\ = \int_a^b p(t)x^\sigma(t)\left(T_{\Delta,\alpha}x(t)\right)^\gamma\Delta^\alpha t - \left[Q_b(t)\left(x(t)\right)^{\gamma+1}\right]_a^b + \int_a^b Q_b(t)T_{\Delta,\alpha}\left(x(t)^{\gamma+1}\right)\Delta^\alpha t.\end{aligned}$$

By applying the condition $x(a) = 0$ and $Q_b(b) = 0$, we obtain

$$\begin{aligned}\int_a^b r(t)\left(T_{\Delta,\alpha}x(t)\right)^{\gamma+1}\Delta^\alpha t = \\ \int_a^b p(t)x^\sigma(t)\left(T_{\Delta,\alpha}x(t)\right)^\gamma\Delta^\alpha t + \int_a^b Q_b(t)T_{\Delta,\alpha}\left(x(t)^{\gamma+1}\right)\Delta^\alpha t.\end{aligned} \tag{3.23}$$

Applying (2.2), we acquire

$$T_{\Delta,\alpha}\left(x(t)^{\gamma+1}\right) = (\gamma+1)\left(T_{\Delta,\alpha}x(t)\right)\int_0^1 \left[h.x^\sigma(t) + (1-h)x(t)\right]^\gamma dh.$$

By applying inequality (2.4), it comes that

$$\begin{aligned}\left|T_{\Delta,\alpha}(x(t)^{\gamma+1})\right| &\leq (\gamma+1)|T_{\Delta,\alpha}x(t)| \int_0^1 |hx^\sigma(t)+(1-h)x(t)|^\gamma\, dh \\ &\leq (\gamma+1)2^{\gamma-1}|T_{\Delta,\alpha}x(t)| \int_0^1 \left(|hx^\sigma(t)|^\gamma + |(1-h)x(t)|^\gamma\right)\, dh \\ &= (\gamma+1)2^{\gamma-1}|T_{\Delta,\alpha}x(t)| \left[|x^\sigma(t)|^\gamma \int_0^1 h^\gamma\, dh + |x(t)|^\gamma \int_0^1 (1-h)^\gamma\, dh\right] \\ &= 2^{\gamma-1}|T_{\Delta,\alpha}x(t)||x^\sigma(t)|^\gamma + 2^{\gamma-1}|T_{\Delta,\alpha}x(t)||x(t)|^\gamma \\ &= 2^{\gamma-1}\left(|x^\sigma(t)|^\gamma + |x(t)|^\gamma\right)|T_{\Delta,\alpha}x(t)| \\ &\leq 2^{\gamma-1}|x^\sigma(t)+x(t)|^\gamma |T_{\Delta,\alpha}x(t)|. \end{aligned} \tag{3.24}$$

From (3.23) and (3.24), we have

$$\begin{aligned}&\int_a^b r(t)\,|T_{\Delta,\alpha}x(t)|^{\gamma+1}\,\Delta^\alpha t \\ &\qquad \leq \int_a^b |p(t)||x^\sigma(t)|\,|T_{\Delta,\alpha}x(t)|^\gamma\,\Delta^\alpha t \\ &\qquad\quad + 2^{\gamma-1}\int_a^b |Q_b(t)||x^\sigma(t)+x(t)|^\gamma\,|T_{\Delta,\alpha}x(t)|\,\Delta^\alpha t. \end{aligned} \tag{3.25}$$

By applying inequality (3.1) to the integral $\int_a^b \left|Q_b(t)\right||x^\sigma(t)+x(t)|^\gamma\left|T_{\Delta,\alpha}x(t)\right|\Delta^\alpha t$, with $v(t) = \left|Q_b(t)\right|, u(t) = r(t), \lambda = \gamma, \gamma = 1$, we obtain

$$\int_a^b \left|Q_b(t)\right||x^\sigma(t)+x(t)|^\gamma\left|T_{\Delta,\alpha}x(t)\right|\Delta^\alpha t \leq C_1(a,b,\gamma,1)\int_a^b r(t)\left|T_{\Delta,\alpha}x(t)\right|^{\gamma+1}\Delta^\alpha t, \tag{3.26}$$

where

$$\begin{aligned}C_1(a,b,\gamma,1) &= 2^{\gamma-1}\sup_{t\in[a,b]_{\mathbb{T}}}\left(t^{\lambda(\alpha-1)}\mu^\lambda(t)\frac{|Q_b(t)|}{r(t)}\right) \\ &\quad + 2^{2\gamma-1}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{1+\gamma}}\left(\int_a^b \frac{|Q_b(t)|^{\frac{1+\gamma}{\gamma}}}{r(t)^{\frac{1}{\gamma}}}\left[\int_a^t r(s)^{-\frac{1}{\gamma}}\Delta^\alpha s\right]^\gamma \Delta^\alpha t\right)^{\frac{\gamma}{1+\gamma}} \\ &= 2^{\gamma-1}\Omega(b) + 2^{2\gamma-1}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{1+\gamma}}\left(\int_a^b \frac{|Q_b(t)|^{\frac{1+\gamma}{\gamma}}}{r(t)^{\frac{1}{\gamma}}}\left[\int_a^t r(s)^{-\frac{1}{\gamma}}\Delta^\alpha s\right]^\gamma \Delta^\alpha t\right)^{\frac{\gamma}{1+\gamma}}.\end{aligned}$$

By using (2.1), we can see that

$$\int_a^b |p(t)||x^\sigma(t)||T_{\Delta,\alpha}x(t)|^\gamma \Delta^\alpha t = \int_a^b |p(t)||x(t)+t^{\alpha-1}\mu(t)T_{\Delta,\alpha}x(t)||T_{\Delta,\alpha}x(t)|^\gamma \Delta^\alpha t$$
$$\leq \int_a^b |p(t)||x(t)||T_{\Delta,\alpha}x(t)|^\gamma \Delta^\alpha t + \int_a^b t^{\alpha-1}\mu(t)|p(t)||T_{\Delta,\alpha}x(t)|^{\gamma+1}\Delta^\alpha t.$$

By applying inequality (3.13) to the integral $\int_a^b |p(t)||x(t)||T_{\Delta,\alpha}x(t)|^\gamma \Delta^\alpha t$ with $v(t) = |p(t)|$, $u(t) = r(t)$, $\lambda = 1$, we get

$$\int_a^b |p(t)||x(t)||T_{\Delta,\alpha}x(t)|^\gamma \Delta^\alpha t \leq K_1(a,b,1,\gamma)\int_a^b r(t)|T_{\Delta,\alpha}x(t)|^{\gamma+1}\Delta^\alpha t,$$

where

$$K_1(a,b,1,\gamma) = \left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}} \left(\int_a^b \frac{|p(t)|^{\gamma+1}}{(r(t))^\gamma}(R_a(t))^\gamma \Delta^\alpha t\right)^{\frac{1}{\gamma+1}}.$$

Given the hypothesis $0 < t^{\alpha-1}\mu(t)|p(t)| \leq \frac{r(t)}{c}$, it follows

$$\begin{aligned}\int_a^b |p(t)||x^\sigma(t)||T_{\Delta,\alpha}x(t)|^\gamma \Delta^\alpha t &\leq K_1(a;b;1;\gamma)\int_a^b r(t)|T_{\Delta,\alpha}x(t)|^{\gamma+1}\Delta^\alpha t \\ &\quad + \frac{1}{c}\int_a^b r(t)|T_{\Delta,\alpha}x(t)|^{\gamma+1}\Delta^\alpha t.\end{aligned} \tag{3.27}$$

Substituting (3.26) and (3.27) into (3.25), we acquire

$$\begin{aligned}\int_a^b r(t)\left|T_{\Delta,\alpha}x(t)\right|^{\gamma+1}\Delta^\alpha t &\leq \frac{1}{c} + K_1(a;b;1;\gamma)\int_a^b r(t)\left|T_{\Delta,\alpha}x(t)\right|^{\gamma+1}\Delta^\alpha t \\ &\quad + 2^{\gamma-1}C_1(a;b;\gamma;1)\int_a^b r(t)\left|T_{\Delta,\alpha}x(t)\right|^{\gamma+1}\Delta^\alpha t.\end{aligned} \tag{3.28}$$

It follows from (3.28) that

$$\begin{aligned}1-\frac{1}{c} &\leq K_1(a;b;1;\gamma) + 2^{\gamma-1}C_1(a;b;\gamma;1) \\ &= 2^{2\gamma-2}\Omega(b) + \frac{2^{3\gamma-2}}{(1+\gamma)^{\frac{1}{1+\gamma}}}\left(\int_a^b \frac{|Q_b(t)|^{\frac{1+\gamma}{\gamma}}}{(r(t))^{\frac{1}{\gamma}}}(R_a(t))^\gamma \Delta^\alpha t\right)^{\frac{\gamma}{1+\gamma}} \\ &\quad + \left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}}\left(\int_a^b \frac{|p(t)|^{1+\gamma}}{(r(t))^\gamma}(R_a(t))^\gamma \Delta^\alpha t\right)^{\frac{1}{1+\gamma}}.\end{aligned}$$

Which is the inequality in question (3.20).

Remark 3.1. Theorem 3.5 yields sufficient conditions for (1.12) to be dis-focal. Specifically, it guarantees that there is no nontrivial solution x such that $x(a) = T_{\Delta,\alpha}x(b) = 0$ or $T_{\Delta,\alpha}x(a) = x(b) = 0$.

Now, we apply Theorem 3.2 and Theorem 3.4 to determine a lower bound for the spacing between successive generalized zeros of a solution to (1.12). In the following, we assume that $T_{\Delta,\alpha}Q(t) = q(t)$ and there exists a unique $\theta \in (a,b)_{\mathbb{T}}$ such that

$$C(\gamma,1) = C_1(a,\theta,\gamma,1) = C_2(\theta,b,\gamma,1),, \tag{3.29}$$

$$K(1,\gamma) = K_1(a,\theta,1,\gamma) = K_2(\theta,b,1,\gamma), \tag{3.30}$$

where

$$C_1(a,\theta,\gamma,1) = 2^{\gamma-1} \sup_{t\in[a,\theta]_{\mathbb{T}}} \left(t^{\gamma(\alpha-1)}\mu^{\gamma}(t)\frac{|Q(t)|}{r(t)} \right) + 2^{2\gamma-1}\left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}} \left(\int_a^{\theta} \frac{|Q(t)|^{\frac{\gamma+1}{\gamma}}}{r(t)^{\frac{1}{\gamma}}} (R_a(t))^{\gamma}\Delta^{\alpha}t \right)^{\frac{\gamma}{\gamma+\lambda}}.$$

$$C_2(\theta,b,\gamma,1) = 2^{\gamma-1} \sup_{t\in[\theta,b]_{\mathbb{T}}} \left(t^{\gamma(\alpha-1)}\mu^{\gamma}(t)\frac{|Q(t)|}{r(t)} \right) + 2^{2\gamma-1}\left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}} \left(\int_{\theta}^{b} \frac{|Q(t)|^{\frac{\gamma+1}{\gamma}}}{r(t)^{\frac{1}{\gamma}}} (R_b(t))^{\gamma}\Delta^{\alpha}t \right)^{\frac{\gamma}{\gamma+1}}.$$

$$K_1(a,\theta,1,\gamma) = \left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}} \left(\int_a^{\theta} \frac{|Q(t)|^{\gamma+1}}{r(t)^{\gamma}} (R_a(t))^{\gamma}\Delta^{\alpha}t \right)^{\frac{1}{1+\gamma}}.$$

$$K_2(\theta,b,1,\gamma) = \left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}} \left(\int_{\theta}^{b} \frac{|Q(t)|^{\gamma+1}}{r(t)^{\gamma}} (R_b(t))^{\gamma}\Delta^{\alpha}t \right)^{\frac{1}{1+\gamma}}.$$

Theorem 3.6. *Suppose that x is a nontrivial solution to (1.12). If $x(a) = x(b) = 0$, then we have*

$$K(1,\gamma) + 2^{\gamma-1}C(\gamma,1) \geq 1 - \frac{1}{c} \tag{3.31}$$

where $K(1,\gamma)$ and $C(\gamma,1)$ are defined as in (3.29) and (3.30) .

Proof. By multiplying (1.12) by $x^{\sigma}(t)$ and following the same argument as in the proof of Theorem 3.5, we obtain

$$\int_a^b r(t)\left(T_{\Delta,\alpha}x(t)\right)^{\gamma+1}\Delta^{\alpha}t = \int_a^b p(t)x^{\sigma}(t)\left(T_{\Delta,\alpha}x(t)\right)^{\gamma}\Delta^{\alpha}t + \int_a^b T_{\Delta,\alpha}Q(t)\left(x^{\sigma}(t)\right)^{\gamma+1}\Delta^{\alpha}t.$$

By using fractional integration by parts on the right-hand side, we have

$$\int_a^b r(t)\left(T_{\Delta,\alpha}x(t)\right)^{\gamma+1}\Delta^{\alpha}t = \int_a^b p(t)x^{\sigma}(t)\left(T_{\Delta,\alpha}x(t)\right)^{\gamma}\Delta^{\alpha}t + \Big[Q(t)x(t)\Big]_a^b - \int_a^b Q(t)T_{\Delta,\alpha}\left(x(t)\right)^{\gamma+1}\Delta^{\alpha}t.$$

By using the fact that $x(a) = x(b) = 0$, we obtain

$$\int_a^b r(t)\left(T_{\Delta,\alpha}x(t)\right)^{\gamma+1}\Delta^\alpha t = \int_a^b p(t)x^\sigma(t)\left(T_{\Delta,\alpha}x(t)\right)^\gamma \Delta^\alpha t - \int_a^b Q(t)T_{\Delta,\alpha}\left((x(t))^{\gamma+1}\right)\Delta^\alpha t.$$

We have the following inequality

$$\int_a^b r(t)|T_{\Delta,\alpha}x(t)|^{\gamma+1}\Delta^\alpha t \leq \int_a^b |p(t)|\cdot|x^\sigma(t)||T_{\Delta,\alpha}x(t)|^\gamma\Delta^\alpha t + \int_a^b |Q(t)|\left|T_{\Delta,\alpha}\left((x(t))^{\gamma+1}\right)\right|\Delta^\alpha t. \tag{3.32}$$

Applying inequality (3.24), we obtain

$$\int_a^b |Q(t)||T_{\Delta,\alpha}\left((x(t))^{\gamma+1}\right)|\Delta^\alpha t \leq 2^{\gamma-1}\int_a^b |Q(t)|\cdot|x(t)+x^\sigma(t)|^\gamma\cdot|T_{\Delta,\alpha}x(t)|\Delta^\alpha t. \tag{3.33}$$

By applying inequality (3.10) with $v(t) = |Q(t)|$,$u(t) = r(t)$,$\lambda = \gamma$ and $\gamma = 1$, we acquire

$$\int_a^b |Q(t)|\cdot|x(t)+x^\sigma(t)|^\gamma|T_{\Delta,\alpha}x(t)|\Delta^\alpha t \leq C(\gamma,1)\int_a^b r(t)|T_{\Delta,\alpha}x(t)|^{\gamma+1}\Delta^\alpha t. \tag{3.34}$$

If we combine (3.33) and (3.34), we obtain

$$\int_a^b |Q(t)|\cdot|T_{\Delta,\alpha}\left((x(t))^{\gamma+1}\right)|\Delta^\alpha t \leq 2^{\gamma-1}C(\gamma,1)\int_a^b r(t)|T_{\Delta,\alpha}x(t)|^{\gamma+1}\Delta^\alpha t. \tag{3.35}$$

Following the same reasoning as the proof of Theorem 3.5, we obtain

$$\int_a^b |p(t)|\quad|x^\sigma(t)||T_{\Delta,\alpha}x(t)|^\gamma\Delta^\alpha t \leq \int_a^b |p(t)||x(t)||T_{\Delta,\alpha}x(t)|^\gamma\Delta^\alpha t + \int_a^b t^{\alpha-1}\mu(t)|p(t)|\quad|T_{\Delta,\alpha}x(t)|^{\gamma+1}\Delta^\alpha t. \tag{3.36}$$

By combining the assumption $0 < t^{\alpha-1}\mu(t)|p(t)| \leq \frac{r(t)}{c}$, with inequality (3.36), we get

$$\int_a^b |p(t)||x^\sigma(t)||T_{\Delta,\alpha}x(t)|^\gamma\Delta^\alpha t \leq \int_a^b |p(t)||x(t)||T_{\Delta,\alpha}x(t)|^\gamma\Delta^\alpha t + \frac{1}{c}\int_a^b r(t)|T_{\Delta,\alpha}x(t)|^{\gamma+1}\Delta^\alpha t. \tag{3.37}$$

Applying inequality (3.19) with $v(t) = |p(t)|$, $u(t) = r(t)$ and $\lambda = 1$, we acquire

$$\int_a^b |p(t)||x(t)||T_{\Delta,\alpha}x(t)|^\gamma\Delta^\alpha t \leq K(1,\gamma)\int_a^b r(t)|T_{\Delta,\alpha}x(t)|^{\gamma+1}\Delta^\alpha t. \tag{3.38}$$

By combining (3.32), (3.35), (3.37) and (3.38), we obtain

$$\int_a^b r(t)|T_{\Delta,\alpha}x(t)|^{\gamma+1}\Delta^\alpha t \leq \left(K(1,\gamma)+2^{\gamma-1}C(\gamma,1)+\frac{1}{c}\right)\int_a^b r(t)|T_{\Delta,\alpha}x(t)|^{\gamma+1}\Delta^\alpha t.$$

After canceling the common term $\int_a^b r(t)|T_{\Delta,\alpha}x(t)|^{\gamma+1}\Delta^\alpha t$, from both sides of this inequality, we obtain the desired result (3.31).

Remark 3.2. Theorem 3.6 provides sufficient conditions for the dis-conjugacy of (1.12), that is, conditions ensuring that there is no nontrivial solution x satisfies $x(a) = x(b) = 0$.

4. Conclusions

In this work, we have successfully established some new inequalities of Hardy-Opial via conformable fractional calculus on timescales. By utilizing the properties of conformable fractional calculus and the theory of timescales, our results generalize and extend various existing inequalities found in the literature. Furthermore, we demonstrated the utility of these results by providing lower bounds for the distance between successive zeros of solutions to specific conformable fractional dynamic equations. These findings offer new criteria for the dis-focality and dis-conjugacy of linear dynamic systems, opening further possibilities for research into the qualitative behavior of fractional dynamic equations on general timescales.

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Conflict of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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