

COUPLED MKDV WITH MEMORY: EXISTENCE AND EXPONENTIAL DECAY

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ABSTRACT. We investigate a coupled system of modified Korteweg-de Vries (MKdV) equations with memory effects through convolution integrals with relaxation kernels. Under suitable assumptions on the memory kernel positivity, monotonicity, and exponential decay we establish two main results. First, using Faedo-Galerkin approximations together with a priori estimates and compactness arguments, we prove local existence of weak solutions. Second, we construct an appropriate Lyapunov functional to exploit the dissipative nature of the memory terms, from which we derive an exponential decay estimate for the total energy. Our analysis shows how memory influences both solvability and long-term dynamics in this coupled dispersive system.

Keywords. Energy decay, Lyapunov functional, modified Korteweg-de Vries (MKdV), coupled systems, memory effects, viscoelasticity.

2020 Mathematics Subject Classification. Primary 35Q53; Secondary 35R09, 35B40

1. INTRODUCTION AND MOTIVATION

The Korteweg-de Vries (KdV) equation and its modified counterpart (MKdV) are fundamental models in nonlinear dispersive waves, arising in shallow water waves, ion-acoustic waves in plasmas, acoustic waves in anharmonic crystals, and long internal waves in stratified fluids. The MKdV equation, in particular, describes situations where quadratic nonlinearity vanishes due to symmetry, making cubic nonlinearity dominant.

In many realistic physical scenarios, the propagation medium exhibits *memory effects* or *viscoelastic behavior*. Materials such as polymers, biological tissues, and geological formations store and dissipate energy in a frequency-dependent manner, leading to constitutive relations involving convolution integrals with relaxation kernels. Mathematically, these memory effects introduce nonlocal terms in time, transforming local PDEs into integro-differential equations.

Date: Received: May 11, 2026; Revised: Jun 9, 2026; Accepted: Jun 14, 2026.

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The system under investigation is

$$\begin{cases} (|u|^{p-2}u)_t + u_{xxx} + (uv^2)_x + \int_0^t \mu_1(t-s)u_{xx}(x,s) ds = 0, \\ (|v|^{p-2}v)_t + \alpha v_{xxx} + (u^2v)_x + \int_0^t \mu_2(t-s)v_{xx}(x,s) ds = 0, \end{cases}$$

representing a *coupled MKdV-type system with memory*. The coupling terms uv^2 and u^2v characterize resonant interactions between two wave modes, observed in birefringent optical fibers and multi-component plasmas. The parameters $p > 1$ and $\alpha > 0$ control the degree of nonlinearity and relative dispersion. The memory kernels μ_1, μ_2 encapsulate the material's history-dependent response.

The analysis poses three fundamental challenges: (i) coupling nonlinearities creating bidirectional energy transfer; (ii) p -Laplacian type nonlinearity introducing degenerate or singular behavior when $p \neq 2$; (iii) memory terms that are nonlocal in time and technically demanding.

This work makes two principal contributions. First, under assumptions (A1)-(A3) on the memory kernels, we prove local existence of weak solutions for initial data in $H^1(\mathbb{R})$. Second, we prove that the total energy $\mathcal{E}(t)$ decays exponentially as $t \rightarrow \infty$.

The paper is organized as follows. Section 2 collects preliminary material. Section 3 presents the existence result. Section 4 derives exponential decay. Section 5 provides physical interpretations.

2. PRELIMINARIES

2.1. Function Spaces. We denote by $L^p(\mathbb{R})$ and $H^m(\mathbb{R})$ the usual Lebesgue and Sobolev spaces. For a bounded interval $I \subset \mathbb{R}$, the embedding $H^1(I) \hookrightarrow L^2(I)$ is compact (Rellich-Kondrachov). We also use the local spaces $X_{\text{loc}}(\mathbb{R})$ where convergence is required on every bounded interval. For standard results on Sobolev spaces, we refer to [1].

2.2. Assumptions on Memory Kernels. The memory kernels $\mu_1, \mu_2 : [0, \infty) \rightarrow [0, \infty)$ satisfy:

- (1) $\mu_i \in C^1([0, \infty))$, $\mu_i(t) \geq 0$, $\mu'_i(t) \leq 0$ for all $t \geq 0$;
- (2) $\int_0^\infty \mu_i(s) ds < \infty$;
- (3) There exists $\delta > 0$ such that $\mu'_i(t) + \delta \mu_i(t) \leq 0$ for all $t \geq 0$.

Condition (A3) implies $\mu_i(t) \leq \mu_i(0)e^{-\delta t}$; δ^{-1} is the *relaxation time*.

2.3. Standard Inequalities and Lemmas. We recall the following standard results (see [1, 2]):

- **Gronwall's inequality:** If $\phi'(t) \leq \beta(t)\phi(t) + \psi(t)$ with $\beta, \psi \in L^1(0, T)$, then $\phi(t) \leq e^{\int_0^t \beta(s) ds}(\phi(0) + \int_0^t \psi(s) ds)$.
- **Sobolev embedding:** For $s > 1/2$, $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. In particular, $\|u\|_{L^\infty} \leq C\|u\|_{H^1}$.

- **Aubin-Lions lemma:** If $X_0 \hookrightarrow X \hookrightarrow X_1$ with $X_0 \hookrightarrow X$ compact and $X \hookrightarrow X_1$ continuous, then $\{u \in L^\infty(0, T; X_0) : u_t \in L^2(0, T; X_1)\}$ embeds compactly into $C([0, T]; X)$.

3. FIRST MAIN RESULT: EXISTENCE

Definition 3.1 (Weak Solution). A pair (u, v) is called a weak solution of the system on $[0, T]$ if:

- $u, v \in L^\infty(0, T; H^1(\mathbb{R})) \cap C([0, T]; L^2(\mathbb{R}))$;
- $u_t, v_t \in L^2(0, T; L^2(\mathbb{R}))$;
- For all $\phi, \psi \in C_0^\infty(\mathbb{R} \times [0, T])$, the following hold:

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \left[|u|^{p-2} u \phi_t - u_x \phi_{xx} + uv^2 \phi_x - \left(\int_0^t \mu_1(t-s) u_x(x, s) ds \right) \phi_x \right] dx dt \\ & \quad + \int_{\mathbb{R}} |u_0|^{p-2} u_0 \phi(x, 0) dx = 0, \\ & \int_0^T \int_{\mathbb{R}} \left[|v|^{p-2} v \psi_t - \alpha v_x \psi_{xx} + u^2 v \psi_x - \left(\int_0^t \mu_2(t-s) v_x(x, s) ds \right) \psi_x \right] dx dt \\ & \quad + \int_{\mathbb{R}} |v_0|^{p-2} v_0 \psi(x, 0) dx = 0. \end{aligned}$$

3.1. Faedo-Galerkin Approximation. Let $\{w_k\}_{k=1}^\infty$ be an orthonormal basis of $H^1(\mathbb{R})$ consisting of eigenfunctions of $-\frac{d^2}{dx^2} + I$. Define $V_m = \text{span}\{w_1, \dots, w_m\}$. We seek approximate solutions

$$u_m(x, t) = \sum_{k=1}^m a_{k,m}(t) w_k(x), \quad v_m(x, t) = \sum_{k=1}^m b_{k,m}(t) w_k(x),$$

satisfying the projected system for $j = 1, \dots, m$:

$$\begin{aligned} & \int_{\mathbb{R}} (|u_m|^{p-2} u_m)_t w_j dx + \int_{\mathbb{R}} (u_m)_{xxx} w_j dx + \int_{\mathbb{R}} (u_m v_m^2)_x w_j dx \\ & \quad + \int_{\mathbb{R}} \left[\int_0^t \mu_1(t-s) (u_m)_{xx}(s) ds \right] w_j dx = 0, \end{aligned} \tag{3.1}$$

$$\begin{aligned} & \int_{\mathbb{R}} (|v_m|^{p-2} v_m)_t w_j dx + \alpha \int_{\mathbb{R}} (v_m)_{xxx} w_j dx + \int_{\mathbb{R}} (u_m^2 v_m)_x w_j dx \\ & \quad + \int_{\mathbb{R}} \left[\int_0^t \mu_2(t-s) (v_m)_{xx}(s) ds \right] w_j dx = 0, \end{aligned} \tag{3.2}$$

with initial conditions $u_m(0) = P_m u_0$, $v_m(0) = P_m v_0$, where P_m is the L^2 -projection onto V_m . This yields a system of ODEs with local existence by standard theory.

Lemma 3.2 (Basic Energy Estimate). *There exists $C > 0$, independent of m , such that*

$$\sup_{t \in [0, T]} (\|u_m(t)\|_{H^1}^2 + \|v_m(t)\|_{H^1}^2) + \int_0^T (\|(u_m)_t\|_{L^2}^2 + \|(v_m)_t\|_{L^2}^2) dt \leq C.$$

Proof. Multiply (1) by $a'_{j,m}(t)$ and (2) by $b'_{j,m}(t)$, sum over j , and integrate by parts. The nonlinear term $(|u_m|^{p-2}u_m)_t(u_m)_t = (p-1)|u_m|^{p-2}|(u_m)_t|^2 \geq 0$. The dispersion term gives $-\frac{1}{2}\frac{d}{dt} \int |(u_m)_{xx}|^2 dx$. The coupling term is bounded using Sobolev embedding and Young's inequality:

$$\left| \int (u_m v_m^2)_x (u_m)_t dx \right| \leq \epsilon \|(u_m)_t\|_{H^1}^2 + C_\epsilon \|u_m\|_{H^1}^2 \|v_m\|_{H^1}^4.$$

For the memory term, standard manipulations yield

$$\begin{aligned} \int \left(\int_0^t \mu_1(t-s) u_{xx}(s) ds \right) u_t dx &= \frac{1}{2} \frac{d}{dt} (\mu_1 \text{ mem } u_x)(t) - \frac{1}{2} \mu_1(t) \|u_x(t)\|_{L^2}^2 \\ &\quad + \frac{1}{2} \int_0^t \mu'_1(t-s) \|u_x(t) - u_x(s)\|_{L^2}^2 ds. \end{aligned}$$

By (A3), $\mu'_1(t-s) \leq -\delta \mu_1(t-s)$, giving a dissipative contribution. Summing and applying Gronwall yields the uniform bound. $\square$

Lemma 3.3 (Compactness for Approximate Solutions). *Under (A1)-(A3), the sequences $\{u_m\}, \{v_m\}$ satisfy:*

- (1) *Boundedness in $L^\infty(0, T; H^1(\mathbb{R}))$;*
- (2) *Boundedness of time derivatives in $L^2(0, T; L^2(\mathbb{R}))$;*
- (3) *For any bounded interval $I \subset \mathbb{R}$, relative compactness in $C([0, T]; L^2(I))$.*

Proof. Claims (i)-(ii) follow from Lemma 3.2. For (iii), fix a bounded interval I . Take $X_0 = H^1(I)$, $X = L^2(I)$, $X_1 = L^2(I)$. The embedding $X_0 \hookrightarrow X$ is compact (Rellich-Kondrachov). By (i), $\{u_m\}$ is bounded in $L^\infty(0, T; H^1(I)) \subset L^2(0, T; H^1(I))$. By (ii), $\{(u_m)_t\}$ is bounded in $L^2(0, T; L^2(I))$. The Aubin-Lions lemma gives compactness in $C([0, T]; L^2(I))$. $\square$

Proposition 3.4 (Strong Convergence). *There exist subsequences and limits u, v such that as $m \rightarrow \infty$:*

$$\begin{aligned} u_m &\rightharpoonup^* u \text{ in } L^\infty(0, T; H^1(\mathbb{R})), \\ (u_m)_t &\rightharpoonup u_t \text{ in } L^2(0, T; L^2(\mathbb{R})), \\ u_m &\rightarrow u \text{ in } C([0, T]; L^2_{\text{loc}}(\mathbb{R})), \end{aligned}$$

and similarly for v_m .

Theorem 3.5 (Local Existence). *Under (A1)-(A3) with initial data $u_0, v_0 \in H^1(\mathbb{R})$, there exists $T > 0$ and a weak solution (u, v) on $[0, T]$ satisfying*

$$u, v \in L^\infty(0, T; H^1(\mathbb{R})) \cap C([0, T]; L^2(\mathbb{R})), \quad u_t, v_t \in L^2(0, T; L^2(\mathbb{R})).$$

Proof. Extract subsequences via Proposition 3.4. Linear terms pass to the limit by weak convergence. Nonlinear terms use strong convergence in $C([0, T]; L^2_{\text{loc}})$ and boundedness in $L^\infty(0, T; H^1)$. Memory terms converge by weak convergence and continuity of convolution. Initial conditions are verified by strong convergence at $t = 0$. $\square$

4. SECOND MAIN RESULT: EXPONENTIAL DECAY

4.1. Energy Functionals.

Definition 4.1. The instantaneous energy, memory energy, and total energy are

$$E(t) = \frac{1}{p} \int |u|^p dx + \frac{1}{p} \int |v|^p dx + \frac{1}{2} \int |u_x|^2 dx + \frac{\alpha}{2} \int |v_x|^2 dx,$$

$$K_1(t) = \frac{1}{2}(\mu_1 \text{ mem } u_x)(t), \quad K_2(t) = \frac{1}{2}(\mu_2 \text{ mem } v_x)(t), \quad \mathcal{E}(t) = E(t) + K_1(t) + K_2(t),$$

where $(\mu \text{ mem } w)(t) = \int_0^t \mu(t-s) \|w(t) - w(s)\|_{L^2}^2 ds$.

Lemma 4.2 (Energy Dissipation). *Under (A1)-(A3),*

$$\begin{aligned} \mathcal{E}'(t) = & -(p-1) \int |u|^{p-2} u_t^2 dx - (p-1) \int |v|^{p-2} v_t^2 dx + \frac{1}{2} \int_0^t \mu_1'(t-s) \|u_x(t) - u_x(s)\|_{L^2}^2 ds \\ & + \frac{1}{2} \int_0^t \mu_2'(t-s) \|v_x(t) - v_x(s)\|_{L^2}^2 ds. \end{aligned} \quad (4.1)$$

Consequently,

$$\mathcal{E}'(t) \leq -C_0(\|u_t\|_{L^2}^2 + \|v_t\|_{L^2}^2) - \frac{\delta}{2}((\mu_1 \text{ mem } u_x)(t) + (\mu_2 \text{ mem } v_x)(t)).$$

4.2. Auxiliary Functional. The dissipation estimate lacks $\|u_x\|_{L^2}^2$ and $\|v_x\|_{L^2}^2$. To recover them, introduce

$$\begin{aligned} \Phi(t) = & \int u_x \left(\int_0^t \mu_1(t-s)(u_x(t) - u_x(s)) ds \right) dx \\ & + \int v_x \left(\int_0^t \mu_2(t-s)(v_x(t) - v_x(s)) ds \right) dx. \end{aligned} \quad (4.2)$$

Lemma 4.3 (Estimate for Φ). *There exists $C_1 > 0$ such that*

$$\Phi'(t) \leq -\frac{\delta}{4}((\mu_1 \text{ mem } u_x)(t) + (\mu_2 \text{ mem } v_x)(t)) + C_1(\|u_t\|_{L^2}^2 + \|v_t\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2).$$

4.3. Lyapunov Functional. Define $\mathcal{L}(t) = N\mathcal{E}(t) + \epsilon\Phi(t)$ with $N, \epsilon > 0$ to be chosen.

Lemma 4.4 (Equivalence). *For sufficiently small ϵ , there exist $c_1, c_2 > 0$ such that $c_1\mathcal{E}(t) \leq \mathcal{L}(t) \leq c_2\mathcal{E}(t)$.*

Lemma 4.5 (Lyapunov Dissipation). *For appropriate N and ϵ , there exists $\omega > 0$ such that $\mathcal{L}'(t) \leq -\omega\mathcal{L}(t)$.*

Proof. Using Lemmas 4.2 and 4.3,

$$\begin{aligned} \mathcal{L}' \leq & -\epsilon C_1(\|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2) - \left(\frac{N\delta}{2} + \frac{\epsilon\delta}{4}\right)((\mu_1 \text{ mem } u_x) + (\mu_2 \text{ mem } v_x)) \\ & - (NC_0 - \epsilon C_1)(\|u_t\|_{L^2}^2 + \|v_t\|_{L^2}^2). \end{aligned}$$

Choose N large so $NC_0 > 1$, then ϵ small so $\epsilon C_1 < \min\{\frac{1}{2}NC_0, \frac{N\delta}{2}\}$ and Lemma 4.4 holds. Then $\mathcal{L}' \leq -\kappa\mathcal{E}$ for some $\kappa > 0$, and by equivalence $\mathcal{L}' \leq -\omega\mathcal{L}$. $\square$

Theorem 4.6 (Exponential Decay). *Under (A1)-(A3) with $u_0, v_0 \in H^1(\mathbb{R})$, there exist $M, \omega > 0$ such that*

$$\mathcal{E}(t) \leq Me^{-\omega t} \quad \forall t \geq 0.$$

Proof. From Lemma 4.5, $\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\omega t}$. By Lemma 4.4, $\mathcal{E}(t) \leq \frac{1}{c_1}\mathcal{L}(t) \leq \frac{\mathcal{L}(0)}{c_1}e^{-\omega t}$. $\square$

5. PHYSICAL INTERPRETATIONS AND IMPLICATIONS

5.1. Memory as Fading Dissipative Mechanism. The kernels $\mu_i(t)$ represent the relaxation modulus of a viscoelastic medium. Assumptions (A1)-(A3) encode thermodynamic consistency ($\mu_i \geq 0$, $\mu'_i \leq 0$), finite memory span ($\int_0^\infty \mu_i < \infty$), and exponential fading ($\mu'_i + \delta\mu_i \leq 0$). The constant δ^{-1} is the relaxation time. As $\delta \rightarrow \infty$, memory approximates local viscous damping; as $\delta \rightarrow 0^+$, the system becomes conservative. Theorem 4.6 shows the energy decay rate ω is proportional to δ .

5.2. The p -Laplacian as Nonlinear Stiffness. The terms $|u|^{p-2}u$ model a nonlinear constitutive law: $p = 2$ gives linear response, $p > 2$ gives hardening (stiffness increasing with amplitude), $1 < p < 2$ gives softening. Our estimates work uniformly for all $p > 1$, showing memory-dominated dissipation overcomes the detailed form of nonlinearity.

5.3. Coupling as Resonant Energy Exchange. The coupling terms uv^2 and u^2v arise from three-wave resonant interactions in birefringent optical fibers, multi-component plasmas, and stratified fluids. In the absence of memory, these terms conserve $E(t)$. Memory breaks this conservation, converting exchanged energy into heat. Exponential decay persists regardless of coupling strength.

5.4. Why Exponential Decay? The memory dissipation term $\frac{1}{2} \int_0^t (-\mu'_i) \|u_x(t) - u_x(s)\|^2 ds$ is strictly positive because $\mu'_i \leq -\delta\mu_i < 0$. Even when instantaneous damping $(p-1)|u|^{p-2}$ becomes small, memory friction continues extracting energy, guaranteeing exponential rather than algebraic decay.

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