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A STUDY OF (α, β)-INTUITIONISTIC FUZZY ORDERS

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Abstract. In this work, we establish some properties of (α, β)-intuitionistic fuzzy order relations. Firstly, we prove that the infimum of any family of (α, β)-intuitionistic fuzzy orders defined on a non-empty set $\mathcal{X}$ is also an (α, β)-intuitionistic fuzzy order on it. Secondly, we present a characterization of a class of total (α, β)-intuitionistic fuzzy order relations compatibles with the usual algebraic structures on the real line $\mathbb{R}$.

Keywords. Fuzzy set, intuitionistic fuzzy set, α-fuzzy order relations, (α, β)-intuitionistic fuzzy order relations.
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1. Introduction

Fuzzy set theory and fuzzy ordering, introduced by Zadeh [17, 18], have revolutionized our understanding of uncertainty and imprecision. Building upon this foundation, many researchers have explored various extensions and generalizations of fuzzy sets and fuzzy ordering [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 20]. In this respect, the first author of the present paper and Zedam [14] studied the properties of α-fuzzy order relations which later were generalized in [15] to (α, β)-intuitionistic fuzzy order relations. In this paper we aim to extend the significant obtained results in [14] to the more general framework of (α, β)-intuitionistic fuzzy order relations. For that, we will establish some properties of these generalized intuitionistic fuzzy orders.

We organize this work as follows. In the second section, we recall useful definitions and results that we will need later. In the third section, we prove that if $(\alpha, \beta) \in]0, 1] \times [0, 1[$ with $\alpha + \beta \leq 1$ and $(\mathcal{R}_i)_{i \in I}$ is a family of (α, β)-intuitionistic fuzzy order relations on $\mathcal{X}$, then the intuitionistic fuzzy relation $\mathcal{R} = (\mu_\mathcal{R}, \nu_\mathcal{R})$ defined by:

$$\mu_\mathcal{R}(a, b) = \inf_{i \in I}\{\mu_{\mathcal{R}_i}(a, b)\} \text{ and } \nu_\mathcal{R}(a, b) = \sup_{i \in I}\{\nu_{\mathcal{R}_i}(a, b)\} \text{ for every } (a, b) \in \mathcal{X} \times \mathcal{X}$$

is an (α, β)-intuitionistic fuzzy order relation on $\mathcal{X}$ (see Theorem 3.1). In the fourth section, we show that if for every $i \in \{1, ..., n\}$, $\mathcal{R}_i = (\mu_{\mathcal{R}_i}, \nu_{\mathcal{R}_i})$ is compatible with the vector algebraic structures on $\mathcal{X}_i$, so $\mathcal{V} = (\mu_\mathcal{V}, \nu_\mathcal{V})$ defined on the

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real vector space $\mathcal{X} = \prod_{i=1}^{n} \mathcal{X}_i$ by:

$$\mu_{\mathcal{V}}((a_1, ..., a_n), (b_1, ..., b_n)) = \min_{i \in \{1,...,n\}} \{\mu_{\mathcal{R}_i}(a_i, b_i)\}$$

and

$$\nu_{\mathcal{V}}((a_1, ..., a_n), (b_1, ..., b_n)) = \max_{i \in \{1,...,n\}} \{\nu_{\mathcal{R}_i}(a_i, b_i)\}$$

is an (α, β)-intuitionistic fuzzy order relation compatible with the vector algebraic structures on $\mathcal{X}$ (see Theorem 4.2). In the fifth section, we characterize a class of total (α, β)-intuitionistic fuzzy order relations compatibles with the usual algebraic structures on the real line $\mathbb{R}$ (see Theorem 5.8).

2. Preliminaries

Let $\mathcal{X}$ be a nonempty set.

- In [17], Zadeh defined a fuzzy set $\mathcal{A}$ by the membership function $\mu_{\mathcal{A}} : \mathcal{X} \to [0, 1]$. For every $a \in \mathcal{X}$, the value $\mu_{\mathcal{A}}(a)$ is the degree of membership of a in $\mathcal{A}$. We shall write $\mathcal{A}$ as $\mathcal{A} = \mu_{\mathcal{A}}$. Any fuzzy set $\mathcal{R} = \mu_{\mathcal{R}}$ of $\mathcal{X} \times \mathcal{X}$ is called a fuzzy relation on $\mathcal{X}$.

- In [1], Antanassov extended the concept of fuzzy set to intuitionistic fuzzy set. An intuitionistic fuzzy set $\mathcal{A}$ of $\mathcal{X}$ is characterized by two functions $\mu_{\mathcal{A}}, \nu_{\mathcal{A}} : \mathcal{X} \to [0, 1]$ satisfying, for every $a \in \mathcal{X}$ the following condition: $0 \leq \mu_{\mathcal{A}}(a) + \nu_{\mathcal{A}}(a) \leq 1$. For every $a \in \mathcal{X}$, the values $\mu_{\mathcal{A}}(a)$ and $\nu_{\mathcal{A}}(a)$ are respectively the degree of membership and non-membership of a in $\mathcal{A}$. In the sequel, we shall write $\mathcal{A}$ as $\mathcal{A} = (\mu_{\mathcal{A}}, \nu_{\mathcal{A}})$. Every intuitionistic fuzzy set $\mathcal{R} = (\mu_{\mathcal{R}}, \nu_{\mathcal{R}})$ of $\mathcal{X} \times \mathcal{X}$ is called an intuitionistic fuzzy relation on $\mathcal{X}$.

Definition 2.1. [12]. Let $\mathcal{X}$ be a nonempty set, $\alpha \in]0, 1]$ and $\mathcal{R} = \mu_{\mathcal{R}}$ be a fuzzy relation on $\mathcal{X}$.

1) $\mathcal{R}$ is said to be α-fuzzy reflexive if for every $a \in \mathcal{X}$, $\mu_{\mathcal{R}}(a, a) = \alpha$;

2) $\mathcal{R}$ is called to be α-fuzzy antisymmetric if for every $(a, b) \in \mathcal{X} \times \mathcal{X}$ satisfying $\mu_{\mathcal{R}}(a, b) + \mu_{\mathcal{R}}(b, a) > \alpha$, then $a = b$;

3) $\mathcal{R}$ is said to be α-fuzzy transitive if for every $(a, c) \in \mathcal{X} \times \mathcal{X}$, $\mu_{\mathcal{R}}(a, c) \geq \sup_{b \in \mathcal{X}} [\min\{\mu_{\mathcal{R}}(a, b), \mu_{\mathcal{R}}(b, c)\}]$.

4) $\mathcal{R}$ is called to be an α–fuzzy order relation on $\mathcal{X}$ if $\mathcal{R}$ is α-fuzzy reflexive, antisymmetric and transitive.

Definition 2.2. [15]. Let $\mathcal{X}$ be a nonempty set, $(\alpha, \beta) \in]0, 1] \times [0, 1[$ with $\alpha + \beta \leq 1$ and let $\mathcal{R} = (\mu_{\mathcal{R}}, \nu_{\mathcal{R}})$ be an intuitionistic fuzzy relation on $\mathcal{X}$.

1) $\mathcal{R}$ is said to be (α, β)-intuitionistic fuzzy reflexive if for every $a \in \mathcal{X}$, $\mu_{\mathcal{R}}(a, a) = \alpha$ and $\nu_{\mathcal{R}}(a, a) = \beta$;

2) $\mathcal{R}$ is called to be (α, β)-intuitionistic fuzzy antisymmetric if for every $(a, b) \in \mathcal{X} \times \mathcal{X}$ satisfying $\mu_{\mathcal{R}}(a, b) + \mu_{\mathcal{R}}(b, a) > \alpha$ and $\nu_{\mathcal{R}}(a, b) + \nu_{\mathcal{R}}(b, a) < \beta + 1$, then $a = b$;

3) $\mathcal{R}$ is said to be (α, β)-intuitionistic fuzzy transitive if for every $(a, c) \in \mathcal{X} \times \mathcal{X}$, $\mu_{\mathcal{R}}(a, c) \geq \sup_{b \in \mathcal{X}} \{\min\{\mu_{\mathcal{R}}(a, b), \mu_{\mathcal{R}}(b, c)\}\}$ and

$\nu_{\mathcal{R}}(a,c) \leq \inf_{b\in\mathcal{X}}\{\max\{\nu_{\mathcal{R}}(a,b),\nu_{\mathcal{R}}(b,c)\}\}$.

4) $\mathcal{R}$ is called to be an (α,β)–intuitionistic fuzzy order relation on $\mathcal{X}$ if $\mathcal{R}$ is (α,β)–intuitionistic fuzzy reflexive, antisymmetric and transitive.

Note that $\mathcal{R}$ is (α,β)-intuitionistic fuzzy antisymmetry if

$$\big[(\forall(a,b)\in\mathcal{X}\times\mathcal{X})(a\neq b)\Rightarrow(\mu_{\mathcal{R}}(a,b)+\mu_{\mathcal{R}}(b,a)\leq\alpha \text{ or } \nu_{\mathcal{R}}(a,b)+\nu_{\mathbf{R}}(b,a)\geq\beta+1)\big]. \tag{2.1}$$

If $\mathcal{R}$ is an (α,β)-intuitionistic fuzzy order relation on $\mathcal{X}$, then $\mathcal{X}$ is called an (α,β)-intuitionistic fuzzy ordered set and we denote it by $(\mathcal{X},\mathcal{R})$.

An (α,β)-intuitionistic fuzzy order relation $\mathcal{R}$ on $\mathcal{X}$ is said to be total if for all $(a,b)\in\mathcal{X}\times\mathcal{X}$, we have either

$$\Big(\mu_{\mathcal{R}}(a,b)>\frac{\alpha}{2} \text{ and } \nu_{\mathcal{R}}(a,b)<\frac{\beta+1}{2}\Big) \text{ or } \Big(\mu_{\mathcal{R}}(b,a)>\frac{\alpha}{2} \text{ and } \nu_{\mathcal{R}}(b,a)<\frac{\beta+1}{2}\Big).$$

If $\mathcal{R}$ is a total (α,β)-intuitionistic fuzzy order relation on $\mathcal{X}$, then $(\mathcal{X},\mathcal{R})$ is an (α,β)-intuitionistic fuzzy chain.

Let B be a nonempty subset of $\mathcal{X}$.

1) We say that an element $u\in\mathcal{X}$ is an upper bound of B if $\mu_{\mathcal{R}}(a,u)>\dfrac{\alpha}{2}$ and $\nu_{\mathcal{R}}(a,u)<\dfrac{\beta+1}{2}$ for every $a\in B$. When u is an upper bound of B and $u\in B$, then u is called a greatest element of B and we write $u=\max_{\mathcal{R}}(B)$.

2) The subset B has a a lower bound if there exists an element $v\in\mathcal{X}$ satisfying $\mu_{\mathcal{R}}(v,a)>\dfrac{\alpha}{2}$ and $\nu_{\mathcal{R}}(v,a)<\dfrac{\beta+1}{2}$ for every $a\in B$. If v is a lower bound of B and $v\in B$, then v is called a least element of B and we write $v=\min_{\mathcal{R}}(B)$.

3) If the set of all upper bounds of B has a least element, m, we say that B has a supremum and we write $m=\sup(B)$.

4) When the set of all lower bounds of B has a greatest element, ℓ, we say that B has an infimum and we write $\ell=\inf(B)$.

Examples 2.3.

(i) Let $(\alpha,\beta)\in]0,1]\times[0,1[$ with $\alpha+\beta\leq 1$ and let $A=\{a,b,c\}$ be a set. Let $\mathcal{R}=(\mu_{\mathcal{R}},\nu_{\mathcal{R}})$ be the following intuitionistic fuzzy relation defined on A by:

$\mu_{\mathcal{R}}(.,.)$	a	b	c
a	α	$0,6\alpha$	$0,5\alpha$
b	$0,3\alpha$	α	$0,4\alpha$
c	$0,2\alpha$	$0,2\alpha$	α

and

$\nu_{\mathcal{R}}(.,.)$	a	b	c
a	β	$\frac{\beta+3}{4}$	$\frac{\beta+4}{5}$
b	$\frac{\beta+8}{9}$	β	$\frac{\beta+6}{7}$
c	$\frac{\beta+9}{10}$	$\frac{\beta+9}{10}$	β

Then, $\mathcal{R}$ is an (α,β)-intuitionistic fuzzy order relation on A.

(ii) Let $\alpha\in]0,1]$ and $\mathcal{R}=(\mu_{\mathcal{R}},\nu_{\mathcal{R}})$ be the following intuitionistic fuzzy relation defined on $\mathbb{R}$ by:

$$\mu_{\mathcal{R}}(a,b)=\begin{cases}\alpha & \text{if } a=b\\ \alpha\big(1-\dfrac{a}{b}\big) & \text{if } 0\leq a<b\\ 0 & \text{if } b<a\\ \alpha\big(1-\dfrac{b}{a}\big) & \text{if } a<b\leq 0\\ \alpha & \text{if } a<0 \text{ and } b>0\end{cases}$$

and

$$\nu_{\mathcal{R}}(a,b)=\begin{cases}1-\alpha & \text{if } a=b\\ 1-\alpha\big(1-\dfrac{a}{b}\big) & \text{if } 0\leq a<b\\ 1 & \text{if } b<a\\ 1-\alpha\big(1-\dfrac{b}{a}\big) & \text{if } a<b\leq 0\\ 1-\alpha & \text{if } a<0 \text{ and } b>0.\end{cases}$$

Then, $\mathcal{R}$ is an $(\alpha, 1-\alpha)$-intuitionistic fuzzy order relation on $\mathbb{R}$.

3. Constructing new (α,β)-intuitionistic fuzzy order relations

In this section, we present some techniques for building new (α,β)-intuitionistic fuzzy order relations.

Theorem 3.1. *Let $(\alpha,\beta)\in]0,1]\times[0,1[$ with $\alpha+\beta\leq 1$ and let $\mathcal{X}$ be a nonempty set. If $(\mathcal{R}_i)_{i\in I}$ is a family of (α,β)-intuitionistic fuzzy order relations on $\mathcal{X}$, then the intuitionistic fuzzy relation $\mathcal{R}=(\mu_{\mathcal{R}},\nu_{\mathcal{R}})$ defined by:*
$\mu_{\mathcal{R}}(a,b)=\inf\limits_{i\in I}\{\mu_{\mathcal{R}_i}(a,b)\}$ and $\nu_{\mathcal{R}}(a,b)=\sup\limits_{i\in I}\{\nu_{\mathcal{R}_i}(a,b)\}$ for every $(a,b)\in\mathcal{X}\times\mathcal{X}$ is an (α,β)-intuitionistic fuzzy order relation on $\mathcal{X}$.

Proof. Let $(\mathcal{R}_i)_{i\in I}$ be a family of (α,β)-intuitionistic fuzzy order relations defined on $\mathcal{X}$.

1) Since for every $a\in\mathcal{X}$, we have

$$\mu_{\mathcal{R}_i}(a,a)=\alpha \text{ and } \nu_{\mathcal{R}_i}(a,a)=\beta \text{ for every } i\in I, \tag{3.1}$$

then we get

$$\mu_{\mathcal{R}}(a,a)=\alpha \text{ and } \nu_{\mathcal{R}}(a,a)=\beta. \tag{3.2}$$

Therefore, $\mathcal{R}$ is (α,β)-intuitionistic fuzzy reflexive.

2) Let $(a,b)\in\mathcal{X}\times\mathcal{X}$ such that $a\neq b$ and let $i_0\in I$. Then, we have

$$\mu_{\mathcal{R}}(a,b)+\mu_{\mathcal{R}}(b,a)\leq \mu_{\mathcal{R}_{i_0}}(a,b)+\mu_{\mathcal{R}_{i_0}}(b,a) \tag{3.3}$$

and

$$\nu_{\mathcal{R}}(a,b)+\nu_{\mathcal{R}}(b,a)\geq \nu_{\mathcal{R}_{i_0}}(a,b)+\nu_{\mathcal{R}_{i_0}}(b,a). \tag{3.4}$$

As $\mathcal{R}_{i_0}$ is (α,β)-intuitionistic fuzzy antisymmetric, then we obtain

$$\mu_{\mathcal{R}_{i_0}}(a,b)+\mu_{\mathcal{R}_{i_0}}(b,a)\leq\alpha \text{ or } \nu_{\mathcal{R}_{i_0}}(a,b)+\nu_{\mathcal{R}_{i_0}}(b,a)\geq\beta+1. \tag{3.5}$$

Thus, we have

$$\mu_{\mathcal{R}}(a,b)+\mu_{\mathcal{R}}(b,a)\leq\alpha \text{ or } \nu_{\mathcal{R}}(a,b)+\nu_{\mathcal{R}}(b,a)\geq\beta+1. \tag{3.6}$$

Therefore, $\mathcal{R}$ is (α,β)-intuitionistic fuzzy antisymmetric.

3) Let $a,b,c\in\mathcal{X}$ and let $i\in I$. So, by our definition of $\mathcal{R}$ we have

$$\begin{cases}\min\{\mu_{\mathcal{R}}(a,b),\mu_{\mathcal{R}}(b,c)\}\leq\mu_{\mathcal{R}}(a,b)\leq\mu_{\mathcal{R}_i}(a,b)\\ \min\{\mu_{\mathcal{R}}(a,b),\mu_{\mathcal{R}}(b,c)\}\leq\mu_{\mathcal{R}}(b,c)\leq\mu_{\mathcal{R}_i}(b,c)\end{cases}\tag{3.7}$$

and

$$\begin{cases}\max\{\nu_{\mathcal{R}}(a,b),\nu_{\mathcal{R}}(b,c)\}\geq\nu_{\mathcal{R}}(a,b)\geq\nu_{\mathcal{R}_i}(a,b)\\ \max\{\nu_{\mathcal{R}}(a,b),\nu_{\mathcal{R}}(b,c)\}\geq\nu_{\mathcal{R}}(b,c)\geq\nu_{\mathcal{R}_i}(b,c).\end{cases}\tag{3.8}$$

Hence, for every $b\in\mathcal{X}$ and $i\in I$ we obtain

$$\begin{cases}\min\{\mu_{\mathcal{R}}(a,b),\mu_{\mathcal{R}}(b,c)\}\leq\min\{\mu_{\mathcal{R}_i}(a,b),\mu_{\mathcal{R}_i}(b,c)\}\\ \max\{\nu_{\mathcal{R}}(a,b),\nu_{\mathcal{R}}(b,c)\}\geq\max\{\nu_{\mathcal{R}_i}(a,b),\nu_{\mathcal{R}_i}(b,c)\}.\end{cases}\tag{3.9}$$

Since for every $i\in I$, $\mathcal{R}_i$ is (α,β)-intuitionistic fuzzy transitive, so we obtain

$$\begin{cases}\min\{\mu_{\mathcal{R}}(a,b),\mu_{\mathcal{R}}(b,c)\}\leq\mu_{\mathcal{R}_i}(a,c) & \forall b\in\mathcal{X}\text{ and }\forall i\in I\\ \max\{\nu_{\mathcal{R}}(a,b),\nu_{\mathcal{R}}(b,c)\}\geq\nu_{\mathcal{R}_i}(a,c) & \forall b\in\mathcal{X}\text{ and }\forall i\in I.\end{cases}\tag{3.10}$$

So, for every $i\in I$

$$\mu_{\mathcal{R}_i}(a,c)\text{ is an upper bound of }\Big\{\min\{\mu_{\mathcal{R}}(b,c),\mu_{\mathcal{R}}(a,b)\}:b\in\mathcal{X}\Big\}\tag{3.11}$$

and

$$\nu_{\mathcal{R}_i}(a,c)\text{ is a lower bound of }\Big\{\max\{\nu_{\mathcal{R}}(b,c),\nu_{\mathcal{R}}(a,b)\}:b\in\mathcal{X}\Big\}.\tag{3.12}$$

Then, we get

$$\begin{cases}\sup_{b\in\mathcal{X}}\Big\{\min\{\mu_{\mathcal{R}}(a,b),\mu_{\mathcal{R}}(b,c)\}\Big\}\leq\mu_{\mathcal{R}_i}(a,c) & \forall i\in I\\ \inf_{b\in\mathcal{X}}\Big\{\max\{\nu_{\mathcal{R}}(a,b),\nu_{\mathcal{R}}(b,c)\}\Big\}\geq\nu_{\mathcal{R}_i}(a,c) & \forall i\in I.\end{cases}\tag{3.13}$$

Hence, we have

$$\sup_{b\in\mathcal{X}}\Big\{\min\{\mu_{\mathcal{R}}(a,b),\mu_{\mathcal{R}}(b,c)\}\Big\}\text{ is a lower bound of }\Big\{\mu_{\mathcal{R}_i}(a,c):i\in I\Big\}\tag{3.14}$$

and

$$\inf_{b\in\mathcal{X}}\Big\{\max\{\nu_{\mathcal{R}}(a,b),\nu_{\mathcal{R}}(b,c)\}\Big\}\text{ is an upper bound of }\Big\{\nu_{\mathcal{R}_i}(a,c):i\in I\Big\}.\tag{3.15}$$

Then, we obtain

$$\sup_{b\in\mathcal{X}}\Big\{\min\{\mu_{\mathcal{R}}(a,b),\mu_{\mathcal{R}}(b,c)\}\Big\}\leq\mu_{\mathcal{R}}(a,c)\tag{3.16}$$

and

$$\inf_{b\in\mathcal{X}}\Big\{\max\{\nu_{\mathcal{R}}(a,b),\nu_{\mathcal{R}}(b,c)\}\Big\}\geq\nu_{\mathcal{R}}(a,c).\tag{3.17}$$

Then, $\mathcal{R}$ is (α,β)-intuitionistic fuzzy transitive. Hence, $\mathcal{R}$ is an (α,β)-intuitionistic fuzzy order relation on $\mathcal{X}$. $\square$

By using Theorem 3.1, we obtain the following consequence.

Corollary 3.2. *Let $(\mathcal{X}, \mathcal{R})$ be a nonempty (α, β)-intuitionistic fuzzy ordered set. Then, the fuzzy relation $\mathcal{V}$ defined on $\mathcal{X}$ by:*

$$\mu_{\mathcal{V}}(a,b) = \min\{\mu_{\mathcal{R}}(a,b), \mu_{\mathcal{R}}(b,a)\} \text{ and } \nu_{\mathcal{V}}(a,b) = \max\{\nu_{\mathcal{R}}(a,b), \nu_{\mathcal{R}}(b,a)\}$$

for every $(a,b) \in \mathcal{X} \times \mathcal{X}$ is an (α, β)-intuitionistic fuzzy order relation on $\mathcal{X}$.

Using a proof similar to that of Theorem 3.1, we obtain the following result.

Theorem 3.3. *Let $(\mathcal{X}_i, \mathcal{R}_i)_{i \in \{1,\dots,n\}}$ be a finite family of nonempty (α, β)-intuitionistic fuzzy ordered sets. Then the intuitionistic fuzzy relation $\mathcal{V} = (\mu_{\mathcal{V}}, \nu_{\mathcal{V}})$ defined on $\mathcal{X} = \prod_{i=1}^{n} \mathcal{X}_i$ by:*

$$\mu_{\mathcal{V}}((a_1,\dots,a_n),(b_1,\dots,b_n)) = \min_{i \in \{1,\dots,n\}} \{\mu_{\mathcal{R}_i}(a_i, b_i)\}$$

and

$$\nu_{\mathcal{V}}((a_1,\dots,a_n),(b_1,\dots,b_n)) = \max_{i \in \{1,\dots,n\}} \{\nu_{\mathcal{R}_i}(a_i, b_i)\}$$

is an (α, β)-intuitionistic fuzzy order relation on $\mathcal{X}$.

From Theorem 3.3, we obtain the following corollary.

Corollary 3.4. *Let $(X, \mathcal{R})$ be a nonempty (α, β)-intuitionistic fuzzy ordered set. Then, the intuitionistic fuzzy relation $\mathcal{V} = (\mu_{\mathcal{V}}, \nu_{\mathcal{V}})$ defined on $\mathcal{X}^n$ by:*

$$\mu_{\mathcal{V}}((a_1,\dots,a_n),(b_1,\dots,b_n)) = \min_{i \in \{1,\dots,n\}} \{\mu_{\mathcal{R}}(a_i, b_i)\}$$

and

$$\nu_{\mathcal{V}}((a_1,\dots,a_n),(b_1,\dots,b_n)) = \max_{i \in \{1,\dots,n\}} \{\nu_{\mathcal{R}}(a_i, b_i)\}$$

is an (α, β)-intuitionistic fuzzy order relation on $\mathcal{X}^n$.

4. Compatibility of (α, β)-intuitionistic fuzzy order relations with the algebraic structures of vector spaces

In this section, we study the compatibility of (α, β)-intuitionistic fuzzy order relations with the algebraic structures of vector spaces. We begin with the following definition.

Definition 4.1. Let $\mathcal{R}$ be an (α, β)-intuitionistic fuzzy order relation defined on a real vector space $\mathcal{X}$.

1) $\mathcal{R}$ is said to be compatible with the addition on $\mathcal{X}$ if for every $(a_1, b_1), (a_2, b_2) \in \mathcal{X}^2$, we have:

$$\left(\left(\mu_{\mathcal{R}}(a_1, b_1) > \frac{\alpha}{2} \text{ and } \mu_{\mathcal{R}}(a_2, b_2) > \frac{\alpha}{2} \right) \Rightarrow \left(\mu_{\mathcal{R}}(a_1 + a_2, b_1 + b_2) > \frac{\alpha}{2} \right) \right) \quad (4.1)$$

and

$$\left(\left(\nu_{\mathcal{R}}(a_1,b_1) < \frac{\beta+1}{2} \text{ and } \nu_{\mathcal{R}}(a_2,b_2) < \frac{\beta+1}{2}\right) \Rightarrow \left(\nu_{\mathcal{R}}(a_1+a_2,b_1+b_2) < \frac{\beta+1}{2}\right)\right). \tag{4.2}$$

2) $\mathcal{R}$ is said to be compatible with the multiplication by scalars on $\mathcal{X}$ if for all $(a,b) \in \mathcal{X}\times\mathcal{X}$ and $\lambda > 0$, we have:

$$\left(\mu_{\mathcal{R}}(a,b) > \frac{\alpha}{2} \Rightarrow \mu_{\mathcal{R}}(\lambda a,\lambda b) > \frac{\alpha}{2}\right) \text{ and } \left(\nu_{\mathcal{R}}(a,b) < \frac{\beta+1}{2} \Rightarrow \nu_{\mathcal{R}}(\lambda a,\lambda b) < \frac{\beta+1}{2}\right). \tag{4.3}$$

Next, we shall prove the following result.

Theorem 4.2. *Let $(\mathcal{X}_i,\mathcal{R}_i)_{i\in\{1,\dots,n\}}$ be a finite family of (α,β)-intuitionistic fuzzy ordered real vector spaces and let $\mathcal{V}=(\mu_{\mathcal{V}},\nu_{\mathcal{V}})$ be the intuitionistic fuzzy relation defined on $\mathcal{X}=\prod_{i=1}^{n}\mathcal{X}_i$ by*

$$\mu_{\mathcal{V}}((a_1,...,a_n),(b_1,...,b_n)) = \min_{i\in\{1,\dots,n\}}\{\mu_{\mathcal{R}_i}(a_i,b_i)\} \text{ and}$$
$$\nu_{\mathcal{V}}((a_1,...,a_n),(b_1,...,b_n)) = \max_{i\in\{1,\dots,n\}}\{\nu_{\mathcal{R}_i}(a_i,b_i)\}.$$

Then, we have

(i) *if for every $i\in\{1,...,n\}$, $\mathcal{R}_i$ is compatible with the addition on $\mathcal{X}_i$, then $\mathcal{V}$ is an (α,β)-intuitionistic fuzzy order relation compatible with the addition on $\mathcal{X}$;*

(ii) *if for each $i\in\{1,...,n\}$, $\mathcal{R}_i$ is compatible with the multiplication by scalars on $\mathcal{X}_i$, then $\mathcal{V}$ is an (α,β)-intuitionistic fuzzy order relation compatible with the multiplication by scalars on $\mathcal{X}$.*

(iii) *if for every $i\in\{1,...,n\}$, $\mathcal{R}_i$ is compatible with the addition and the multiplication by scalars on $\mathcal{X}_i$, then $\mathcal{V}$ is an (α,β)-intuitionistic fuzzy order relation compatible with the addition and the multiplication by scalars on $\mathcal{X}$.*

Proof. Let $(\mathcal{X}_i,\mathcal{R}_i)_{i\in\{1,\dots,n\}}$ be a finite family of nonempty (α,β)-intuitionistic fuzzy ordered real vector spaces and let $\mathcal{V}=(\mu_{\mathcal{V}},\nu_{\mathcal{V}})$ be the intuitionistic fuzzy relation defined on $\mathcal{X}=\prod_{i=1}^{n}\mathcal{X}_i$ by

$$\mu_{\mathcal{V}}((a_1,...,a_n),(b_1,...,b_n)) = \min_{i\in\{1,\dots,n\}}\{\mu_{\mathcal{R}_i}(a_i,b_i)\} \tag{4.4}$$

and

$$\nu_{\mathcal{V}}((a_1,...,a_n),(b_1,...,b_n)) = \max_{i\in\{1,\dots,n\}}\{\nu_{\mathcal{R}_i}(a_i,b_i)\}. \tag{4.5}$$

From Theorem 3.3, $\mathcal{V}=(\mu_{\mathcal{V}},\nu_{\mathcal{V}})$ is an (α,β)-intuitionistic fuzzy order relation on the real vector space $\mathcal{X}$.

(i) Assume that for each $i\in\{1,...,n\}$, $\mathcal{R}_i$ is compatible with the addition on $\mathcal{X}_i$. Let $(a_1,...,a_n),(b_1,...,b_n),(c_1,...,c_n),(d_1,...,d_n)\in\mathcal{X}$ such that

$$\mu_{\mathcal{V}}((a_1,...,a_n),(b_1,...,b_n)) > \frac{\alpha}{2}, \tag{4.6}$$

$$\mu_{\mathcal{V}}((c_1, ..., c_n), (d_1, ..., d_n)) > \frac{\alpha}{2}, \tag{4.7}$$

$$\nu_{\mathcal{V}}((a_1, ..., a_n), (b_1, ..., b_n)) < \frac{\beta+1}{2} \tag{4.8}$$

and

$$\nu_{\mathcal{V}}((c_1, ..., c_n), (d_1, ..., d_n)) < \frac{\beta+1}{2}. \tag{4.9}$$

Then by using (4.6), (4.7), (4.8) and (4.9) for every $i \in \{1, ..., n\}$, we get

$$\mu_{\mathcal{R}_i}(a_i, b_i) > \frac{\alpha}{2}, \tag{4.10}$$

$$\mu_{\mathcal{R}_i}(c_i, d_i) > \frac{\alpha}{2}, \tag{4.11}$$

$$\nu_{\mathcal{R}_i}(a_i, b_i) < \frac{\beta+1}{2} \tag{4.12}$$

and

$$\nu_{\mathcal{R}_i}(c_i, d_i) < \frac{\beta+1}{2}. \tag{4.13}$$

Since for every $i \in \{1, ..., n\}$, $\mathcal{R}_i$ is compatible with the addition on $\mathcal{X}_i$, so by using (4.10), (4.11), (4.12) and (4.13) we obtain

$$\mu_{\mathcal{R}_i}(a_i + c_i, b_i + d_i) > \frac{\alpha}{2} \text{ and } \nu_{\mathcal{R}_i}(a_i + c_i, b_i + d_i) < \frac{\beta+1}{2} \text{ for all } i \in \{1, ..., n\}. \tag{4.14}$$

Hence, we get

$$\mu_{\mathcal{V}}((a_1 + c_1, ..., a_n + c_n), (b_1 + d_1, ..., b_n + d_n)) = \min_{i\in\{1,...,n\}} \{\mu_{\mathcal{R}_i}(a_i + c_i, b_i + d_i)\} > \frac{\alpha}{2} \tag{4.15}$$

and

$$\nu_{\mathcal{V}}((a_1+c_1, ..., a_n+c_n), (b_1+d_1, ..., b_n+d_n)) = \max_{i\in\{1,...,n\}} \{\nu_{\mathcal{R}_i}(a_i+c_i, b_i+d_i)\} < \frac{\beta+1}{2}. \tag{4.16}$$

Thus, $\mathcal{V}$ is compatible with the addition on $\mathcal{X}$.

(ii) Assume that for every $i \in \{1, ..., n\}$, $\mathcal{R}_i$ is compatible with the multiplication by scalars on $\mathcal{X}_i$. Let $\lambda > 0$ and $(a_1, ..., a_n), (b_1, ..., b_n)$ in $\mathcal{X}$ such that

$$\mu_{\mathcal{V}}((a_1, ..., a_n), (b_1, ..., b_n)) > \frac{\alpha}{2} \text{ and } \nu_{\mathcal{V}}((a_1, ..., a_n), (b_1, ..., b_n)) < \frac{\beta+1}{2}. \tag{4.17}$$

Then by our definition of $\mathcal{V}$, for every $i \in \{1, ..., n\}$ we get

$$\mu_{\mathcal{R}_i}(a_i, b_i) > \frac{\alpha}{2} \text{ and } \nu_{\mathcal{R}_i}(a_i, b_i) < \frac{\beta+1}{2}. \tag{4.18}$$

As for every $i \in \{1, ..., n\}$, $\mathcal{R}_i$ is compatible with the multiplication by scalars on $\mathcal{X}_i$, so from (4.18) we obtain

$$\mu_{\mathcal{R}_i}(\lambda a_i, \lambda b_i) > \frac{\alpha}{2} \text{ and } \nu_{\mathcal{R}_i}(\lambda a_i, \lambda b_i) < \frac{\beta+1}{2} \text{ for every } i \in \{1, ..., n\}. \tag{4.19}$$

Hence, we get

$$\mu_{\mathcal{V}}((\lambda a_1, ..., \lambda a_n), (\lambda b_1, ..., \lambda b_n)) = \min_{i \in \{1,...,n\}} \{\mu_{\mathcal{R}_i}(\lambda a_i, \lambda b_i)\} > \frac{\alpha}{2} \tag{4.20}$$

and

$$\nu_{\mathcal{V}}((\lambda a_1, ..., \lambda a_n), (\lambda b_1, ..., \lambda b_n)) = \max_{i \in \{1,...,n\}} \{\nu_{\mathcal{R}_i}(\lambda a_i, \lambda b_i)\} < \frac{\beta + 1}{2}. \tag{4.21}$$

Thus, $\mathcal{V}$ is compatible with the multiplication by scalars on $\mathcal{X}$.
(iii) From (i) and (ii) we deduce (iii). □

Combining Corollary 3.4 and Theorem 4.2, we get the following result.

Corollary 4.3. *Let $(\mathcal{X}, \mathcal{R})$ be an (α, β)-intuitionistic fuzzy ordered real vector space. Assume that $\mathcal{R}$ is compatible with the addition and the multiplication by scalars on $\mathcal{X}$. Then the intuitionistic fuzzy relation $\mathcal{V} = (\mu_{\mathcal{V}}, \nu_{\mathcal{V}})$ defined on $\mathcal{X}^n$ by*

$$\mu_{\mathcal{V}}((a_1, ..., a_n), (b_1, ..., b_n)) = \min_{i \in \{1,...,n\}} \{\mu_{\mathcal{R}}(a_i, b_i)\} \text{ and}$$
$$\nu_{\mathcal{V}}((a_1, ..., a_n), (b_1, ..., b_n)) = \max_{i \in \{1,...,n\}} \{\nu_{\mathcal{R}}(a_i, b_i)\}$$

is an (α, β)-intuitionistic fuzzy order relation compatible with the addition and the multiplication by scalars on $\mathcal{X}^n$.

5. A characterization of a class of total (α, β)-intuitionistic fuzzy order relations that are compatible with the usual algebraic structures on $\mathbb{R}$

In this section, we give a characterization of a class of total (α, β)-intuitionistic fuzzy order relations that are compatible with the usual algebraic structures on $\mathbb{R}$. To this end, we first need the following definitions.

Definition 5.1. Let $\mathcal{R}$ be an (α, β)-intuitionistic fuzzy order relation on $\mathbb{R}$ and $a \in \mathbb{R}$.

1) If $\mu_{\mathcal{R}}(0, a) > \frac{\alpha}{2}$ and $\nu_{\mathcal{R}}(0, a) < \frac{\beta+1}{2}$, then a is called a $\mathcal{R}$-positive real number. The set of all $\mathcal{R}$-positive real numbers is denoted by $\mathbb{R}^+_{(\alpha,\beta)}$.

2) If $\mu_{\mathcal{R}}(a, 0) > \frac{\alpha}{2}$ and $\nu_{\mathcal{R}}(a, 0) < \frac{\beta+1}{2}$, then a is called a $\mathcal{R}$-negative real number and the set of all $\mathcal{R}$-negative real numbers is denoted by $\mathbb{R}^-_{(\alpha,\beta)}$.

Next, we introduce the following two classes $\mathcal{C}_{(\alpha,\beta)}$ and $\mathcal{C}'_{(\alpha,\beta)}$ of (α, β)-intuitionistic fuzzy order relations on $\mathbb{R}$.

Definition 5.2. Let $(\alpha, \beta) \in]0, 1] \times [0, 1[$ and let $\mathcal{C}_{(\alpha,\beta)}$ be the set of all (α, β)-intuitionistic fuzzy order relations $\mathcal{R}$ defined on $\mathbb{R}$ satisfying the two following conditions:

(i) for all $a, b, c \in \mathbb{R}$, we have

$$\mu_{\mathcal{R}}(a, b) \leq \mu_{\mathcal{R}}(a + c, b + c) \text{ and } \nu_{\mathcal{R}}(a, b) \geq \nu_{\mathcal{R}}(a + c, b + c); \tag{5.1}$$

(ii) for every $a, b \in \mathbb{R}$ and $\lambda > 0$,

$$\mu_{\mathcal{R}}(a, b) \leq \mu_{\mathcal{R}}(\lambda a, \lambda b) \text{ and } \nu_{\mathcal{R}}(a, b) \geq \nu_{\mathcal{R}}(\lambda a, \lambda b). \tag{5.2}$$

Definition 5.3. Let $(\alpha, \beta) \in]0, 1] \times [0, 1[$ and let $\mathcal{C}'_{(\alpha,\beta)}$ be the set of all (α, β)-intuitionistic fuzzy order relations $\mathcal{R}$ defined on $\mathbb{R}$ satisfying the two following conditions:

(i) for every $a, b, c \in \mathbb{R}$,

$$\mu_{\mathcal{R}}(a, b) \leq \mu_{\mathcal{R}}(a + c, b + c) \text{ and } \nu_{\mathcal{R}}(a, b) \geq \nu_{\mathcal{R}}(a + c, b + c); \tag{5.3}$$

(ii) for all $a, b \in \mathbb{R}$ and $\gamma > 0$,

$$\mu_{\mathcal{R}}(a, b) \geq \mu_{\mathcal{R}}(\gamma a, \gamma b) \text{ and } \nu_{\mathcal{R}}(a, b) \leq \nu_{\mathcal{R}}(\gamma a, \gamma b). \tag{5.4}$$

Next, we shall show that for every $(\alpha, \beta) \in]0, 1] \times [0, 1[$, we have $\mathcal{C}_{(\alpha,\beta)} = \mathcal{C}'_{(\alpha,\beta)}$.

Proposition 5.4. *Let $(\alpha, \beta) \in]0, 1] \times [0, 1[$. Then, we have $\mathcal{C}_{(\alpha,\beta)} = \mathcal{C}'_{(\alpha,\beta)}$.*

Proof. 1) Suppose that $\mathcal{R} \in \mathcal{C}_{(\alpha,\beta)}$. Then from (5.1), for every $a, b, c \in \mathbb{R}$ we have

$$\mu_{\mathcal{R}}(a, b) \leq \mu_{\mathcal{R}}(a + c, b + c) \text{ and } \nu_{\mathcal{R}}(a, b) \geq \nu_{\mathcal{R}}(a + c, b + c). \tag{5.5}$$

Now, let $(a, b) \in \mathbb{R}^2$ and $\gamma > 0$. Then from (5.2), we have

$$\mu_{\mathcal{R}}(\gamma a, \gamma b) \leq \mu_{\mathcal{R}}\Big(\frac{1}{\gamma}(\gamma a), \frac{1}{\gamma}(\gamma b)\Big) \text{ and } \nu_{\mathcal{R}}(\gamma a, \gamma b) \geq \nu_{\mathcal{R}}\Big(\frac{1}{\gamma}(\gamma a), \frac{1}{\gamma}(\gamma b)\Big). \tag{5.6}$$

Hence, we get

$$\mu_{\mathcal{R}}(\gamma a, \gamma b) \leq \mu_{\mathcal{R}}(a, b) \text{ and } \nu_{\mathcal{R}}(\gamma a, \gamma b) \geq \nu_{\mathcal{R}}(a, b). \tag{5.7}$$

So, $\mathcal{R}$ satisfies (5.3) and (5.4). Therefore, $\mathcal{R}$ is an element of $\mathcal{C}'_{(\alpha,\beta)}$. Thus, we have

$$\mathcal{C}_{(\alpha,\beta)} \subseteq \mathcal{C}'_{(\alpha,\beta)}. \tag{5.8}$$

2) Suppose that $\mathcal{R}$ is an element of $\mathcal{C}'_{(\alpha,\beta)}$. Then from (5.3), for every $a, b, c \in \mathbb{R}$ we have

$$\mu_{\mathcal{R}}(a, b) \leq \mu_{\mathcal{R}}(a + c, b + c) \text{ and } \nu_{\mathcal{R}}(a, b) \geq \nu_{\mathcal{R}}(a + c, b + c). \tag{5.9}$$

Now, let $(a, b) \in \mathbb{R}^2$ and $\lambda > 0$. Then from (5.4), we obtain

$$\mu_{\mathcal{R}}(\lambda a, \lambda b) \geq \mu_{\mathcal{R}}\Big(\frac{1}{\lambda}(\lambda a), \frac{1}{\lambda}(\lambda b)\Big) \text{ and } \nu_{\mathcal{R}}(\lambda a, \lambda b) \leq \nu_{\mathcal{R}}\Big(\frac{1}{\lambda}(\lambda a), \frac{1}{\lambda}(\lambda b)\Big). \tag{5.10}$$

So, we get

$$\mu_{\mathcal{R}}(\lambda a, \lambda b) \geq \mu_{\mathcal{R}}(a, b) \text{ and } \nu_{\mathcal{R}}(\lambda a, \lambda b) \leq \nu_{\mathcal{R}}(a, b). \tag{5.11}$$

Hence, $\mathcal{R}$ satisfies (5.1) and (5.2). Thus, $\mathcal{R}$ is an element of $\mathcal{C}_{(\alpha,\beta)}$. Then,

$$\mathcal{C}'_{(\alpha,\beta)} \subseteq \mathcal{C}_{(\alpha,\beta)}. \tag{5.12}$$

Therefore from (5.8) and (5.12), we deduce that $\mathcal{C}_{(\alpha,\beta)} = \mathcal{C}'_{(\alpha,\beta)}$. □

In what follows we shall prove that every element of $\mathcal{C}_{(\alpha,\beta)}$ is compatible with the usual algebraic structures on $\mathbb{R}$.

Proposition 5.5. *Let $(\alpha, \beta) \in]0, 1] \times [0, 1[$ and $\mathcal{R}$ be an element of $\mathcal{C}_{(\alpha,\beta)}$. Then,*

(i) *for every $(a_1, b_1), (a_2, b_2) \in \mathbb{R}^2$, we have*

$$\mu_{\mathcal{R}}(a_1+a_2,b_1+b_2)\geq \min\{\mu_{\mathcal{R}}(a_1,b_1),\mu_{\mathcal{R}}(a_2,b_2)\}$$

and

$$\nu_{\mathcal{R}}(a_1+a_2,b_1+b_2)\leq \max\{\nu_{\mathcal{R}}(a_1,b_1),\nu_{\mathcal{R}}(a_2,b_2)\}.$$

(ii) $\mathcal{R}$ *is compatible with the addition and the multiplication by scalars on* $\mathbb{R}$.

Proof. Let $(\alpha,\beta)\in]0,1]\times[0,1[$ and $\mathcal{R}\in\mathcal{C}_{(\alpha,\beta)}$.
(i) Let $(a_1,b_1),(a_2,b_2)\in\mathbb{R}^2$. By (α,β)-intuitionistic fuzzy transitivity, we have

$$\begin{cases}\mu_{\mathcal{R}}(a_1+a_2,b_1+b_2)\geq \min\{\mu_{\mathcal{R}}(a_1+a_2,b_1+a_2),\mu_{\mathcal{R}}(b_1+a_2,b_1+b_2)\}\\ \nu_{\mathcal{R}}(a_1+a_2,b_1+b_2)\leq \max\{\nu_{\mathcal{R}}(a_1+a_2,b_1+a_2),\nu_{\mathcal{R}}(b_1+a_2,b_1+b_2)\}.\end{cases} \tag{5.13}$$

Since $\mathcal{R}\in\mathcal{C}_{(\alpha,\beta)}$, then from (5.1) we get

$$\begin{cases}\mu_{\mathcal{R}}(a_1,b_1)\leq \mu_{\mathcal{R}}(a_1+a_2,b_1+a_2) \text{ and } \mu_{\mathcal{R}}(a_2,b_2)\leq \mu_{\mathcal{R}}(b_1+a_2,b_1+b_2)\\ \nu_{\mathcal{R}}(a_1,b_1)\geq \nu_{\mathcal{R}}(a_1+a_2,b_1+a_2) \text{ and } \nu_{\mathcal{R}}(a_2,b_2)\geq \nu_{\mathcal{R}}(b_1+a_2,b_1+b_2).\end{cases}$$

Then, we have

$$\begin{cases}\min\{\mu_{\mathcal{R}}(a_1,b_1),\mu_{\mathcal{R}}(a_2,b_2)\}\leq \mu_{\mathcal{R}}(a_1+a_2,b_1+a_2)\\ \min\{\mu_{\mathcal{R}}(a_1,b_1),\mu_{\mathcal{R}}(a_2,b_2)\}\leq \mu_{\mathcal{R}}(b_1+a_2,b_1+b_2)\end{cases} \tag{5.14}$$

and

$$\begin{cases}\max\{\nu_{\mathcal{R}}(a_1,b_1),\nu_{\mathcal{R}}(a_2,b_2)\}\geq \nu_{\mathcal{R}}(a_1+a_2,b_1+a_2)\\ \max\{\nu_{\mathcal{R}}(a_1,b_1),\nu_{\mathcal{R}}(a_2,b_2)\}\geq \nu_{\mathcal{R}}(b_1+a_2,b_1+b_2).\end{cases} \tag{5.15}$$

Hence, we obtain

$$\begin{cases}\min\{\mu_{\mathcal{R}}(a_1+a_2,b_1+a_2),\mu_{\mathcal{R}}(b_1+a_2,b_1+b_2)\}\geq \min\{\mu_{\mathcal{R}}(a_1,b_1),\mu_{\mathcal{R}}(a_2,b_2)\}\\ \max\{\nu_{\mathcal{R}}(a_1+a_2,b_1+a_2),\nu_{\mathcal{R}}(b_1+a_2,b_1+b_2)\}\leq \max\{\nu_{\mathcal{R}}(a_1,b_1),\nu_{\mathcal{R}}(a_2,b_2)\}.\end{cases} \tag{5.16}$$

By using (5.13) and (5.16), we get for every $(a_1,b_1),(a_2,b_2)\in\mathbb{R}^2$

$$\begin{cases}\mu_{\mathcal{R}}(a_1+a_2,b_1+b_2)\geq \min\{\mu_{\mathcal{R}}(a_1,b_1),\mu_{\mathcal{R}}(a_2,b_2)\}\\ \nu_{\mathcal{R}}(a_1+a_2,b_1+b_2)\leq \max\{\nu_{\mathcal{R}}(a_1,b_1),\nu_{\mathcal{R}}(a_2,b_2)\}.\end{cases}$$

(ii) Let $(a_1,b_1),(a_2,b_2)\in\mathbb{R}^2$, such that

$$\begin{cases}\mu_{\mathcal{R}}(a_1,b_1)>\dfrac{\alpha}{2} \text{ and } \mu_{\mathcal{R}}(a_2,b_2)>\dfrac{\alpha}{2}\\ \nu_{\mathcal{R}}(a_1,b_1)<\dfrac{\beta+1}{2} \text{ and } \nu_{\mathcal{R}}(a_2,b_2)<\dfrac{\beta+1}{2}.\end{cases}$$

Then, we get

$$\begin{cases}\min\{\mu_{\mathcal{R}}(a_2,b_2),\mu_{\mathcal{R}}(a_1,b_1)\}>\dfrac{\alpha}{2}\\ \max\{\nu_{\mathcal{R}}(a_1,b_1),\nu_{\mathcal{R}}(a_2,b_2)\}<\dfrac{\beta+1}{2}.\end{cases} \tag{5.17}$$

By (i) above, we know that

$$\begin{cases}\mu_{\mathcal{R}}(a_1+a_2,b_1+b_2)\geq \min\{\mu_{\mathcal{R}}(a_2,b_2),\mu_{\mathcal{R}}(a_1,b_1)\}\\ \nu_{\mathcal{R}}(a_1+a_2,b_1+b_2)\leq \max\{\nu_{\mathcal{R}}(a_1,b_1),\nu_{\mathcal{R}}(a_2,b_2)\}.\end{cases} \tag{5.18}$$

So by using (5.17) and (5.18), we get

$$\mu_{\mathcal{R}}(a_1+a_2,b_1+b_2) > \frac{\alpha}{2} \text{ and } \nu_{\mathcal{R}}(a_1+a_2,b_1+b_2) < \frac{\beta+1}{2}. \tag{5.19}$$

Thus, $\mathcal{R}$ is compatible with the addition on $\mathbb{R}$.

Now, let $\lambda > 0$ and $a,b\in\mathbb{R}$ such that

$$\mu_{\mathcal{R}}(a,b) > \frac{\alpha}{2} \text{ and } \nu_{\mathcal{R}}(a,b) < \frac{\beta+1}{2}. \tag{5.20}$$

As $\mathcal{R}\in\mathcal{C}_{(\alpha,\beta)}$, then from (5.2) we have

$$\mu_{\mathcal{R}}(a,b) \le \mu_{\mathcal{R}}(\lambda a,\lambda b) \text{ and } \nu_{\mathcal{R}}(a,b) \ge \nu_{\mathcal{R}}(\lambda a,\lambda b). \tag{5.21}$$

Hence, we get

$$\mu_{\mathcal{R}}(\lambda a,\lambda b) > \frac{\alpha}{2} \text{ and } \nu_{\mathcal{R}}(\lambda a,\lambda b) < \frac{\beta+1}{2}. \tag{5.22}$$

Therefore, $\mathcal{R}$ is compatible with the addition and the multiplication by scalars on $\mathbb{R}$. □

Next, we shall establish some technical results for every element of $\mathcal{C}_{(\alpha,\beta)}$.

Proposition 5.6. *Let $(\alpha,\beta)\in]0,1]\times[0,1[$ and $\mathcal{R}\in\mathcal{C}_{(\alpha,\beta)}$. Then,*
(i) *for all $a,b\in\mathbb{R}$, we have*
$\mu_{\mathcal{R}}(a,b)=\mu_{\mathcal{R}}(0,b-a)=\mu_{\mathcal{R}}(a-b,0)$ and $\nu_{\mathcal{R}}(a,b)=\nu_{\mathcal{R}}(0,b-a)=\nu_{\mathcal{R}}(a-b,0)$;
(ii) *for any $a\in]0,+\infty[$, we have $\mu_{\mathcal{R}}(0,a)=\mu_{\mathcal{R}}(0,1)$ and $\nu_{\mathcal{R}}(0,a)=\nu_{\mathcal{R}}(0,1)$;*
(iii) *for every $a\in]0,+\infty[$, we have $\mu_{\mathcal{R}}(a,0)=\mu_{\mathcal{R}}(1,0)$ and $\nu_{\mathcal{R}}(a,0)=\nu_{\mathcal{R}}(1,0)$;*
(iv) *for any $a,b\in\mathbb{R}$, if $a<b$, then $\mu_{\mathcal{R}}(a,b)=\mu_{\mathcal{R}}(0,1)$ and $\nu_{\mathcal{R}}(a,b)=\nu_{\mathcal{R}}(0,1)$;*
(v) *for all $a,b\in\mathbb{R}$, if $a>b$, then $\mu_{\mathcal{R}}(a,b)=\mu_{\mathcal{R}}(1,0)$ and $\nu_{\mathcal{R}}(a,b)=\nu_{\mathcal{R}}(1,0)$;*
(vi) $\mathbb{R}^+_{(\alpha,\beta)}\cap\mathbb{R}^-_{(\alpha,\beta)}=\{0\}$.

Proof. Let $\mathcal{R}\in\mathcal{C}_{(\alpha,\beta)}$.
(i) Let $a,b\in\mathbb{R}$. Since $\mathcal{R}\in\mathcal{C}_{(\alpha,\beta)}$, then from (5.1) we have

$$\mu_{\mathcal{R}}(a-b,0) \le \mu_{\mathcal{R}}\Big(a-b+b,0+b\Big) \le \mu_{\mathcal{R}}\Big(a+(-a),b+(-a)\Big) \tag{5.23}$$

and

$$\nu_{\mathcal{R}}(a-b,0) \ge \nu_{\mathcal{R}}\Big(a-b+b,0+b\Big) \ge \nu_{\mathcal{R}}\Big(a+(-a),b+(-a)\Big). \tag{5.24}$$

Hence, we get

$$\mu_{\mathcal{R}}(a-b,0) \le \mu_{\mathcal{R}}(a,b) \le \mu_{\mathcal{R}}(0,b-a) \tag{5.25}$$

and

$$\nu_{\mathcal{R}}(a-b,0) \ge \nu_{\mathcal{R}}(a,b) \ge \nu_{\mathcal{R}}(0,b-a). \tag{5.26}$$

Since $\mathcal{R}\in\mathcal{C}_{(\alpha,\beta)}$, then

$$\mu_{\mathcal{R}}(0,b-a) \le \mu_{\mathcal{R}}\Big(0+a,b-a+a\Big) \le \mu_{\mathcal{R}}\Big(a+(-b),b+(-b)\Big) \tag{5.27}$$

and

$$\nu_{\mathcal{R}}(0,b-a) \ge \nu_{\mathcal{R}}\Big(0+a,b-a+a\Big) \ge \nu_{\mathcal{R}}\Big(a+(-b),b+(-b)\Big). \tag{5.28}$$

Then, we get

$$\mu_{\mathcal{R}}(0,b-a) \leq \mu_{\mathcal{R}}(a,b) \leq \mu_{\mathcal{R}}(a-b,0) \tag{5.29}$$

and

$$\nu_{\mathcal{R}}(0,b-a) \geq \nu_{\mathcal{R}}(a,b) \geq \nu_{\mathcal{R}}(a-b,0). \tag{5.30}$$

By combining (5.25) and (5.29) we obtain

$$\mu_{\mathcal{R}}(0,b-a) = \mu_{\mathcal{R}}(a,b) = \mu_{\mathcal{R}}(a-b,0).$$

Using (5.26) and (5.30), we get

$$\nu_{\mathcal{R}}(0,b-a) = \nu_{\mathcal{R}}(a,b) = \nu_{\mathcal{R}}(a-b,0).$$

(ii) Let $a \in]0,+\infty[$. As $\mathcal{R} \in \mathcal{C}_{(\alpha,\beta)}$, then from (5.2) we have

$$\mu_{\mathcal{R}}(0,1) \leq \mu_{\mathcal{R}}\Big(a \times 0, a \times 1\Big) \text{ and } \nu_{\mathcal{R}}(0,1) \geq \nu_{\mathcal{R}}\Big(a \times 0, a \times 1\Big). \tag{5.31}$$

So, we get

$$\mu_{\mathcal{R}}(0,1) \leq \mu_{\mathcal{R}}(0,a) \text{ and } \nu_{\mathcal{R}}(0,1) \geq \nu_{\mathcal{R}}(0,a). \tag{5.32}$$

As $\mathcal{R} \in \mathcal{C}_{(\alpha,\beta)}$, then

$$\mu_{\mathcal{R}}(0,a) \leq \mu_{\mathcal{R}}\Big(\frac{1}{a} \times 0, \frac{1}{a} \times a) \text{ and } \nu_{\mathcal{R}}(0,a) \geq \nu_{\mathcal{R}}\Big(\frac{1}{a} \times 0, \frac{1}{a} \times a\Big).$$

Hence, we obtain

$$\mu_{\mathcal{R}}(0,a) \leq \mu_{\mathcal{R}}(0,1) \text{ and } \nu_{\mathcal{R}}(0,a) \geq \nu_{\mathcal{R}}(0,1). \tag{5.33}$$

Therefore, by using (5.32) and (5.33), we get

$$\mu_{\mathcal{R}}(0,a) = \mu_{\mathcal{R}}(0,1) \text{ and } \nu_{\mathcal{R}}(0,a) = \nu_{\mathcal{R}}(0,1). \tag{5.34}$$

(iii) Let $a \in]0,+\infty[$. As $\mathcal{R} \in \mathcal{C}_{(\alpha,\beta)}$, then from (5.2) we get

$$\mu_{\mathcal{R}}(1,0) \leq \mu_{\mathcal{R}}(a \times 1, a \times 0) \ \text{ and } \nu_{\mathcal{R}}(1,0) \geq \nu_{\mathcal{R}}\Big(a \times 1, a \times 0\Big).$$

So, we obtain

$$\mu_{\mathcal{R}}(1,0) \leq \mu_{\mathcal{R}}(a,0) \text{ and } \nu_{\mathcal{R}}(1,0) \geq \nu_{\mathcal{R}}(a,0). \tag{5.35}$$

On the other hand, we have

$$\mu_{\mathcal{R}}(a,0) \leq \mu_{\mathcal{R}}\Big(\frac{1}{a} \times a, \frac{1}{a} \times 0\Big) \text{ and } \nu_{\mathcal{R}}(a,0) \geq \nu_{\mathcal{R}}\Big(\frac{1}{a} \times a, \frac{1}{a} \times 0\Big).$$

Hence, we get

$$\mu_{\mathcal{R}}(a,0) \leq \mu_{\mathcal{R}}(1,0) \text{ and } \nu_{\mathcal{R}}(a,0) \geq \nu_{\mathcal{R}}(1,0). \tag{5.36}$$

Therefore by using (5.35) and (5.36), for all $a \in]0,+\infty[$ we get

$$\mu_{\mathcal{R}}(a,0) = \mu_{\mathcal{R}}(1,0) \text{ and } \nu_{\mathcal{R}}(a,0) = \nu_{\mathcal{R}}(1,0). \tag{5.37}$$

(iv) Let $a, b \in \mathbb{R}$ such that $a < b$. Then, $0 < b-a$. By (i) and (ii), we get

$$\mu_{\mathcal{R}}(a,b) = \mu_{\mathcal{R}}(0,b-a) = \mu_{\mathcal{R}}(0,1) \text{ and } \nu_{\mathcal{R}}(a,b) = \nu_{\mathcal{R}}(0,b-a) = \nu_{\mathcal{R}}(0,1).$$

(v) Let $a, b \in \mathbb{R}$ such that $b < a$. So, $0 < a-b$. From (i) and (iii), we get

$$\mu_{\mathcal{R}}(a,b) = \mu_{\mathcal{R}}(a-b,0) = \mu_{\mathcal{R}}(1,0) \text{ and } \nu_{\mathcal{R}}(a,b) = \nu_{\mathcal{R}}(a-b,0) = \nu_{\mathcal{R}}(1,0).$$

(vi) Let $a \in \mathbb{R}^+_{(\alpha,\beta)} \cap \mathbb{R}^-_{(\alpha,\beta)}$. Then, we have

$$\mu_{\mathcal{R}}(0,a) > \frac{\alpha}{2}, \nu_{\mathcal{R}}(0,a) < \frac{\beta+1}{2}, \mu_{\mathcal{R}}(a,0) > \frac{\alpha}{2} \text{ and } \nu_{\mathcal{R}}(a,0) < \frac{\beta+1}{2}.$$

So, we get

$$\nu_{\mathcal{R}}(a,0) + \mu_{\mathcal{R}}(0,a) > \alpha \text{ and } \nu_{\mathcal{R}}(a,0) + \nu_{\mathcal{R}}(0,a) < \beta + 1.$$

So, by (α,β)-intuitionistic fuzzy antisymmetry we obtain $a = 0$.
So, we get

$$\mathbb{R}^+_{(\alpha,\beta)} \cap \mathbb{R}^-_{(\alpha,\beta)} \subseteq \{0\}.$$

On the other hand, we know that

$$\mu_{\mathcal{R}}(0,0) = \alpha > \frac{\alpha}{2} \text{ and } \nu_{\mathcal{R}}(0,0) = \beta < \frac{\beta+1}{2}.$$

Then, we obtain

$$\{0\} \subseteq \mathbb{R}^+_{(\alpha,\beta)} \cap \mathbb{R}^-_{(\alpha,\beta)}.$$

Therefore, we deduce that

$$\mathbb{R}^+_{(\alpha,\beta)} \cap \mathbb{R}^-_{(\alpha,\beta)} = \{0\}.$$

□

Next, we give the following result.

Proposition 5.7. *Let $(\alpha,\beta) \in]0,1] \times [0,1[$ and let $\mathcal{R}$ be an element of $\mathcal{C}_{(\alpha,\beta)}$. Then,*
(i) *if $\mathcal{R}$ is a total (α,β)-intuitionistic fuzzy order on $\mathbb{R}$, then $\mathbb{R} = \mathbb{R}^+_{(\alpha,\beta)} \cup \mathbb{R}^-_{(\alpha,\beta)}$;*
(ii) *$\mathcal{R}$ is total on $\mathbb{R}$ if and only if*

$$\Big(\mu_{\mathcal{R}}(0,1) > \frac{\alpha}{2} \textit{ and } \nu_{\mathcal{R}}(0,1) < \frac{\beta+1}{2}\Big) \textit{ or } \Big(\mu_{\mathcal{R}}(1,0) > \frac{\alpha}{2} \textit{ and } \nu_{\mathcal{R}}(1,0) < \frac{\beta+1}{2}\Big).$$

Proof. Let $\mathcal{R} \in \mathcal{C}_{(\alpha,\beta)}$.
(i) Assume that $\mathcal{R}$ is a total (α,β)-intuitionistic fuzzy order relation on $\mathbb{R}$. So, for every $a \in \mathbb{R}$, we have

$$\Big(\mu_{\mathcal{R}}(0,a) > \frac{\alpha}{2} \text{ and } \nu_{\mathcal{R}}(0,a) < \frac{\beta+1}{2}\Big) \text{ or } \Big(\mu_{\mathcal{R}}(a,0) > \frac{\alpha}{2} \text{ and } \nu_{\mathcal{R}}(a,0) < \frac{\beta+1}{2}\Big).$$

Hence, we get $a \in \mathbb{R}^+_{(\alpha,\beta)}$ or $a \in \mathbb{R}^-_{(\alpha,\beta)}$. Thus, we get $\mathbb{R} \subseteq \mathbb{R}^+_{(\alpha,\beta)} \cup \mathbb{R}^-_{(\alpha,\beta)}$. On the other hand by our definition, we know that $\mathbb{R}^+_{(\alpha,\beta)} \cup \mathbb{R}^-_{(\alpha,\beta)} \subseteq \mathbb{R}$. Therefore, we deduce that $\mathbb{R} = \mathbb{R}^+_{(\alpha,\beta)} \cup \mathbb{R}^-_{(\alpha,\beta)}$.
(ii) Assume that $\mathcal{R}$ is a total (α,β)-intuitionistic fuzzy order relation defined on $\mathbb{R}$. So, we have

$$\Big(\mu_{\mathcal{R}}(0,1) > \frac{\alpha}{2} \text{ and } \nu_{\mathcal{R}}(0,1) < \frac{\beta+1}{2}\Big) \text{ or } \Big(\mu_{\mathcal{R}}(1,0) > \frac{\alpha}{2} \text{ and } \nu_{\mathcal{R}}(1,0) < \frac{\beta+1}{2}\Big).$$

Conversely, let $a, b \in \mathbb{R}$ with $a \neq b$. So, $a < b$ or $a > b$.
First case : $\mu_{\mathcal{R}}(0,1) > \frac{\alpha}{2}$ and $\nu_{\mathcal{R}}(0,1) < \frac{\beta+1}{2}$.
If $a < b$. Then by Proposition 5.6, we get

$$\mu_{\mathcal{R}}(a,b) = \mu_{\mathcal{R}}(0,1) > \frac{\alpha}{2} \text{ and } \nu_{\mathcal{R}}(a,b) = \nu_{\mathcal{R}}(0,1) < \frac{\beta+1}{2}.$$

If $a > b$, then by Proposition 5.6, we have

$$\mu_{\mathcal{R}}(b,a) = \mu_{\mathcal{R}}(0,1) > \frac{\alpha}{2} \text{ and } \nu_{\mathcal{R}}(b,a) = \nu_{\mathcal{R}}(0,1) < \frac{\beta+1}{2}.$$

Therefore, $\mathcal{R}$ is a total (α,β)-intuitionistic fuzzy order relation defined on $\mathbb{R}$.
The proof of the second case when $\mu_{\mathcal{R}}(1,0) > \frac{\alpha}{2}$ and $\nu_{\mathcal{R}}(1,0) < \frac{\beta+1}{2}$ is similar to the first case above. □

Using Propositions 5.5, 5.6 and 5.7, we obtain the main result in this section.

Theorem 5.8. *Let $(\alpha,\beta) \in]0,1] \times [0,1[$ and let $\mathcal{R} = (\mu_{\mathcal{R}}, \nu_{\mathcal{R}})$ be an element of $\mathcal{C}_{(\alpha,\beta)}$. Then, $\mathcal{R}$ is defined by*

$$\mu_{\mathcal{R}}(a,b) = \begin{cases} \alpha & \text{if } a = b \\ \mu_{\mathcal{R}}(0,1) & \text{if } a < b \\ \mu_{\mathcal{R}}(1,0) & \text{if } a > b \end{cases} \quad \text{and} \quad \nu_{\mathcal{R}}(a,b) = \begin{cases} \beta & \text{if } a = b \\ \nu_{\mathcal{R}}(0,1) & \text{if } a < b \\ \nu_{\mathcal{R}}(1,0) & \text{if } a > b. \end{cases}$$

Moreover, if $\left(\mu_{\mathcal{R}}(0,1) > \frac{\alpha}{2} \text{ and } \nu_{\mathcal{R}}(0,1) < \frac{\beta+1}{2}\right)$ or $\left(\mu_{\mathcal{R}}(1,0) > \frac{\alpha}{2} \text{ and } \nu_{\mathcal{R}}(1,0) < \frac{\beta+1}{2}\right)$, then $\mathcal{R}$ is a total (α,β)-intuitionistic fuzzy order relation on $\mathbb{R}$.

6. Conclusion

In conclusion, this study establishes key properties of (α,β)-intuitionistic fuzzy order relations defined on real vector spaces. Notably, we characterize a class of total (α,β)-intuitionistic fuzzy order relations compatible with the addition and the multiplication on the real line $\mathbb{R}$.

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